

Yukawa's $\psi^+\psi\phi$ interactions

The lagrangian of this theory is given by,

$$\mathcal{L} = \mathcal{L}_{\text{Scalar } \phi} + \mathcal{L}_{\text{Complex Scalar } \psi} - g \psi^+ \psi \phi$$

where,

$$\mathcal{L}_{\text{Scalar } \phi} = \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} \mu^2 \phi^2$$

$$\mathcal{L}_{\text{Complex Scalar } \psi} = (\partial^\mu \psi^+)(\partial_\mu \psi) - m^2 \psi^+ \psi$$

This lagrangian describes the interaction between the ϕ & ψ fields with coupling constant of 'g'. The mass of ϕ particles is ' μ ' & the mass of ψ particles is 'm'.

Then the Hamiltonian density is,

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\text{Scalar } \phi} + \hat{\mathcal{H}}_{\text{Complex Scalar } \psi} + \hat{\mathcal{H}}_{\text{int}}$$

Thus $\hat{H}_0 = \hat{H}_{\text{scalar}} \phi + \hat{H}_{\text{scalar}}^{\text{complex}} \psi$ gives the non-interacting part which is canonically quantizable.

$\hat{H}_{\text{int}} = g \hat{\psi}^\dagger \hat{\psi} \hat{\phi}$ is the Yukawa's interaction part of the Hamiltonian.

But if we go to the interaction picture, then the field operators $\hat{\psi}_I(x)$, $\hat{\psi}_I^\dagger(x)$ & $\hat{\phi}_I(x)$ are governed by the canonically quantizable part $\hat{H}_0$. So, the mode expansion for $\hat{\psi}_I$ & $\hat{\phi}_I$ in the interaction picture is given by,

$$\hat{\psi}_I(x) = \frac{\int d^3p [\hat{a}(\vec{p}) e^{-ipx} + \hat{b}^\dagger(\vec{p}) e^{+ipx}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\psi}_I^\dagger(x) = \frac{\int d^3p [\hat{a}^\dagger(\vec{p}) e^{+ipx} + \hat{b}(\vec{p}) e^{-ipx}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\& \phi_{\mathbf{I}}(x) = \frac{\int d^3k [\hat{c}(\vec{q}) e^{-i k x} + \hat{c}^\dagger(\vec{q}) e^{+i k x}]}{\sqrt{(2\pi)^3 (2\mathcal{E}_k)}}$$

Now, the equations that $\hat{\Psi}_{\mathbf{I}}, \hat{\Psi}_{\mathbf{I}}^\dagger$ & $\hat{\phi}_{\mathbf{I}}$ solve is the usual K.G equation got by $\hat{H}_0$ part. Thus, the dispersion relation for the particles described by $\Psi_{\mathbf{I}}$ & $\phi_{\mathbf{I}}$ are given by $E_p^2 = |\vec{p}|^2 + m^2$ and $\mathcal{E}_k^2 = |\vec{k}|^2 + \mu^2$ respectively.

Here $\hat{a}/\hat{a}^\dagger$ destroy/create Ψ particle
 $\hat{b}/\hat{b}^\dagger$ destroy/create Ψ anti-particle
 & $\hat{c}/\hat{c}^\dagger$ destroy/create ϕ particle.

$\hat{H}_0 = \int d^3x \hat{H}_0$ is given by,

$$\int d^3p \left(\mathcal{E}_p \hat{c}^\dagger(\vec{p}) \hat{c}(\vec{p}) + E_p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p})] \right)$$

The vacuum state of $\hat{H}_0$ is $|0\rangle$ where,

$$\underline{\hat{a}(\vec{p})|0\rangle = \hat{b}(\vec{p})|0\rangle = \hat{c}(\vec{p})|0\rangle = 0}$$

From now we drop the subscript 'I' for the field operators but still mean that they are in the interaction picture.

The feynmann propagator for the ϕ particle is given by,

$$\Delta_\phi(x, y) = \langle 0 | T [\hat{\phi}(x) \hat{\phi}(y)] | 0 \rangle = \overline{\hat{\phi}(x) \hat{\phi}(y)}$$

And in momentum space,

$$\Delta_\phi(x, y) = \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}$$

where the poles are taken as per the feynmann's prescription.

$$\overline{\hat{\phi}(y) \hat{C}^+(\vec{k})} = \frac{\int d^3 k' e^{-ik'y} \langle 0 | \hat{C}(\vec{k}') \hat{C}^+(\vec{k}) | 0 \rangle}{\sqrt{(2\pi)^3 (2E_{k'})}}$$

$$\rightarrow \overline{\hat{\phi}(x) \hat{C}^+(\vec{k})} = \frac{e^{-ikx}}{\sqrt{(2\pi)^3 (2E_k)}}$$

This gives the probability amplitude of a ϕ particle of four momentum k entering a space-time point at ' x '.

$$\text{Similarly, } \overline{\hat{C}(\vec{k})} \hat{\phi}(x) = \frac{e^{ikx}}{\sqrt{(2\pi)^3 (2E_k)}}$$

which gives the probability amplitude of a ϕ particle of four momentum k leaving from a space-time point ' x '.

The feynmann propagator for the ψ particle is given by,

$$\Delta_\psi(x, y) = \langle 0 | T [\hat{\psi}(x) \hat{\psi}^\dagger(y)] | 0 \rangle = \overline{\hat{\psi}(x) \hat{\psi}^\dagger(y)}$$

And in momentum space,

$$\Delta_\psi(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon}$$

where the poles are taken as per the feynmann's prescription.

- $\overline{\hat{\psi}(x)} \hat{a}^+(\vec{p}) = \frac{e^{-ipx}}{\sqrt{(2\pi)^3(2E_p)}}$

This gives the probability amplitude of a ψ particle of four momentum 'p' to enter a space-time point at 'x'.

- $\hat{a}(\vec{p}) \hat{\psi}^+(x) = \frac{e^{ipx}}{\sqrt{(2\pi)^3(2E_p)}}$ gives the

probability amplitude of a ψ particle of four momentum 'p' to leave from a space-time point 'x'.

- $\overline{\hat{\psi}^+(x)} \hat{b}^+(\vec{p}) = \frac{e^{-ipx}}{\sqrt{(2\pi)^3(2E_p)}}$ gives the

probability amplitude of a ψ antiparticle of four momentum 'p' to enter space-time point at 'x'.

- $\overbrace{\hat{b}(\vec{p}) \hat{\psi}(x)} = \frac{e^{ipx}}{\sqrt{(2\pi)^3(2E_p)}}$ gives the

probability amplitude of a ψ antiparticle of four momentum 'p' to leave from the space-time point 'x'.

Thus among all possible contractions between field operators ϕ, ψ & $\psi^\dagger$ only six contraction combinations

$$\overbrace{\hat{\phi}(x) \hat{c}^\dagger(\vec{k})}, \overbrace{\hat{c}(\vec{k}) \hat{\phi}(x)}, \overbrace{\hat{\psi}(x) \hat{a}^\dagger(\vec{p})},$$

$$\overbrace{\hat{a}(\vec{p}) \hat{\psi}^\dagger(x)}, \overbrace{\hat{\psi}^\dagger(x) \hat{b}^\dagger(\vec{p})} \neq \overbrace{\hat{b}(\vec{p}) \hat{\psi}(x)}$$

Survive and all other combinations are 0.

Then $\overbrace{\hat{\psi}(x) \hat{\psi}^\dagger(y)} \Rightarrow \psi$ particle going from space time point 'y' to 'x'.

And $\overbrace{\hat{\psi}^\dagger(x) \hat{\psi}(y)} \Rightarrow \psi$ anti-particle going from space time point 'y' to 'x'.

S-matrix expansion

The interacting part of hamiltonian in the interaction picture is given by,

$$\hat{H}_I = g \hat{\psi}^\dagger \hat{\psi} \hat{\phi}$$

This is what appears in the dyson's series. The S-matrix operator is given by,

$$\hat{S} = T \left[\exp \left(-ig \int d^4x \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \right) \right]$$

We shall try to calculate the scattering amplitude for process of one ψ particle entering with the four momentum 'p' & one ψ -particle to leave with the four momentum 'q'.

$\rightarrow A = \langle q | \hat{S} | p \rangle \rightarrow$ Scattering Amplitude
where $\langle q |, | p \rangle$ are 4-momentum states.

$$\rightarrow \mathcal{A} = (2\pi)^3 2\sqrt{E_p E_q} \langle 0 | \hat{a}(\vec{v}) \hat{S} \hat{a}^\dagger(\vec{p}) | 0 \rangle$$

$$\text{But } \hat{S} = 1 + T \left((-ig) \int d^4x \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \right. \\ \left. + \frac{(-ig)^2}{2!} \int d^4x d^4y \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \right. \\ \left. + \dots \right)$$

$$\rightarrow \mathcal{A} = (2\pi)^3 2\sqrt{E_p E_q} \left(\langle 0 | \hat{a}(\vec{v}) \hat{a}^\dagger(\vec{p}) | 0 \rangle \right. \\ + (-ig) \int d^4x \langle 0 | T [\hat{a}(\vec{v}) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{a}^\dagger(\vec{p})] | 0 \rangle \\ + \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0 | T \left[\hat{a}(\vec{v}) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \right. \\ \left. \times \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}) \right] | 0 \rangle \\ \left. + \dots \right)$$

Thus we can write the total amplitude as sum of ordered terms of '-ig'

$$\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}^{(n)}$$

$$A^{(0)} = (2\pi)^3 2E_p \delta(\vec{q} - \vec{p}) \equiv \uparrow$$

Note that the $A^{(1)}$ term has Wick's contractions for an odd number of operators so, it is zero.

Consider the second order terms $A^{(2)}$,

$$A^{(2)} = (2\pi)^3 2\sqrt{E_p E_q} \frac{(-ig)^2}{2!} \int d^4x d^4y$$

$$\times \langle 0 | T [\hat{a}(\vec{q}) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p})] | 0 \rangle$$

There are a total of 105 terms in the Wick's expansion but not all of them survive. The only ones to remain are those terms which consists of the

type, $\overbrace{\hat{a}(\vec{p}) \hat{\psi}^\dagger(x)}$, $\overbrace{\hat{\psi}(x) \hat{a}^\dagger(\vec{p})}$,

$\overbrace{\hat{\psi}(x) \hat{\psi}^\dagger(y)}$, $\overbrace{\hat{\psi}^\dagger(x) \hat{\psi}(y)}$, & $\overbrace{\hat{\phi}(x) \hat{\phi}(y)}$

One such surviving term is given by the following contraction.

$$\overbrace{\hat{a}(\vec{v})\hat{\psi}^+(x)} \quad \overbrace{\hat{\psi}(x)\hat{\phi}(x)} \quad \overbrace{\hat{\psi}^+(y)\hat{\psi}(y)} \quad \overbrace{\hat{\phi}(y)\hat{a}^+(\vec{p})}$$

This term represents,

Called as an oyster diagram

But due to symmetry in $x \neq y$ vertices, we can observe that

$$\overbrace{\hat{a}(\vec{v})\hat{\psi}^+(x)} \quad \overbrace{\hat{\psi}(x)\hat{\phi}(x)} \quad \overbrace{\hat{\psi}^+(y)\hat{\psi}(y)} \quad \overbrace{\hat{\phi}(y)\hat{a}^+(\vec{p})}$$

gives exactly the same type of diagram with $x \neq y$ interchanged. So, we can consider two of such terms to represent the same diagram which cancels with the $2!$ in the denominator of $A^{(2)}$.

$\rightarrow \mathcal{A}_{\text{oyster}} =$

When expanded we get

$$\begin{aligned}
 \mathcal{A}_{\text{oyster}} &= (2\pi)^3 2\sqrt{E_p E_q} (-ig)^2 \\
 &\times \int d^4x d^4y \overline{\hat{a}(\vec{v})} \hat{\psi}^+(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^+(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^+(\vec{p}) \\
 &= (2\pi)^3 2\sqrt{E_p E_q} (-ig)^2 \int d^4x d^4y \frac{e^{iqx} e^{-ipy}}{(2\pi)^3 2\sqrt{E_p E_q}} \\
 &\times \int \frac{d^4k_1}{(2\pi)^4} \frac{i e^{-ik_1(x-y)}}{k_1^2 - m^2 + i\varepsilon} \int \frac{d^4k_2}{(2\pi)^4} \frac{i e^{-ik_2(x-y)}}{k_2^2 - \mu^2 + i\varepsilon}
 \end{aligned}$$

Now as seen in previous theory, the $(2\pi)^3 2\sqrt{E_p E_q}$ factors gets cancelled.

$$\rightarrow A_{\text{oyster}} = (-ig)^2 \int d^4x d^4y e^{iqx} e^{-ipy}$$

$$\times \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i e^{-ik_1(x-y)}}{k_1^2 - m^2 + i\epsilon} \frac{i e^{-ik_2(x-y)}}{k_2^2 - \mu^2 + i\epsilon}$$

$$= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - \mu^2 + i\epsilon}$$

$$\times \int d^4x d^4y e^{iqx} e^{-ipy} e^{-ik_1(x-y)} e^{-ik_2(x-y)}$$

$$= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - \mu^2 + i\epsilon}$$

$$\times \int d^4x d^4y e^{i(q-k_1-k_2)x} e^{-i(p-k_1-k_2)y}$$

(Perform y integral)

$$= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - \mu^2 + i\epsilon}$$

$$\times \int d^4x e^{i(q-k_1-k_2)x} (2\pi)^4 \delta^{(4)}(p-k_1-k_2)$$

(Perform k_2 integral
& replace $k_1 \rightarrow k$)

$$\rightarrow A_{\text{oyster}} = (-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - \mu^2 + i\epsilon} \times \int d^4 x e^{i(q-p)x}$$

(Perform $d^4 x$ integral)

Thus we get in the momentum space as,

$$A_{\text{oyster}} = (2\pi)^4 \delta^{(4)}(q-p) (-ig)^2 \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p-k)^2 - \mu^2 + i\epsilon}$$

$$A_{\text{oyster}} = \text{Diagram}$$

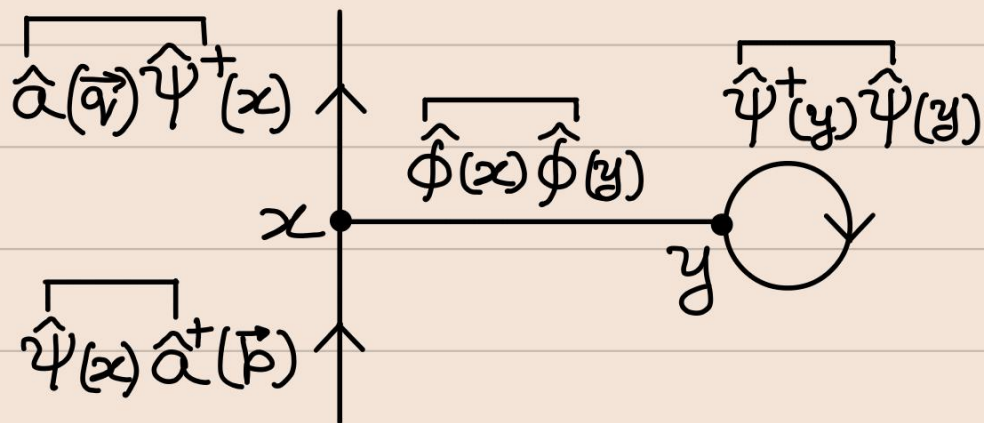
The lines with an arrow $\uparrow$ represent the ψ particle with $\overline{\psi}\psi$ contraction.

and $\downarrow$ represents $\overline{\psi^+}\psi$ contraction that represent ψ anti particle. Then, lastly the lines with no arrows will represent $\overline{\phi}\phi$ contraction.

Another surviving term in $A^{(2)}$ expansion is given by the following contraction.

$$\overbrace{\hat{a}(\vec{a})\hat{\psi}^+(x)} \quad \overbrace{\hat{\psi}(x)\hat{\phi}(x)} \quad \overbrace{\hat{\psi}^+(y)\hat{\psi}(y)} \quad \overbrace{\hat{\phi}(y)\hat{a}^+(\vec{b})}$$

This term represents Tadpole diagram



But due to symmetry in $x \neq y$ vertices, we can observe that

$$\overbrace{\hat{a}(\vec{a})\hat{\psi}^+(x)} \quad \overbrace{\hat{\psi}(x)\hat{\phi}(x)} \quad \overbrace{\hat{\psi}^+(y)\hat{\psi}(y)} \quad \overbrace{\hat{\phi}(y)\hat{a}^+(\vec{b})}$$

gives exactly the same type of diagram with $x \neq y$ interchanged. So, we can consider two of such terms to represent the same diagram which cancels with the $2!$ in the denominator of $A^{(2)}$.

$$\rightarrow \mathcal{A}_{\text{tadpole}} =$$

When this term is expanded, we get,

$$\begin{aligned} \mathcal{A}_{\text{tadpole}} &= (2\pi)^3 2 \sqrt{E_p E_q} (-ig)^2 \\ &\times \int d^4x d^4y \overbrace{\hat{a}(\vec{q}) \hat{\psi}^+(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^+(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^+(\vec{p})}^{\text{Diagram}} \\ &= (2\pi)^3 2 \sqrt{E_p E_q} (-ig)^2 \int d^4x d^4y \frac{e^{iqx} e^{-ipx}}{(2\pi)^3 2 \sqrt{E_p E_q}} \\ &\times \int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \int \frac{d^4k_2}{(2\pi)^4} \frac{i e^{-ik_2(x-y)}}{k_2^2 - \mu^2 + i\epsilon} \\ &= (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - \mu^2 + i\epsilon} \\ &\times \int d^4x d^4y e^{iqx} e^{-ipx} e^{-ik_2(x-y)} \end{aligned}$$

$$\begin{aligned}
&= (-ig)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - \mu^2 + i\varepsilon} \\
&\quad \times \int d^4 x d^4 y e^{i(q-p-k_2)x} e^{ik_2 y} \\
&\quad \left(\text{Perform } y \text{ integral} \right)
\end{aligned}$$

$$\begin{aligned}
&= (-ig)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - \mu^2 + i\varepsilon} \\
&\quad \times \int d^4 x e^{i(q-p-k_2)x} (2\pi)^4 \delta^{(4)}(k_2) \\
&\quad \left(\text{Perform } k_2 \text{ integral} \right. \\
&\quad \left. \& \text{ replace } k_1 \rightarrow k \right)
\end{aligned}$$

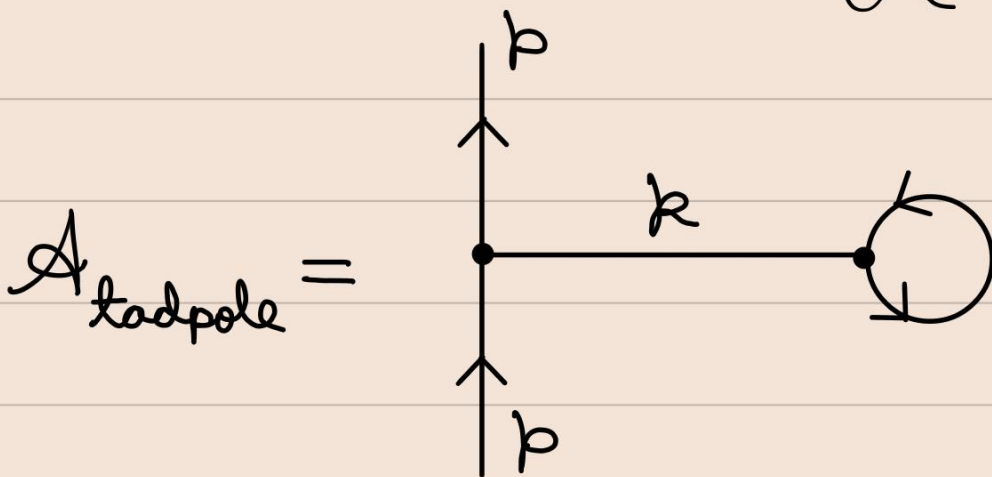
$$\begin{aligned}
&= (-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{0^2 - \mu^2 + i\varepsilon} \\
&\quad \times \int d^4 x e^{i(q-p)x}
\end{aligned}$$

$$\begin{aligned}
&= (2\pi)^4 \delta^{(4)}(q-p) (-ig)^2 \\
&\quad \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{0^2 - \mu^2 + i\varepsilon}
\end{aligned}$$

Thus the loop of $\overline{\hat{\psi}(y)}\hat{\psi}(y)$ or $\overline{\hat{\psi}(x)}\hat{\psi}(x)$ has no momentum which implies that it has no ψ particles or $\bar{\psi}$ anti particles entering / leaving the vertexe.

Thus we get in the momentum space as,

$$A_{\text{tadpole}} = (2\pi)^4 \delta^{(4)}(q-p) (-ig)^2 \frac{i}{0^2 - \mu^2 + i\varepsilon} \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

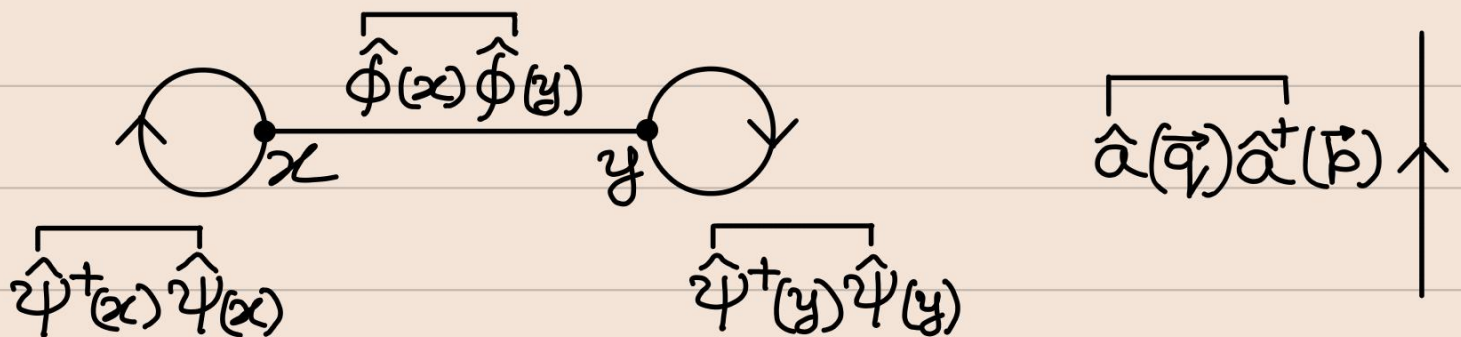


Notice that the symmetry factors for this theory are all '1' for every feynmann diagram.

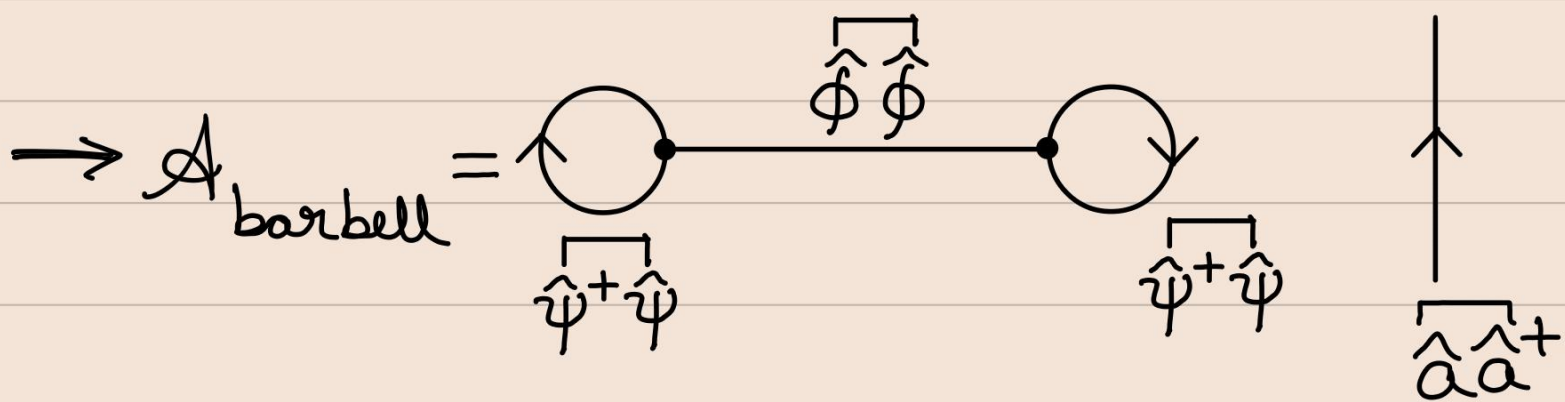
There are also disconnected diagrams possible in 2nd order scattering amplitudes. One such diagram is given by the below contraction,

$$\overbrace{\hat{a}(\vec{q}) \hat{\psi}^+(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^+(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^+(\vec{p})}^{\text{Barbell diagram}}$$

This term represents Bar-bell diagram



Again due to Symmetry in x & y , interchanging them results in same diagram. Thus two Wick's contractions give out the same type of Bar-bell diagram. Thus it cancels with the $2!$ in the denominator of $A^{(2)}$.



When this term is expanded, we get,

$$\begin{aligned}
 \mathcal{A}_{\text{barbell}} &= (2\pi)^3 2\sqrt{E_p E_q} (-ig)^2 \\
 &\times \int d^4x d^4y \hat{a}(\vec{v}) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}) \\
 &= (2\pi)^3 2\sqrt{E_p E_q} \delta(\vec{v} - \vec{p}) (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \\
 &\times \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - m^2 + i\epsilon} \frac{i}{k_3^2 - \mu^2 + i\epsilon} \\
 &\times \int d^4x d^4y e^{-ik_3(x-y)} \\
 &= (2\pi)^3 2E_p \delta(\vec{v} - \vec{p}) (-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \\
 &\times \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - m^2 + i\epsilon} \frac{i}{0^2 - \mu^2 + i\epsilon} \underbrace{\int d^4x}_{(2\pi)^4 \delta(0)}
 \end{aligned}$$

$$\begin{aligned} \Rightarrow A_{\text{barbell}} &= (2\pi)^3 2E_p \delta(\vec{q} - \vec{p}) (-ig)^2 \frac{i}{0^2 - \mu^2 + i\epsilon} \\ &\times \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - m^2 + i\epsilon} \\ &\times (2\pi)^4 \delta(0) \end{aligned}$$

Thus we see that every vacuum bubble piece of a diagram carries the $(2\pi)^4 \delta(0)$ factor.

If $(2\pi)^3 2E_p \delta(\vec{q} - \vec{p}) \equiv \uparrow$, and

$$\begin{aligned} \text{Diagram} &\equiv (2\pi)^4 \delta(0) (-ig)^2 \frac{i}{0^2 - \mu^2 + i\epsilon} \\ &\times \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{i}{k_1^2 - m^2 + i\epsilon} \frac{i}{k_2^2 - m^2 + i\epsilon} \end{aligned}$$

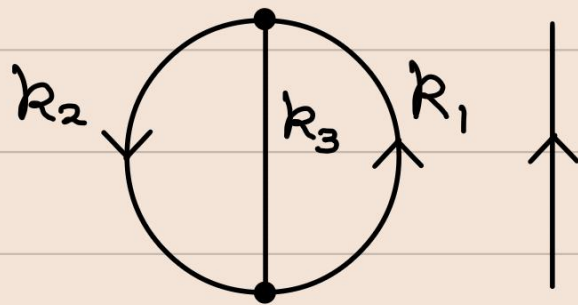
$$\text{Then } A_{\text{barbell}} = \uparrow \text{Diagram}$$

is the product of two disconnected diagrams.

The last type of vacuum diagram possible in the 2nd order expansion is given by the following Wick's contraction

$$\hat{a}(\vec{v}) \hat{\psi}^+(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^+(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^+(\vec{p})$$

which give the following diagram.



$$\begin{aligned}
 &= (2\pi)^3 2 \sqrt{E_p E_q} \delta(\vec{v} - \vec{p}) (-ig)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4} \\
 &\quad \times \int d^4 x d^4 y \frac{i e^{-ik_1(x-y)}}{k_1^2 - m^2 + i\varepsilon} \frac{i e^{ik_2(x-y)}}{k_2^2 - m^2 + i\varepsilon} \frac{i e^{-ik_3(x-y)}}{k_3^2 - \mu^2 + i\varepsilon} \\
 &= (2\pi)^3 2 \sqrt{E_p E_q} \delta(\vec{v} - \vec{p}) (-ig)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4} \\
 &\quad \times \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - \mu^2 + i\varepsilon} \\
 &\quad \times \int d^4 x d^4 y e^{i(k_2 - k_1 - k_3)x} e^{i(k_1 - k_2 - k_3)y}
 \end{aligned}$$

$$= (2\pi)^3 2E_p \delta(\vec{q} - \vec{p}) (-ig)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4}$$

$$\times \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{k_3^2 - \mu^2 + i\varepsilon}$$

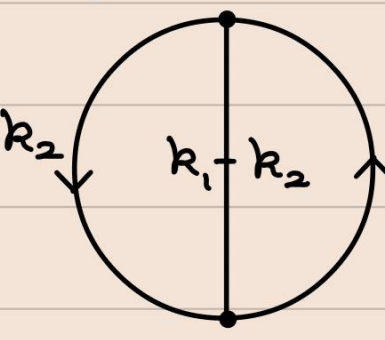
$$\times \int d^4 x e^{i(k_2 - k_1 - k_3)x} (2\pi)^4 \delta^{(4)}(k_1 - k_2 - k_3)$$

$$= (2\pi)^3 2E_p \delta(\vec{q} - \vec{p}) (-ig)^2 (2\pi)^4 \delta(0) \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4}$$

$$\times \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon} \frac{i}{(k_1 - k_2)^2 - \mu^2 + i\varepsilon}$$

$$\rightarrow \text{Diagram} = (2\pi)^3 2E_p \delta(\vec{q} - \vec{p})$$

2



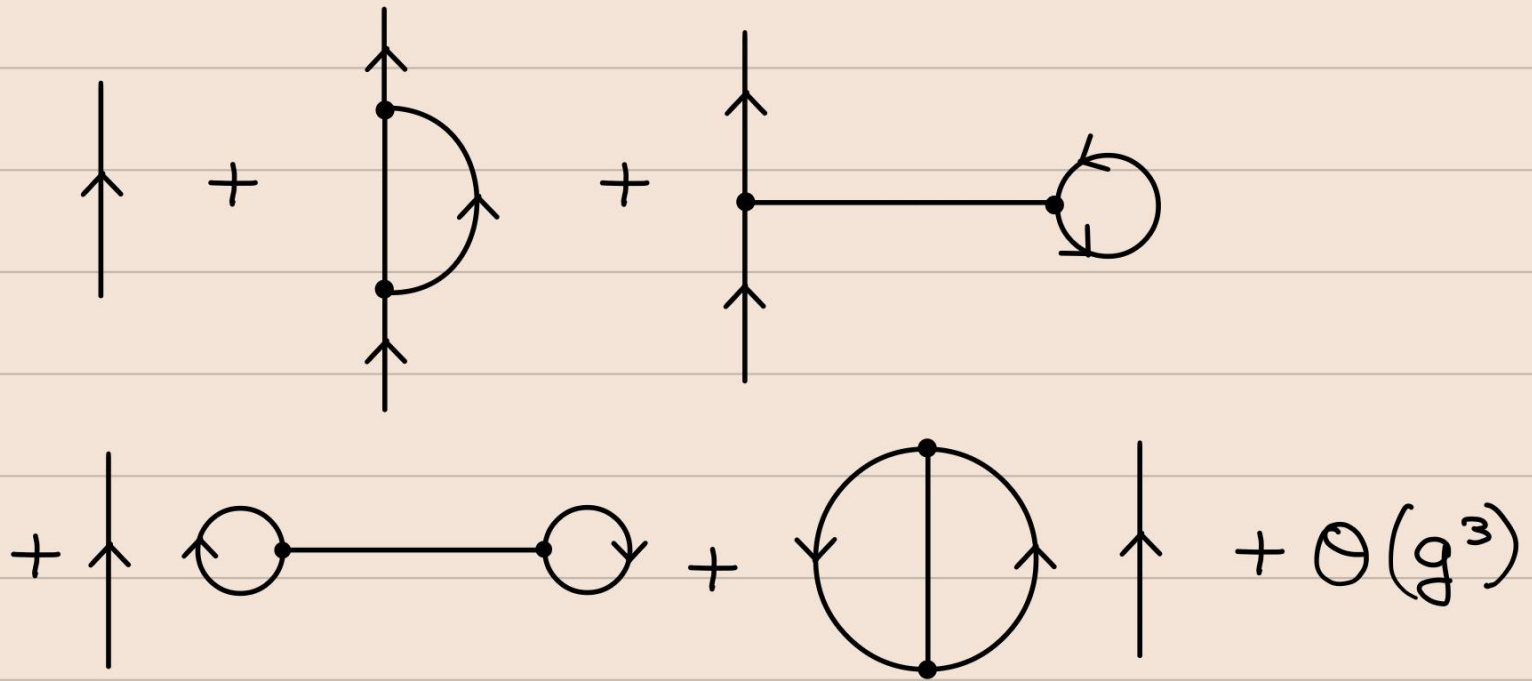
$$= (2\pi)^4 \delta(0) (-ig)^2 \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4}$$

$$\times \frac{i}{k_1^2 - m^2 + i\varepsilon} \frac{i}{k_2^2 - m^2 + i\varepsilon}$$

$$\times \frac{i}{(k_1 - k_2)^2 - \mu^2 + i\varepsilon}$$

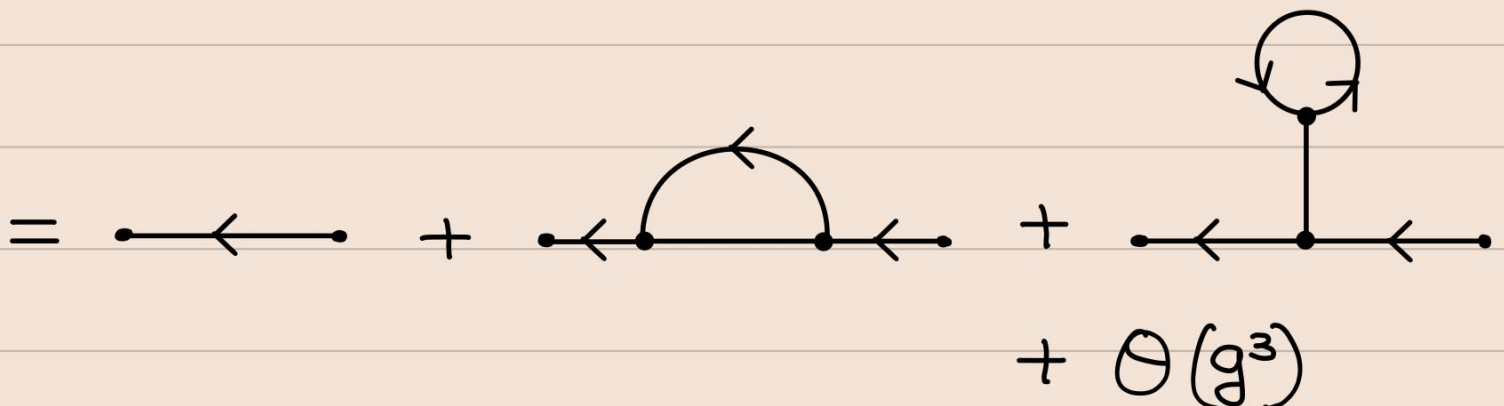
Thus we get

$$A = (2\pi)^3 2\sqrt{E_p E_q} \langle 0 | \hat{a}(\vec{q}) \hat{S} \hat{a}^\dagger(\vec{p}) | 0 \rangle \quad \text{as}$$



It should be observed that the 2 point correlation function is given by

$$\langle \Omega | T[\hat{\psi}(x) \hat{\psi}^\dagger(y)] | \Omega \rangle = \sum \left(\begin{array}{l} \text{Sum of all} \\ \text{Connected pieces} \\ \text{with two} \\ \text{external points} \end{array} \right)$$



Feynman rules for Scalar Yukawa theory

- a) Each vertex contributes a factor of $-ig$, which connects to three lines. Two of which are for ' ψ ' propagators & one for ϕ propagator
- b) Each ϕ particle line has a factor of $\frac{i}{k^2 - \mu^2 + i\epsilon}$ which carries a four momentum ' k '. Similarly the ψ -particle / anti particle propagator line has a factor of $\frac{i}{k^2 - m^2 + i\epsilon}$ with four momentum ' k '.
- c) Impose momentum conservation at all vertices. And every undetermined momenta has an integral with measure $\int \frac{d^4k}{(2\pi)^4}$
- d) Incoming & outgoing external lines has a factor of 1.

e) All Symmetry factors are 1.

f) For diagrams connected with external lines include overall four-momentum conserving dirac delta func. of $(2\pi)^4 \delta^{(4)}(q-p)$.

g) Every vacuum bubble piece carries a factor of $(2\pi)^4 \delta(0)$. And the line $\begin{matrix} q \\ \uparrow \\ p \end{matrix} \hat{a} \hat{a}^+$ is $(2\pi)^3 2E_p \delta(\vec{q}-\vec{p})$.

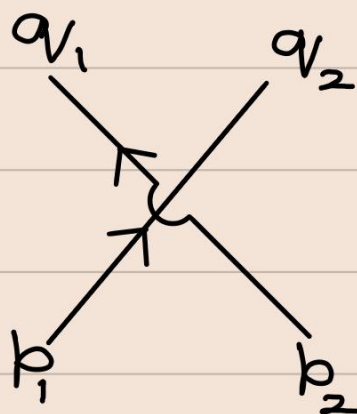
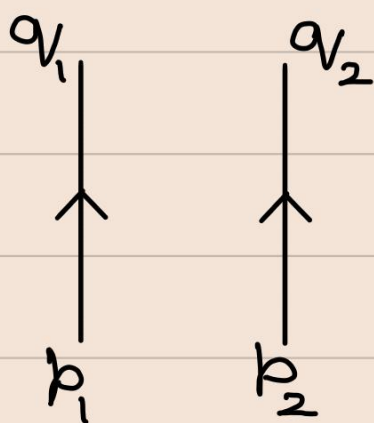
h) ϕ lines have no arrows. But ψ lines have arrows whose direction is based on if it is a particle or anti particle. The anti-particle has arrow in opposite to momentum w.r.t the time (sort of going back in time). ψ particle lines have arrow in the direction of increasing time, along the momentum direction.

Two Particle scattering Amplitudes

Here we shall consider two ψ particles entering & two ψ particles leaving. So the S -matrix element we need is,

$$A(p_1, p_2, q_1, q_2) = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \\ \times \langle 0 | \hat{a}(\vec{q}_1) \hat{a}(\vec{q}_2) \hat{S} \hat{a}^\dagger(\vec{p}_1) \hat{a}^\dagger(\vec{p}_2) | 0 \rangle$$

Here the 0^{th} order terms are given by,



And all 1^{st} order terms are zero since VEV of odd number (7) of operators can not have any saturated Wick's contraction terms.

2nd order Scattering processes

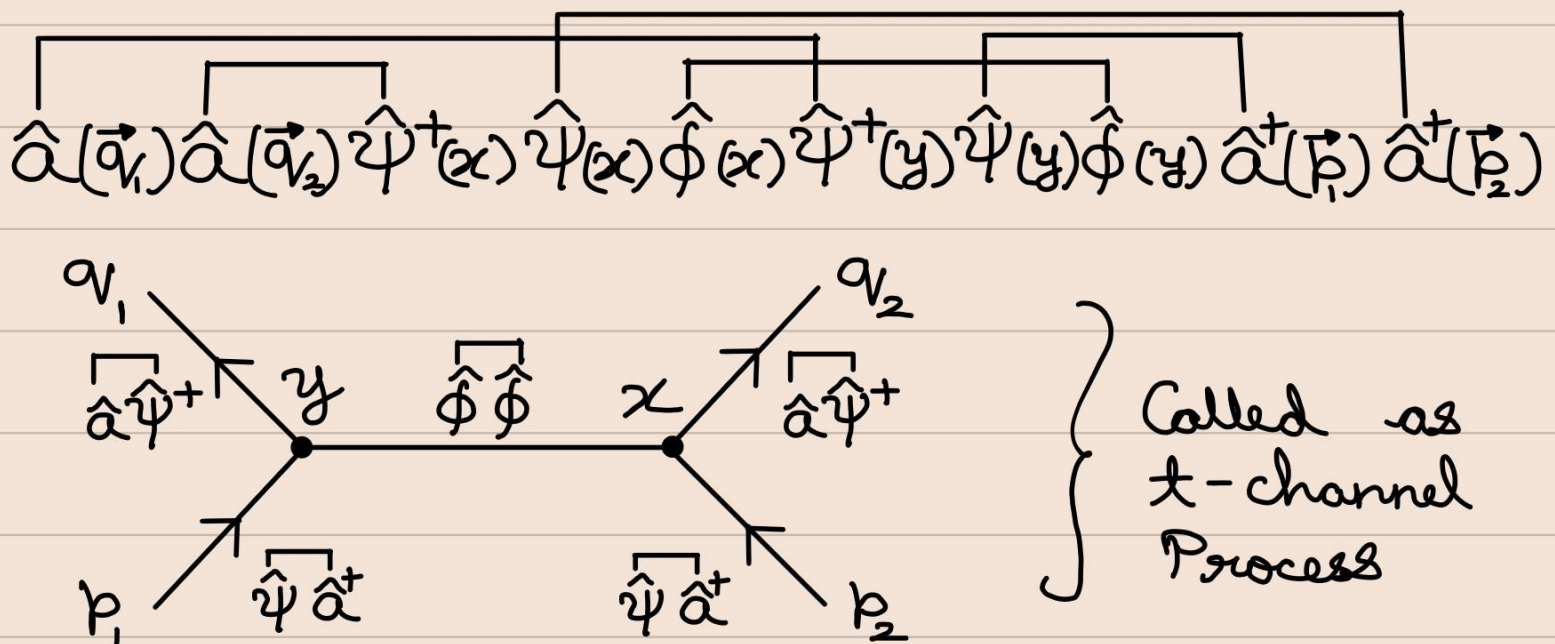
The second order Scattering amplitude is,

$$A^{(2)}(p_1, p_2, q_1, q_2) = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}$$

$$\times \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0 | T \left[\hat{a}(\vec{q}_1) \hat{a}(\vec{q}_2) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \right. \\ \left. \times \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(p_1) \hat{a}^\dagger(p_2) \right] | 0 \rangle$$

Here we shall look into the interacting diagrams where two ψ particles mix using a ϕ particle.

One such diagram is given by the following contraction.



t-channel Scattering

The amplitude for t-channel scattering is given by,

$$A_t = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!}$$

$$\begin{aligned} & \times \int d^4x d^4y \hat{a}(\vec{q}_1) \hat{a}(\vec{q}_2) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}_1) \hat{a}^\dagger(\vec{p}_2) \\ & = (-ig)^2 \int d^4x d^4y e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 x} \Delta_\phi(x, y) \end{aligned}$$

Note that interchanging x & y result in the same process. So, it has twice the weight which cancel with $2!$ in denominator

On going to momentum space, we get

$$\begin{aligned} A_t & = (-ig)^2 \int d^4x d^4y e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 x} \\ & \quad \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\varepsilon} \end{aligned}$$

$$= (-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4 x e^{-i(k+k_2-q_2)x} \\ \times \int d^4 y e^{-i(p_1-k-q_1)y}$$

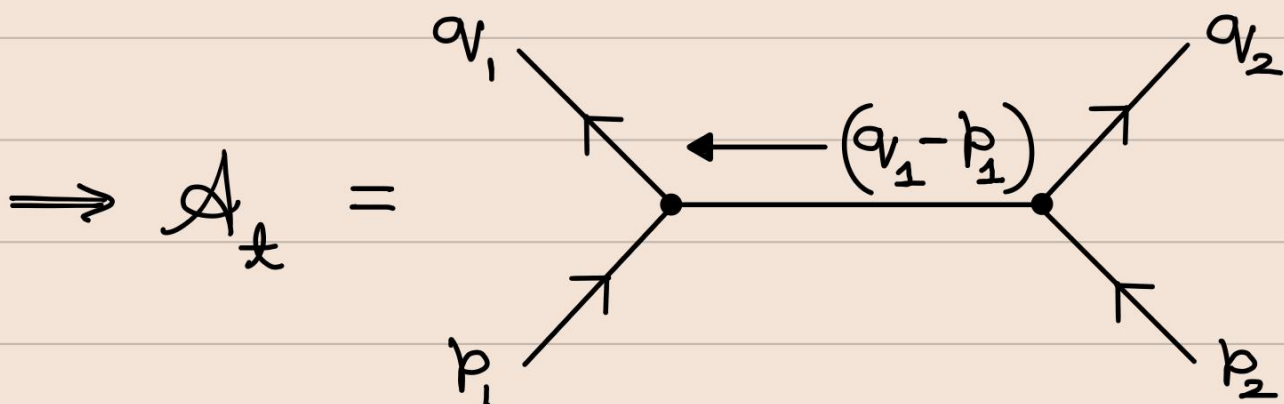
$$= (-ig)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4 x e^{-i(k+k_2-q_2)x} \\ \times (2\pi)^4 \underbrace{\delta^{(4)}(p_1 - k - q_1)}_{k = -(q_1 - p_1)}$$

(Perform 'k' integral)

$$= (-ig)^2 \frac{i}{(q_1 - p_1)^2 - \mu^2 + i\varepsilon} \int d^4 x e^{-i(p_1 - q_1 + k_2 - q_2)x}$$

If we denote $t = (q_1 - p_1)^2$, then we get the amplitude of t -channel process as,

$$\mathcal{A}_t = (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2) (-ig)^2 \frac{i}{t - \mu^2 + i\varepsilon}$$



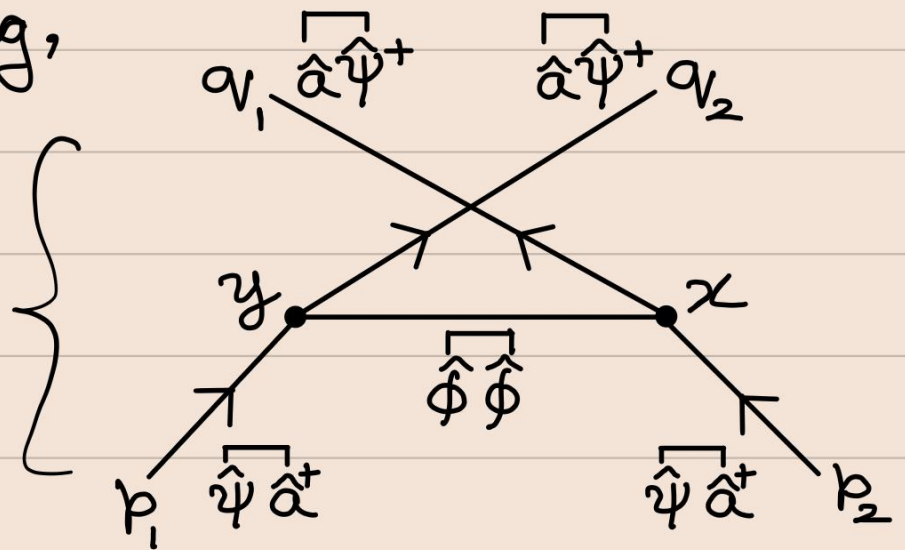
u-channel Scattering

Another similar scattering process is got by the below contraction.

$$\hat{a}(\vec{q}_1) \hat{a}(\vec{q}_2) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}_1) \hat{a}^\dagger(\vec{p}_2)$$

This is similar to t-channel, except that the ψ particles interchange after ϕ particle interaction. It is represented by the following,

Called as
u-channel
Process



The amplitude for u-channel scattering is,

$$\mathcal{A}_u = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!}$$

$$\times \int d^4x d^4y \hat{a}(\vec{q}_1) \hat{a}(\vec{q}_2) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}_1) \hat{a}^\dagger(\vec{p}_2)$$

$$= (-ig)^2 \int d^4x d^4y e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x} \Delta_\phi(x, y)$$

Note that interchanging x & y result in the same process. So, it has twice the weight which cancel with $2!$ in denominator. On going to momentum space, we get,

$$A_u = (-ig)^2 \int d^4x d^4y e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x} \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon}$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \int d^4x e^{-i(k+p_2-q_1)x} \times \int d^4y e^{-i(p_1-k-q_2)y}$$

(Perform ' x ' integral)

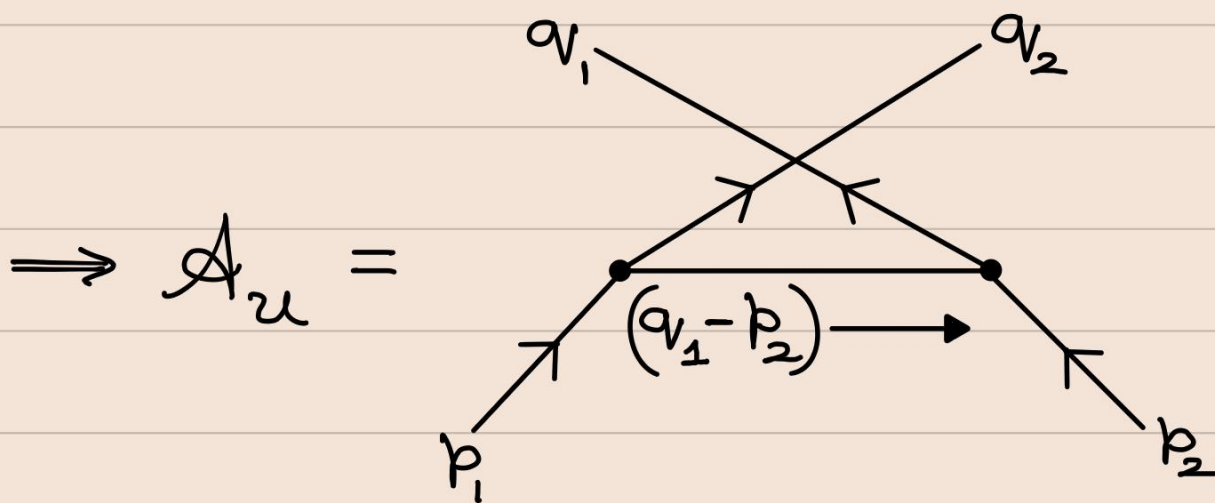
$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} (2\pi)^4 \delta^{(4)}(\overbrace{k+p_2-q_1}^{k=q_1-p_2}) \times \int d^4y e^{-i(p_1-k-q_2)y}$$

(Perform ' k ' integral)

$$\rightarrow \mathcal{A}_u = (-ig)^2 \frac{i}{(q_1 - p_2)^2 - \mu^2 + i\epsilon} \int d^4x e^{-i(p_1 - q_1 + p_2 - q_2)x}$$

If we denote $u = (q_1 - p_2)^2$, then we get the amplitude of u -channel process as,

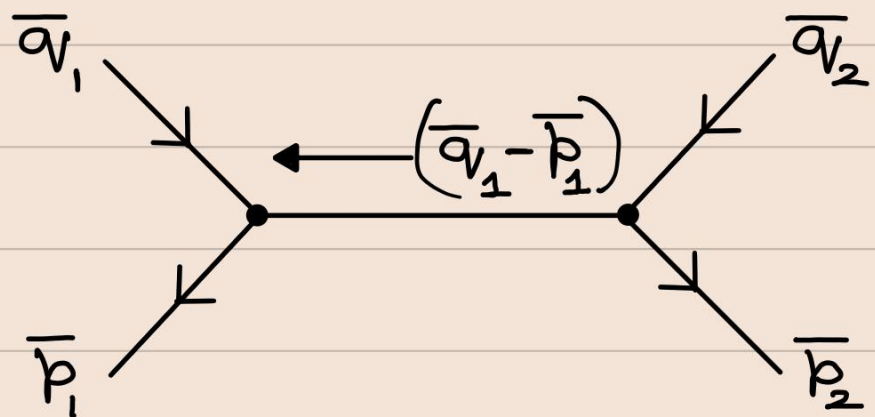
$$\mathcal{A}_u = (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2) (-ig)^2 \frac{i}{u - \mu^2 + i\epsilon}$$



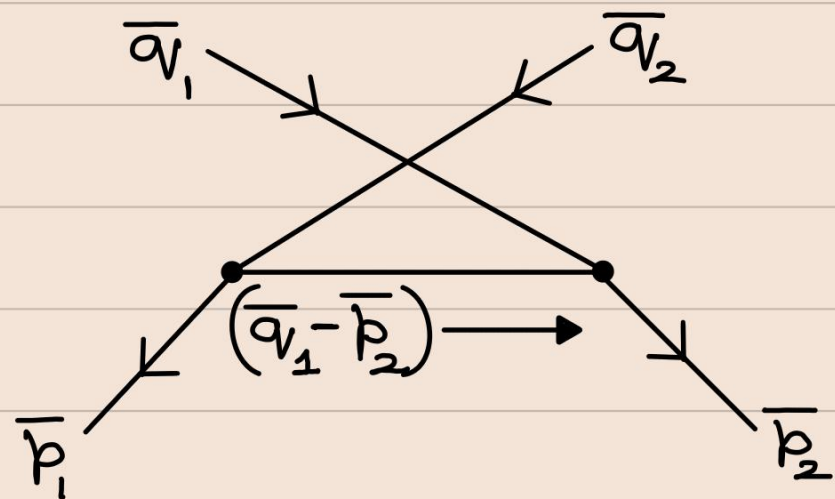
These two processes shows a simple interaction between two Ψ -particles mediated by the ϕ particle. Thus, the ϕ field acts as the force carrier between ' Ψ ' particles. Even if we had two Ψ -anti particles entering & leaving, still we would get exactly same diagrams but with all arrows reversed.

Two Anti-Particle Scattering Amplitudes

If consider two ψ anti-particles entering & two ψ anti-particles leaving, we will get the following 2nd order interacting feynman diagrams.

$$\mathcal{A}_{\bar{t}} =$$


The diagram shows two incoming anti-particles from the top, labeled $\bar{q}_1$ and $\bar{q}_2$, and two outgoing anti-particles from the bottom, labeled $\bar{p}_1$ and $\bar{p}_2$. The lines are oriented downwards, indicating the flow of anti-particles. A horizontal internal line connects two vertices, with an arrow pointing from right to left and the label $(\bar{q}_1 - \bar{p}_1)$ above it.

$$\mathcal{A}_{\bar{u}} =$$


The diagram shows two incoming anti-particles from the top, labeled $\bar{q}_1$ and $\bar{q}_2$, and two outgoing anti-particles from the bottom, labeled $\bar{p}_1$ and $\bar{p}_2$. The lines are oriented downwards. A horizontal internal line connects two vertices, with an arrow pointing from left to right and the label $(\bar{q}_1 - \bar{p}_2)$ above it.

The bar's on the 4-momentum variables is to indicate that they represent an anti-particle. Also, all arrows reversed, against the time direction.

Interaction b/w One ψ particle and One $\bar{\psi}$ anti-particle

Here we try to look into S-matrix elements of the form $\langle q, \bar{q}_2 | \hat{S} | p, \bar{p}_2 \rangle$
 $\neq \langle \bar{q}, q_2 | \hat{S} | p, \bar{p}_2 \rangle$.

In second order level, $\langle q, \bar{q}_2 | \hat{S} | p, \bar{p}_2 \rangle$ is given by,

$$A^{(2)}(p, \bar{p}_2, q, \bar{q}_2) = (2\pi)^6 4 \sqrt{E_p E_{p_2} E_q E_{q_2}} \\ \times \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0 | T \left[\hat{a}(\vec{q}_1) \hat{b}(\vec{q}_2) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \right. \\ \left. \times \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}_1) \hat{b}^\dagger(\vec{p}_2) \right] | 0 \rangle$$

Then based on summing wick's terms we get the following processes.

$$\begin{array}{cccccccccc} & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ \hat{a}(\vec{q}_1) & \hat{b}(\vec{q}_2) & \hat{\psi}^\dagger(x) & \hat{\psi}(x) & \hat{\phi}(x) & \hat{\psi}^\dagger(y) & \hat{\psi}(y) & \hat{\phi}(y) & \hat{a}^\dagger(\vec{p}_1) & \hat{b}^\dagger(\vec{p}_2) \end{array}$$

gives the t-channel process.

$$\mathcal{A}_t(p_1, \bar{p}_2, q_1, \bar{q}_2) =$$

$$= (-ig)^2 \int d^4x d^4y e^{-ip_1 y} e^{-i\bar{p}_2 x} e^{i\bar{q}_2 x} e^{-iq_1 y} \times \Delta_\phi(x, y)$$

$$= (-ig)^2 \int d^4x d^4y \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} \times e^{-ip_1 y} e^{-i\bar{p}_2 x} e^{i\bar{q}_2 x} e^{iq_1 y}$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \times \int d^4x e^{-i(k + \bar{p}_2 - \bar{q}_2)x} \int d^4y e^{i(k - p_1 + q_1)y}$$

(Perform 'y' integral)

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \times \int d^4x e^{-i(k + \bar{p}_2 - \bar{q}_2)x} (2\pi)^4 \underbrace{\delta^{(4)}(k - p_1 + q_1)}_{k = -(q_1 - p_1)}$$

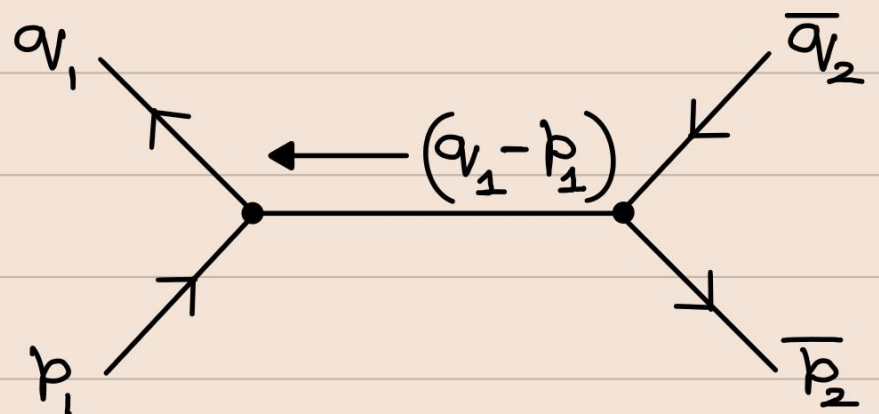
(Perform k-integral)

$$= (-ig)^2 \frac{i}{(q_1 - p_1)^2 - \mu^2 + i\varepsilon} \times \int d^4x e^{i(q_1 + \bar{q}_2 - p_1 - \bar{p}_2)x}$$

$$= (2\pi)^4 \delta^{(4)}(q_1 + \bar{q}_2 - p_1 - \bar{p}_2) (-ig)^2 \frac{i}{(q_1 - p_1)^2 - \mu^2 + i\varepsilon}$$

If we denote $t = (q_1 - p_1)^2$, we get

$$\mathcal{A}_t = (2\pi)^4 \delta^{(4)}(q_1 + \bar{q}_2 - p_1 - \bar{p}_2) (-ig)^2 \frac{i}{t - \mu^2 + i\varepsilon}$$

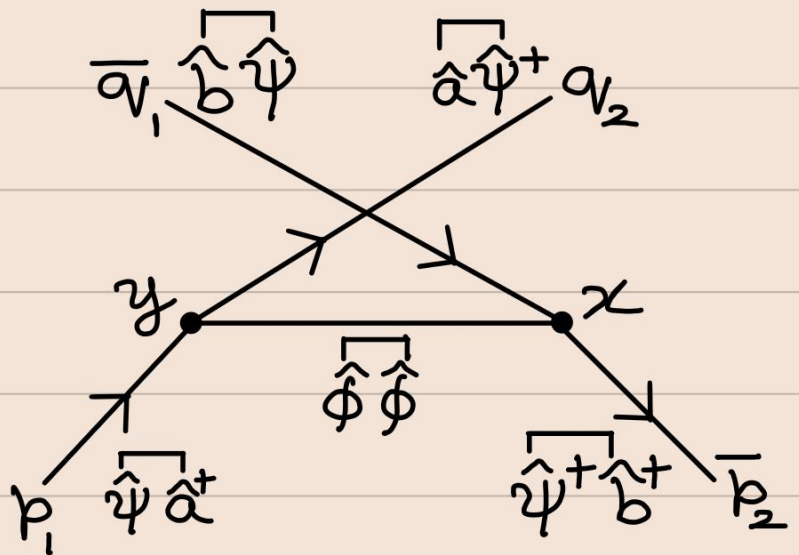
$$\Rightarrow \mathcal{A}_t(p_1, \bar{p}_2, q_1, \bar{q}_2) =$$


Thus even here we observe t -channel process in the matrix element of $\langle q_1, \bar{q}_2 | \hat{S} | p_1, \bar{p}_2 \rangle$. Note that since the ψ particle & anti particle are distinguishable now, there is no u -channel process in this S -matrix expansion.

But in $\langle \bar{q}_1, q_2 | \hat{S} | p_1, \bar{p}_2 \rangle$, there is no t -channel scattering, and has u -channel scattering given by the term containing

$$\hat{b}(\bar{q}_1) \hat{a}(q_2) \hat{\psi}^+(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^+(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^+(\bar{p}_1) \hat{b}^+(\bar{p}_2)$$

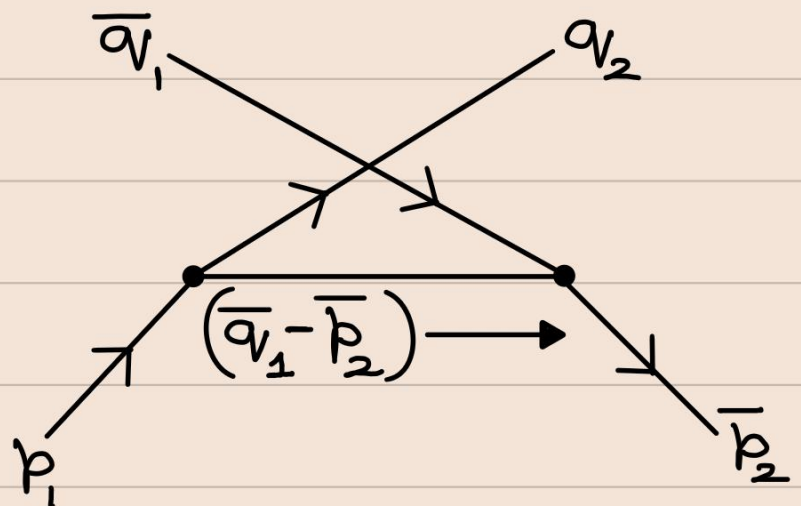
This represents
the adjacent
 u -channel Process
b/w ψ particle &
 ψ anti particle



If we denote $u = (\bar{q}_1 - \bar{p}_2)^2$, we get

$$\mathcal{A}_t = (2\pi)^4 \delta^{(4)}(\bar{q}_1 + q_2 - p_1 - \bar{p}_2) (-ig)^2 \frac{i}{u - \mu^2 + i\epsilon}$$

$$\Rightarrow \mathcal{A}_u(p_1, \bar{p}_2, \bar{q}_1, q_2) =$$



S-channel process

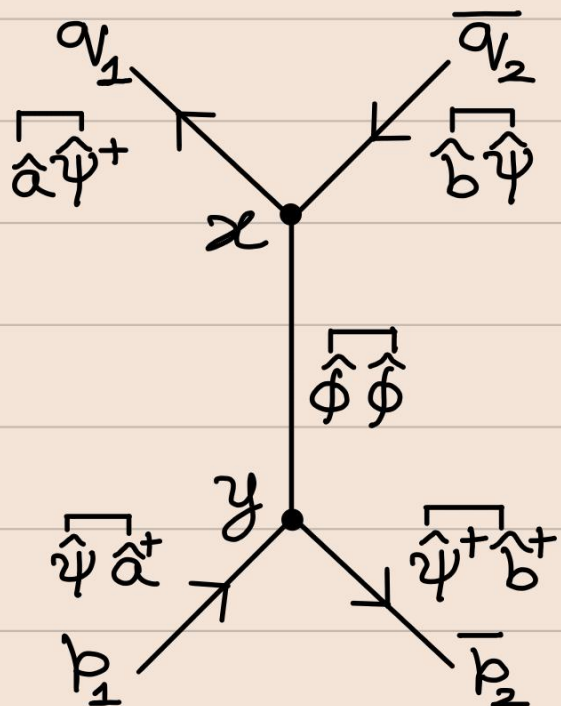
This represents the Pair annihilation & Pair creation processes b/w ψ particle & ψ -anti particle, mediated by ϕ particle.

In the 2nd order expansion of the $\langle q_1, \bar{q}_2 | \hat{S} | p_1, \bar{p}_2 \rangle$, the following Wick's contraction term

$$\hat{a}(\vec{q}_1) \hat{b}(\vec{q}_2) \hat{\psi}^\dagger(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}^\dagger(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}^\dagger(\vec{p}_1) \hat{b}^\dagger(\vec{p}_2)$$

will represent the S-channel process.

Interchanging x & y label won't matter, so, it cancels the $2!$ in the denominator.



$$\mathcal{A}_2(p_1, \bar{p}_2, q_1, \bar{q}_2) = (-ig)^2 \int d^4x d^4y \Delta_\phi(x, y) \\ \times e^{-ip_1 y} e^{-i\bar{p}_2 y} e^{i\bar{q}_2 x} e^{-iq_1 x}$$

$$= (-ig)^2 \int d^4x d^4y \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} \\ \times e^{-ip_1 y} e^{-i\bar{p}_2 y} e^{i\bar{q}_2 x} e^{iq_1 x}$$

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon}$$

$$\times \int d^4x e^{-i(k - \bar{q}_2 - q_1)x} \int d^4y e^{i(k - p_1 - \bar{p}_2)y}$$

(Perform 'y' integral)

$$= (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon}$$

$$\times \int d^4x e^{-i(k - \bar{q}_2 - q_1)x} \underbrace{(2\pi)^4 \delta^{(4)}(k - p_1 - \bar{p}_2)}_{k = (p_1 + \bar{p}_2)}$$

(Perform k-integral)

$$= (-ig)^2 \frac{i}{(p_1 + \bar{p}_2)^2 - \mu^2 + i\varepsilon} \times \int d^4 x e^{i(q_1 + \bar{q}_2 - p_1 - \bar{p}_2)x}$$

$$= (2\pi)^4 \delta^{(4)}(q_1 + \bar{q}_2 - p_1 - \bar{p}_2) (-ig)^2 \frac{i}{(p_1 + \bar{p}_2)^2 - \mu^2 + i\varepsilon}$$

If we denote $s = (p_1 + \bar{p}_2)^2$, we get

$$\mathcal{A}_t = (2\pi)^4 \delta^{(4)}(q_1 + \bar{q}_2 - p_1 - \bar{p}_2) (-ig)^2 \frac{i}{s - \mu^2 + i\varepsilon}$$

$$\Rightarrow \mathcal{A}_s(p_1, \bar{p}_2, q_1, \bar{q}_2) =$$

Note that $\langle \bar{q}_1, q_2 | \hat{S} | p_1, \bar{p}_2 \rangle$ also has an s -channel process which has the exact same amplitude as that of s -channel process in $\langle q_1, \bar{q}_2 | \hat{S} | p_1, \bar{p}_2 \rangle$ expansion.

Thus here, a ψ particle - antiparticle pair annihilates to form a ϕ particle such that the net 4-momenta are conserved. And then, the ϕ particle will form ψ particle - antiparticle pair, such that the overall 4-momenta is conserved.

So here the force carrying particle has spin zero (scalar) & no any charge, but has a non-zero mass ' μ '. This type of particle is called as Yukawa's Force carrying Particle.

Yukawa suggested that the potential mediated by this particle should be of the form,

$$U(\vec{r}) \propto \frac{e^{-\frac{|\vec{r}|}{\lambda_c}}}{4\pi|\vec{r}|}$$

where $\lambda_\phi = \frac{\hbar}{\mu_\phi c}$ is the Compton

wavelength of the ϕ particle. This is actually the inverse Fourier transform

of the time independent Klein Gordon propagator that solves,

$$(-\nabla^2 + \lambda_\phi^2) \mathcal{U}(\vec{r}) = \delta^3(\vec{r})$$

Thus, Yukawa's potential is actually the field propagator describing the propagation of virtual scalar field particles also called as Yukawa's force carriers.

Since the propagators are off mass shell by an amount of ' $i\varepsilon$ ', the propagator has a factor of $\exp\left[-\frac{|\vec{r}| \mu_\phi c}{\hbar}\right]$ in it. Thus higher the mass, exponentially the potential decays. Thus the Yukawa's forces are very short ranged of the order of ' λ_ϕ '.