

The Transition Matrix ($\hat{T}$) and Invariant Matrix element (M)

The T -matrix is defined by,

$$\hat{S} = \hat{I} + i\hat{T}$$

So, the interacting parts are collected by the T -matrix.

Since every term in the expansion of either $\hat{S}$ or $\hat{T}$ which includes some non-trivial interaction will always have a factor of overall 4-momentum conserving delta function, we define the Invariant matrix element ' M ' by the following eqⁿ.

$$\langle q_1, q_2, \dots | i\hat{T} | p_1, p_2, \dots \rangle = (2\pi)^4 \delta^{(4)} \left[\sum_j q_j - \sum_j p_j \right] iM$$

We can show that this matrix element ' M ' is analogous to the scattering amplitude of Single particle Quantum mech.

So the scattering cross-section for the scattering of four identical particles, we can write

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|M|^2}{64\pi^2 E_{CM}^2}$$

(CM $\rightarrow$ center of mass frame)

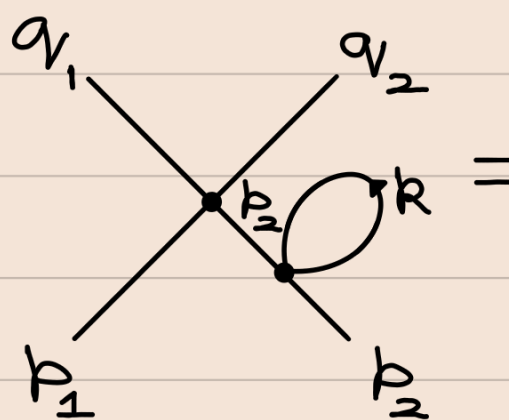
Here M is got by finding $\langle q_1, q_2 | i\hat{T} | p_1, p_2 \rangle$.

Since the scattering cross sections depend only on absolute magnitude of ' M ', the phase factor contributions are not relevant. So the disconnected pieces or the vacuum bubbles which adds up only to a phase factor are irrelevant. (as seen in 2-point correlation of ϕ^4 theory).

So for now we can consider only the fully connected diagrams & ignore the vacuum bubbles.

Also, M is derived from the transition or the interacting matrix. So the diagrams which doesn't have any interaction b/w two particle channels are irrelevant. So we should consider only the fully connected diagrams where all external legs are connected to each other.

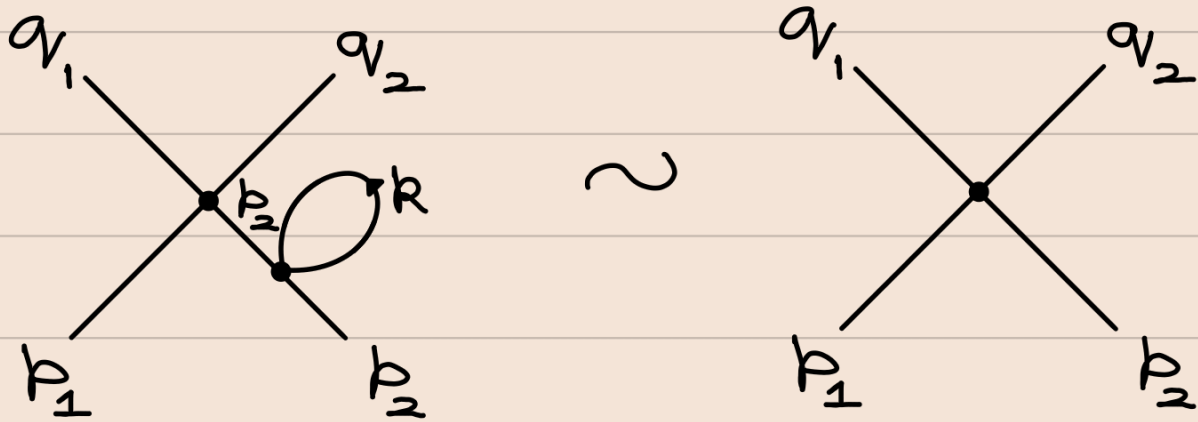
But we should also be alert about the unwanted infinities that occur in certain diagrams. For example, consider the following diagram in ϕ^4 theory,



$$= (2\pi)^4 \delta(q_1 + q_2 - p_1 - p_2) \frac{(-i\lambda)^2}{2} \times \frac{i}{(p_2^2 - m^2 + i\epsilon)} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{(k^2 - m^2 + i\epsilon)}$$

Here the term $\frac{i}{(p_2^2 - m^2 + i\epsilon)}$ is problematic.

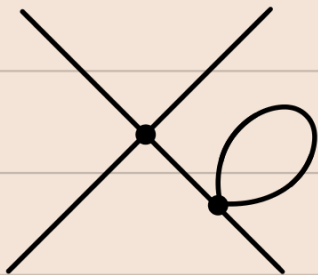
Since ' p_2 ' is a real particle, it is always on-shell. So p_2 is exactly equal to m^2 . And thus this term goes to ' ∞ '. Also this term is unnecessary for the scattering cross section, since it adds no extra interactions b/w the two particle channels. So, as far as scattering is concerned,



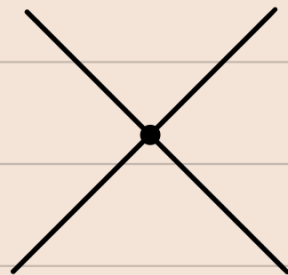
The extra loop adds no significant contribution to the scattering process.

Thus we define a procedure called as 'Amputation', which removes such ' ∞ ' causing terms.

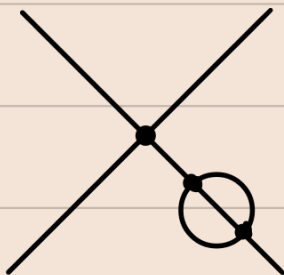
a) 2.0g.



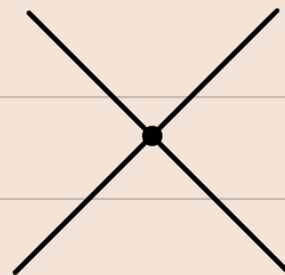
Amputate →



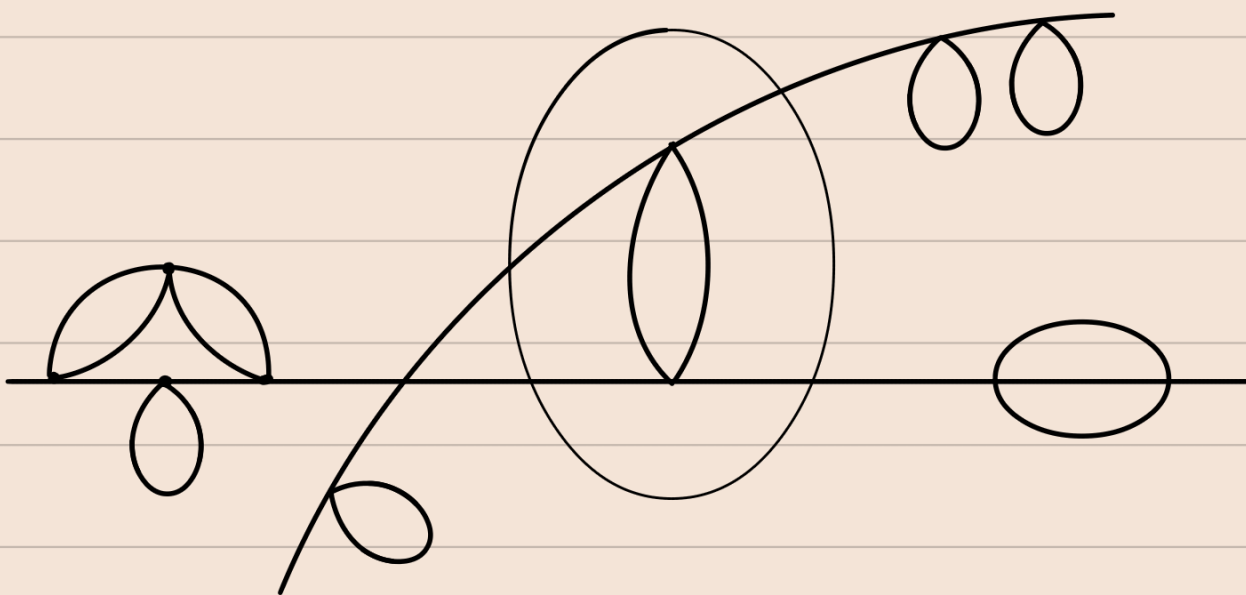
b)



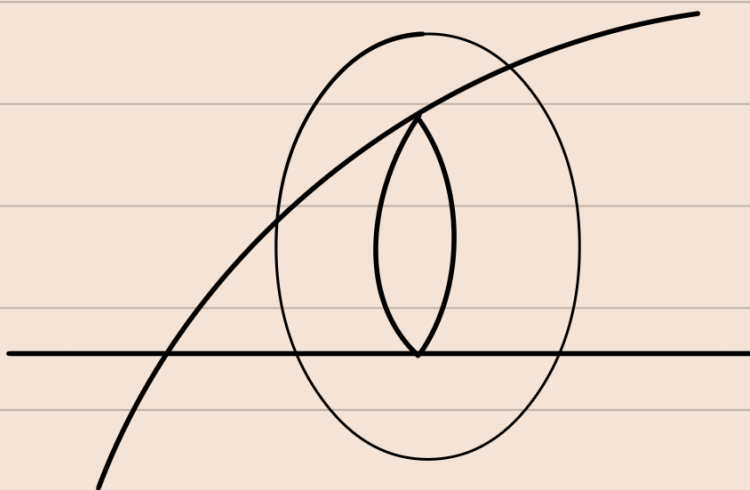
Amputate →



c)



Amputate ↓



Thus we shall define the invariant amplitude 'M' in terms of the relevant feynmann diagrams as follows,

$$i\mathcal{M}(2\pi)^4 \delta(q_1 + q_2 - p_1 - p_2)$$
$$= \sum \left(\begin{array}{l} \text{Sum of all fully connected and} \\ \text{Amputated Feynman diagrams with} \\ p_1, p_2 \text{ incoming \& } q_1, q_2 \text{ outgoing} \end{array} \right)$$

Scattering Cross section for Scalar Yukawa

In the Scalar Yukawa theory, where two indistinguishable ψ particles are scattered in t -channel process, we have the amplitude as,

$$i\mathcal{M}_t = (-ig)^2 \frac{i}{t - \mu^2 + i\epsilon}$$

$$\text{where } t = (q_1 - p_1)^2 = k^2$$

In non relativistic case where $|\vec{p}| \ll m$ we can write any four momentum $p \sim (m, \vec{p})$. Thus, $t = -|\vec{q}_1 - \vec{p}_1|^2$

$$\rightarrow t = -|\vec{k}|^2$$

$$\rightarrow i\mathcal{M}_t = (-ig)^2 \frac{i}{-|\vec{k}|^2 - \mu^2 + i\varepsilon}$$

But here ' $i\varepsilon$ ' part is unnecessary.

$$\text{Thus } \underline{\mathcal{M}_t = \frac{g^2}{|\vec{k}|^2 + \mu^2}}$$

But the scattering cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_t = \frac{|\mathcal{M}_t|^2}{64\pi^2 E_{cm}^2}$$

But $E_{cm} = 2m$ here for two ψ 's.

$$\rightarrow \left(\frac{d\sigma}{d\Omega} \right)_t = \frac{1}{256 \pi^2 m^2} \left[\frac{g^2}{|\vec{k}|^2 + \mu^2} \right]^2$$

But in non-relativistic q.mech, using Born's approx., the amplitude for scattering is given by scattering potential $\mathcal{U}(\vec{r})$.

where $\langle \vec{q} | i\hat{T} | \vec{p} \rangle = -i \tilde{\mathcal{U}}(\vec{k}) 2\pi \delta(E_p - E_q)$
and $\tilde{\mathcal{U}}(\vec{k})$ is the fourier transform of $\mathcal{U}(\vec{r})$. i.e.

$$\tilde{\mathcal{U}}(\vec{k}) = \langle \vec{q} | \mathcal{U}(\vec{r}) | \vec{p} \rangle$$

$$\rightarrow \tilde{\mathcal{U}}(\vec{k}) = \int d^3x e^{i(\vec{p} - \vec{q}) \cdot \vec{r}} \mathcal{U}(\vec{r})$$

where $\vec{k} = \vec{p} - \vec{q}$

Thus by comparing two theories, we get

$$\mathcal{U}(\vec{k}) = \frac{-g^2}{|\vec{k}|^2 + \mu^2}$$

$$\rightarrow \mathcal{U}(\vec{r}) = \frac{-g^2 e^{-\mu|\vec{r}|}}{4\pi |\vec{r}|}$$

which is inverse fourier transform of $\mathcal{U}(\vec{k}) = \frac{-g^2}{|\vec{k}|^2 + \mu^2}$.

Thus we have matched the scattering potential of t -channel process as the Yukawa Potential by both QFT in non-relativistic limit & single particle Q.Mech.