The ϕ^4 Theory

The Lagrangian density of the ϕ^4 theory is given by,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\rightarrow \mathcal{L}_{\phi^4} = \mathcal{L}_{free-scalar} - \frac{\lambda}{4!}\phi^4$$

Here the interaction term is $\frac{\lambda\phi^4}{4!}$ which is a Self-interaction term.

The Hamiltonian density is given by

$$\mathcal{H} = \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}\right](\partial_0 \phi) - \mathcal{L}$$

$$\rightarrow \mathcal{H} = \underbrace{\frac{1}{2}\left[(\partial_0\phi)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right]}_{\mathcal{H}_0} + \underbrace{\frac{\lambda\phi^4}{4!}}_{\mathcal{H}_{int.}}$$

Thus the Hamiltonian for ϕ^4 theory can be written as the sum of the free Scalar field Hamiltonian & a Self-interaction λ-term (Interaction Hamiltonian).

λ is called as Coupling constant. To make 'H' positive definite, we need $\lambda > 0$. Also $\mathcal{L}_{\phi 4}$ preserves all the Symmetries of $\mathcal{L}_{scalar}$. In natural units 'λ' is dimensionless.

The eq^n of motion Satisfied by $\phi(x)$ is,

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = -\frac{\lambda \phi^3(x)}{6}$$

Then the Solution for the above eq^n is,

$$\phi_H(x) = \phi_I(x) - \frac{i\lambda}{6}\int d^4y\, \phi^3(y)\, \Delta_F(x,y)$$

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where $(\partial_\mu \partial^\mu + m^2)\Delta_F(x,y) = -i\delta^{(4)}(x-y)$

$\Delta_F(x,y) \rightarrow$ Feynmann propagator.

$\phi_0(x)$ Satisfies $(\partial_\mu \partial^\mu + m^2)\phi_0(x) = 0$

$$\Delta_F(x,y) = \langle 0|\, T\left[\hat{\phi}_I(x)\,\hat{\phi}_I(y)\right]|0\rangle$$

$$\hat{\phi}_I(x) = \int \frac{d^3p\left[\hat{a}(\vec{p})e^{-ik_\mu x^\mu} + \hat{a}^\dagger(\vec{p})e^{+ik_\mu x^\mu}\right]}{\sqrt{(2\pi)^3(2E_p)}}$$

Since $\hat{\phi}_I(x)$ solves the free scalar field eqⁿ, its time evolution is given by the free part of the Hamiltonian. Hence in interaction picture we can use free scalar field itself as the field operator.

But $\hat{\phi}_H$ solves the full eqⁿ of motion Governed by total hamiltonian. Thus it is the Heisenberg picture field operator which we will not use.

Then in interaction picture, the interacting hamiltonian density is given by,

$$\hat{\mathcal{H}}_I = \frac{\lambda}{4!}\,\hat{\phi}_I^4(x)$$

[From now we denote $\hat{\phi}_I \to \hat{\phi}$ discarding the subscript]

S-matrix Expansion

The Scattering operator is given by,

$$\hat{S} = T\left[\exp\left(-i\int d^4x\, \hat{\mathcal{H}}_I(x^\mu)\right)\right]$$

On expanding the exponential we get,

$$\hat{S} = T\left[1 - i\int d^4x\, \hat{\mathcal{H}}_I(x^\mu) + \frac{i^2}{2!}\int d^4x\, d^4y\, \hat{\mathcal{H}}_I(x^\mu)\, \hat{\mathcal{H}}_I(y^\mu) + \cdots\right]$$

$$= 1 - \frac{i\lambda}{4!}\int d^4x\, T\left[\hat{\phi}^4(x)\right] + \frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y\, T\left[\hat{\phi}^4(x)\hat{\phi}^4(y)\right] + \cdots$$

Then the elements of S-matrix is taken with respect to the 4-momentum states $|p\rangle$, which are created by the operators $\hat{\alpha}^\dagger(p) = \sqrt{(2\pi)^3(2E_p)}\,\hat{a}^\dagger(\vec{p})$

$$\hat{\alpha}^\dagger(p)|0\rangle = |p\rangle, \quad \langle q|p\rangle = (2\pi)^3(2E_p)\,\delta(\vec{p}-\vec{q})$$

$$\int \underbrace{\frac{d^3p}{(2\pi)^3(2E_p)}}_{\text{lorentz invariant}} |p\rangle\langle p| = 1$$

Where as the 3-momentum states got by $\hat{a}^\dagger(\vec{p})|0\rangle = |\vec{p}\rangle$ & $\langle\vec{p}|\vec{q}\rangle = \delta^3(\vec{p}-\vec{q})$ will satisfy $\int d^3p |\vec{p}\rangle\langle\vec{p}| = 1$. But this is not lorentz invariant measure.

Thus any general S-matrix element is given by, "$\langle q|\hat{S}|p\rangle$" which gives the scattering amplitude of a particle initially at 4-momentum 'p' & later at 'q'.

$$\mathcal{A} = \langle q|\hat{S}|p\rangle \rightarrow \text{Scattering Amplitude}$$
$$\rightarrow \mathcal{A} = (2\pi)^3 2\sqrt{E_p E_q}\langle 0|\hat{a}(\vec{q})\hat{S}\hat{a}^\dagger(\vec{p})|0\rangle$$

Since $\hat{S} = \hat{U}_I(\infty, -\infty)$, the operator $\hat{a}^\dagger(\vec{p})$ creates particle at time '$-\infty$' & the operator $\hat{a}(\vec{q})$ destroys particles at time '$+\infty$', so there is no need of an explicit time ordering of them.

$$\rightarrow \mathcal{A} = (2\pi)^3 2\sqrt{E_p E_q}\, T\Big\{ \langle 0|\hat{a}(\vec{q})\hat{a}^\dagger(\vec{p})|0\rangle$$

$$-\frac{i\lambda}{4!}\int d^4x \langle 0|\hat{a}(\vec{q})\hat{\phi}^4(x)\hat{a}^\dagger(\vec{p})|0\rangle$$

$$+\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \langle 0|\hat{a}(\vec{q})\hat{\phi}^4(x)\hat{\phi}^4(y)\hat{a}^\dagger(\vec{p})|0\rangle$$

$$-\frac{i^3\lambda^3}{3!(4!)^3}\int d^4x d^4y d^4z \langle 0|\hat{a}(\vec{q})\hat{\phi}^4(x)\hat{\phi}^4(y)\hat{\phi}^4(z)\hat{a}^\dagger(\vec{p})|0\rangle$$

$$+ \cdots \Big\}$$

Since the Scattering amplitude is a sum of terms ordered by powers of 'λ' we can label the terms as follows.

$$\mathcal{A} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + \cdots = \sum_{n=0}^{\infty} \mathcal{A}^{(n)}$$

where $\mathcal{A}^{(n)}$ is term proportional to λ^n.

$$\mathcal{A}^{(0)} = (2\pi)^3 2\sqrt{E_p E_q}\, \langle 0|\hat{a}(\vec{q})\hat{a}^\dagger(\vec{p})|0\rangle$$
$$= (2\pi)^3 2E_p \delta^3(\vec{p}-\vec{q})$$

$$\rightarrow \underline{\mathcal{A}^{(0)} = (2\pi)^3 2E_p \delta^3(\vec{p}-\vec{q})}$$

Then consider the $A^{(1)}$ term,

$$A^{(1)} = -\frac{i\lambda}{4!}\int d^4x \langle 0|T[\hat{a}(\vec{q})\phi^4(x)\hat{a}^+(\vec{p})]|0\rangle$$

$\langle 0|T[\hat{a}(\vec{q})\phi^4(x)\hat{a}^+(\vec{p})]|0\rangle$ can be expanded using wick's theorem. And Since it is a vacuum expectation value of '6' terms, we get $\frac{6!}{2^3 \times 3!} = 15$ terms that Survive after Wick's expansion.

$$\langle 0|T[\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^+(\vec{p})]|0\rangle$$

= Contractions of the type

$$\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\,\overbrace{\hat{\phi}(x)\hat{\phi}(x)}\,\overbrace{\hat{\phi}(x)\hat{a}^+(\vec{p})}$$

in 12 distinct but equivalent ways

+ Contractions of the type

$$\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)\,\hat{\phi}(x)\hat{\phi}(x)\,\hat{\phi}(x)\hat{a}^+(\vec{p})}$$

in 3 distinct but equivalent ways

These equivalent ways have same numerical values. Thus

$$\mathcal{A}^{(1)} = -\frac{i\lambda}{4!}(2\pi)^3 2\sqrt{E_p E_q}$$

$$\times \int d^4x \left(12\, \overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\, \overbrace{\hat{\phi}(x)\hat{\phi}(x)}\, \overbrace{\hat{\phi}(x)\hat{a}^\dagger(\vec{p})} \right.$$

$$\left. +3\, \overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\, \overbrace{\hat{\phi}(x)}\, \overbrace{\hat{\phi}(x)\hat{\phi}(x)}\, \overbrace{\hat{a}^\dagger(\vec{p})} \right)$$

Consider $\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\, \overbrace{\hat{\phi}(x)\hat{\phi}(x)}\, \overbrace{\hat{\phi}(x)\hat{a}^\dagger(\vec{p})}$

$$= \langle 0|\hat{a}(\vec{q})\hat{\phi}(x)|0\rangle \langle 0|T[\hat{\phi}(x)\hat{\phi}(x)]|0\rangle \times \langle 0|\hat{\phi}(x)\hat{a}^\dagger(\vec{p})|0\rangle$$

But $\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle = \Delta_F(x,y)$ is the Feynmann propagator which gives the amplitude for a particle from y^μ to go to x^μ.

& $\langle 0|\hat{\phi}(x)\hat{a}^\dagger(\vec{p})|0\rangle =$

$$\int \frac{d^3k \langle 0|\hat{a}(\vec{k})\hat{a}^\dagger(\vec{p})|0\rangle e^{-ik_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_k)}}$$

$$\rightarrow \langle 0|\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})|0\rangle = \frac{e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\& \quad \langle 0|\hat{a}(\vec{q})\hat{\phi}(x)|0\rangle = \frac{e^{iq_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_q)}}$$

$$\rightarrow \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$$

$$= \frac{e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}} \Delta_F(x,x) \frac{e^{iq_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_q)}}$$

$$= \frac{e^{i(q_\mu - p_\mu)x^\mu}\Delta_F(x,x)}{(2\pi)^3 2\sqrt{E_p E_q}}$$

$$\rightarrow \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$$

$$= \frac{e^{i(q_\mu - p_\mu)x^\mu}\Delta_F(x,x)}{(2\pi)^3 2\sqrt{E_p E_q}}$$

Note that the denominator $(2\pi)^3 2\sqrt{E_p E_q}$ cancells with the numerator in the S-matrix esepansion.

$$\rightarrow \int d^4x \; \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$$

$$= \int d^4x \; \frac{e^{i(q_\mu - p_\mu)x^\mu}}{(2\pi)^3 2\sqrt{E_p E_q}} \; \Delta_F(x,x)$$

Then, consider $\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$

$$= \langle 0|\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})|0\rangle \langle 0|T[\hat{\phi}(x)\hat{\phi}(x)]|0\rangle^2$$

$$= \delta(\vec{q}-\vec{p})\left[\Delta_F(x,x)\right]^2$$

$$\rightarrow \int d^4x \; \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$$

$$= \delta(\vec{q}-\vec{p}) \int d^4x \left[\Delta_F(x,x)\right]^2$$

Thus we get $A^{(1)}$ as,

$$A^{(1)} = -\frac{i\lambda}{4!}(2\pi)^3 2\sqrt{E_p E_q}$$

$$\times \int d^4x \left(12\,\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p}) \right.$$

$$\left. + 3\,\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p}) \right)$$

$$\mathcal{A}^{(1)} = -\frac{i\lambda}{2}\int d^4x\, e^{i(q_\mu - p_\mu)x^\mu}\Delta_F(x,x)$$

$$-\frac{i\lambda}{8}(2\pi)^3 2\sqrt{E_p E_q}\,\delta(\vec{q}-\vec{p})\int d^4x\left[\Delta_F(x,x)\right]^2$$

we can write, on-shell delta func.

$$(2\pi)^3 2\sqrt{E_p E_q}\,\delta(\vec{q}-\vec{p}) = \overbrace{\delta^4(q^\mu - p^\mu)}$$

$$= (2\pi)^4\int d^4z\, e^{i(q_\mu - p_\mu)z^\mu}$$

$\longrightarrow$

$$\mathcal{A}^{(1)} = -\frac{i\lambda}{2}\int d^4x\, e^{i(q_\mu - p_\mu)x^\mu}\Delta_F(x,x)$$

$$-\frac{i\lambda}{8}\int d^4z\, e^{i(q_\mu - p_\mu)z^\mu}(2\pi)^4\int d^4x\left[\Delta_F(x,x)\right]^2$$

If you notice, the two terms have few features in common.

a) Since they are first order expansion of S-matrix they have one $(-i\lambda)$ factor multiplying them.

b) Each term have incoming & outgoing plane waves corresponding to the four momenta of incoming & outgoing particle.

c) They have propagator terms which describle the probability amplitudes & tell what the particle does.

Now we can further simplify the terms & get everything in terms of the momentum.

We can perform the d^4x integrals & write propagator as,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

with $(k^0)^2 - E_k^2 = k^2 - m^2 = (k^0)^2 - |\vec{k}|^2 - m^2$,
& Feynmann prescription for pole orientation.

Thus we get $\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$

$$= \frac{1}{(2\pi)^3 2\sqrt{E_p E_q}} \int \frac{d^4k}{(2\pi)^4} \frac{i e^{i(q_\mu - p_\mu)x^\mu}}{k^2 - m^2 + i\varepsilon}$$

$$\rightarrow \int d^4x \; \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$$

$$= \frac{1}{(2\pi)^3 2\sqrt{E_p E_q}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

$$\times \underbrace{\int d^4x \; e^{i(q_\mu - p_\mu)x^\mu}}_{(2\pi)^4 \delta^{(4)}(q^\mu - p^\mu)}$$

$$\rightarrow \int d^4x \; \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$$

$$= \frac{(2\pi)^4 \delta^{(4)}(q^\mu - p^\mu)}{(2\pi)^3 2\sqrt{E_p E_q}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

Note that the denominator $(2\pi)^3 2\sqrt{E_p E_q}$ cancells with the numerator in the S-matrix esepansion.

Then, consider $\hat{a}(\vec{q})\,\overbrace{\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)}\,\hat{a}^{\dagger}(\vec{p})$

$= \langle 0|\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})|0\rangle \langle 0|T[\hat{\phi}(x)\hat{\phi}(x)]|0\rangle^2$

$= \delta(\vec{q}-\vec{p})\left[\Delta_F(x,x)\right]^2$

$= \delta(\vec{q}-\vec{p})\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{(k^2-m^2+i\varepsilon)}\frac{i}{(k^2-m^2+i\varepsilon)}$

It has no dependence on 'x^{μ}'.

$\Longrightarrow \int d^4x\; \hat{a}(\vec{q})\,\overbrace{\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)}\,\hat{a}^{\dagger}(\vec{p})$

$= \delta(\vec{q}-\vec{p})\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{(k_1^2-m^2+i\varepsilon)}\frac{i}{(k_2^2-m^2+i\varepsilon)}$

$\times \int d^4x\, \underbrace{e^{-i[(k_1+k_2)-(k_1+k_2)]x}}_{(2\pi)^4\delta(0)}$

$\Longrightarrow \int d^4x\; \hat{a}(\vec{q})\,\overbrace{\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)}\,\hat{a}^{\dagger}(\vec{p})$

$= \delta(\vec{q}-\vec{p})\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{(i)^2(2\pi)^4\delta(k_1+k_2=0)}{(k_1^2-m^2+i\varepsilon)(k_2^2-m^2+i\varepsilon)}$

$\delta(k_1+k_2=0)$ implies that k_1 & k_2 can take any value, such that $k_1+k_2=0$.

Thus we get $\mathcal{A}^{(1)}$ as,

$$\mathcal{A}^{(1)} = -\frac{i\lambda}{4!}(2\pi)^3 2\sqrt{E_p E_q}$$

$$\times \int d^4x \left(12\, \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{+}(\vec{p}) \right.$$

$$\left. + 3\, \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{+}(\vec{p}) \right)$$

$$= -\frac{i\lambda}{4!}(2\pi)^3 2\sqrt{E_p E_q}$$

$$\times \left(12 \frac{(2\pi)^4 \delta^{(4)}(q^\mu - p^\mu)}{(2\pi)^3 2\sqrt{E_p E_q}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \right.$$

$$\left. + 3\,\delta(\vec{q}-\vec{p}) \times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{(i)^2}{(k_1^2 - m^2 + i\varepsilon)(k_2^2 - m^2 + i\varepsilon)} \right)$$

$$= \frac{(-i\lambda)}{2} (2\pi)^4 \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

$$+ \frac{(-i\lambda)}{8} (2\pi)^3 2E_p\, \delta(\vec{q}-\vec{p}) \times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{(i)^2 (2\pi)^4 \delta(k_1 + k_2 = 0)}{(k_1^2 - m^2 + i\varepsilon)(k_2^2 - m^2 + i\varepsilon)}$$

But $(2\pi)^3 2E_p\, \delta(\vec{q}-\vec{p}) = \delta^4(q^\mu - p^\mu)$

Thus $A^{(1)}$ is finally given by,

$$= (2\pi)^4 \delta^{(4)}(q^\mu - p^\mu) \left\{ \frac{(-i\lambda)}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \right.$$

$$\left. + \frac{(-i\lambda)}{8} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{(i)^2 \delta(k_1 + k_2 = 0)}{(k_1^2 - m^2 + i\varepsilon)(k_2^2 - m^2 + i\varepsilon)} \right\}$$

If you notice, the two terms have few features in common.

a) Since they are first order expansion of S-matrix they have one $(-i\lambda)$ factor multiplying them.

b) Each term have a lorentz invariant Energy-momentum conserving dirac-delta function multiplied with $(2\pi)^4$

c) Each Propagator term has gave rise to a 4-integral of an unconstrained momentum 'k, k_1, k_2'.

Next we shall look into the second order $\mathcal{A}^{(2)}$ term given by,

$$\mathcal{A}^{(2)} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \right.$$
$$\left. \times\langle 0|T\left[\hat{a}(\vec{q})\hat{\phi}^4(x)\hat{\phi}^4(y)\hat{a}^\dagger(\vec{p})\right]|0\rangle\right\}$$

Here we need to perform the Wick's expansion of

$$\langle 0|T\left[\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\right.$$
$$\left.\times\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^\dagger(\vec{p})\right]|0\rangle$$

It has VEV of 10 time ordered operators. So a total of $\frac{10!}{2^5 5!} = 945$ terms that survive after Wick's expansion. But performing it by brute force is not needed as we can use the technique of feynmann Diagrams & write the numerically distinct terms diagrammatically

Feynman Diagrams for ϕ^4 theory

A Feynman diagram will represent an amplitude in the expansion of S-matrix which describe interactions (Self interaction in the case of ϕ^4 theory) & the number of interactions represents to which order the term belong to.

Since the interaction term is $[\phi(x)]^4$, we see that any interaction vertex has 4 line segments. The Wick's contractions joins the legs of interaction to each other.

The factor $\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$ represents an external line segment joining an interaction vertex at position 'x' it represents a particle with momentum '$\vec{p}$' entering 'x^{μ}'.

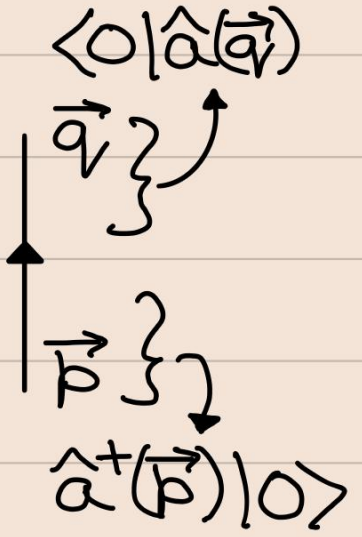
$\hat{\phi}(x)\hat{\phi}(y) = \Delta_F(x, y)$ represents a line segment joining two vertex x & y.

$\overbrace{\hat{\phi}(x)\hat{\phi}(x)} = \Delta_F(x,x)$ represents a line segment as a loop from x to x.

$\overbrace{\hat{a}(\vec{q})\hat{\phi}(y)}$ represents an external outgoing line segment from the interaction vertex at 'y', which means a particle with momentum '$\vec{q}$' is leaving from 'y'.

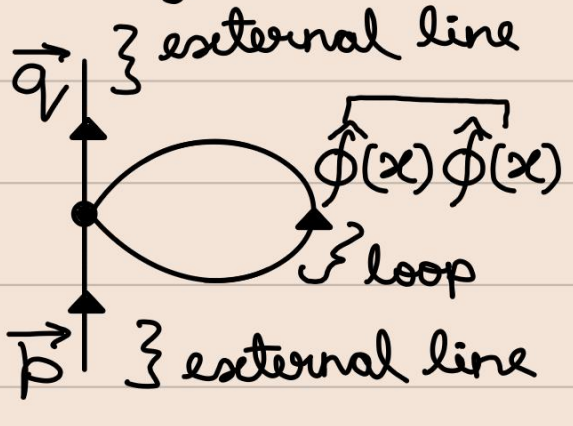
Thus the zeroth order term without any interaction is represented by just a line segment.

$$A^{(0)} = (2\pi)^3 2E_p \langle 0|\hat{a}(\vec{q})\hat{a}^+(\vec{p})|0\rangle =$$

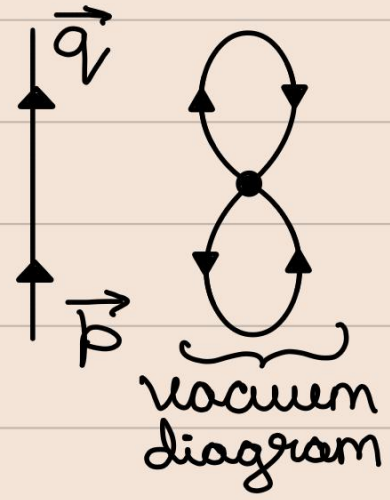


$\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\overbrace{\hat{\phi}(x)\hat{\phi}(x)}\overbrace{\hat{\phi}(x)\hat{a}^+(\vec{p})}$ term gives rise to the following diagram

Connected diagram with one loop



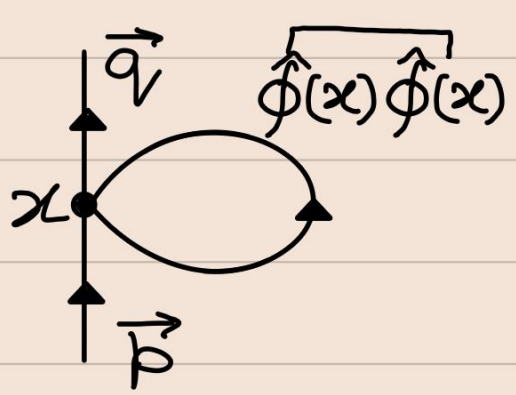
$\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})$ term gives rise to the adjacent feynman diagram



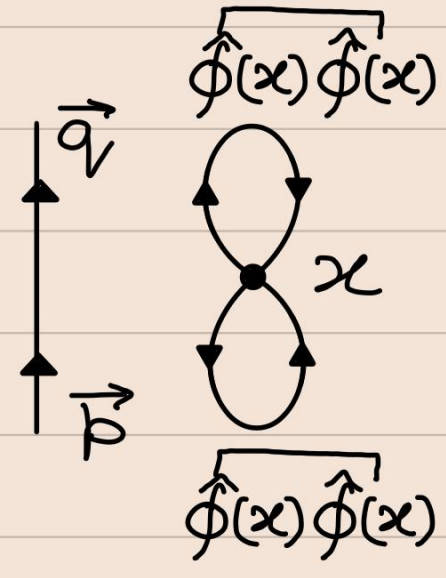
Vacuum diagram has no external lines. In the above figure, the two loop vacuum diagram is multiplied to the straight line (0^{th} order term). Hence the above is a disconnected diagram with two connected parts.

The above two diagrams with one vertex represents the 1^{st} order terms in the S-matrix expansion. Thus $A^{(1)}$ is given by,

$A^{(1)} =$



$+$



Feynman rules for ϕ^4 theory in Position Space

a) Every vertex is connected to four lines (External or internal) and it contributes to a factor of '$-i\lambda$'

b) External lines contribute to the On-mass shell real/observable particles the incoming particles contribute a factor of $e^{-ip_\mu x^\mu}$ (incoming waves), and the outgoing particles contribute a factor of $e^{iq_\mu x^\mu}$ (outgoing waves).

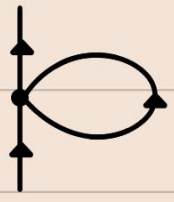
c) Each internal line b/w two vertices 'x' & 'y' contributes to the feynmann propagator factor $\Delta_F(x, y)$, they represent off-mass shell particles because of their deviation by poles in the complex plane, they are virtual particles which are not observables.

d) Integrate over the positions of all the vertices.

e) To get correct coefficient in front of the term divide by appropriate symmetry factor 'D'

Symmetry factors (for ϕ^4 theory)

For any feynmann diagram we should be able to tell what number 'D' we should use to divide by, when working out the Amplitude. This factor arizes by number of ways there are to produce a given diagram through contractions, divided by 4!.

As seen before, in the final amplitude expression for $A^{(1)}$ terms,  $\rightarrow D = 2$

&  $\rightarrow D = 8$, lets see how.

"If there are m-ways of arranging vertices and internal lines (Propagators) to give identical parts of a diagram (keeping external lines fixed & without cutting propagator lines), then we get a factor $D_i = m$ for i^{th} part. And the symmetry factor is given by product of all D_i's of a diagram".

General rules

a) A loop (propagator whose two ends join one vertex) gives a factor of $D_i = 2$.

b) Every pair of vertices directly joined by 'n' propagators have $D_i = n!$

These rules are for ϕ^4 - theory only. Other theories have different rules.

Feynman rules for ϕ^4 theory in Momentum Space

a) Each vertex contribute to a factor of '$-i\lambda$'

b) Eexternal lines that contribute to the On-mass shell particles will have a factor of '1'.

c) Every internal line is a feynman propagator with a momentum factor 'k' flowing along it given by,

$$\text{'}\tilde{\Delta}_F(k) = \frac{i}{k^2 - m^2 + i\varepsilon}\text{'}$$

($i\varepsilon$: slightly off shell)

with $(k^0)^2 - E_k^2 = k^2 - m^2 = (k^0)^2 - |\vec{k}|^2 - m^2$,
& Feynmann prescription for pole orientation.

d) Force the momentum conservation at every vertex where Sum of momenta entering the vertex is equal to the Sum of momenta leaving.

e) Integrate over all unconstrained momenta 'k' by $\int \frac{d^4k}{(2\pi)^4}$

f) Include an overall energy-momentum conserving delta function $(2\pi)^4 \delta^{(4)}(p-q)$ where 'p' & 'q' are momenta of particles entering & leaving respectively.

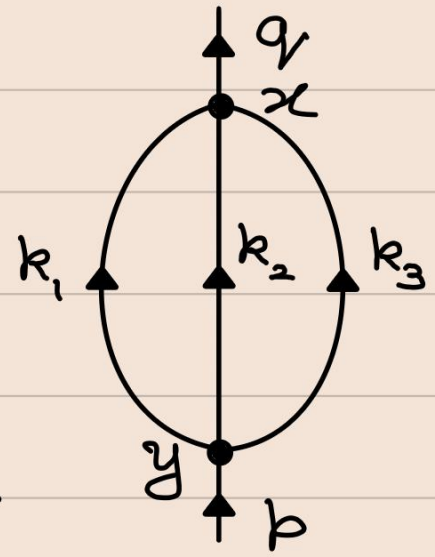
g) Divide by the Symmetry factor 'D' appropriately.

h) A Vacuum bubble carries a factor of '$\delta(0)$'.

(Note: A loop is always an unconstrained momenta.)

Calculation of 2nd order terms

a) Saturn Diagram



This diagram has two vertices. Hence, it is a second order diagram.

It appears in $A^{(2)}$ term

$$A^{(2)} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times \langle 0|T\left[\hat{a}(\vec{q})\hat{\phi}^4(x)\hat{\phi}^4(y)\hat{a}^+(\vec{p})\right]|0\rangle\right\}$$

from the following Wick's contraction

$$\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^+(\vec{p})$$

Particle leaving from 'x' with $\vec{q}$ momentum

Three propagator lines connecting the two vertices in 3! ways.

Particle entering at 'y' with $\vec{p}$ momentum

It can be shown that 192 terms in wick's contractions gives rise to Saturn diagram. Thus,

$$\underset{(\text{Saturn})}{\mathcal{A}^{(2)}} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times 192 \right.$$

$$\left. \times \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})\right\}$$

$$\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p}) = \frac{e^{-ip_\mu y^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\hat{a}(\vec{q})\hat{\phi}(x) = \frac{e^{iq_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_q)}}$$

$$\hat{\phi}(x)\hat{\phi}(y) = \Delta_F(x,y)$$

So, in position space we get,

$$\underset{(\text{Saturn})}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{6}\int d^4x\, d^4y\, e^{-ip_\mu y^\mu} e^{iq_\mu x^\mu}\left[\Delta_F(x,y)\right]^3$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{\text{(Saturn)}}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{6} \int d^4x\, d^4y\, e^{-ipy}\, e^{iqx}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i\, e^{-ik_1(x-y)}}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i\, e^{-ik_2(x-y)}}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i\, e^{-ik_3(x-y)}}{(k_3^2 - m^2 + i\varepsilon)}$$

$$= \frac{(-i\lambda)^2}{6} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y\, e^{-ipy}\, e^{iqx}\, e^{-i[k_1 + k_2 + k_3](x-y)}$$

$$= \frac{(-i\lambda)^2}{6}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_2^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

$$\times\int d^4x\, d^4y\, e^{-i(p-k_1-k_2-k_3)y}\, e^{i(q-k_1-k_2-k_3)x}$$

(Integrate over 'y')

$$= \frac{(-i\lambda)^2}{6}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_2^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

$$\times\int d^4x\, e^{i(q-k_1-k_2-k_3)x}\,(2\pi)^4\delta(p-k_1-k_2-k_3)$$

(Integrate over 'k_3')

$$= \frac{(-i\lambda)^2}{6}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}\ \frac{i}{(k_2^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(p-k_1-k_2)^2-m^2+i\varepsilon}\int d^4x\, e^{i(q-p)x}$$

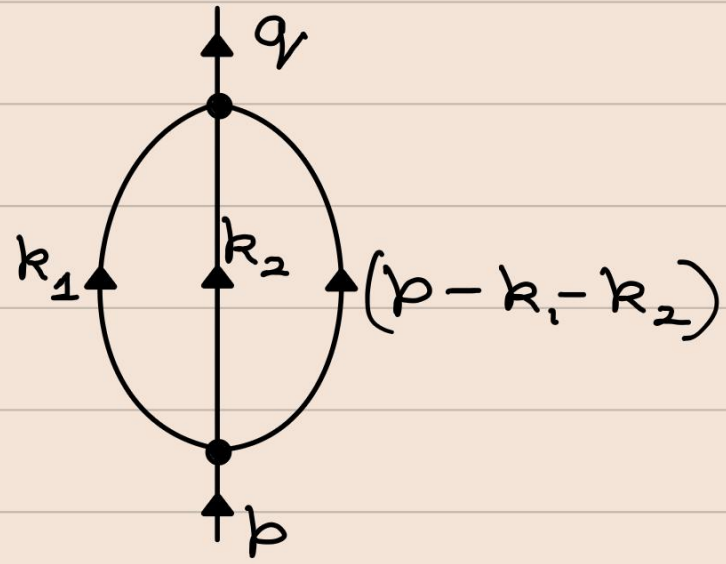
(Integrate over 'x')

Thus, finally in momentum space we get,

$$\underset{(\text{Saturn})}{\mathcal{A}^{(2)}} = (2\pi)^4 \delta^{(4)}(q-p) \frac{(-i\lambda)^2}{6} \int \frac{d^4k_1\, d^4k_2}{(2\pi)^4 (2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

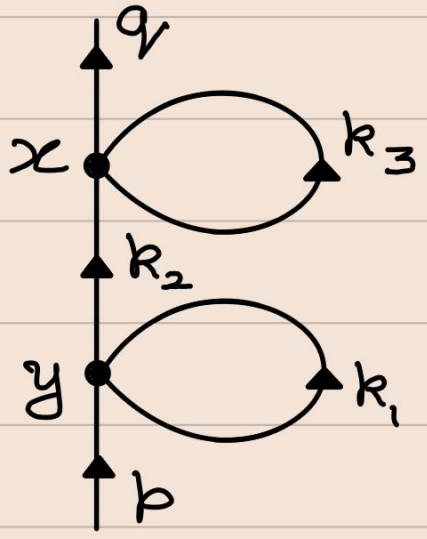
$$\times \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{\left[(p-k_1-k_2)^2 - m^2 + i\varepsilon\right]}$$

$\underset{(\text{Saturn})}{\mathcal{A}^{(2)}}$ =



The above amplitude can also be got exactly by feynman rules.

b) Two-bubble Propagator



It appears in $A^{(2)}$ term from the following Wick's contraction.

$$\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\;\overbrace{\hat{\phi}(x)\hat{\phi}(x)}\;\overbrace{\hat{\phi}(x)\hat{\phi}(y)}\;\overbrace{\hat{\phi}(y)\hat{\phi}(y)}\;\overbrace{\hat{\phi}(y)\hat{a}^{+}(\vec{p})}$$

Particle leaving from 'x' with $\vec{q}$ momentum	One loop at vertex 'x'	Propagator from 'y' to 'x'	One loop at vertex 'y'	Particle entering at 'y' with $\vec{p}$ momentum

It's symmetry factor is $D = 4$, that arises because of two loops, where each loop has $D_i = 2$.

It can be shown that 288 terms in wick's contractions gives rise to Two-bubble Propagator diagram. Thus,

$$\mathcal{A}^{(2)}_{(2\text{-bubble})} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times 288\right.$$

$$\left.\times \overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\overbrace{\hat{\phi}(x)}\overbrace{\hat{\phi}(x)}\overbrace{\hat{\phi}(x)}\overbrace{\hat{\phi}(y)}\overbrace{\hat{\phi}(y)}\overbrace{\hat{\phi}(y)}\overbrace{\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})}\right\}$$

$$\overbrace{\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})} = \frac{e^{-ip_\mu y^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)} = \frac{e^{iq_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_q)}}$$

$$\overbrace{\hat{\phi}(x)\hat{\phi}(y)} = \Delta_F(x,y)$$

So, in position space we get,

$$\mathcal{A}^{(2)}_{(2\text{-bubble})} = \frac{(-i\lambda)^2}{4}\int d^4x\, d^4y\, e^{-ip_\mu y^\mu} e^{iq_\mu x^\mu} \Delta_F(x,y) \times \Delta_F(x,x)\,\Delta_F(y,y)$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{(2\text{-bubble})}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{4} \int d^4x\, d^4y\, e^{-ipy} e^{iqx}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i e^{-ik_2(x-y)}}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$$= \frac{(-i\lambda)^2}{4} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y\, e^{-ipy} e^{iqx} e^{-ik_2(x-y)}$$

$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_2^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

$$\times \int d^4x\, d^4y\, e^{-i(p-k_2)y}\, e^{i(q-k_2)x}$$

(Integrate over 'y')

$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_2^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

$$\times \int d^4x\ e^{i(q-k_2)x}\,(2\pi)^4\delta(p-k_2)$$

(Integrate over 'k_2')

$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(p^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)} \int d^4x\, e^{i(q-p)x}$$

[replace dummy variable $k_3 \to k_2$ and
Integrate over 'x']

$$= \frac{(-i\lambda)^2}{4} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

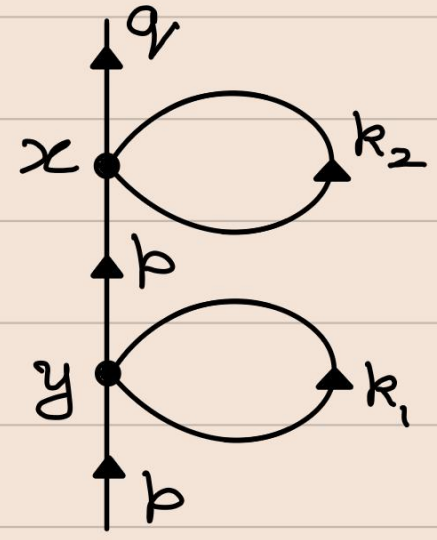
$$\times \frac{i}{(p^2 - m^2 + i\varepsilon)} (2\pi)^4 \delta(q - p)$$

Thus, finally in momentum space we get,

$$\mathcal{A}^{(2)}_{(2\text{-bubble})} = (2\pi)^4 \delta^{(4)}(q - p) \frac{(-i\lambda)^2}{4} \frac{i}{(p^2 - m^2 + i\varepsilon)}$$

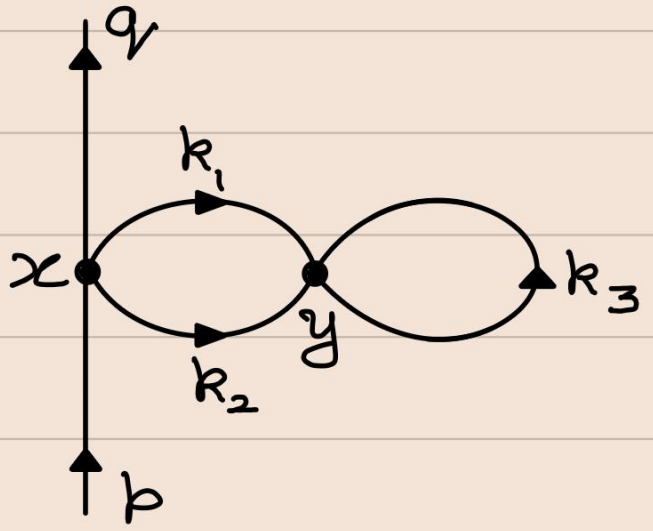
$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

$\mathcal{A}^{(2)}_{(2\text{-bubble})} =$



The above amplitude can also be got exactly by feynman rules.

c) Connected Ring-loop Propagator



It appears in $A^{(2)}$ term from the following Wick's contraction.

Particle entering at 'x' with momentum $\vec{k}$

$$\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{k})$$

Particle leaving from 'x' with $\vec{q}$ momentum

Propagators from 'y' to 'x'

One Bubble at vertex 'y'

It's Symmetry factor is $D = 4$, that arizes because of one loop ($D_i = 2$) and two internal propagators from 'x' to 'y' whose $D_i = 2!$

It can be shown that 288 terms in wick's contractions gives rise to Connected Ring-loop Propagator diagram.

$$\mathcal{A}^{(2)}_{(\text{ring-loop})} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times 288\right.$$

$$\left. \times \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})\right\}$$

$$\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p}) = \frac{e^{-ip_\mu y^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\hat{a}(\vec{q})\hat{\phi}(x) = \frac{e^{iq_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_q)}}$$

$$\hat{\phi}(x)\hat{\phi}(y) = \Delta_F(x,y)$$

So, in position space we get,

$$\mathcal{A}^{(2)}_{(\text{ring-loop})} = \frac{(-i\lambda)^2}{4}\int d^4x\, d^4y\, e^{-ip_\mu y^\mu} e^{iq_\mu x^\mu}\, \Delta_F(x,y) \times \Delta_F(x,y)\,\Delta_F(y,y)$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{(\text{ring-loop})}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{4} \int d^4x\, d^4y\, e^{-ipy} e^{iqx}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i e^{-ik_1(x-y)}}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i e^{-ik_2(x-y)}}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$$= \frac{(-i\lambda)^2}{4} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y\, e^{-ipy} e^{iqx} e^{-ik_1(x-y)} e^{-ik_2(x-y)}$$

$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_2^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

$$\times \int d^4x\, d^4y\, e^{-i(p-k_1-k_2)y}\, e^{i(q-k_1-k_2)x}$$

(Integrate over 'y')

$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_2^2-m^2+i\varepsilon)} \times \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

$$\times \int d^4x\ e^{i(q-k_1-k_2)x}\,(2\pi)^4\,\delta(p-k_1-k_2)$$

(Integrate over 'k_2')

$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\ \frac{i}{(k_1^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(p-k_1)^2-m^2+i\varepsilon} \times \frac{i}{(k_3^2-m^2+i\varepsilon)} \int d^4x\, e^{i(q-p)x}$$

[replace dummy variable $k_3 \to k_2$ and Integrate over 'x']

$$= \frac{(-i\lambda)^2}{4} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

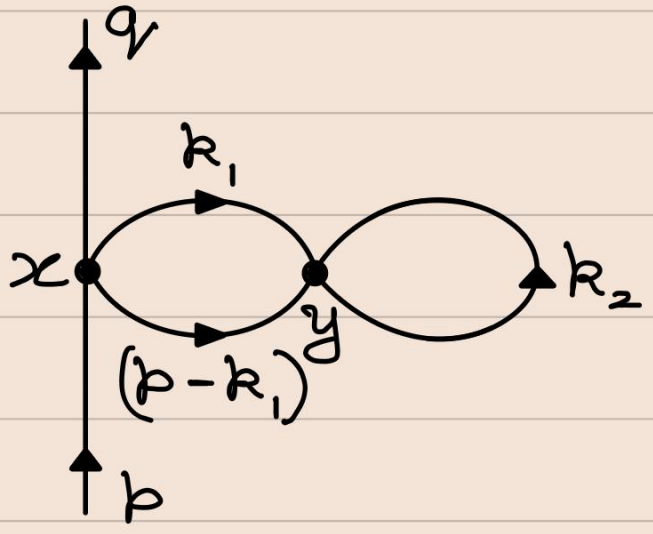
$$\times \frac{i}{\left[(p-k_1)^2 - m^2 + i\varepsilon\right]} (2\pi)^4 \delta(q-p)$$

Thus, finally in momentum space we get,

$$\underset{(\text{ring-loop})}{\mathcal{A}^{(2)}} = (2\pi)^4 \delta^{(4)}(q-p) \frac{(-i\lambda)^2}{4} \int \frac{d^4k_1 \, d^4k_2}{(2\pi)^4 (2\pi)^4}$$

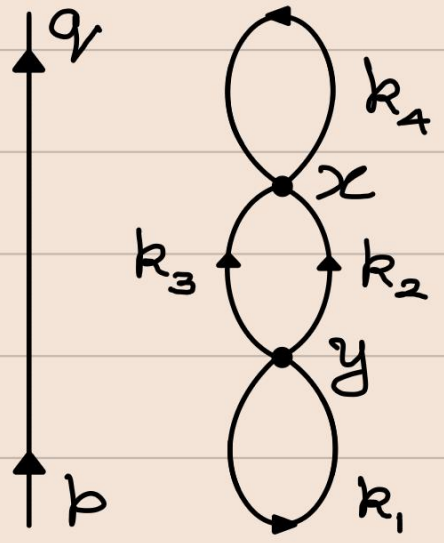
$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{\left[(p-k_1)^2 - m^2 + i\varepsilon\right]} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

$\underset{(\text{ring-loop})}{\mathcal{A}^{(2)}} =$

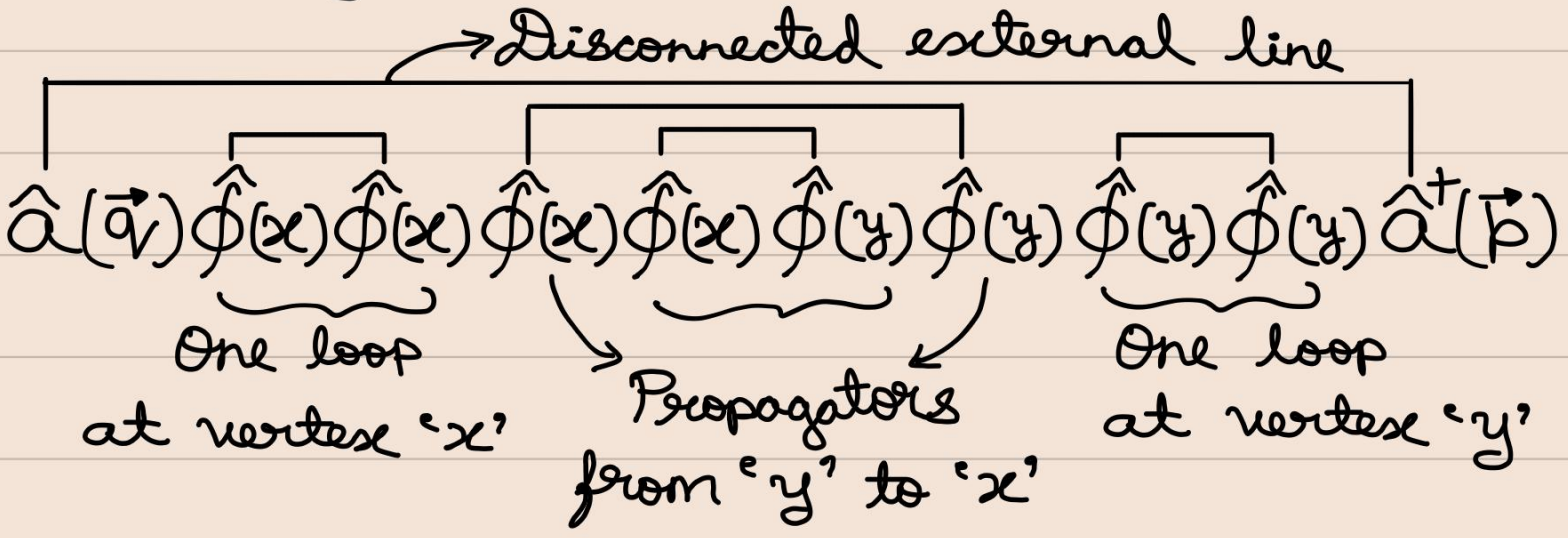


The above amplitude can also be got exactly by feynman rules.

d) Two-vertex- Triple - Bubble



It appears in $A^{(2)}$ term from the following Wick's contraction.



It's symmetry factor is $D = 16$, that arises because of two loop ($D_i = 2 \times 2$), two internal propagators from 'x' to 'y' whose $D_i = 2!$ & one π rotation will keep it invariant so $D_i = 2$ (Interchange of x & y will give same diagram).

It can be shown that 72 terms in wick's contractions gives rise to Two-vertex-Double-Bubble diagram.

$$\mathcal{A}^{(2)}_{\text{(triple bubble)}} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times 72 \right.$$

$$\left. \times \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})\right\}$$

$$\overbrace{\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})} = \langle 0|\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})|0\rangle = \delta^{(3)}(\vec{q}-\vec{p})$$

$$\overbrace{\hat{\phi}(x)\hat{\phi}(y)} = \Delta_F(x,y)$$

$$(2\pi)^3 2\sqrt{E_p E_q}\,\delta^{(3)}(\vec{q}-\vec{p}) = \delta^{(4)}(q^\mu - p^\mu)$$

So, in position space we get,

$$\mathcal{A}^{(2)}_{\text{(triple bubble)}} = \frac{(-i\lambda)^2}{16}\delta^{(4)}(q^\mu - p^\mu)\int d^4x\, d^4y\, \Delta_F(y,y)\Delta_F(x,x) \times \Delta_F(x,y)\Delta_F(x,y)$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{(\text{triple bubble})}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{16} \mathcal{Z}^{(4)}(q^\mu - p^\mu) \int d^4x\, d^4y$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\frac{i e^{-ik_2(x-y)}}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i e^{-ik_3(x-y)}}{(k_3^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$= \frac{(-i\lambda)^2}{16} \mathcal{Z}^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y\; e^{-ik_2(x-y)} e^{-ik_3(x-y)}$$

$$= \frac{(-i\lambda)^2}{16} \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y \; e^{-i(k_2 + k_3)x} \, e^{i(k_2 + k_3)y}$$

(Integrate over 'y')

$$= \frac{(-i\lambda)^2}{16} \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x \; e^{-i(k_2 + k_3)x} \quad (2\pi)^4 \, \delta(k_2 + k_3)$$

(Integrate over 'k_3')

$$= \frac{(-i\lambda)^2}{16} \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x \; \underbrace{e^{-i(0)x}}_{(2\pi)^4 \, \delta(0)}$$

[replace dummy variable $k_4 \to k_3$]

$$= \frac{(-i\lambda)^2}{16}(2\pi)^4 \mathcal{Z}^{(4)}(q-p)\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}$$

$$\times \mathcal{Z}(0)\,\frac{i}{(k_1^2-m^2+i\varepsilon)}\left[\frac{i}{(k_2^2-m^2+i\varepsilon)}\right]^2\frac{i}{(k_3^2-m^2+i\varepsilon)}$$

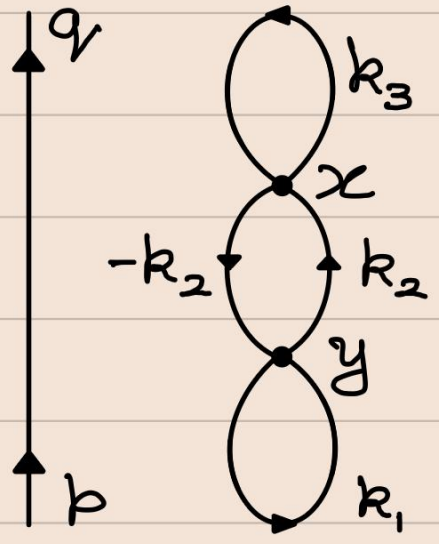
Thus, finally in momentum space we get,

$$\underset{\text{(triple bubble)}}{\mathcal{A}^{(2)}} = (2\pi)^4\mathcal{Z}^{(4)}(q-p)\,\frac{(-i\lambda)^2}{16}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}$$

$$\times \mathcal{Z}(0)\,\frac{i}{(k_1^2-m^2+i\varepsilon)}\left[\frac{i}{(k_2^2-m^2+i\varepsilon)}\right]^2\frac{i}{(k_3^2-m^2+i\varepsilon)}$$

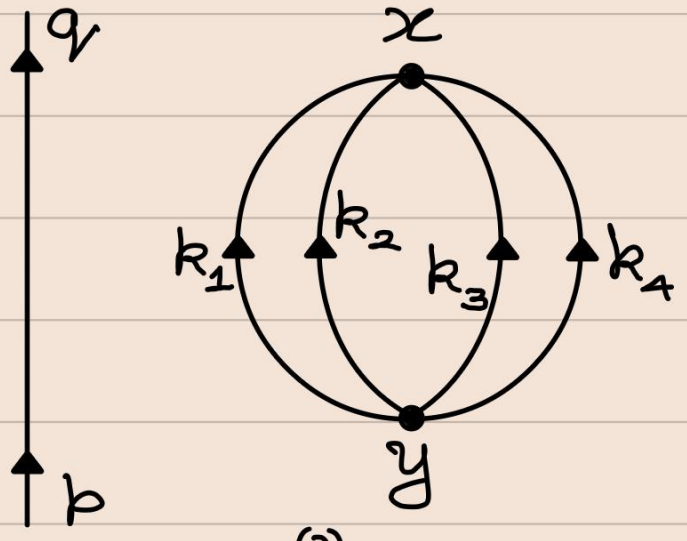
Vacuum Bubble

$\underset{\text{(double bubble)}}{\mathcal{A}^{(2)}} =$



The above amplitude can also be got exactly by feynman rules.

e) Basket ball diagram



It appears in $A^{(2)}$ term from the following Wick's contraction.



It's symmetry factor is $D = 4! \times 2 = 48$.
It has 4 internal propagators, so, $D_i = 4!$
& diagram remains invariant under one π-rotation (Interchanging, x & y once will keep the diagram invariant, $D_i = 2$)

It can be shown that 24 terms in wick's contractions gives rise to Basket ball diagram.

$$\underset{\text{(Basket ball)}}{\mathcal{A}^{(2)}} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times 24 \right.$$

$$\left. \times \hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})\right\}$$

$$\overbrace{\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})} = \langle 0|\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})|0\rangle = \delta^{(3)}(\vec{q}-\vec{p})$$

$$\overbrace{\hat{\phi}(x)\hat{\phi}(y)} = \Delta_F(x,y)$$

$$(2\pi)^3 2\sqrt{E_p E_q}\,\delta^{(3)}(\vec{q}-\vec{p}) = \delta^{(4)}(q^\mu - p^\mu)$$

So, in position space we get,

$$\underset{\left(\text{Basket ball}\right)}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{48}\,\delta^{(4)}(q^\mu - p^\mu)\int d^4x\, d^4y \left[\Delta_F(x,y)\right]^4$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{\text{(Basketball)}}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{48} Z^{(4)}(q^\mu - p^\mu) \int d^4x\, d^4y$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4} \frac{i e^{-ik_1(x-y)}}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\frac{i e^{-ik_2(x-y)}}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i e^{-ik_3(x-y)}}{(k_3^2 - m^2 + i\varepsilon)} \times \frac{i e^{-ik_4(x-y)}}{(k_4^2 - m^2 + i\varepsilon)}$$

$$= \frac{(-i\lambda)^2}{48} Z^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y\, e^{-ik_1(x-y)} e^{-ik_2(x-y)}$$

$$\times e^{-ik_3(x-y)} e^{-ik_4(x-y)}$$

$$= \frac{(-i\lambda)^2}{48} \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\, d^4y\; e^{-i(k_1+k_2+k_3+k_4)x}\, e^{i(k_1+k_2+k_3+k_4)y}$$

(Integrate over 'y')

$$= \frac{(-i\lambda)^2}{48} \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$\times \int d^4x\; e^{-i(k_1+k_2+k_3+k_4)x}\, (2\pi)^4 \delta(k_1+k_2+k_3+k_4)$$

(Integrate over 'k_4')

$$= \frac{(-i\lambda)^2}{48} \delta^{(4)}(q^\mu - p^\mu) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{\left([k_1 + k_2 + k_3]^2 - m^2 + i\varepsilon\right)}$$

$$\times \int d^4x\; \underbrace{e^{-i(0)x}}_{(2\pi)^4 \delta(0)}$$

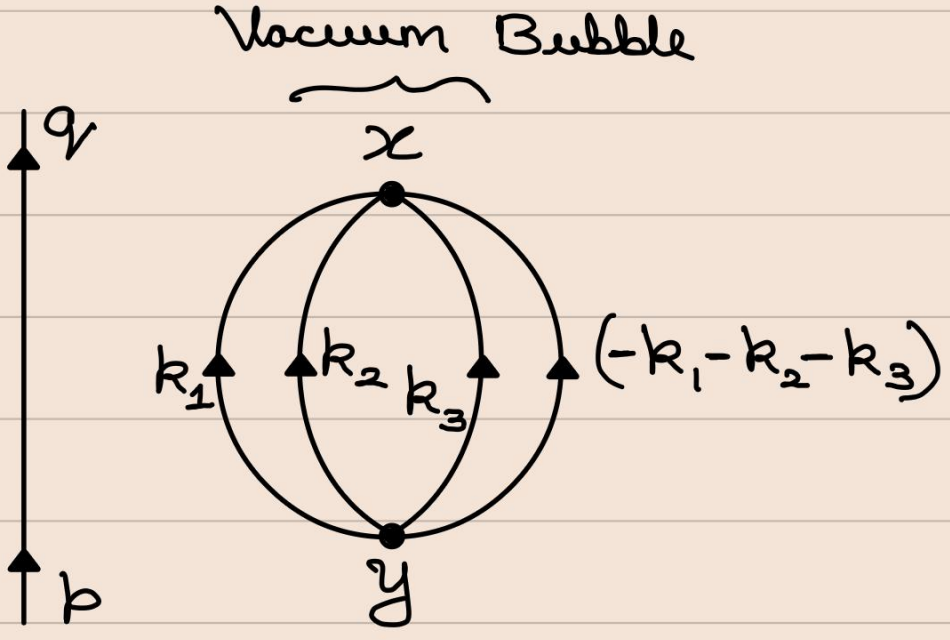
Thus, finally in momentum space we get,

$$\mathcal{A}^{(2)}_{\left(\begin{smallmatrix}\text{Basket}\\ \text{ball}\end{smallmatrix}\right)} = (2\pi)^4 \delta^{(4)}(q-p)\,\frac{(-i\lambda)^2}{48}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}$$

$$\times \delta(0)\,\frac{i}{(k_1^2-m^2+i\varepsilon)}\,\frac{i}{(k_2^2-m^2+i\varepsilon)}\,\frac{i}{(k_2^2-m^2+i\varepsilon)}$$

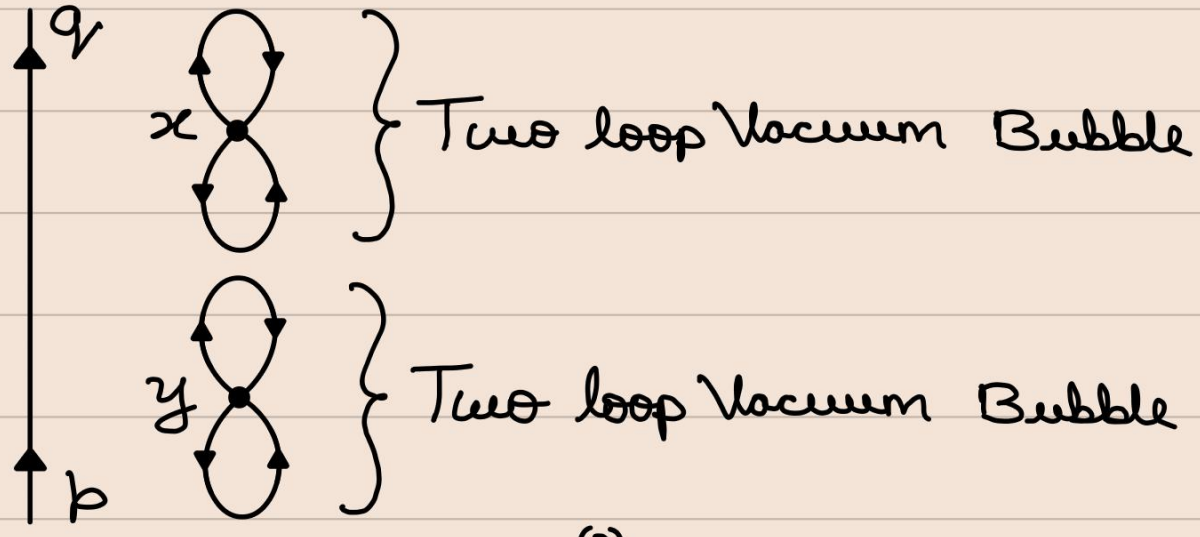
$$\times \frac{i}{\left([k_1+k_2+k_3]^2-m^2+i\varepsilon\right)}$$

$\mathcal{A}^{(2)}_{\left(\begin{smallmatrix}\text{Basket}\\ \text{ball}\end{smallmatrix}\right)}$ =

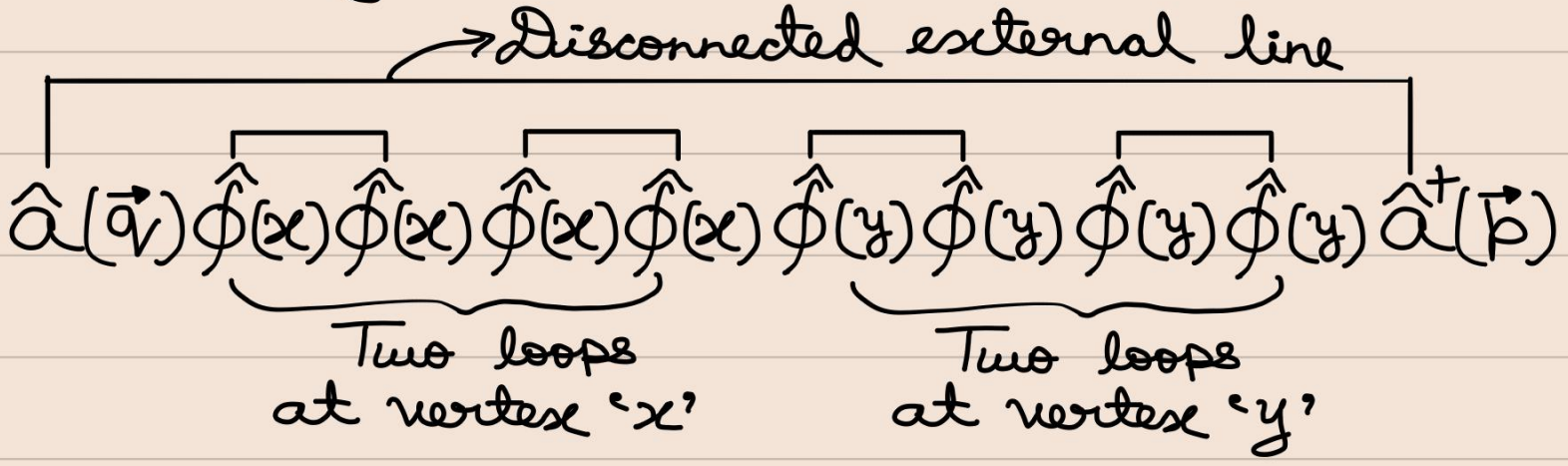


The above amplitude can also be got exactly by feynman rules.

f)



It appears in $A^{(2)}$ term from the following Wick's contraction.



It's symmetry factor is $D = 128$, that arizes because of four loops. Each loop has $D_i = 2$. Then rotating a particular two loop bubble gives $D_i = 2$. So, one two loop bubble gives a factor of 8. Each centered at x & y. Then upon interchanging x & y, results in same overall diagram. Thus total symmetry factor is $8 \times 8 \times 2 = 128$.

It can be shown that 9 terms in wick's contractions gives rise to Two-disconnected-Double-Bubble diagram.

$$\mathcal{A}^{(2)}_{(\text{double bubble})} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times 128 \right.$$

$$\left. \times \overbrace{\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})}\right\}$$

$$\overbrace{\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})} = \langle 0|\hat{a}(\vec{q})\hat{a}^{\dagger}(\vec{p})|0\rangle = \delta^{(3)}(\vec{q}-\vec{p})$$

$$\overbrace{\hat{\phi}(x)\hat{\phi}(y)} = \Delta_F(x,y)$$

$$(2\pi)^3 2\sqrt{E_p E_q}\,\delta^{(3)}(\vec{q}-\vec{p}) = \delta^{(4)}(q^\mu - p^\mu)$$

So, in position space we get,

$$\mathcal{A}^{(2)}_{(\text{double bubble})} = \frac{(-i\lambda)^2}{128}\delta^{(4)}(q^\mu - p^\mu)\int d^4x\, d^4y \left[\Delta_F(y,y)\,\Delta_F(x,x)\right]^2$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{\left(\begin{smallmatrix}\text{double}\\ \text{bubble}\end{smallmatrix}\right)}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{128} \mathcal{Z}^{(4)}(q^\mu - p^\mu) \int d^4x\, d^4y$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\frac{i}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$$= \frac{(-i\lambda)^2}{128} (2\pi)^4 \mathcal{Z}^{(4)}(q-p) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$$\times [\mathcal{Z}(0)]^2 \times \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

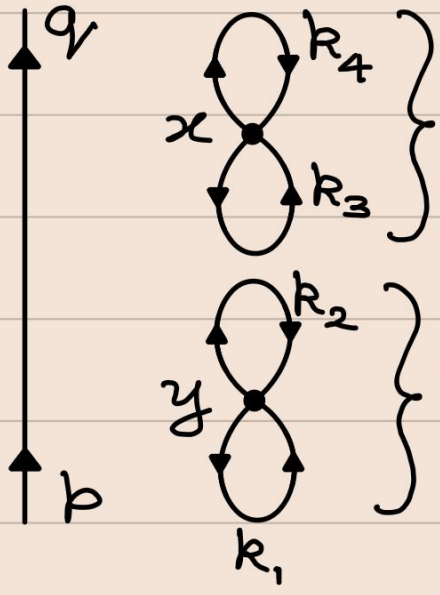
Thus, finally in momentum space we get,

$$\mathcal{A}^{(2)}_{(\text{double bubble})} = (2\pi)^4 \delta^{(4)}(q-p) \frac{(-i\lambda)^2}{128} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{d^4k_4}{(2\pi)^4}$$

$$\times [\delta(0)]^2 \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

$$\times \frac{i}{(k_3^2 - m^2 + i\varepsilon)} \frac{i}{(k_4^2 - m^2 + i\varepsilon)}$$

$\mathcal{A}^{(2)}_{(\text{double bubble})}$ =

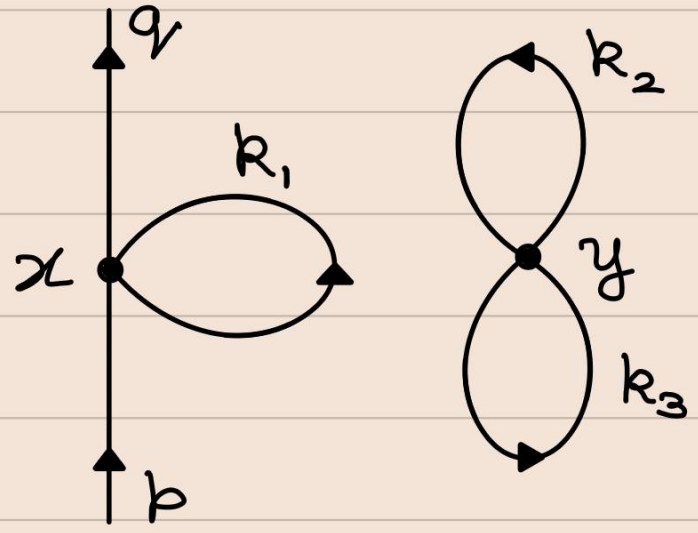


Two loop Vacuum Bubble

Two loop Vacuum Bubble

The above amplitude can also be got exactly by feynman rules.

g)



[Two disconnected first order diagrams]

It appears in $A^{(2)}$ term from the following Wick's contraction.

External particle entering vertex 'x'

$$\hat{a}(\vec{q})\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})$$

Particle leaving vertex 'x'

A loop at vertex 'x'

Two loops at vertex 'y'

It's Symmetry factor is $D = 16$, that arises because of three loops, where each loop has $D_i = 2$. Then on interchanging x & y, results in same overall diagram. Thus total Symmetry factor is 8×2.

It can be shown that 72 terms in wick's contractions gives rise to Two-disconnected-Double-Bubble diagram.

$$\mathcal{A}^{(2)}_{\left(\substack{\text{Two } 1^{st} \\ \text{order}}\right)} = (2\pi)^3 2\sqrt{E_p E_q}\left\{\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \times (72)\right.$$

$$\left. \times \overbrace{\hat{a}(\vec{q})\hat{\phi}(x)}\,\overbrace{\hat{\phi}(x)}\,\overbrace{\hat{\phi}(x)}\,\overbrace{\hat{\phi}(x)\hat{\phi}(y)}\,\overbrace{\hat{\phi}(y)}\,\overbrace{\hat{\phi}(y)}\,\overbrace{\hat{\phi}(y)\hat{a}^{\dagger}(\vec{p})}\right\}$$

$$\overbrace{\hat{\phi}(x)\hat{a}^{\dagger}(\vec{p})} = \frac{e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\overbrace{\hat{a}(\vec{q})\hat{\phi}(x)} = \frac{e^{iq_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_q)}}$$

$$\overbrace{\hat{\phi}(x)\hat{\phi}(y)} = \Delta_F(x,y)$$

So, in position space we get,

$$\mathcal{A}^{(2)}_{\left(\substack{\text{Two } 1^{st} \\ \text{order}}\right)} = \frac{(-i\lambda)^2}{16}\int d^4x\, d^4y\, e^{-ip_\mu x^\mu} e^{iq_\mu x^\mu} \Delta_F(x,x) \times \Delta_F(y,y)\,\Delta_F(y,y)$$

In momentum space,

$$\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$$

$$\underset{\left(\text{Two } 1^{st} \text{ order}\right)}{\mathcal{A}^{(2)}} = \frac{(-i\lambda)^2}{16} \int d^4x \, e^{i(q_\mu - p_\mu)x^\mu} \underbrace{\int d^4y}_{\mathcal{Z}(0)}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\frac{i}{(k_2^2 - m^2 + i\varepsilon)} \times \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

(Perform x integral)

$$= \frac{(-i\lambda)^2}{16} (2\pi)^4 \mathcal{Z}^{(4)}(q-p) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

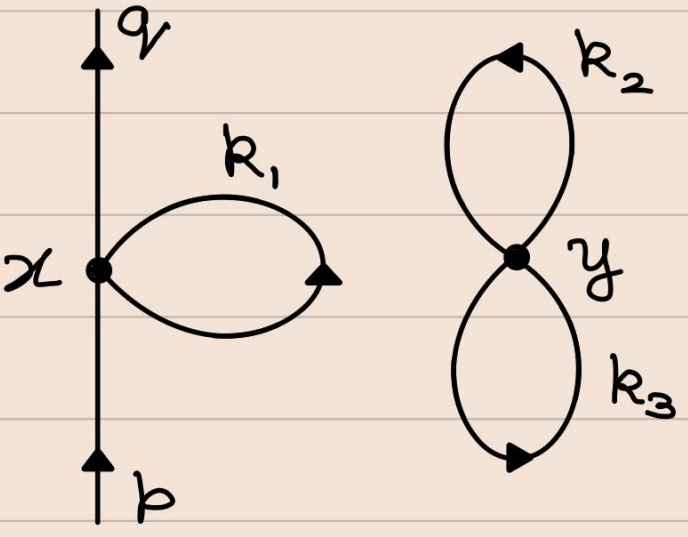
$$\times [\mathcal{Z}(0)] \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

Thus, finally in momentum space we get,

$$\mathcal{A}^{(2)}_{(\text{Two } 1^{st} \text{ order})} = (2\pi)^4 \delta^{(4)}(q-p) \frac{(-i\lambda)^2}{16} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4}$$

$$\times [\delta(0)] \frac{i}{(k_1^2 - m^2 + i\varepsilon)} \frac{i}{(k_2^2 - m^2 + i\varepsilon)} \frac{i}{(k_3^2 - m^2 + i\varepsilon)}$$

$\mathcal{A}^{(2)}_{(\text{Two } 1^{st} \text{ order})}$ =



The above amplitude can also be got exactly by feynman rules.

Feynman Diagrams for 2 particle Scattering

Here we look at scattering amplitude for two particles entering, undergo self / interparticle interactions & then two particles leave.

Using S-matrix, we can write the two particle scattering amplitude as,

$$\mathcal{A}(p_1, p_2, q_1, q_2) = \langle q_1, q_2 | \hat{S} | p_1, p_2 \rangle$$

$$= (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}$$

$$\times \langle 0 | \hat{a}(\vec{q}_1) \hat{a}(\vec{q}_2) \hat{S} \hat{a}^\dagger(\vec{p}_2) \hat{a}^\dagger(\vec{p}_1) | 0 \rangle$$

This describes two particles with 4-momenta p_1 & p_2 enter at the infinite past & two particles with 4-momenta q_1 & q_2 leave at the infinite future.

The Scattering operator is given by,

$$\hat{S} = T\left[\exp\left(-i\int d^4x\, \hat{H}_I(x^\mu)\right)\right]$$

On expanding the exponential we get,

$$\hat{S} = 1 - \frac{i\lambda}{4!}\int d^4x\, T\left[\hat{\phi}^4(x)\right] + \frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y\, T\left[\hat{\phi}^4(x)\hat{\phi}^4(y)\right] + \cdots$$

$$\rightarrow \underset{(p_1,p_2,q_1,q_2)}{\mathcal{A}} = (2\pi)^6 \mathcal{A}\sqrt{E_{p_1}E_{p_2}E_{q_1}E_{q_2}} \times T\Big\{\langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)|0\rangle$$

$$-\frac{i\lambda}{4!}\int d^4x \langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{\phi}^4(x)\hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)|0\rangle$$

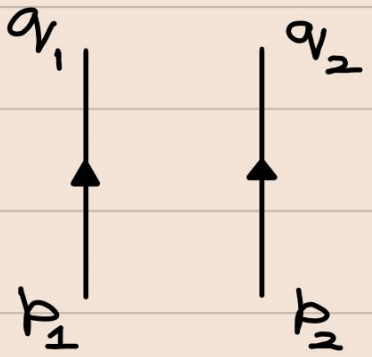
$$+\frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y \langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{\phi}^4(x)\hat{\phi}^4(y)\hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)|0\rangle + \cdots\Big\}$$

$$\rightarrow \underset{(p_1,p_2,q_1,q_2)}{\mathcal{A}} = \mathcal{A}^{(0)} + \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + \cdots = \sum_{n=0}^{\infty} \mathcal{A}^{(n)}$$

where $\mathcal{A}^{(n)}$ is term proportional to λ^n

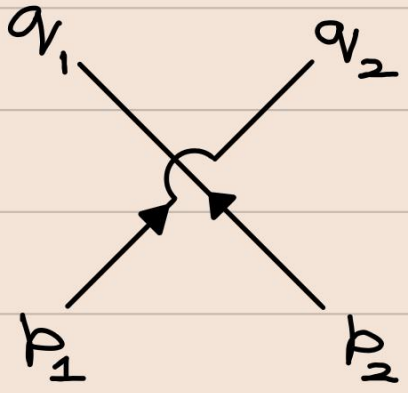
0^{th} order scattering

a)



$$= (2\pi)^6 A \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}\, \delta^{(3)}(\vec{q}_1 - \vec{p}_1)\, \delta^{(3)}(\vec{q}_2 - \vec{p}_2)$$

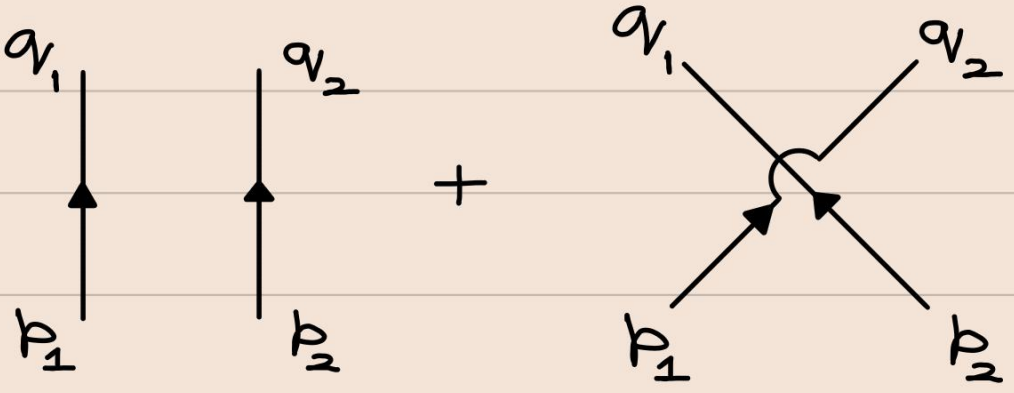
b)



$$= (2\pi)^6 A \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}\, \delta^{(3)}(\vec{q}_1 - \vec{p}_2)\, \delta^{(3)}(\vec{q}_2 - \vec{p}_1)$$

$$\mathcal{A}^{(0)} = (2\pi)^6 A \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}$$
$$\times \langle 0 | T[\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)] | 0 \rangle$$

$\rightarrow \mathcal{A}^{(0)} =$

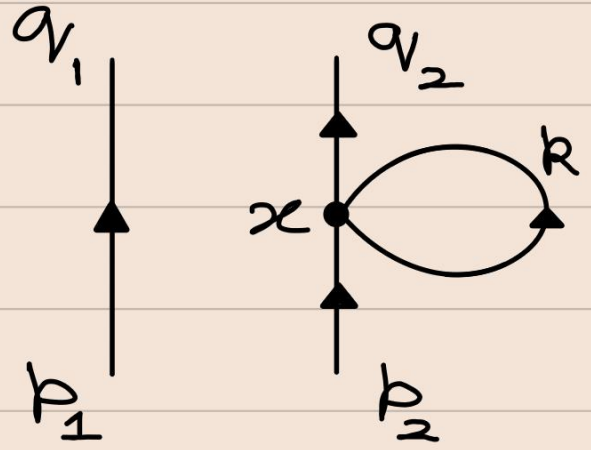


1st order scattering

$$\mathcal{A}^{(1)} = (2\pi)^6 4\sqrt{E_{p_1}E_{p_2}E_{q_1}E_{q_2}}\left[-\frac{i\lambda}{4!}\right]$$

$$\times \int d^4x \langle 0|T[\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{\phi}^4(x)\hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_1)]|0\rangle$$

a)



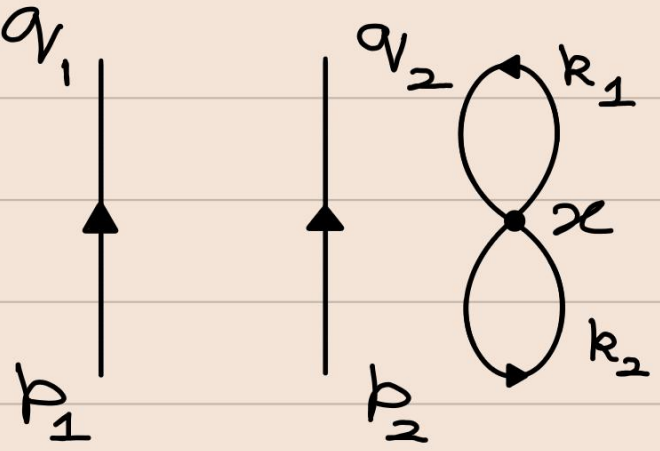
$$= (2\pi)^3 2\sqrt{E_{p_1}E_{q_1}}\,\delta^{(3)}(\vec{q}_1 - \vec{p}_1)$$

$$\times\left[-\frac{i\lambda}{2}\right]\int d^4x\, e^{-ip_2x} e^{iq_2x}\, \Delta_F(x,x)$$

$$= (2\pi)^3 2\sqrt{E_{p_1}E_{q_1}}\,\delta^{(3)}(\vec{q}_1 - \vec{p}_1)$$

$$\times\left[-\frac{i\lambda}{2}\right](2\pi)^4\delta^{(4)}(q_2^\mu - p_2^\mu)\int\frac{d^4k}{(2\pi)^4}\frac{i}{(k^2 - m^2 + i\varepsilon)}$$

b)



$$= (2\pi)^6 4\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}\, \delta^{(3)}(\vec{q}_1 - \vec{p}_1)\delta^{(3)}(\vec{q}_2 - \vec{p}_2)$$

$$\times \left[-\frac{i\lambda}{8}\right] \int d^4x\, \Delta_F(x,x)\Delta_F(x,x)$$

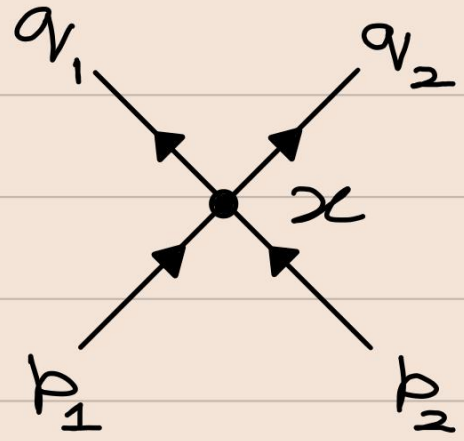
$$= (2\pi)^6 4\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}\, \delta^{(3)}(\vec{q}_1 - \vec{p}_1)\delta^{(3)}(\vec{q}_2 - \vec{p}_2)$$

$$\times \left[-\frac{i\lambda}{8}\right] (2\pi)^4 \delta(0) \int \frac{d^4k_1}{(2\pi)^4} \frac{i}{(k_1^2 - m^2 + i\varepsilon)}$$

$$\times \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{(k_2^2 - m^2 + i\varepsilon)}$$

The previous two 1st order diagrams are just extensions of single particle feynmann diagram where one 0th order external line is just multiplied.

c)



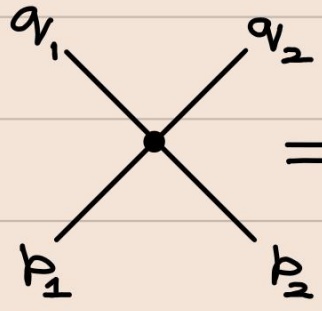
$$= (2\pi)^6 4\sqrt{E_{p_1}E_{p_2}E_{q_1}E_{q_2}}\left[\frac{-i\lambda}{4!}\right] \times (24)$$

$$\times \int d^4x \langle 0|\hat{a}(\vec{q}_1)\hat{a}(\vec{q}_2)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(x)\hat{a}^+(\vec{p}_2)\hat{a}^+(\vec{p}_1)|0\rangle$$

(24 Wick's contractions give same value.)

$$= (-i\lambda)\int d^4x\, e^{-ip_2x} e^{iq_2x} e^{-ip_1x} e^{iq_1x}$$

$\longrightarrow$



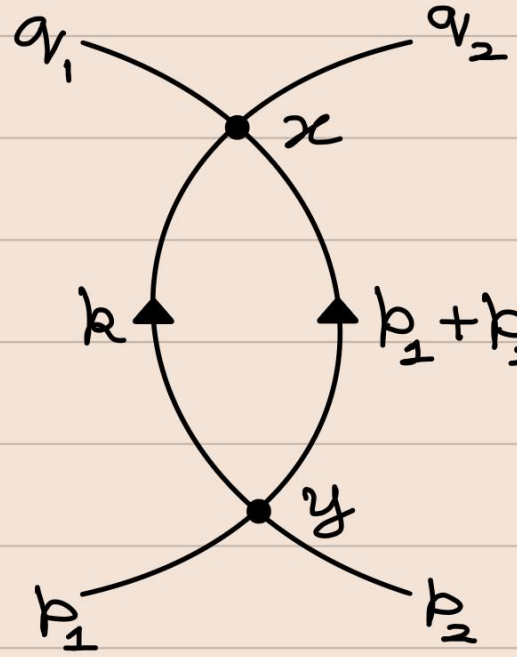
$$= (-i\lambda)(2\pi)^4 \delta^{(4)}(q_1^\mu + q_2^\mu - p_1^\mu - p_2^\mu)$$

This 1st order diagram just highlights conservation of total 4-momenta in the two particle scattering process.

2nd order scattering

Some non-trivial 2nd order scattering processes which mixes two incoming particles are as follows.

a)



$= A^{(2)}_{fish}$

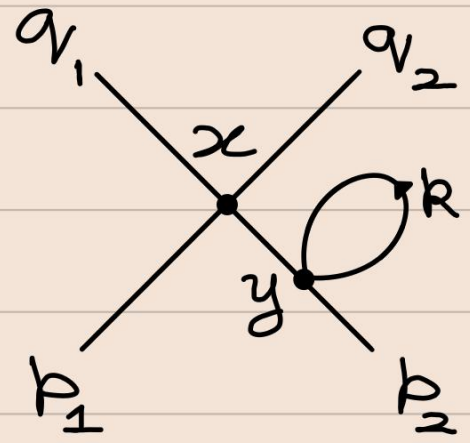
$(D = 2)$

This is called as the fish diagram.

$$A^{(2)}_{fish} = \frac{(-i\lambda)^2}{2}\int d^4x\, d^4y\, e^{-i(p_1+p_2)y} e^{i(q_1+q_2)x} \times [\Delta_F(x,y)]^2$$

$$= (2\pi)^4 \delta^{(4)}(q_1+q_2-p_1-p_2)\frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4} \times \frac{i}{(k^2-m^2+i\varepsilon)} \frac{i}{[(p_1+p_2-k)^2-m^2+i\varepsilon]}$$

b)

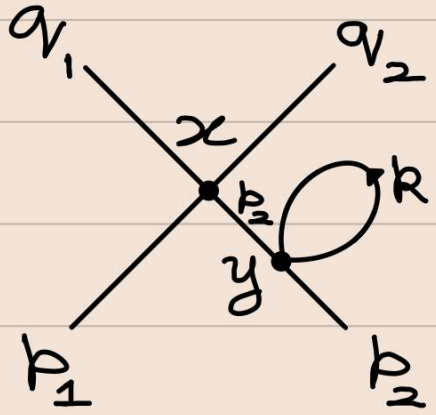


$$= \frac{(-i\lambda)^2}{2} \int d^4x\, d^4y\, e^{-ip_2 y}\, e^{i(q_1+q_2-p_1)x} \times \Delta_F(x,y)\,\Delta_F(y,y)$$

On going to momentum space, we get,

$$= \frac{(-i\lambda)^2}{2} \int d^4x\, d^4y\, e^{-ip_2 y}\, e^{i(q_1+q_2-p_1)x} \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \int \frac{d^4k'}{(2\pi)^4} \frac{i\, e^{-ik'(x-y)}}{k'^2-m^2+i\varepsilon}$$

Thus we can finally write,



$$= (2\pi)^4 \delta(q_1+q_2-p_1-p_2) \frac{(-i\lambda)^2}{2} \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2-m^2+i\varepsilon)} \frac{i}{(p_2^2-m^2+i\varepsilon)}$$

2 Point Correlation Functions

The two point correlation function is defined as the Ground state expectation value of time ordered product of the field operators. It is given by,

$$\langle \Omega | T[\hat{\phi}_I(x)\hat{\phi}_I(y)] | \Omega \rangle$$

Note that $|\Omega\rangle$ is the Ground state of the interacting theory. Where as $|0\rangle$ is the Ground state / vacuum state of the free theory without any interactions.

$\hat{\phi}_I(x)$ is the field operator of the interacting theory in interaction picture.

This correlation function gives the amplitude for propagation of a particle or excitation from space-time point 'y' to 'x', in the presence of interactions.

Consider free vacuum $|0\rangle$ which is the Ground state of $\hat{H}_0$. Then, using Gell-Mann-Low theorem we can express the two point correlation function as,

$$\langle\Omega|T[\hat{\phi}_I(x)\hat{\phi}_I(y)]|\Omega\rangle$$

$$= \frac{\langle 0|T\left[\hat{\phi}_I(x)\hat{\phi}_I(y)\, e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\right]|0\rangle}{\langle 0|T\left[e^{-i\int d^4x\, \hat{\mathcal{H}}_I(x)}\right]|0\rangle}$$

Now we can relate this to the S-matrix, thus we get,

$$\langle\Omega|T[\hat{\phi}_I(x)\hat{\phi}_I(y)]|\Omega\rangle = \frac{\langle 0|T\left[\hat{\phi}_I(x)\hat{\phi}_I(y)\,\hat{S}\right]|0\rangle}{\langle 0|\hat{S}|0\rangle}$$

Thus perturbative expansion of the S-matrix is used to calculate the perturbative expansion of Correlators.

[From now $\phi_I(x)$ is denoted as $\phi(x)$]

Consider the numerator $\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{S}]|0\rangle$ where $\hat{S}$ is given by,

$$\hat{S} = T\left[\exp\left(-i\int d^4x \frac{\lambda\hat{\phi}^4(x)}{4!}\right)\right]$$

On expanding the exponential we get,

$$\hat{S} = 1 - \frac{i\lambda}{4!}\int d^4x\, T[\hat{\phi}^4(x)] + \frac{i^2\lambda^2}{2!(4!)^2}\int d^4x\, d^4y\, T[\hat{\phi}^4(x)\hat{\phi}^4(y)] + \cdots$$

$$\rightarrow \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{S}]|0\rangle = \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle + \left(-\frac{i\lambda}{4!}\right)\int d^4z\, \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z)]|0\rangle + \frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2\int d^4z\, d^4\omega\, \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z)\hat{\phi}^4(\omega)]|0\rangle + \cdots$$

Now, $\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle = \Delta_F(x,y)$,

$\rightarrow \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)]|0\rangle \equiv$ 

We can use Wick's theorem to expand $\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z)]|0\rangle$

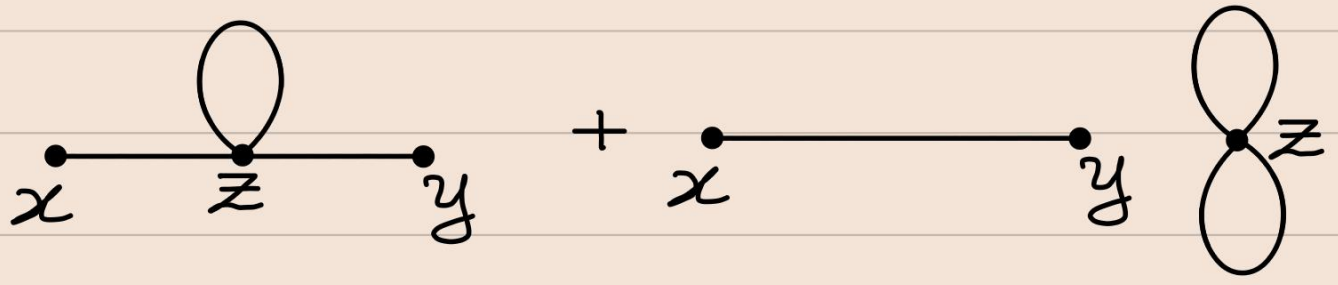
$$= 12\ \overbrace{\hat{\phi}(x)\hat{\phi}(z)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(z)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(y)}$$

$$+ 3\ \overbrace{\hat{\phi}(x)\hat{\phi}(y)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(z)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(z)}$$

Thus the 1st order term becomes,

$$\left(\frac{-i\lambda}{4!}\right)\int d^4z\ \langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}^4(z)]|0\rangle$$

$=$



where,



$$= \frac{(-i\lambda)}{2}\int d^4z\ \overbrace{\hat{\phi}(x)\hat{\phi}(z)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(z)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(y)}$$



$$= \frac{(-i\lambda)}{8}\int d^4z\ \overbrace{\hat{\phi}(x)\hat{\phi}(y)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(z)}\ \overbrace{\hat{\phi}(z)\hat{\phi}(z)}$$

Thus we see that these espressions resemble the feynman diagrams in the S-matrix espansion, except that now the incoming & outgoing particles have definite position.

- Each propagator b/w two vertices is a line,

$$\overbrace{\hat{\phi}(z)\hat{\phi}(w)} = \Delta_F(z,w) =$$

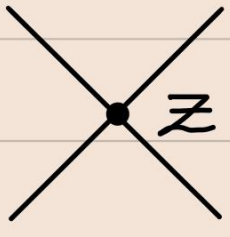


$$\overbrace{\hat{\phi}(z)\hat{\phi}(z)} = \Delta_F(z,z) =$$



→ loop

- Every internal vertex is represented by,



$$= (-i\lambda)\int d^4z$$

- Each external point, 'x':

$$\overbrace{\hat{\phi}(x)} =$$

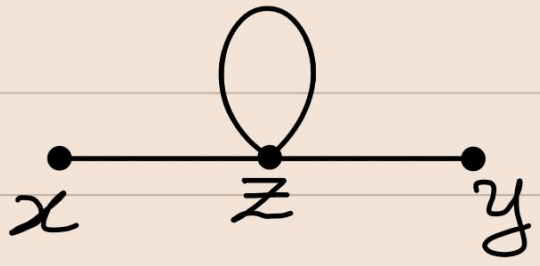


$$\equiv 1$$

Consider expressing the 1st order term in momentum space.

where $\Delta_F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\varepsilon}$

So, we get,



$$= \frac{(-i\lambda)}{2} \int d^4z \, \overbrace{\hat{\phi}(x)\hat{\phi}(z)} \, \overbrace{\hat{\phi}(z)\hat{\phi}(z)} \, \overbrace{\hat{\phi}(z)\hat{\phi}(y)}$$

$$= \frac{(-i\lambda)}{2} \int d^4z \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-z)}}{p^2 - m^2 + i\varepsilon}$$

$$\times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(z-z)}}{k^2 - m^2 + i\varepsilon} \int \frac{d^4q}{(2\pi)^4} \frac{i e^{-iq(z-y)}}{q^2 - m^2 + i\varepsilon}$$

($p \rightarrow$ 4-momenta of particle entering from 'x'
$k \rightarrow$ 4-momenta of particle looping at 'z'
$q \rightarrow$ 4-momenta of particle leaving to 'y')

$$= \frac{(-i\lambda)}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\;\frac{i}{p^2-m^2+i\varepsilon}$$

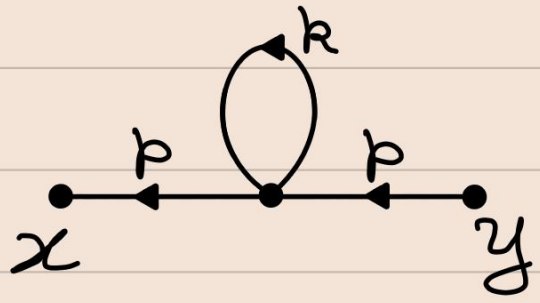
$$\times \frac{i}{k^2-m^2+i\varepsilon}\;\frac{i}{q^2-m^2+i\varepsilon}$$

$$\times e^{-ipx}e^{iqy}\underbrace{\int d^4z\; e^{-i(q-p)z}}_{(2\pi)^4\,\delta^{(4)}(q-p)}$$

$$= \frac{(-i\lambda)}{2}\int \frac{d^4p}{(2\pi)^4}\,e^{-ipx}e^{ipy}\left(\frac{i}{p^2-m^2+i\varepsilon}\right)^2$$

$$\times \int \frac{d^4k}{(2\pi)^4}\;\frac{i}{k^2-m^2+i\varepsilon}$$

$\Longrightarrow$

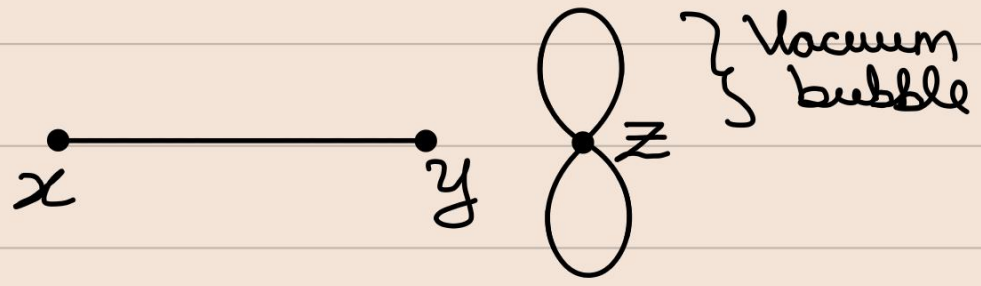


is given by

$$= \frac{(-i\lambda)}{2}\int \frac{d^4p}{(2\pi)^4}\,e^{-ipx}e^{ipy}\left(\frac{i}{p^2-m^2+i\varepsilon}\right)^2$$

$$\times \int \frac{d^4k}{(2\pi)^4}\;\frac{i}{k^2-m^2+i\varepsilon}$$

Then, consider this diagram,



$$= \frac{(-i\lambda)}{8}\int d^4z\, \hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)$$

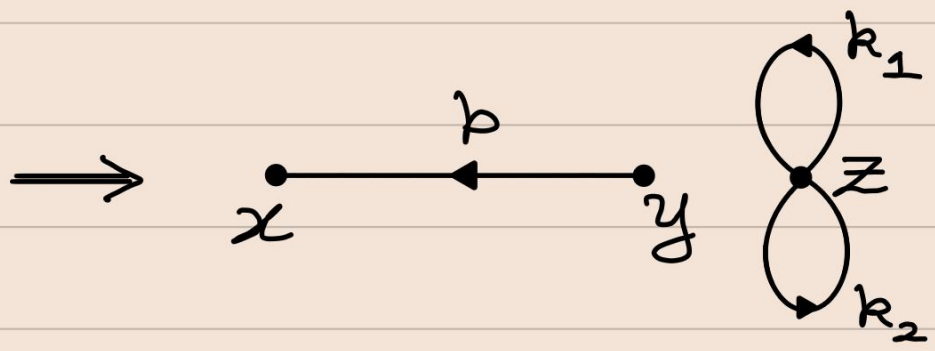
$$= \frac{(-i\lambda)}{8}\int d^4z \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2-m^2+i\varepsilon} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_2^2-m^2+i\varepsilon}$$

$$= \frac{(-i\lambda)}{8}\int \frac{d^4p}{(2\pi)^4} e^{-ipx} e^{ipy} \frac{i}{p^2-m^2+i\varepsilon}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2-m^2+i\varepsilon} \frac{i}{k_2^2-m^2+i\varepsilon}$$

$$\times \underbrace{\int d^4z}_{(2\pi)^4 \delta(0)}$$

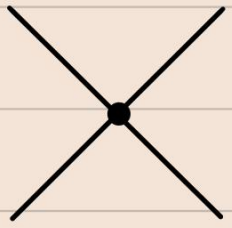


is given by

$$= (2\pi)^4 \delta(0) \frac{(-i\lambda)}{8} \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ipx} e^{ipy}}{p^2 - m^2 + i\varepsilon}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon}$$

Thus we can generalize the rules in momentum space as,

- Propagator: $\xleftarrow[k]{} = \frac{i}{k^2 - m^2 + i\varepsilon}$
- Vertex:  $= -i\lambda$ [Enforce momentum Conservation]
- External points carry a plane wave factor

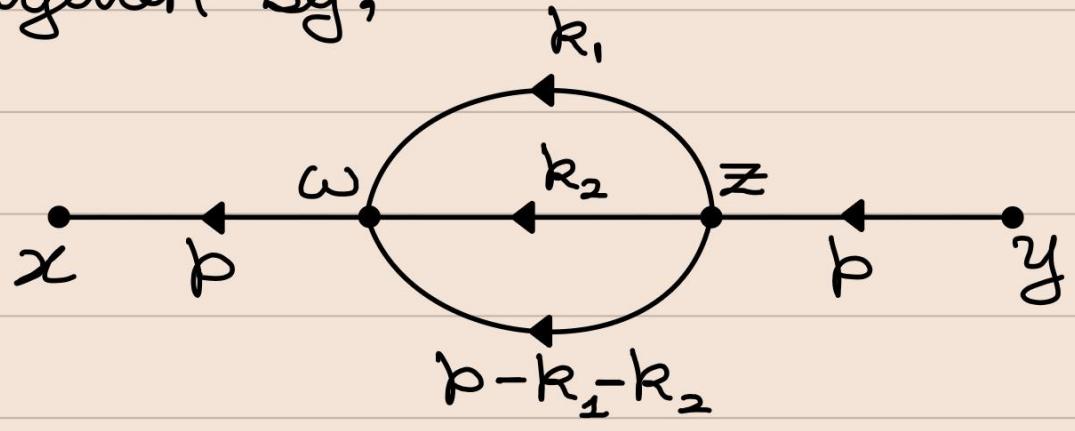
$$\xleftarrow[x]{p} = e^{-ipx}$$

$$\xleftarrow[y]{q} = e^{iqy}$$

- Integrate over all undetermined momenta 'k', $\int \frac{d^4k}{(2\pi)^4}$
- Divide by the symmetry factor
- Every vacuum bubble, which is a disconnected piece of a diagram, that is not connected to esternal point will carry a factor of $(2\pi)^4 \delta(0)$

Thus the second order diagrams are given by,

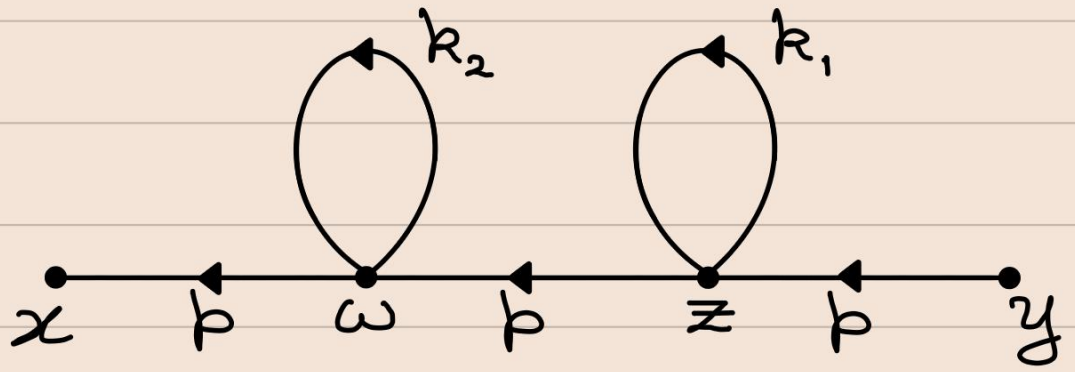
a)



$$= \frac{(-i\lambda)^2}{6} \int \frac{d^4p\, d^4k_1\, d^4k_2\, e^{-ipx} e^{ipy}}{(2\pi)^4(2\pi)^4(2\pi)^4} \left(\frac{i}{p^2-m^2+i\varepsilon}\right)^2$$

$$\times \left(\frac{i}{k_1^2-m^2+i\varepsilon}\right)\left(\frac{i}{k_2^2-m^2+i\varepsilon}\right)\left(\frac{i}{(p-k_1-k_2)^2-m^2+i\varepsilon}\right)$$

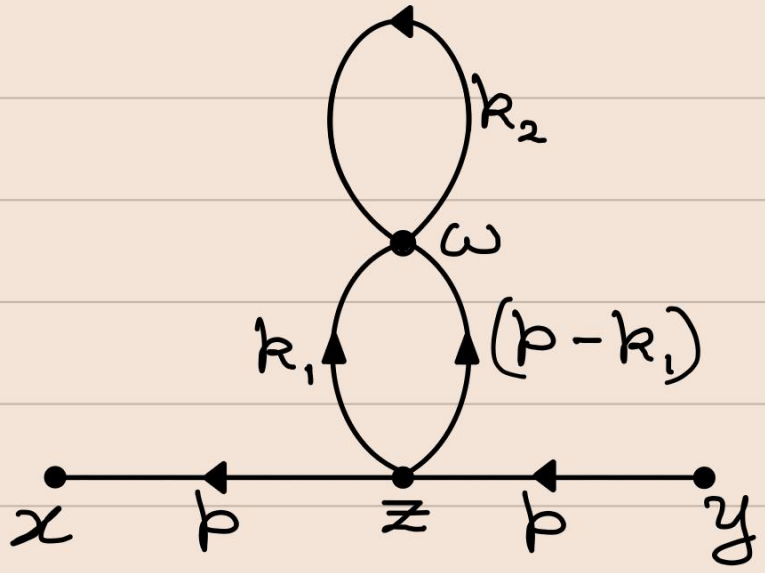
b)



$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4p\,d^4k_1\,d^4k_2\,e^{-ipx}e^{ipy}}{(2\pi)^4(2\pi)^4(2\pi)^4}\left(\frac{i}{p^2-m^2+i\varepsilon}\right)^3$$

$$\times\left(\frac{i}{k_1^2-m^2+i\varepsilon}\right)\left(\frac{i}{k_2^2-m^2+i\varepsilon}\right)$$

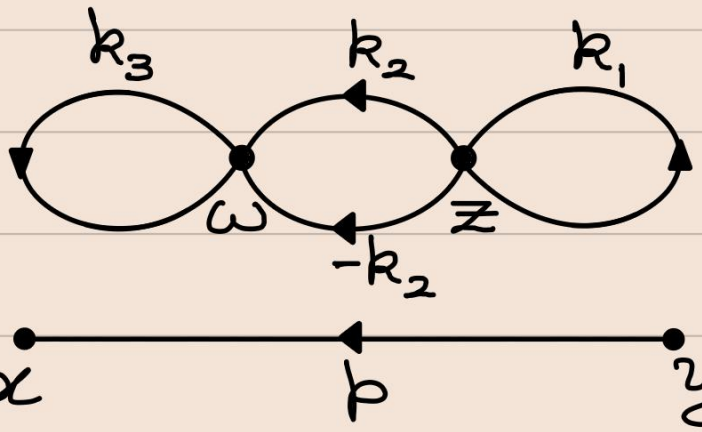
c)



$$= \frac{(-i\lambda)^2}{4}\int \frac{d^4p\,d^4k_1\,d^4k_2\,e^{-ipx}e^{ipy}}{(2\pi)^4(2\pi)^4(2\pi)^4}\left(\frac{i}{p^2-m^2+i\varepsilon}\right)^2$$

$$\times \frac{i}{(k_1^2-m^2+i\varepsilon)}\,\frac{i}{\left[(p-k_1)^2-m^2+i\varepsilon\right]}\,\frac{i}{(k_2^2-m^2+i\varepsilon)}$$

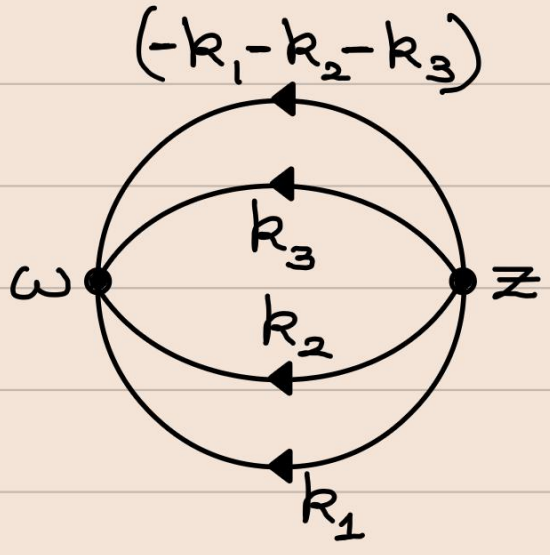
d)

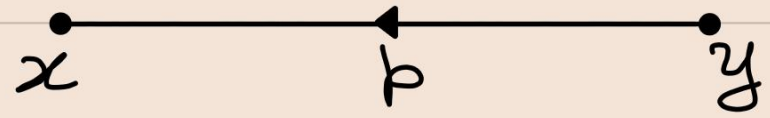


$$= \frac{(-i\lambda)^2}{16}\int \frac{d^4p}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\;\frac{i\,e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$$

$$\times(2\pi)^4\delta(0)\frac{i}{(k_1^2-m^2+i\varepsilon)}\left[\frac{i}{(k_2^2-m^2+i\varepsilon)}\right]^2\frac{i}{(k_3^2-m^2+i\varepsilon)}$$

e)



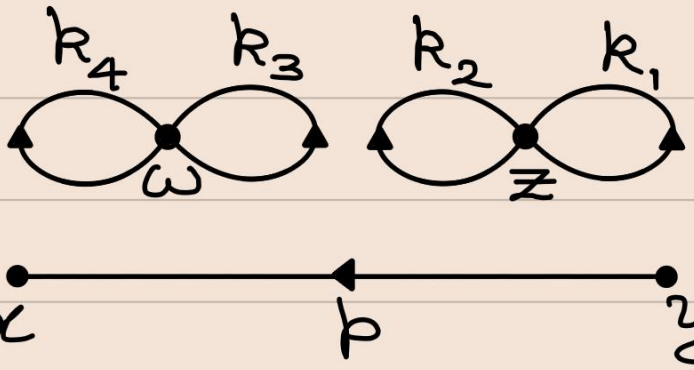


$$= \frac{(-i\lambda)^2}{48}\int \frac{d^4p}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\;\frac{i\,e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$$

$$\times(2\pi)^4\delta(0)\frac{i}{(k_1^2-m^2+i\varepsilon)}\frac{i}{(k_2^2-m^2+i\varepsilon)}\frac{i}{(k_2^2-m^2+i\varepsilon)}$$

$$\times\frac{i}{\left(\left[k_1+k_2+k_3\right]^2-m^2+i\varepsilon\right)}$$

f)

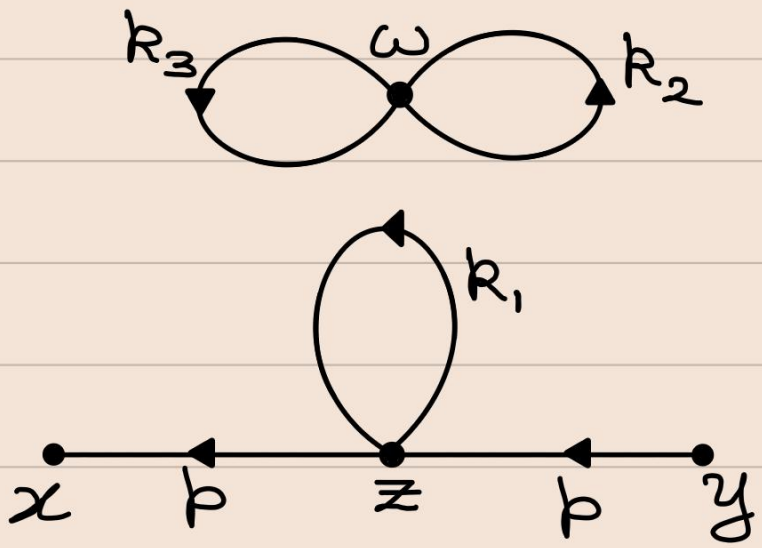


$$= \frac{(-i\lambda)^2}{128}\int \frac{d^4p\, d^4k_1 d^4k_2 d^4k_3 d^4k_4}{(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}$$

$$\times\left[(2\pi)^4 \delta(0)\right]^2 \frac{i}{(k_1^2-m^2+i\varepsilon)} \frac{i}{(k_2^2-m^2+i\varepsilon)}$$

$$\times \frac{i}{(k_3^2-m^2+i\varepsilon)} \frac{i}{(k_4^2-m^2+i\varepsilon)}$$

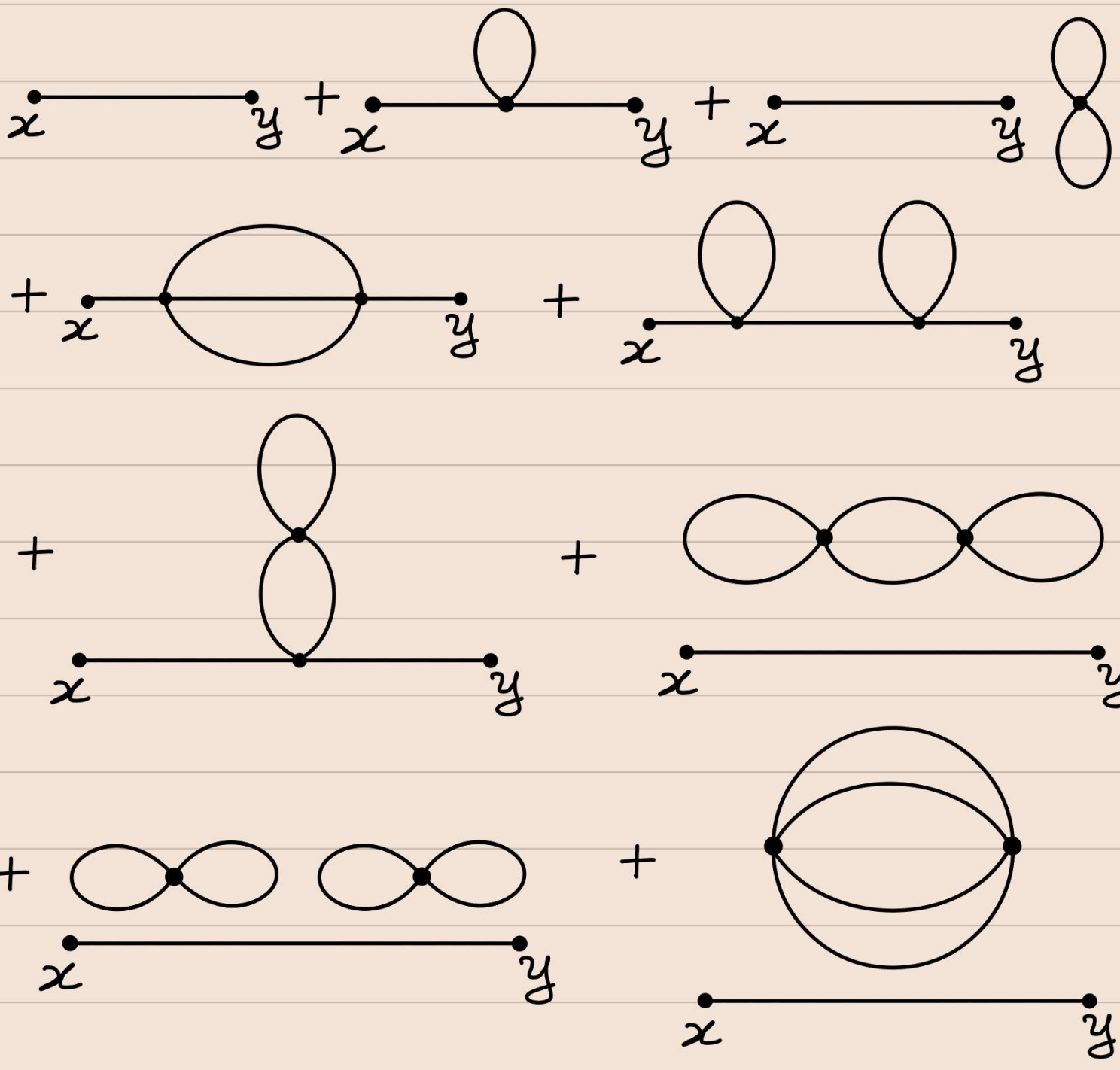
g)



$$= \frac{(-i\lambda)^2}{16}\int \frac{d^4p\, d^4k_1 d^4k_2 d^4k_3}{(2\pi)^4(2\pi)^4(2\pi)^4(2\pi)^4} e^{-ip(x-y)} \frac{(i)^2}{(p^2-m^2+i\varepsilon)^2}$$

$$\times\left[(2\pi)^4 \delta(0)\right] \frac{i}{(k_1^2-m^2+i\varepsilon)} \frac{i}{(k_2^2-m^2+i\varepsilon)} \frac{i}{(k_3^2-m^2+i\varepsilon)}$$

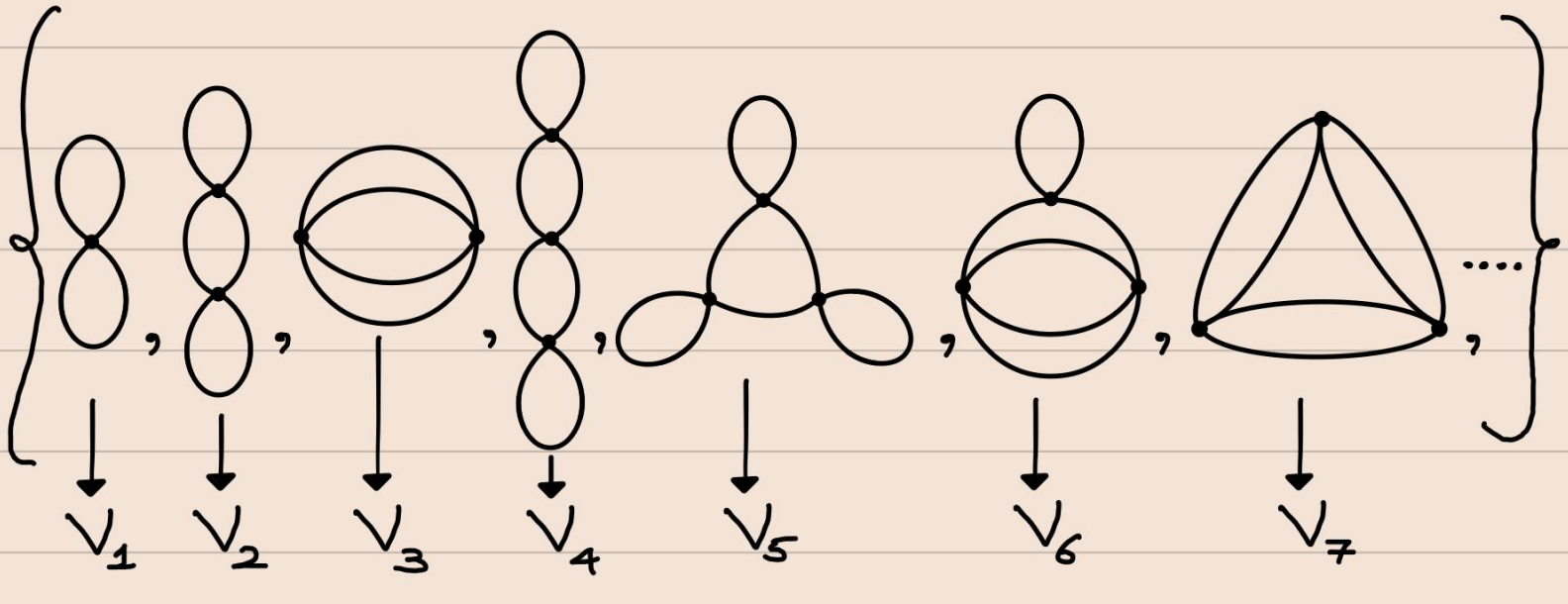
Thus we can express $\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{S}]|0\rangle$ as an infinite sum of following diagrams



The presence of vacuum bubbles is not troublesome since it can be factored out from diagrams containing external points 'x' & 'y'. So the problem of $(2\pi)^4 \delta(0)$ factor can be treated.

Any diagram will have one piece that which is CONNECTED to the external points. Hence, called as 'Connected' piece. And then there are many Vacuum bubble or disconnected pieces where each piece will carry a $(2\pi)^4 \delta(0)$ factor.

Let all possible disconnected pieces be collected in a set $\{V_i\}$ shown below,



Suppose that a diagram has 'n_i' number of a particular disconnected piece 'V_i'. So, then the value of that diagram will be of the form,

$$\left(\begin{matrix}\text{Value of}\\ \text{Connected piece}\end{matrix}\right) \times \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

The $\frac{1}{n!}$ is to take care of the Symmetry factor coming from the interchange of n_i copies of V_i.

Then, Sum of all possible such diagrams is the numerator $\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{S}]|0\rangle$,

$$= \sum_{\{\text{connected}\}} \sum_{\{n_1, n_2, \cdots\}} \left(\begin{matrix}\text{Value of}\\ \text{Connected piece}\end{matrix}\right) \times \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

$\{\text{connected}\}$ → all possible connected pieces.

$\{n_1, n_2, \cdots\}$ → all ordered set $\{n_i\}$

$$= \sum_{\{\text{connected}\}} \left(\begin{matrix}\text{Value of}\\ \text{Connected piece}\end{matrix}\right) \sum_{\{n_i\}} \prod_i \frac{1}{n_i!}(V_i)^{n_i}$$

$$= \sum_{\{\text{connected}\}} \left(\begin{matrix}\text{Value of} \\ \text{Connected piece}\end{matrix}\right) \prod_i \underbrace{\sum_{n_i=1}^{\infty} \frac{1}{n_i!} (V_i)^{n_i}}_{e^{V_i}}$$

$$= \sum_{\{\text{connected}\}} \left(\begin{matrix}\text{Value of} \\ \text{Connected piece}\end{matrix}\right) \prod_i e^{V_i}$$

$$= \sum_{\{\text{connected}\}} \left(\begin{matrix}\text{Value of} \\ \text{Connected piece}\end{matrix}\right) e^{\sum_i V_i}$$

Thus we get,

$$\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{S}]|0\rangle = \sum_{\{\text{connected}\}} \left(\begin{matrix}\text{Value of} \\ \text{Connected piece}\end{matrix}\right) e^{\sum_i V_i}$$

Thus we have showed that the sum of all possible feynman diagrams of ϕ^4 interaction of scalar fields with two external lines (including all connected & vacuum bubbles) is equal to the sum of all connected pieces multiplied with exponential of sum of all vacuum bubbles.

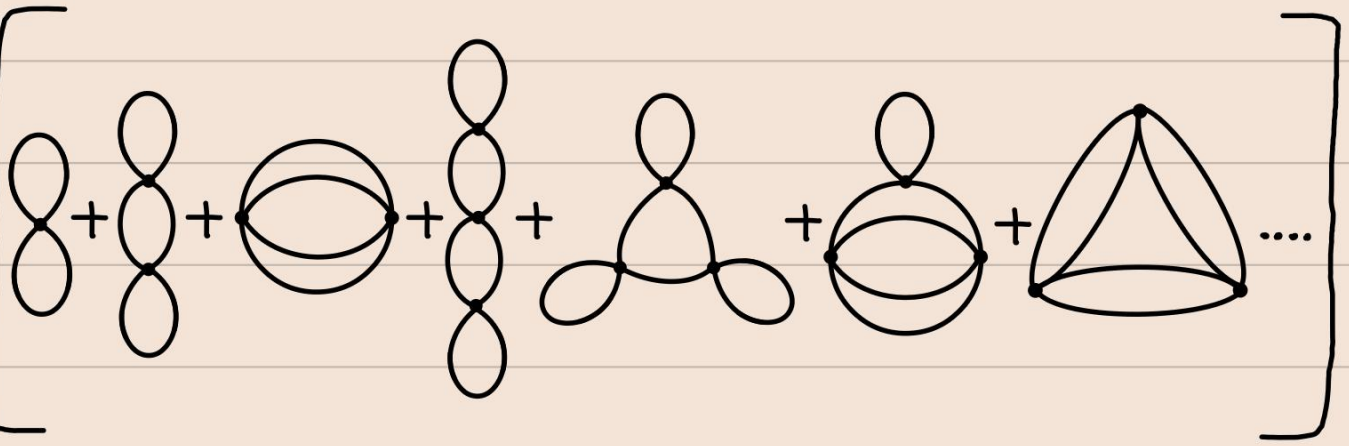
In a similar manner we can show that,

$$\langle 0|\hat{S}|0\rangle = \langle 0|T\left[e^{-i\int d^4x \frac{\lambda\phi^4}{4!}}\right]|0\rangle$$

can be written as the exponential of the sum of all vacuum bubbles.

$$\rightarrow \langle 0|\hat{S}|0\rangle = e^{\sum_i V_i}$$

$$= \exp\Bigg[\quad + \quad + \quad + \quad + \quad + \quad + \quad \ldots \Bigg]$$

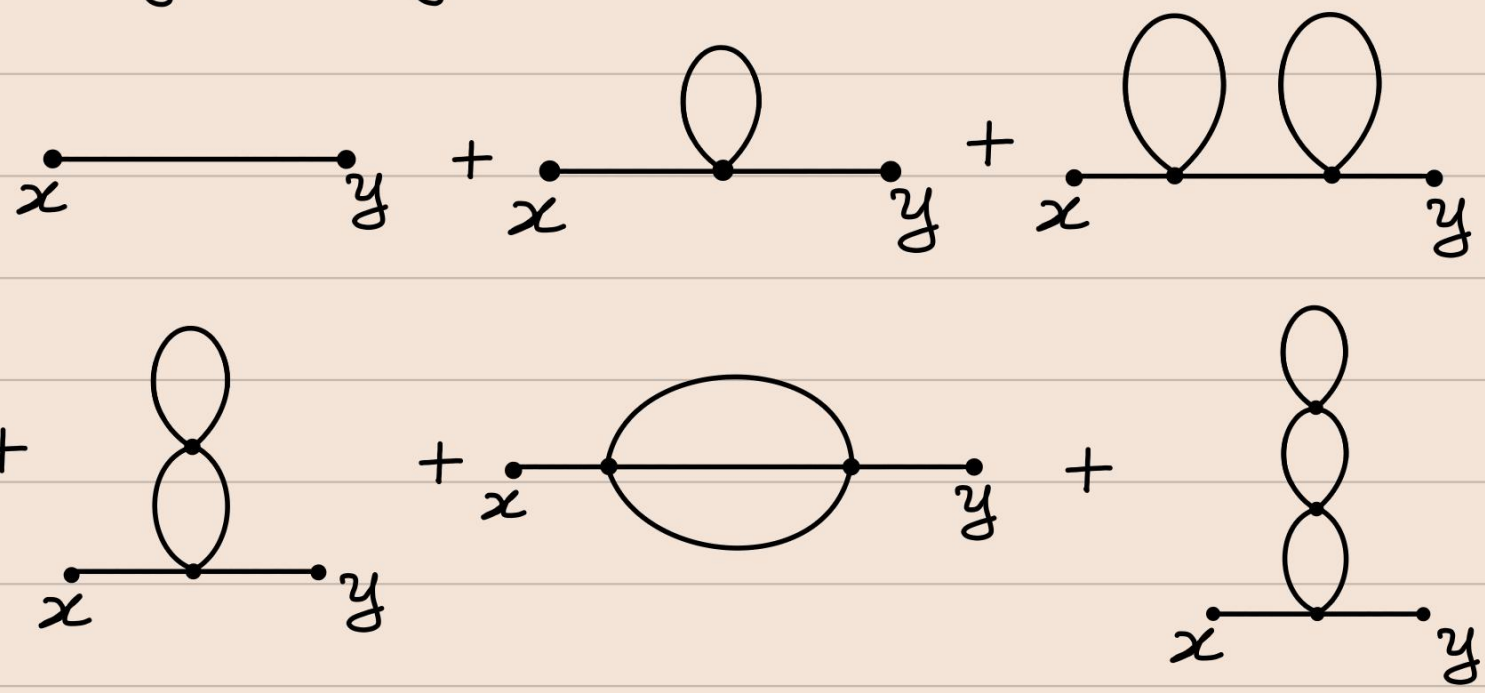


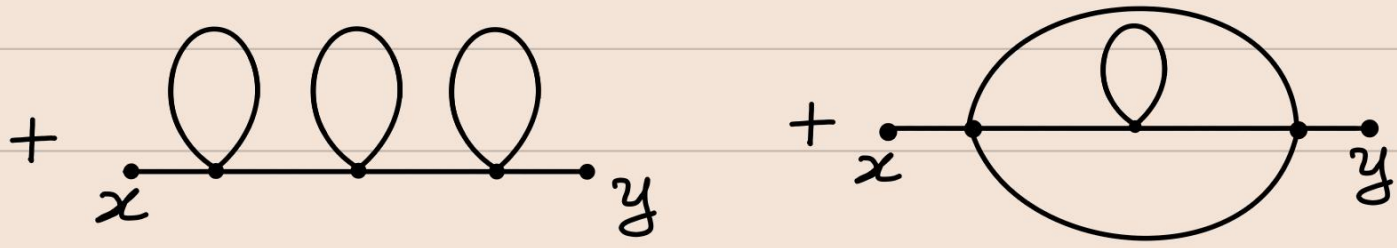
Thus the two point correlation function for ϕ^4 interaction of scalar fields is

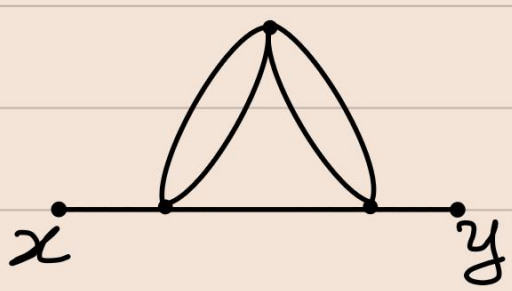
$$\langle \Omega|T[\hat{\phi}(x)\hat{\phi}(y)]|\Omega\rangle = \frac{\langle 0|T[\hat{\phi}(x)\hat{\phi}(y)\hat{S}]|0\rangle}{\langle 0|\hat{S}|0\rangle}$$

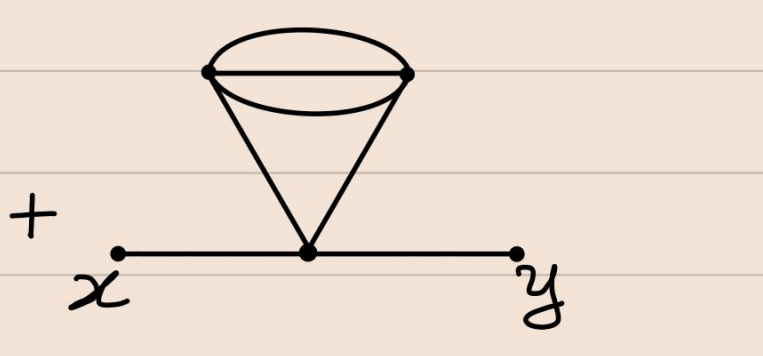
$$\rightarrow \boxed{\langle \Omega|T[\hat{\phi}(x)\hat{\phi}(y)]|\Omega\rangle = \sum \left(\begin{array}{c}\text{Sum of all}\\ \text{Connected pieces}\\ \text{with two}\\ \text{external points}\end{array}\right)}$$

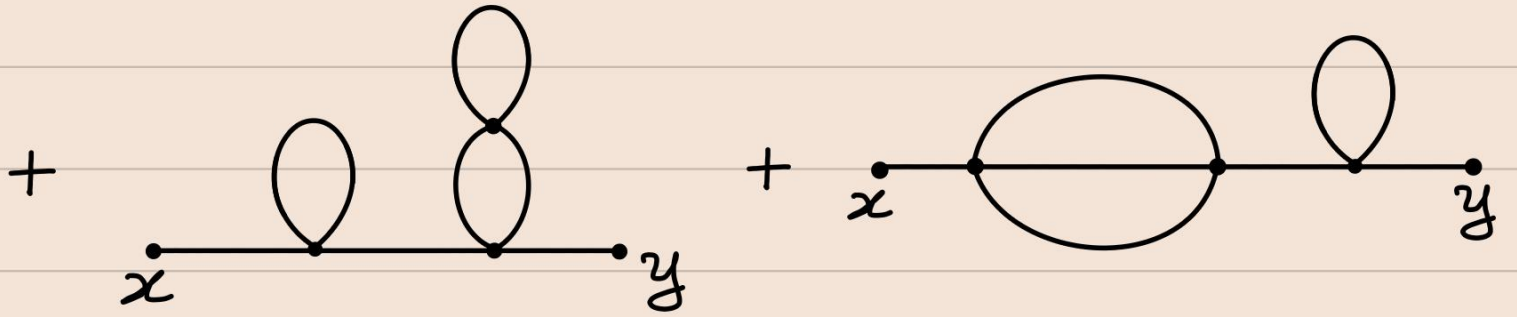
Thus diagrammatically, $\langle \Omega | T[\hat{\phi}(x)\hat{\phi}(y)] | \Omega \rangle$ is given by,

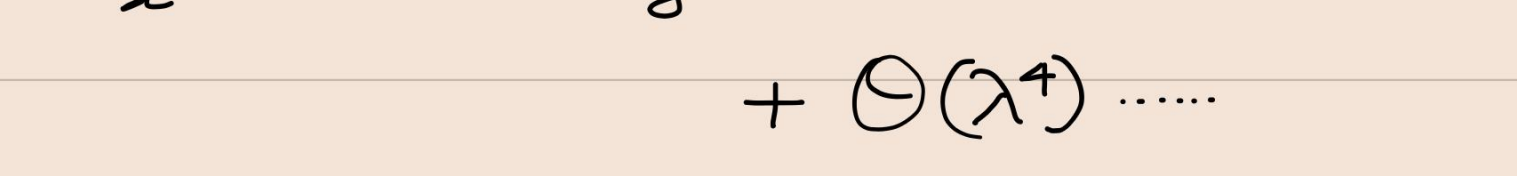












$+ \Theta(\lambda^4) \ldots\ldots$

The higher Correlation functions can be got by Similar Analysis.

$\langle \Omega | T[\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\ldots\ldots\hat{\phi}(x_n)] | \Omega \rangle$ is given by, $\sum$ (Sum of all Connected pieces with 'n' external points)

Note that in ϕ^4 theory, all correlation functions of odd number of fields are zero, which can be seen by Wick's theorem, also diagrammatically, it is impossible to draw an allowed diagram with odd number of external points where every vertex has exactly 4 lines attached.

$$\longrightarrow \langle \Omega | T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(z)] | \Omega \rangle = 0$$

But $\langle \Omega | T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(w)] | \Omega \rangle$ is the 4 point correlation function given by the Sum of all the connected diagrams with 4 external points.

$\rightarrow \langle \Omega | T[\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(z)\hat{\phi}(w)] | \Omega \rangle =$

