

Interaction Picture & Dyson Series

Interacting theories are those where the Lagrangian density / Hamiltonian density often contains two parts. A Solvable part which describes the free particles of the canonically quantizable theory, hence it is called the free part. And then there is an interacting part, which makes the theory unsolvable, it describes the interactions between particles including their creation/annihilation processes.

Generally the free / canonically quantized Hamiltonian is time independent, we denote it by $\hat{H}_0$. Then the interacting part is given by $\hat{H}'$ which is usually time dependent & makes the total $\hat{H}$ unsolvable, where,

$$\hat{H}_S = \hat{H}_0 + \hat{H}'_S$$

Subscript 'S' refers to Schrodinger picture.

We shall define the interaction picture ket by $|\psi_I(t)\rangle = e^{i\hat{H}_0 t} |\psi_S(t)\rangle$

$$\rightarrow i \frac{\partial}{\partial t} |\psi_I(t)\rangle = e^{i\hat{H}_0 t} \left[-\hat{H}_0 + i \frac{\partial}{\partial t} \right] |\psi_S(t)\rangle$$

But $\hat{H} |\psi_S(t)\rangle = i \frac{\partial}{\partial t} |\psi_S(t)\rangle$ is the time evolution action for ket in the Schrodinger Picture. Thus,

$$\begin{aligned} i \frac{\partial}{\partial t} |\psi_I(t)\rangle &= e^{i\hat{H}_0 t} [-\hat{H}_0 + \hat{H}_S] |\psi_S(t)\rangle \\ &= e^{i\hat{H}_0 t} \hat{H}'_S |\psi_S(t)\rangle \\ &= e^{i\hat{H}_0 t} \hat{H}'_S e^{-i\hat{H}_0 t} |\psi_I(t)\rangle \end{aligned}$$

Defining $\hat{H}'_I(t) = e^{i\hat{H}_0 t} \hat{H}'_S e^{-i\hat{H}_0 t}$ as the time evolution action of $\hat{H}'$ in the interaction picture, we get

$$\underline{i \frac{\partial}{\partial t} |\psi_I(t)\rangle = \hat{H}'_I(t) |\psi_I(t)\rangle}$$

Thus a ket in interaction picture evolves by the action of interacting hamiltonian.

Now similarly how $\hat{H}'_I$ evolves by action of only the $\hat{H}_0 \rightarrow$ solvable part, we shall define all time independent operators in the Schrodinger picture to evolve in the interaction picture by the action of $\hat{H}_0$.

$$\rightarrow \underline{\hat{O}_I(t) = e^{i\hat{H}_0 t} \hat{O}_S e^{-i\hat{H}_0 t}}$$

$$\Rightarrow i \frac{\partial}{\partial t} \hat{O}_I(t) = -\hat{H}_0 e^{i\hat{H}_0 t} \hat{O}_S e^{-i\hat{H}_0 t} + e^{i\hat{H}_0 t} \hat{O}_S e^{-i\hat{H}_0 t} \hat{H}_0$$

$$\Rightarrow = -\hat{H}_0 \hat{O}_I(t) + \hat{O}_I(t) \hat{H}_0$$

$$\Rightarrow \underline{i \frac{\partial}{\partial t} \hat{O}_I(t) = [\hat{O}_I(t), \hat{H}_0]}$$

Thus in the interaction picture, both states & operators evolve in time.

All operators in interaction picture evolve by the free part of the Hamiltonian,

$$\hat{\mathcal{O}}_I(t) = e^{i\hat{H}_0 t} \hat{\mathcal{O}}_S e^{-i\hat{H}_0 t}$$

$$\& i \frac{\partial}{\partial t} \hat{\mathcal{O}}_I(t) = [\hat{\mathcal{O}}_I(t), \hat{H}_0]$$

Then, the state kets evolve by the interaction part of Hamiltonian written in interaction picture.

$$\text{i.e. } i \frac{\partial}{\partial t} |\psi_I(t)\rangle = \hat{H}'_I(t) |\psi_I(t)\rangle$$

$$\text{where } \hat{H}'_I(t) = e^{i\hat{H}_0 t} \hat{H}'_S e^{-i\hat{H}_0 t}$$

Now we are equipped to get the expansion of Time evolution operator Perturbatively.

Expansion of Time evolution Operator

The time evolution operator in the interaction picture is defined by,

$$|\psi_I(t, t_0)\rangle = \hat{U}_I(t, t_0) |\psi_I(t_0, t_0)\rangle$$

$$\text{But } i \frac{\partial}{\partial t} |\psi_I(t, t_0)\rangle = \hat{H}'_I(t) |\psi_I(t, t_0)\rangle$$

$$\rightarrow i \frac{\partial}{\partial t} \hat{U}_I(t, t_0) = \hat{H}'_I(t) \hat{U}_I(t, t_0)$$

So we need to solve this differential eqⁿ subject to the initial condition,

$$\hat{U}_I(t_0, t_0) = \hat{I}$$

We can directly integrate the differential equation subject to above initial condition

$$\rightarrow \hat{U}_I(t, t_0) = \hat{I} - i \int_{t_0}^t dt' \hat{H}'_I(t') \hat{U}_I(t', t_0)$$

Until now the expression is exact.

We can solve the above eqⁿ iteratively

$$\begin{aligned}\hat{U}_I(t, t_0) &= \hat{\mathbb{I}} - i \int_{t_0}^t dt_1 \hat{H}'_I(t_1) \hat{U}_I(t_1, t_0) \\&= \hat{\mathbb{I}} - i \int_{t_0}^t dt_1 \hat{H}'_I(t_1) \left[\hat{\mathbb{I}} - i \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_2) \hat{U}_I(t_2, t_0) \right] \\&= \hat{\mathbb{I}} + (-i) \int_{t_0}^t dt_1 \hat{H}'_I(t_1) \\&\quad + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{U}_I(t_2, t_0) \\&= \hat{\mathbb{I}} + (-i) \int_{t_0}^t dt_1 \hat{H}'_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2) \\&\quad + (-i)^3 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3) \hat{U}_I(t_3, t_0)\end{aligned}$$

Thus as an infinite series, we get,

$$\begin{aligned}
 \hat{U}_I(t, t_0) = & \hat{\mathbb{I}} + (-i) \int_{t_0}^t dt_1 \hat{H}'_I(t_1) + \\
 & + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2) \\
 & + (-i)^3 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3) \\
 & + (-i)^4 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \int_{t_0}^{t_3} dt_4 \hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3) \hat{H}'_I(t_4) \\
 & + \dots + (-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \dots \int_{t_0}^{t_n} dt_n \hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3) \\
 & \quad \times \dots \times \hat{H}'_I(t_n)
 \end{aligned}$$

+... So on.

This expression can be further simplified using time ordering which we will demonstrate.

Consider the second term given by

$$(-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2)$$

$$= \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2) + \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2)$$

$$= \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2) + \frac{(-i)^2}{2} \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \hat{H}'_I(t_2) \hat{H}'_I(t_1)$$

Swap t_1 & t_2
dummy indices

(In 1st term $t_1 > t_2$ throughout & in the
2nd term $t_2 > t_1$ throughout)

$$= \frac{(-i)^2}{2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \Theta(t_1 - t_2) \hat{H}'_I(t_1) \hat{H}'_I(t_2) + \frac{(-i)^2}{2} \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \Theta(t_2 - t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_1)$$

$$= \frac{(-i)^2}{2} \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 \Theta(t_1 - t_2) \hat{H}'_I(t_1) \hat{H}'_I(t_2)$$

$$+ \frac{(-i)^2}{2} \int_{t_0}^t \int_{t_0}^t dt_2 dt_1 \Theta(t_2 - t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_1)$$

(Heaviside step functions make sure that the integration limit is taken care of.)

$$= \frac{(-i)^2}{2} \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 \left[\Theta(t_1 - t_2) \hat{H}'_I(t_1) \hat{H}'_I(t_2) + \Theta(t_2 - t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_1) \right]$$

$$= \frac{(-i)^2}{2} \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 T \left[\hat{H}'_I(t_1) \hat{H}'_I(t_2) \right]$$

$$\rightarrow (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}'_I(t_1) \hat{H}'_I(t_2)$$

$$= \frac{(-i)^2}{2} \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 T \left[\hat{H}'_I(t_1) \hat{H}'_I(t_2) \right]$$

Similarly,

$$(-i)^n \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_n} dt_n \hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3) \cdots \hat{H}'_I(t_n)$$

can be written as,

$$\frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \cdots \int_{t_0}^{t_n} dt_n \mathcal{T} [\hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3) \cdots \hat{H}'_I(t_n)]$$

Thus the unitary operator can be written compactly as,

$$\hat{U}_I(t, t_0) \equiv \mathcal{T} \left[\exp \left(-i \int_{t_0}^t d\tau \hat{H}'_I(\tau) \right) \right]$$

Its expansion is given by,

$$\begin{aligned} \hat{\mathbb{I}} &+ (-i) \int_{t_0}^t dt_1 \hat{H}'_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \mathcal{T} [\hat{H}'_I(t_1) \hat{H}'_I(t_2)] \\ &+ \frac{(-i)^3}{3!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \mathcal{T} [\hat{H}'_I(t_1) \hat{H}'_I(t_2) \hat{H}'_I(t_3)] \end{aligned}$$

Scattering Matrix / S-Matrix

The S-matrix is like a special case of time evolution operator which will describe a scattering experiment.

In usual scattering, we start with well separated particles which are fired at each other, they have their own dynamics governed by $\hat{H}_0$ & also in addition to that they have other interactions governed by $\hat{H}'_S$ (Interacting hamiltonian in Schrodinger picture). Then the particles recede from each other & end up well separated.

But this process becomes much more simpler if we look at this in the Interaction picture because the interactions play a role only for a very short time duration when particles are close.

So we can think that the interaction part of hamiltonian is turned on & off smoothly & slowly as time proceeds.

Observe that $\hat{H}'(t = \pm\infty) = 0$, where only free Hamiltonian is present & then we just have the Heisenberg picture.

Then the system state can be described in the time independent kets which are eigen states of $\hat{H}_0$.

Thus $|\psi_{\pm}(t = \pm\infty)\rangle = |\psi_0\rangle$ are the states which can be built from the free vacuum state $|0\rangle$, which are the eigen kets of $\hat{H}_0$. But during the interaction process, $H'_{\pm}(t)$ is not zero. So the state of the system evolve in time in complicated ways. But at the end of scattering, $H'_{\pm}(t)$ goes to zero. Thus the states stop evolving & become an eigen state of $\hat{H}_0$.

So Scattering matrix is defined as that operator which gives the probability amplitudes of a particular eigen ket of $\hat{H}_0$, to transform to another / same eigen ket of $\hat{H}_0$ after the interaction process.

Thus the elements of S-matrix are given by,

$$\langle \phi_0 | \hat{S} | \psi_0 \rangle$$

But $|\psi_0\rangle = |\psi_I(t = -\infty)\rangle$ and

$|\phi_0\rangle = |\phi_I(t = +\infty)\rangle$

$$\Rightarrow \underline{\langle \phi_0 | \hat{S} | \psi_0 \rangle = \langle \phi_0 | \hat{U}_I(+\infty, -\infty) | \psi_0 \rangle}$$

Thus the elements of S-matrix are

$$\langle \phi_0 | T \left[\exp \left(-i \int_{-\infty}^{\infty} dt \hat{H}'_I(t) \right) \right] | \psi_0 \rangle$$

where $|\psi_0\rangle$ & $|\phi_0\rangle$ are eigen states of the Solvable part of Hamiltonian ($\hat{H}_0$).

So we can identify the S -matrix to be the unitary time evolution op. in the interaction picture which will focus only on the $\hat{H}'_{\text{I}}$ part.

$$\rightarrow \hat{S} = \hat{U}_{\text{I}}(+\infty, -\infty) = T \left[\exp \left(-i \int_{-\infty}^{\infty} dt \hat{H}'_{\text{I}}(t) \right) \right]$$

$$\rightarrow \hat{S} = T \left[\exp \left(-i \int d^4x \hat{\mathcal{H}}_{\text{I}}(x^\mu) \right) \right]$$

($\hat{H}'_{\text{I}}(t) = \int d^3x \hat{\mathcal{H}}_{\text{I}}(x^\mu)$). Then, on expanding $\hat{S}$ above, we get,

$$\begin{aligned} \hat{S} = & \hat{\mathbb{I}} - i \int d^4x \hat{\mathcal{H}}_{\text{I}}(x) \\ & + \frac{i^2}{2} \int d^4x d^4y T [\hat{\mathcal{H}}_{\text{I}}(x) \hat{\mathcal{H}}_{\text{I}}(y)] \\ & - \frac{i^3}{3!} \int d^4x d^4y d^4z T [\hat{\mathcal{H}}_{\text{I}}(x) \hat{\mathcal{H}}_{\text{I}}(y) \hat{\mathcal{H}}_{\text{I}}(z)] \\ & + \dots \end{aligned}$$

Provided $\hat{\mathcal{H}}_{\text{I}}$ is small compared to $\hat{\mathcal{H}}_0$, we can consider only first few terms, this is the basis of perturbative expansion.

Wick's theorem

Whenever we have to evaluate vacuum expectation value of a large string of time ordered operators, we can simplify it by using Wick's theorem.

We have seen that any field operator can be written as a sum of Creation part & an annihilation part.

$$\Rightarrow \hat{A} = \hat{A}^- + \hat{A}^+ \quad \& \quad \hat{B} = \hat{B}^- + \hat{B}^+$$

So,

$$\begin{aligned}\hat{A}\hat{B} &= (\hat{A}^- + \hat{A}^+)(\hat{B}^- + \hat{B}^+) \\ &= \hat{A}^-\hat{B}^- + \hat{A}^+\hat{B}^- + \hat{A}^-\hat{B}^+ + \hat{A}^+\hat{B}^+\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathcal{N}[\hat{A}\hat{B}] &= \hat{A}^-\hat{B}^- + \hat{A}^+\hat{B}^- \pm \hat{B}^+\hat{A}^- + \hat{A}^+\hat{B}^+ \\ &\quad \begin{array}{c} \downarrow \\ +ve \text{ for Bosonic operators} \\ -ve \text{ for Fermionic operators} \end{array}\end{aligned}$$

$$\rightarrow \hat{A}\hat{B} - \mathcal{N}[\hat{A}\hat{B}] = \hat{A}^-\hat{B}^+ \mp \hat{B}^+\hat{A}^-$$

Thus,

$$\hat{A}\hat{B} - \mathcal{N}[\hat{A}\hat{B}] = \begin{cases} [\hat{A}^-, \hat{B}^+], & \text{Bosonic} \\ \{\hat{A}^-, \hat{B}^+\}, & \text{Fermionic} \end{cases}$$

Commutators for bosonic operators or anti commutators for fermionic operators are some complex-numbers. Thus the normal ordered product of two operators differs the usual product of those two operators by just a complex number.

Similarly,

$$\hat{B}\hat{A} - \mathcal{N}[\hat{B}\hat{A}] = \begin{cases} [\hat{B}^-, \hat{A}^+], & \text{Bosonic} \\ \{\hat{B}^-, \hat{A}^+\}, & \text{Fermionic} \end{cases}$$

Also note that,

$$\mathcal{N}[\hat{A}\hat{B}] = \begin{cases} \mathcal{N}[\hat{B}\hat{A}], & \text{Bosonic} \\ -\mathcal{N}[\hat{B}\hat{A}], & \text{Fermionic} \end{cases}$$

Now the time ordered product is

$$T[\hat{A}(x)\hat{B}(y)] = \begin{cases} \hat{A}(x)\hat{B}(y), & x^0 > y^0 \\ \pm \hat{B}(y)\hat{A}(x), & y^0 > x^0 \end{cases}$$

+ve for Bosonic operators

-ve for Fermionic operators

Then, $T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]$ can be calculated as follows.

If $x^0 > y^0$,

$$\begin{aligned} T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)] \\ = \hat{A}(x)\hat{B}(y) - N[\hat{A}(x)\hat{B}(y)] \end{aligned}$$

$$\rightarrow T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]$$

$$= \begin{cases} [\hat{A}^-(x), \hat{B}^+(y)], & \text{Bosonic} \\ \{\hat{A}^-(x), \hat{B}^+(y)\}, & \text{Fermionic} \end{cases}$$

$$\underline{\text{If } y^0 > x^0,}$$

$$T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]$$

$$= \pm \hat{B}(y)\hat{A}(x) - N[\hat{A}(x)\hat{B}(y)]$$

$$= \begin{cases} \hat{B}(y)\hat{A}(x) - N[\hat{B}\hat{A}] \rightarrow \text{Bosonic} \\ -\hat{B}(y)\hat{A}(x) + N[\hat{B}\hat{A}] \rightarrow \text{Fermionic} \end{cases}$$

$$= \begin{cases} [\hat{B}^-(y), \hat{A}^+(x)] \rightarrow \text{Bosonic} \\ -\{\hat{B}^-(y), \hat{A}^+(x)\} \rightarrow \text{Fermionic} \end{cases}$$

Thus we can write for bosons,

$$T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]$$

$$\left(\text{For bosonic operators} \right) = \begin{cases} [\hat{A}^-(x), \hat{B}^+(y)], & x^0 > y^0 \\ [\hat{B}^-(y), \hat{A}^+(x)], & y^0 > x^0 \end{cases}$$

Similarly for fermionic operators,

$$T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]$$

$$\left(\begin{array}{l} \text{For} \\ \text{fermionic} \\ \text{operators} \end{array} \right) = \begin{cases} \{\hat{A}^-(x), \hat{B}^+(y)\}, & x^0 > y^0 \\ -\{\hat{B}^-(y), \hat{A}^+(x)\}, & y^0 > x^0 \end{cases}$$

Now we shall define Contraction of two operators as

$$\overline{\hat{A}(x)\hat{B}(y)} \equiv T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]$$

So for Bosonic operators,

$$\overline{\hat{A}(x)\hat{B}(y)} = \begin{cases} [\hat{A}^-(x), \hat{B}^+(y)], & x^0 > y^0 \\ [\hat{B}^-(y), \hat{A}^+(x)], & y^0 > x^0 \end{cases}$$

& for Fermionic operators,

$$\overline{\hat{A}(x)\hat{B}(y)} = \begin{cases} \{\hat{A}^-(x), \hat{B}^+(y)\}, & x^0 > y^0 \\ -\{\hat{B}^-(y), \hat{A}^+(x)\}, & y^0 > x^0 \end{cases}$$

Now we need to keep in mind that Wick's theorem is valid for only the interaction picture. Since the time dependence of operators is defined by just the free Hamiltonian $\hat{H}_0$, we can always work with the operators in interaction picture, in the presence or absence of interaction Hamiltonian.

Then for operators in interaction picture,
 $\hat{A}^-|0\rangle = 0 = \hat{B}^-|0\rangle$ & $\langle 0|\hat{A}^+ = 0 = \langle 0|\hat{B}^+$

Thus we get, $\langle 0|N[\hat{A}\hat{B}]|0\rangle$

$$= \langle 0|(\hat{A}^-\hat{B}^- + \hat{A}^+\hat{B}^- \pm \hat{B}^+\hat{A}^- + \hat{A}^+\hat{B}^+)|0\rangle$$

$$\rightarrow \underline{\langle 0|N[\hat{A}\hat{B}]|0\rangle = 0}$$

In General vacuum expectation value of Normal ordered product of any number of operators is always Zero.

$$\rightarrow \underline{\langle 0|N[\hat{A}\hat{B}\hat{C}\hat{D}\dots]|0\rangle = 0}$$

Thus we can write $\langle 0 | \widehat{A}(x) \widehat{B}(y) | 0 \rangle$
 $= \langle 0 | T [\widehat{A}(x) \widehat{B}(y)] | 0 \rangle - \underbrace{\langle 0 | N [\widehat{A}(x) \widehat{B}(y)] | 0 \rangle}_0$

$$\rightarrow \langle 0 | \widehat{A}(x) \widehat{B}(y) | 0 \rangle = \langle 0 | T [\widehat{A}(x) \widehat{B}(y)] | 0 \rangle$$

But $\widehat{A}(x) \widehat{B}(y)$ is just a c-number,
 hence

$$\langle 0 | \widehat{A}(x) \widehat{B}(y) | 0 \rangle = \widehat{A}(x) \widehat{B}(y)$$

$$\rightarrow \boxed{\widehat{A}(x) \widehat{B}(y) = \langle 0 | T [\widehat{A}(x) \widehat{B}(y)] | 0 \rangle}$$

Now, we have,

$$T [\widehat{A}(x) \widehat{B}(y)] = N [\widehat{A}(x) \widehat{B}(y)] + \widehat{A}(x) \widehat{B}(y)$$

Or since contraction is not affected
 by order of operator,

$$T [\widehat{A}(x) \widehat{B}(y)] = N \left[\widehat{A}(x) \widehat{B}(y) + \widehat{A}(x) \widehat{B}(y) \right]$$

Thus, for two operators we showed that time ordered product is equal to the sum of normal ordered product and all possible contractions. Similarly, for many operators,

$$\boxed{T[\hat{A}\hat{B}\hat{C}\hat{D}\dots] \text{ is given by,}} \\ \boxed{= N[\hat{A}\hat{B}\hat{C}\hat{D}\dots + \text{All possible Contractions}]}$$

This identity is known as Wick's theorem, which relates time ordered product with normal ordered product and contractions.

Eg: For 3 operators,

$$T[\hat{A}\hat{B}\hat{C}] = N\left[\hat{A}\hat{B}\hat{C} + \overline{\hat{A}\hat{B}}\hat{C} + \overline{\hat{A}\hat{C}}\hat{B} + \hat{A}\overline{\hat{B}\hat{C}}\right]$$

where, $N[\overline{\hat{A}\hat{B}}\hat{C}] = \pm \overline{\hat{A}\hat{C}} N[\hat{B}]$
 +ve for bosonic & -ve for fermionic

For 4 operators,

$$T[\hat{A}\hat{B}\hat{C}\hat{D}] =$$

$$N \left[\begin{aligned} &\hat{A}\hat{B}\hat{C}\hat{D} + \overline{\hat{A}\hat{B}}\hat{C}\hat{D} + \overline{\hat{A}\hat{B}}\overline{\hat{C}\hat{D}} + \overline{\hat{A}\hat{B}\hat{C}\hat{D}} \\ &+ \hat{A}\overline{\hat{B}\hat{C}\hat{D}} + \hat{A}\overline{\hat{B}\hat{C}}\hat{D} + \hat{A}\hat{B}\overline{\hat{C}\hat{D}} \\ &+ \overline{\hat{A}\hat{B}}\overline{\hat{C}\hat{D}} + \overline{\hat{A}\hat{B}}\hat{C}\hat{D} + \overline{\hat{A}\hat{B}\hat{C}}\hat{D} \end{aligned} \right]$$

$$N[\overline{\hat{A}\hat{B}\hat{C}\hat{D}}] = \pm \overline{\hat{A}\hat{C}} N[\hat{B}\hat{D}]$$

+ve for bosonic & -ve for fermionic

We have to note that for fermionic operators, we get -ve signs for every exchange b/w two fermionic operators.

Thus, $N[\overline{\hat{A}\hat{B}\hat{C}\hat{D}}] = \overline{\hat{A}\hat{D}} N[\hat{B}\hat{C}]$ for both fermionic & bosonic operators

$$\text{Similarly, } N[\overline{\hat{A}\hat{B}\hat{C}\hat{D}}] = \pm \overline{\hat{A}\hat{C}} \overline{\hat{B}\hat{D}}$$

Vacuum expectation values of Time ordered String of field operators

Since the vacuum expectation value of Normal ordered product of any number of operators is always Zero.

$$\rightarrow \langle 0 | N[\hat{A} \hat{B} \hat{C} \hat{D} \dots] | 0 \rangle = 0$$

And also since contraction of two operators is just a complex number, then $\langle 0 | (\text{contractions}) | 0 \rangle = (\text{contractions})$

Thus by Wick's theorem, we get

For even number of product, then $\langle 0 | T[\hat{A} \hat{B} \hat{C} \hat{D} \dots] | 0 \rangle =$ All the combinations involving distinct type of contractions of every pair of operators.

For odd number of product, then $\langle 0 | T[\hat{A} \hat{B} \hat{C} \hat{D} \dots] | 0 \rangle = 0$

For 2-operators:

$$\langle 0 | T [\hat{A}(x) \hat{B}(y)] | 0 \rangle = \overline{\hat{A}(x) \hat{B}(y)}$$

For 3-operators:

$$\langle 0 | T [\hat{A}(x) \hat{B}(y) \hat{C}(z)] | 0 \rangle = 0$$

For 4-operators:

$$\begin{aligned} \langle 0 | T [\hat{A}(x) \hat{B}(y) \hat{C}(z) \hat{D}(w)] | 0 \rangle \\ = \overline{\hat{A} \hat{B} \hat{C} \hat{D}} + \overline{\hat{A} \hat{B} \hat{C}} \hat{D} + \overline{\hat{A} \hat{B}} \hat{C} \hat{D} \end{aligned}$$

For 6-operators:

$$\begin{aligned} \langle 0 | T [\hat{A} \hat{B} \hat{C} \hat{D} \hat{E} \hat{F}] | 0 \rangle \\ = \overline{\hat{A} \hat{B} \hat{C} \hat{D} \hat{E} \hat{F}} \pm \overline{\hat{A} \hat{B} \hat{C} \hat{E}} \overline{\hat{D} \hat{F}} + \overline{\hat{A} \hat{B} \hat{C} \hat{F}} \overline{\hat{D} \hat{E}} \\ \pm \overline{\hat{A} \hat{C}} \overline{\hat{B} \hat{D}} \overline{\hat{E} \hat{F}} + \overline{\hat{A} \hat{C}} \overline{\hat{B} \hat{E}} \overline{\hat{D} \hat{F}} \pm \overline{\hat{A} \hat{C}} \overline{\hat{B} \hat{F}} \overline{\hat{D} \hat{E}} \\ + \overline{\hat{A} \hat{D}} \overline{\hat{B} \hat{C}} \overline{\hat{E} \hat{F}} \pm \overline{\hat{A} \hat{D}} \overline{\hat{B} \hat{E}} \overline{\hat{C} \hat{F}} + \overline{\hat{A} \hat{D}} \overline{\hat{B} \hat{F}} \overline{\hat{C} \hat{E}} \\ \pm \overline{\hat{A} \hat{E}} \overline{\hat{B} \hat{C}} \overline{\hat{D} \hat{F}} + \overline{\hat{A} \hat{E}} \overline{\hat{B} \hat{D}} \overline{\hat{C} \hat{F}} \pm \overline{\hat{A} \hat{E}} \overline{\hat{B} \hat{F}} \overline{\hat{C} \hat{D}} \\ + \overline{\hat{A} \hat{F}} \overline{\hat{B} \hat{C}} \overline{\hat{D} \hat{E}} \pm \overline{\hat{A} \hat{F}} \overline{\hat{B} \hat{D}} \overline{\hat{C} \hat{E}} + \overline{\hat{A} \hat{F}} \overline{\hat{B} \hat{E}} \overline{\hat{C} \hat{D}} \end{aligned}$$

Thus in evaluating Vacuum expectation values of time ordered string of 'n' operators, the total number of terms that survive after Wick's expansion is given by,

$$\frac{n!}{(n/2)! 2^{n/2}} \quad \text{when } n = \text{even}$$

& Zero, when $n = \text{odd}$.

This formula arises (for $n = \text{even}$) because, there are $n!$ ways of arranging n -operators, but since $(n/2)$ pair arrangements are equivalent (Eg: $\overline{AB} \overline{CD} = \overline{CD} \overline{AB}$), so we divide by $(n/2)!$. Then also each pair has two non-distinct order ($\overline{AC} \sim \overline{CA}$), thus we again divide by $2^{n/2}$.

Gell-Mann - Low Theorem & Perturbative expansion of Correlation function

Here we shall see the general method to derive the Correlation functions & prove their relation with S-matrix.

Consider the Lagrangian density of some interacting field theory $\mathcal{L}[\phi(x)]$. Then consider adding a source term to this Lagrangian density, that couples linearly with the field ' $\phi(x)$ '.

$$\rightarrow \mathcal{L} \longrightarrow \mathcal{L}[\phi(x)] + j(x)\phi(x)$$

$$\rightarrow \mathcal{L}[j(x), \phi(x)] = \mathcal{L}[\phi(x)] + j(x)\phi(x)$$

$$\rightarrow \mathcal{H}[j(x), \phi(x)] = \mathcal{H}[\phi(x)] - j(x)\phi(x)$$

where $\mathcal{H}[\phi(x)] = \underbrace{\mathcal{H}_0[\phi(x)]}_{\text{free Hamiltonian}} + \underbrace{\mathcal{H}_{\text{int}}[\phi(x)]}_{\text{Interacting Hamiltonian}}$

We shall define the normalized generating functional given by,

$$Z[j] = \frac{\langle 0 | T \left[e^{-i \int d^4x (\hat{H}_I(x) - j(x) \phi_I(x))} \right] | 0 \rangle}{\langle 0 | T \left[e^{-i \int d^4x \hat{H}_I(x)} \right] | 0 \rangle}$$

where $|0\rangle \rightarrow$ free vacuum (ground of $\hat{H}_0$)
 $\hat{H}_I \rightarrow$ Interacting Hamiltonian in Interaction picture

Then $\lim_{j \rightarrow 0} \frac{1}{(i)^2} \frac{\partial^2 Z[j]}{\partial[j(x)] \partial[j(y)]}$ is

$$\lim_{j \rightarrow 0} \frac{\langle 0 | T \left[\phi_I(x) \phi_I(y) e^{-i \int d^4x (\hat{H}_I(x) - j(x) \phi_I(x))} \right] | 0 \rangle}{\langle 0 | T \left[e^{-i \int d^4x \hat{H}_I(x)} \right] | 0 \rangle}$$

This is defined as the two point correlation function $\langle \Omega | T [\phi_I(x) \phi_I(y)] | \Omega \rangle$,
 where $|\Omega\rangle$ is ground state of H .

The two point correlation function is defined as the Ground state expectation value of time ordered product of the field operators. It is given by,

$$\langle \Omega | T [\hat{\phi}_I(x) \hat{\phi}_I(y)] | \Omega \rangle$$

Note that $|\Omega\rangle$ is the Ground state of the interacting theory. Where as $|0\rangle$ is the Ground state / vacuum state of the free theory without any interactions.

$\hat{\phi}_I(x)$ is the field operator of the interacting theory in interaction picture.

This correlation function gives the amplitude for propagation of a particle or excitation from space-time point 'y' to 'x', in the presence of interactions.

Consider free vacuum $|0\rangle$ which is the ground state of $\hat{H}_0$. Then we can express the two point correlation function as,

$$\begin{aligned} & \langle \Omega | T [\hat{\phi}_{\mathbf{I}}(x) \hat{\phi}_{\mathbf{I}}(y)] | \Omega \rangle \\ &= \frac{\langle 0 | T [\hat{\phi}_{\mathbf{I}}(x) \hat{\phi}_{\mathbf{I}}(y) e^{-i \int d^4x \hat{H}_{\mathbf{I}}(x)}] | 0 \rangle}{\langle 0 | T [e^{-i \int d^4x \hat{H}_{\mathbf{I}}(x)}] | 0 \rangle} \end{aligned}$$

Now we can relate this to the S -matrix, thus we get,

$$\langle \Omega | T [\hat{\phi}_{\mathbf{I}}(x) \hat{\phi}_{\mathbf{I}}(y)] | \Omega \rangle = \frac{\langle 0 | T [\hat{\phi}_{\mathbf{I}}(x) \hat{\phi}_{\mathbf{I}}(y) \hat{S}] | 0 \rangle}{\langle 0 | \hat{S} | 0 \rangle}$$

Thus perturbative expansion of the S -matrix is used to calculate the perturbative expansion of correlators.