

Recap of Dirac Field Theory

We have already seen the Spin- $\frac{1}{2}$ Fermions (Dirac particles) described by the Dirac Spinor field $\psi(x)$ solves Dirac eqⁿ

$$(i\not{\partial} - m)\psi(x) = 0$$

$$\rightarrow \psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}, \psi_A \text{ is } A^{\text{th}} \text{ component of the spinor}$$

$\not{\partial} = \gamma^\mu \partial_\mu$, and in chiral basis,

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \quad \begin{aligned} \sigma^\mu &= (\mathbb{I}_2, \vec{\sigma}) \\ \bar{\sigma}^\mu &= (\mathbb{I}_2, -\vec{\sigma}) \end{aligned}$$

ψ_A is not Hermitian & is complex. Then the Dirac conjugate field $\bar{\psi}(x)$ is defined using the adjoint Dirac eqⁿ

$$\bar{\psi}(x)(i\overleftarrow{\not{\partial}} - m) = 0$$

where $\bar{\Psi}(x) = \underbrace{\Psi^\dagger(x)}_{\text{usual Hermitian conjugate}} \gamma^0$

The dispersion relation that the Dirac field satisfies is given by,

$$\underline{E_p^2 = |\vec{p}|^2 + m^2} \rightarrow \text{mass of fermion}$$

The mode expansion of $\Psi(x)$ & $\bar{\Psi}(x)$ is,

$$\hat{\Psi}(x) = \int \frac{d^3p \sum_{s=1}^2 [\psi^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} + \chi^s(p) \hat{b}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\bar{\Psi}}(x) = \int \frac{d^3p \sum_{s=1}^2 [\bar{\psi}^s(p) \hat{a}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu} + \bar{\chi}^s(p) \hat{b}_s(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

where $\psi^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \xi_s^s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi_s^s \end{bmatrix} \rightarrow \text{Dirac Particle}$

$\chi^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \eta^s \end{bmatrix} \rightarrow \text{Dirac anti-particle}$

$$\mathcal{E}^1 = \eta^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \text{gives physical spin up state } S_z = +\frac{1}{2} \text{ (+ve helicity)}$$

$$\mathcal{E}^2 = \eta^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \text{gives physical spin down state } S_z = -\frac{1}{2} \text{ (-ve helicity)}$$

The creation and annihilation operators are given as follows,

$\hat{a}_1^+(\vec{p}) \rightarrow$ creates a spin up particle along $\vec{p}$ axis & +ve helicity.

$\hat{a}_2^+(\vec{p}) \rightarrow$ creates a spin down particle along $\vec{p}$ axis & -ve helicity.

$\hat{b}_1^+(\vec{p}) \rightarrow$ creates a spin up anti-particle along $\vec{p}$ axis & +ve helicity.

$\hat{b}_2^+(\vec{p}) \rightarrow$ creates a spin down anti-particle along $\vec{p}$ axis & +ve helicity.

The Feynman propagator $\mathcal{S}_F(x, y)$ for Dirac eqⁿ is defined by,

$$[i\gamma^\mu \partial_{x^\mu} - m] \mathcal{S}_F(x, y) = i\delta(x-y) \mathbb{I}_{4 \times 4}$$

The feynman propagator is given by the following fourier integral,

$$\mathcal{S}_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \right] e^{-ip(x-y)}$$

When we perform the p^0 integral with feynman's prescription of poles, we get,

$$\begin{aligned} \mathcal{S}_F(x, y) = & \Theta(t_x - t_y) \int \frac{d^3 p (\not{p} + m)}{(2\pi)^3 (2E_p)} e^{-ip(x-y)} \\ & - \Theta(t_y - t_x) \int \frac{d^3 p (\not{p} - m)}{(2\pi)^3 (2E_p)} e^{ip(x-y)} \end{aligned}$$

$$[\text{where } p \equiv p^\mu = (E_p, \vec{p})]$$

This can be related to the Spinor field.

In previous chapters we have seen that

$$\langle 0 | \hat{\psi}_A(x) \hat{\psi}_B(y) | 0 \rangle = \int \frac{d^3 p (\not{p} + m)_{AB}}{(2\pi)^3 (2E_p)} e^{-ip(x-y)}$$

$$\langle 0 | \hat{\psi}_B(y) \hat{\psi}_A(x) | 0 \rangle = \int \frac{d^3 p (\not{p} - m)_{AB}}{(2\pi)^3 2E_p} e^{-ip(x-y)}$$

Thus we can write Feynmann propagator for Dirac eqⁿ as,

$$\underbrace{\mathcal{S}_F^{AB}(x, y)}_{\text{(Matrix element)}} = \Theta(t_x - t_y) \langle 0 | \hat{\psi}_A(x) \hat{\psi}_B(y) | 0 \rangle - \Theta(t_y - t_x) \langle 0 | \hat{\psi}_B(y) \hat{\psi}_A(x) | 0 \rangle$$

Thus, physically, the free feynmann propagator for Dirac field gives the vacuum state expectation value of

$$\mathcal{S}_F^{AB}(x, y) = \begin{cases} \langle 0 | \hat{\psi}_A(x) \hat{\psi}_B(y) | 0 \rangle, & t_x > t_y \\ -\langle 0 | \hat{\psi}_B(y) \hat{\psi}_A(x) | 0 \rangle, & t_y > t_x \end{cases}$$

Note that $\{\hat{\psi}_A(x)\} \neq \{\hat{\psi}_B(y)\}$ are set of fermionic operators that follow the anti-commutation relations, thus whenever we exchange two fermionic operators, we get a -ve sign.

(Odd no: of exchanges result in -ve sign).

Thus we can derive time ordered product for fermion operators as

$$T[\hat{\psi}_A(x)\hat{\psi}_B(y)] = \begin{cases} \hat{\psi}_A(x)\hat{\psi}_B(y), & x^0 > y^0 \\ -\hat{\psi}_B(y)\hat{\psi}_A(x), & y^0 > x^0 \end{cases}$$

So, the feynmann propagator for fields that solve dirac eqⁿ are written as:

$$S_F^{AB}(x, y) = \langle 0 | T[\hat{\psi}_A(x)\hat{\psi}_B(y)] | 0 \rangle$$

All these above results were derived in earlier chapters.

Wick's Contractions for Fermions

The time ordered vacuum expectation value of two field operators is called as the Wick's contraction.

$$\rightarrow \langle 0 | T [\hat{\psi}_A(x) \hat{\psi}_B(y)] | 0 \rangle = \overbrace{\hat{\psi}_A(x) \hat{\psi}_B(y)}$$

$$\rightarrow \overbrace{\hat{\psi}_A(x) \hat{\psi}_B(y)} = \mathcal{S}_F^{AB}(x, y)$$

But instead of dealing with components we shall use the following convention.

$$T[\underbrace{\hat{\psi}(x) \hat{\psi}(y)}_{4 \times 4 \text{ matrix}}] = \begin{cases} \hat{\psi}(x) \hat{\psi}(y), & x^0 > y^0 \\ - \underbrace{[\hat{\psi}(y)]^T}_{\text{column spinor}} \underbrace{[\hat{\psi}(x)]^T}_{\text{row spinor}}, & y^0 > x^0 \end{cases}$$

$$\rightarrow \underline{\overbrace{\hat{\psi}(x) \hat{\psi}(y)}} = \langle 0 | T [\hat{\psi}(x) \hat{\psi}(y)] | 0 \rangle = \mathcal{S}_F(x, y)$$

Thus we can use the usual Wick's theorem but should keep track of -ve signs.

Ex:-

$$\begin{aligned} \hat{\psi}(x) \hat{\psi}(y) \hat{\bar{\psi}}(z) \hat{\bar{\psi}}(w) &= -\hat{\psi}(x) \hat{\bar{\psi}}(z) \hat{\psi}(y) \hat{\bar{\psi}}(w) \\ &= -\mathcal{S}_F(x, z) \mathcal{S}_F(y, w) \end{aligned}$$

$$\rightarrow \hat{\bar{\psi}}(x) \hat{\bar{\psi}}(y) \hat{\psi}(z) \hat{\psi}(w) = -\mathcal{S}_F(x, z) \mathcal{S}_F(y, w)$$

Based on the mode expansion of the field operators, we can show that, since,

$$\hat{\psi}(x) = \frac{\int d^3p \sum_{s=1}^2 [\bar{u}^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} + \bar{v}^s(p) \hat{b}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\bar{\psi}}(x) = \frac{\int d^3p \sum_{s=1}^2 [\bar{u}^s(p) \hat{a}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu} + \bar{v}^s(p) \hat{b}_s(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\bullet \hat{\psi}(x) \hat{\psi}(y) = 0 = \hat{\bar{\psi}}(x) \hat{\bar{\psi}}(y)$$

$$\bullet \hat{a}_s(\vec{p}) \hat{\psi}(x) = 0 = \hat{\bar{\psi}}(x) \hat{a}_s^\dagger(\vec{p})$$

$$\bullet \hat{b}_s(\vec{p}) \hat{\bar{\psi}}(x) = 0 = \hat{\psi}(x) \hat{b}_s^\dagger(\vec{p})$$

The only surviving contractions are:

$$\bullet \overbrace{\hat{\psi}(x) \hat{a}_s^\dagger(\vec{p})} = \frac{u^s(p) e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\bullet \overbrace{\hat{a}_s(\vec{p}) \hat{\psi}(x)} = \frac{\bar{u}^s(p) e^{ip_\mu x^\mu}}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\bullet \overbrace{\hat{b}_s(\vec{p}) \hat{\psi}(x)} = \frac{v^s(p) e^{ip_\mu x^\mu}}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\bullet \overbrace{\hat{\bar{\psi}}(x) \hat{b}_s^\dagger(\vec{p})} = \frac{\bar{v}^s(p) e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\bullet \underbrace{\hat{\psi}(x) \hat{\bar{\psi}}(y)}_{\text{fermion line}} = \mathcal{S}_F(x, y) = - \underbrace{\hat{\bar{\psi}}(y) \hat{\psi}(x)}_{\text{anti-fermion line}}$$

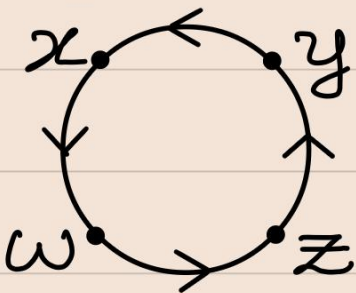
Note that in a complicated string, we can group them by pairing field op. & its conjugate at the same space time point & under complete wick contraction, the pairs commute with other pairs.

Eg:

$$\begin{aligned}
 \text{a) } & [\hat{\psi}(x)\hat{\psi}(x)][\hat{\psi}(y)\hat{\psi}(y)][\hat{\psi}(z)\hat{\psi}(z)][\hat{\psi}(w)\hat{\psi}(w)] \\
 &= + [\hat{\psi}(x)\hat{\psi}(x)][hat{\psi}(z)\hat{\psi}(z)][\hat{\psi}(y)\hat{\psi}(y)][\hat{\psi}(w)\hat{\psi}(w)] \\
 &= \hat{\psi}(x) \mathcal{S}_F(x, z) \mathcal{S}_F(z, y) \mathcal{S}_F(y, w) \hat{\psi}(w)
 \end{aligned}$$

b) But in closed loop of any fermion propagator, we get a factor of (-1) & trace of a product of Propagators

$$\begin{aligned}
 & [\hat{\psi}(x)\hat{\psi}(x)][\hat{\psi}(y)\hat{\psi}(y)][\hat{\psi}(z)\hat{\psi}(z)][\hat{\psi}(w)\hat{\psi}(w)] \\
 &= - \text{Tr} [\hat{\psi}(w)\hat{\psi}(x)\hat{\psi}(x)\hat{\psi}(y)\hat{\psi}(y)\hat{\psi}(z)\hat{\psi}(z)\hat{\psi}(w)] \\
 &= - \text{Tr} [\mathcal{S}_F(w, x) \mathcal{S}_F(x, y) \mathcal{S}_F(y, z) \mathcal{S}_F(z, w)] \\
 &\equiv \text{Diagram} \rightarrow \text{Fermion loop.}
 \end{aligned}$$



Feynman Rules for Fermions in Momentum Space

- An incoming fermion line with a four momentum ' p ' & a spin state ' s ', has a factor $u^s(p)$.
- An outgoing fermion line with a four momentum ' q ' & a spin state ' s ', has a factor $\bar{u}^s(q)$.
- An incoming antifermion line with a four momentum ' p ' & a spin state ' s ', has a factor $\bar{v}^s(p)$.
- An outgoing antifermion line with a four momentum ' q ' & a spin state ' s ', has a factor $v^s(q)$.
- The spinors should have an appropriate inner product based on the line's path.

- Every internal fermion line has a factor of $\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}$ where 'k' is the four momentum carried by the propagator.
- The fermion lines have arrows along the direction of time, and opposite for anti-fermions.
- Trace over the matrix products arising from fermion loop, every fermion loop carries a factor of -1 .
- Multiply ' -1 ' factors appropriately to reflect fermion statistics.

Yukawa's Interaction between Spin- $1/2$ Fermion & Scalar Boson

Here we look into the interactions b/w Spin- $1/2$ fermions described by the Dirac field theory mediated by a massive scalar boson through Yukawa interaction.

The lagrangian density describing this is,

$$\mathcal{L}_{\text{Yukawa}} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Scalar } \phi} - g \bar{\Psi} \Psi \phi$$

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} (i \not{\partial} - m) \Psi$$

$$\mathcal{L}_{\text{Scalar } \phi} = \frac{1}{2} (\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} \mu^2 \phi^2$$

The mass of boson is ' μ ' & the mass of fermion is ' m '. Then the Hamiltonian density is,

$$\hat{H} = \hat{H}_{\text{Dirac}} + \hat{H}_{\text{Scalar } \phi} + \hat{H}_{\text{int}}$$

Thus $\hat{H}_0 = \hat{H}_{\text{Dirac}} + \hat{H}_{\text{scalar } \phi}$ gives the non-interacting part which is canonically quantizable & $\hat{H}_{\text{int}} = g \hat{\bar{\psi}} \hat{\psi} \hat{\phi}$ is the Yukawa's interaction part.

But in the interaction picture, the field operators $\hat{\psi}_{\text{I}}(x)$, $\hat{\bar{\psi}}_{\text{I}}(x)$ & $\hat{\phi}_{\text{I}}(x)$ are governed by the canonically quantizable part ' $\hat{H}_0$ '. So, the mode expansion for field operators in the interaction picture is given by,

$$\hat{\psi}_{\text{I}}(x) = \frac{\int d^3p \sum_{s=1}^2 [\mathcal{U}^s(p) \hat{a}_s(\vec{p}) e^{-ipx} + \mathcal{V}^s(p) \hat{b}_s^\dagger(\vec{p}) e^{ipx}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\bar{\psi}}_{\text{I}}(x) = \frac{\int d^3p \sum_{s=1}^2 [\bar{\mathcal{U}}^s(p) \hat{a}_s^\dagger(\vec{p}) e^{ipx} + \bar{\mathcal{V}}^s(p) \hat{b}_s(\vec{p}) e^{-ipx}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

&

$$\hat{\phi}_{\text{I}}(x) = \frac{\int d^3k [\hat{c}(\vec{q}) e^{-ikx} + \hat{c}^\dagger(\vec{q}) e^{+ikx}]}{\sqrt{(2\pi)^3 (2E_k)}}$$

Now, the equations that $\hat{\Psi}_I$, $\bar{\hat{\Psi}}_I$ and $\hat{\phi}_I$ solve is the usual K.G equation got by $\hat{H}_0$ part. Thus, the dispersion relation for the particles described by Ψ_I 's & ϕ_I are given by $E_p^2 = |\vec{p}|^2 + m^2$ and $E_k^2 = |\vec{k}|^2 + \mu^2$ respectively.

Here $\hat{a} / \hat{a}^\dagger$ destroy / create fermions.
 $\hat{b} / \hat{b}^\dagger$ destroy / create anti-fermions.
 & $\hat{c} / \hat{c}^\dagger$ destroy / create bosons.

The vacuum state of $\hat{H}_0$ is $|0\rangle$ where,

$$\underline{\hat{a}_s(\vec{p})|0\rangle = \hat{b}_s(\vec{p})|0\rangle = \hat{c}(\vec{p})|0\rangle = 0}$$

From now we drop the subscript 'I' for the field operators but still mean that they are in the interaction picture.

The feynmann propagator for the ϕ particle is given by,

$$\Delta_\phi(x, y) = \langle 0 | T [\hat{\phi}(x) \hat{\phi}(y)] | 0 \rangle = \overline{\hat{\phi}(x) \hat{\phi}(y)}$$

And in momentum space,

$$\Delta_\phi(x, y) = \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}$$

where the poles are taken as per the Feynman's prescription.

$$\overline{\hat{\phi}(y) \hat{C}^+(\vec{k})} = \frac{e^{-iq_y k}}{\sqrt{(2\pi)^3 (2E_k)}}$$

This gives the probability amplitude of a ϕ particle of four momentum k entering a space-time point at ' x '.

$$\text{Similarly, } \overline{\hat{C}(\vec{k}) \hat{\phi}(x)} = \frac{e^{ikx}}{\sqrt{(2\pi)^3 (2E_k)}}$$

which gives the probability amplitude of a ϕ particle of four momentum k leaving from a space-time point ' x '.

Scattering of two fermions

Here we shall consider two fermions entering & two fermions leaving. So the S -matrix expansion is $\langle q_1, q_2 | \hat{S} | p_1, p_2 \rangle$

The leading order non-trivial terms which actually induces scattering are given by the second order expansion,

$$A^{(2)}(p_1, p_2, q_1, q_2) = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}}$$

$$\times \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0 | T \left[\hat{a}_{s_1}(\vec{p}_1) \hat{a}_{s_2}(\vec{p}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \right. \\ \left. \times \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1) \right] | 0 \rangle$$

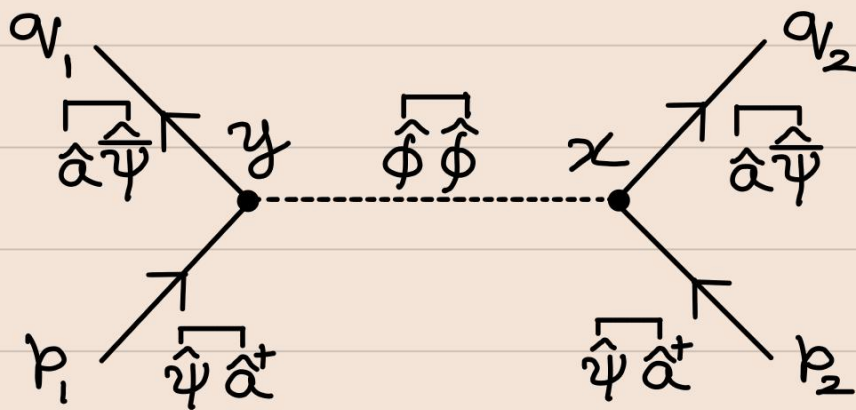
Here we shall look into the interacting diagrams where two fermions interact using a boson.

This gives the similar t -channel and u -channel processes that we saw in the scalar Yukawa theory.

t-channel Scattering

It is given by the following contraction

$$\hat{a}_{s_1}(\vec{p}_1) \hat{a}_{s_2}(\vec{p}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1)$$



The amplitude for t-channel scattering is given by,

$$\mathcal{A}_t = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \hat{a}_{s_1}(\vec{p}_1) \hat{a}_{s_2}(\vec{p}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1)$$

(observe that there are even number of total exchanges)

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. So, we get,

$$\mathcal{A}_t = (-ig)^2 \int d^4x d^4y \bar{u}^{\delta_1}(q_1) e^{iq_1 y} u^{s_1}(p_1) e^{-ip_1 y} \\ \times \bar{u}^{\delta_2}(q_2) e^{iq_2 x} u^{s_2}(p_2) e^{-ip_2 x} \Delta_\phi(x, y)$$

On going to momentum space, we get

$$\mathcal{A}_t = (-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{u}^{\delta_2}(q_2) u^{s_2}(p_2) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\varepsilon} e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 x}$$

$$= (-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{u}^{\delta_2}(q_2) u^{s_2}(p_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4x e^{-i(k+p_2-q_2)x} \\ \times \int d^4y e^{-i(p_1-k-q_1)y}$$

(Perform 'y' integral)

$$\begin{aligned}
&= (-ig)^2 \bar{u}^{\gamma_1}(q_1) u^{s_1}(p_1) \bar{u}^{\gamma_2}(q_2) u^{s_2}(p_2) \\
&\quad \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4 x e^{-i(k+p_2-q_2)x} \\
&\quad \times (2\pi)^4 \underbrace{\delta^{(4)}(p_1 - k - q_1)}_{k = -(q_1 - p_1)}
\end{aligned}$$

(Perform 'k' integral)

$$\begin{aligned}
&= (-ig)^2 \bar{u}^{\gamma_1}(q_1) u^{s_1}(p_1) \bar{u}^{\gamma_2}(q_2) u^{s_2}(p_2) \\
&\quad \times \frac{i}{(q_1 - p_1)^2 - \mu^2} \underbrace{\int d^4 x e^{-i(p_1 - q_1 + p_2 - q_2)x}}_{(2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)}
\end{aligned}$$

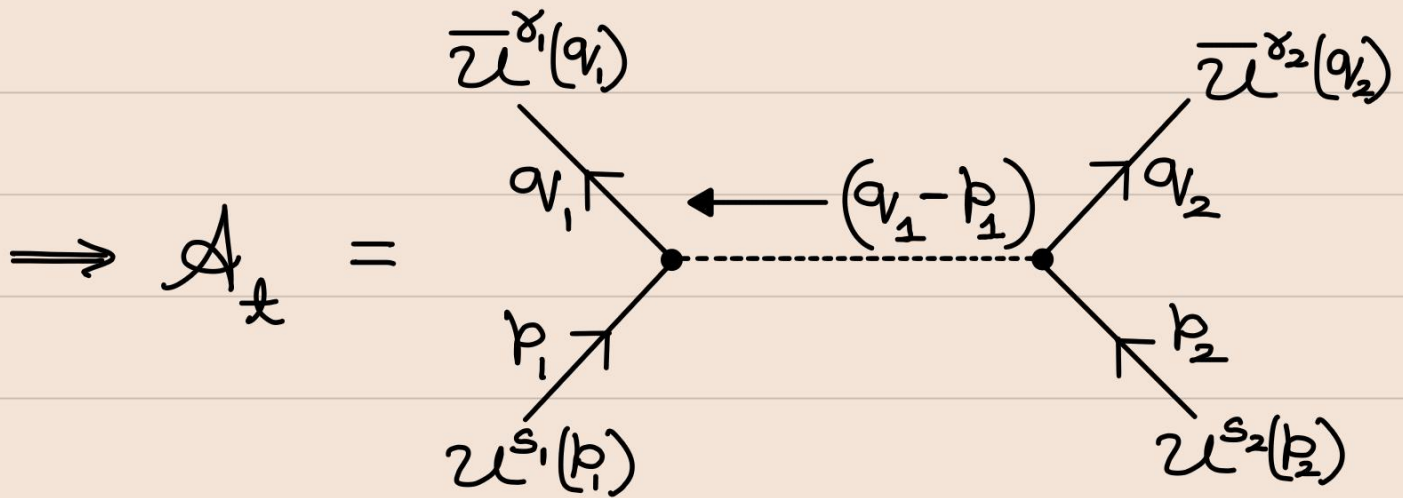
('i\varepsilon' can be neglected as it is not integrated)

If we denote $t = (q_1 - p_1)^2$, then we get the amplitude of t-channel process as,

$$\begin{aligned}
A_t &= (-ig)^2 \bar{u}^{\gamma_1}(q_1) u^{s_1}(p_1) \bar{u}^{\gamma_2}(q_2) u^{s_2}(p_2) \frac{i}{t - \mu^2} \\
&\quad \times (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)
\end{aligned}$$

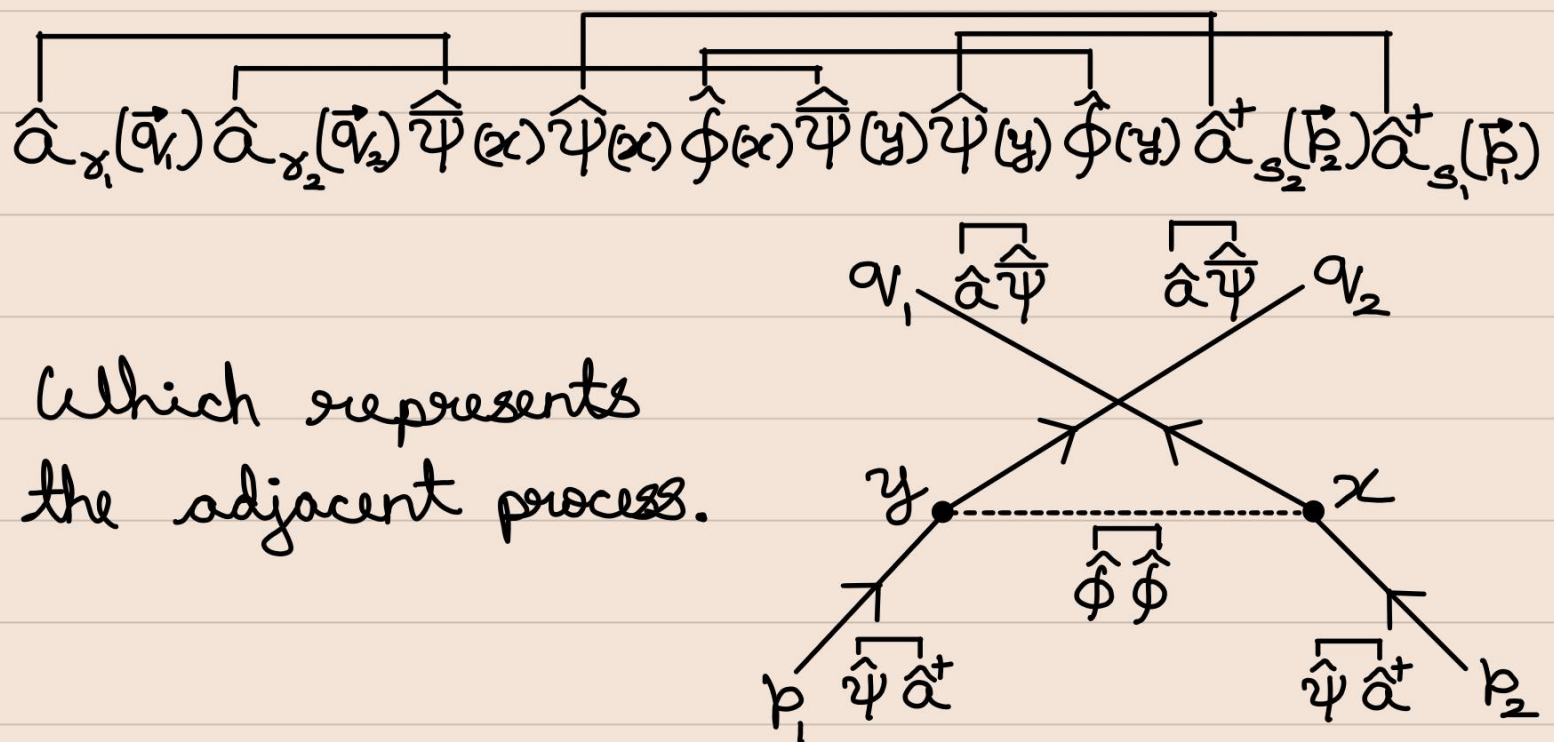
In terms of invariant matrix element we get it to be,

$$i\mathcal{M}_t = -g^2 \frac{i}{t - \mu^2} \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{u}^{\delta_2}(q_2) u^{s_2}(p_2)$$



u-channel Scattering

It is given by the following contraction



The amplitude for u -channel scattering is given by,

$$A_u = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \hat{a}_x(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{a}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1)$$

(there are odd number of total exchanges)

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. So, we get,

$$A_u = -(-ig)^2 \int d^4x d^4y \bar{u}^{\alpha_1}(q_1) e^{iq_1 x} u^{s_2}(p_2) e^{-ip_2 x} \\ \times \bar{u}^{\alpha_2}(q_2) e^{iq_2 y} u^{s_1}(p_1) e^{-ip_1 y} \Delta_{\phi}(x, y)$$

On going to momentum space, we get

$$A_u = -(-ig)^2 \bar{u}^{\alpha_1}(q_1) u^{s_2}(p_2) \bar{u}^{\alpha_2}(q_2) u^{s_1}(p_1) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x}$$

$$\begin{aligned}
&= -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_2}(p_2) \bar{u}^{\delta_2}(q_2) u^{s_1}(p_1) \\
&\quad \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4 x e^{-i(k+p_2-q_1)x} \\
&\quad \times \int d^4 y e^{-i(p_1-k-q_2)y}
\end{aligned}$$

(Perform 'x' integral)

$$\begin{aligned}
&= -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_2}(p_2) \bar{u}^{\delta_2}(q_2) u^{s_1}(p_1) \\
&\quad \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} (2\pi)^4 \delta^{(4)}(k+p_2-q_1) \\
&\quad \times \int d^4 y e^{-i(p_1-k-q_2)y}
\end{aligned}$$

(Perform 'k' integral)

$$\begin{aligned}
&= -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_2}(p_2) \bar{u}^{\delta_2}(q_2) u^{s_1}(p_1) \\
&\quad \times \frac{i}{(q_1-p_2)^2 - \mu^2 + i\varepsilon} \underbrace{\int d^4 y e^{-i(p_1-q_1+p_2-q_2)y}}_{(2\pi)^4 \delta^{(4)}(q_1+q_2-p_1-p_2)}
\end{aligned}$$

We can denote $u = (q_1-p_2)^2$.

('i\varepsilon' can be neglected as it is not integrated)

$$\Rightarrow \mathcal{A}_u = -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_2}(p_2) \bar{u}^{\delta_2}(q_2) u^{s_1}(p_1) \\ \times \frac{i}{u - \mu^2} (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)$$

In terms of invariant matrix element we get it to be,

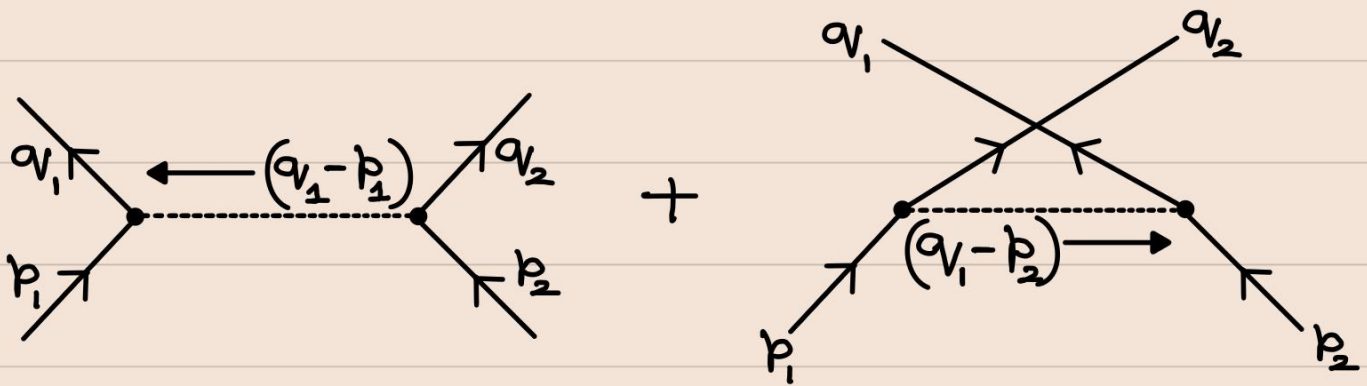
$$i\mathcal{M}_u = g^2 \frac{i}{u - \mu^2} \bar{u}^{\delta_1}(q_1) u^{s_2}(p_2) \bar{u}^{\delta_2}(q_2) u^{s_1}(p_1)$$

$$\Rightarrow \mathcal{A}_u =$$

Thus the full result to the total $\mathcal{M}^{(2)}$ in the second order scattering of two fermions in the Yukawa theory is,

$$i\mathcal{M}^{(2)} = i\mathcal{M}_t + i\mathcal{M}_u$$

Got by the sum of t & u channel processes



$$\rightarrow i\mathcal{M}^{(2)}(p_1, p_2 \rightarrow q_1, q_2)$$

$$= (-ig^2) \left[\frac{1}{(q_1 - p_1)^2 - \mu^2} \bar{u}^{\sigma_1}(q_1) u^{s_1}(p_1) \bar{u}^{\sigma_2}(q_2) u^{s_2}(p_2) - \frac{1}{(q_1 - p_2)^2 - \mu^2} \bar{u}^{\sigma_1}(q_1) u^{s_2}(p_2) \bar{u}^{\sigma_2}(q_2) u^{s_1}(p_1) \right]$$

The -ve sign arises due to the interchange of two fermions after the scattering b/w the t & u processes.

This result is the reflection of the fermi statistics of spin $1/2$ fermions.

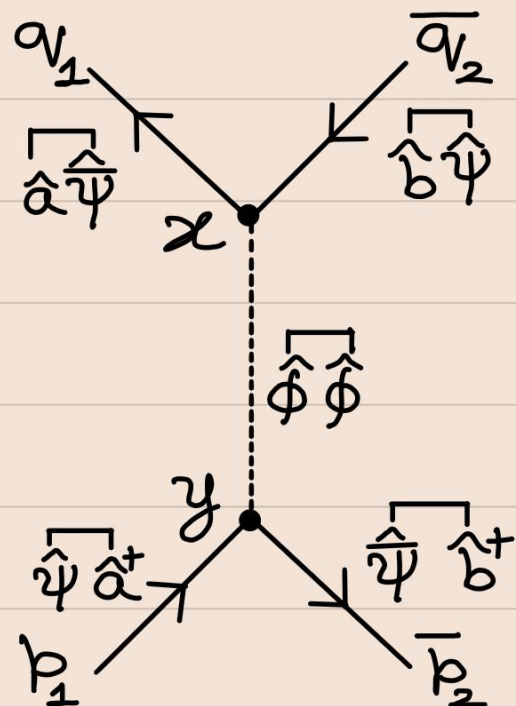
Scattering of fermion & anti-fermion

S-channel Scattering

It appears in the 2nd order terms in the expansion of $\langle q_1, \bar{q}_2 | \hat{S} | p_1, \bar{p}_2 \rangle$, where the anti-fermions have momenta $\bar{p}_2$ & $\bar{q}_2$. This process appears in the following contraction.

$$\hat{a}_{s_1}(\vec{q}_1) \hat{b}_{s_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{b}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1)$$

Interchanging x & y label won't matter, so, it cancels the $2!$ in the denominator.



This represents the Pair annihilation & Pair creation processes b/w fermions & anti fermions, mediated by scalar boson.

$$\rightarrow \mathcal{A}_8 = (-ig)^2 \bar{v}^{s_2}(p_2) u^{s_1}(p_1) \bar{u}^{s_1}(q_1) v^{s_2}(q_2) \\ \times \int d^4x d^4y e^{iq_1 x} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 y} \Delta_\phi(x, y)$$

$$= (-ig)^2 \bar{v}^{s_2}(p_2) u^{s_1}(p_1) \bar{u}^{s_1}(q_1) v^{s_2}(q_2) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} e^{iq_1 x} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 y}$$

$$= (-ig)^2 \bar{v}^{s_2}(p_2) u^{s_1}(p_1) \bar{u}^{s_1}(q_1) v^{s_2}(q_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \int d^4x e^{-i(k - q_2 - q_1)x} \\ \times \int d^4y e^{i(k - p_1 - p_2)y}$$

(Perform 'y' integral)

$$= (-ig)^2 \bar{v}^{s_2}(p_2) u^{s_1}(p_1) \bar{u}^{s_1}(q_1) v^{s_2}(q_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \int d^4x e^{-i(k - q_2 - q_1)x}$$

$$(2\pi)^4 \underbrace{\delta^{(4)}(k - p_1 - p_2)}_{k = (p_1 + p_2)}$$

(Perform k-integral)

$$= (-ig)^2 \frac{i}{(p_1 + p_2)^2 - \mu^2 + i\varepsilon} \bar{\psi}^{s_2}(p_2) \psi^{s_1}(p_1) \bar{u}^{\delta_1}(q_1) \gamma^{\delta_2}(q_2) \times \int d^4x e^{i(q_1 + q_2 - p_1 - p_2)x}$$

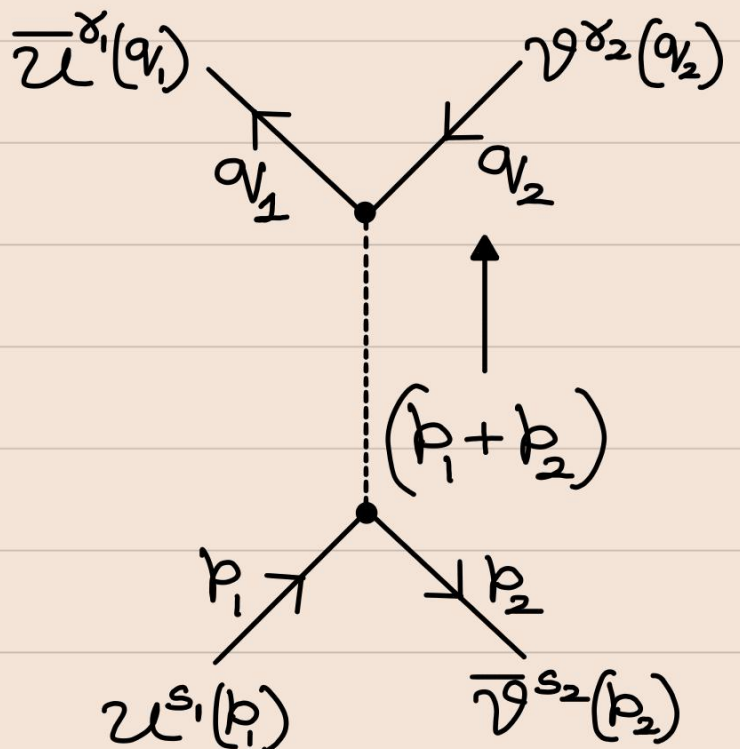
We can denote $\mathcal{S} = (p_1 + p_2)^2$
 (' $i\varepsilon$ ' can be neglected as it is not integrated)

$$\rightarrow \mathcal{A}_{\mathcal{S}} = (-ig)^2 \frac{i}{\mathcal{S} - \mu^2} \bar{\psi}^{s_2}(p_2) \psi^{s_1}(p_1) \bar{u}^{\delta_1}(q_1) \gamma^{\delta_2}(q_2) \times (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)$$

Then, the invariant matrix element is,

$$i\mathcal{M}_{\mathcal{S}} = -g^2 \frac{i}{\mathcal{S} - \mu^2} \bar{\psi}^{s_2}(p_2) \psi^{s_1}(p_1) \bar{u}^{\delta_1}(q_1) \gamma^{\delta_2}(q_2)$$

$$\rightarrow \mathcal{A}_{\mathcal{S}}(p_1, p_2, q_1, q_2) =$$



The s-channel process also occurs in the 2nd order expansion of $\langle \bar{q}_1, q_2 | \hat{S} | \bar{p}_1, p_2 \rangle$ where the channel 1 have anti particle entering & leaving & particles in the channel '2'. Their numerical value is same as compared to the previous one.

But we can also get 't' & 'u' channel processes in scattering of fermion with anti fermion.

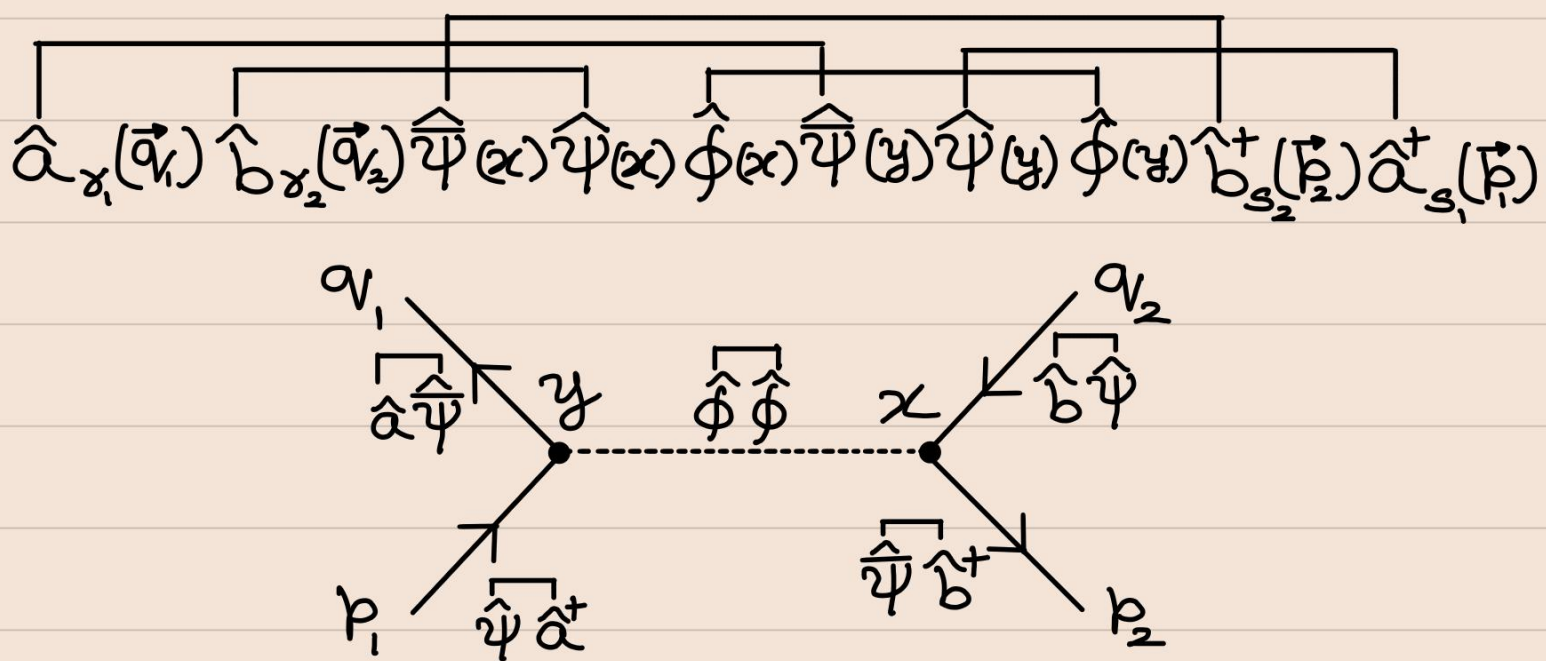
$\langle q_1, \bar{q}_2 | \hat{S} | p_1, \bar{p}_2 \rangle \rightarrow$ has t & s channel
but no u-channel.

$\langle \bar{q}_1, q_2 | \hat{S} | \bar{p}_1, p_2 \rangle \rightarrow$ has u & s channel
but no t-channel.

The Scattering amplitudes are calculated below for the t & u channel.

t-channel Scattering

It appears in the 2nd order terms in the expansion of $\langle q_1, \bar{q}_2 | \hat{S} | p_1, \bar{p}_2 \rangle$, where the anti-fermions have momenta $\bar{p}_2$ & $\bar{q}_2$. This process appears in the following contraction.



The amplitude for t-channel scattering is given by,

$$A_t = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \hat{a}_{x_1}(\vec{q}_1) \hat{b}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{b}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1)$$

(there are odd(3) number of total exchanges)

$$\rightarrow \mathcal{A}_x = -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \\ \times \int d^4x d^4y e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 x} \Delta_\phi(x, y)$$

$$= -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\varepsilon} e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 x}$$

$$= -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4x e^{-i(k - q_2 + p_2)x} \\ \times \int d^4y e^{i(k + q_1 - p_1)y}$$

(Perform 'y' integral)

$$= -(-ig)^2 \bar{u}^{\delta_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4x e^{-i(k - q_2 + p_2)x}$$

$$(2\pi)^4 \underbrace{\delta^{(4)}(k + q_1 - p_1)}_{k = -(q_1 - p_1)}$$

(Perform k-integral)

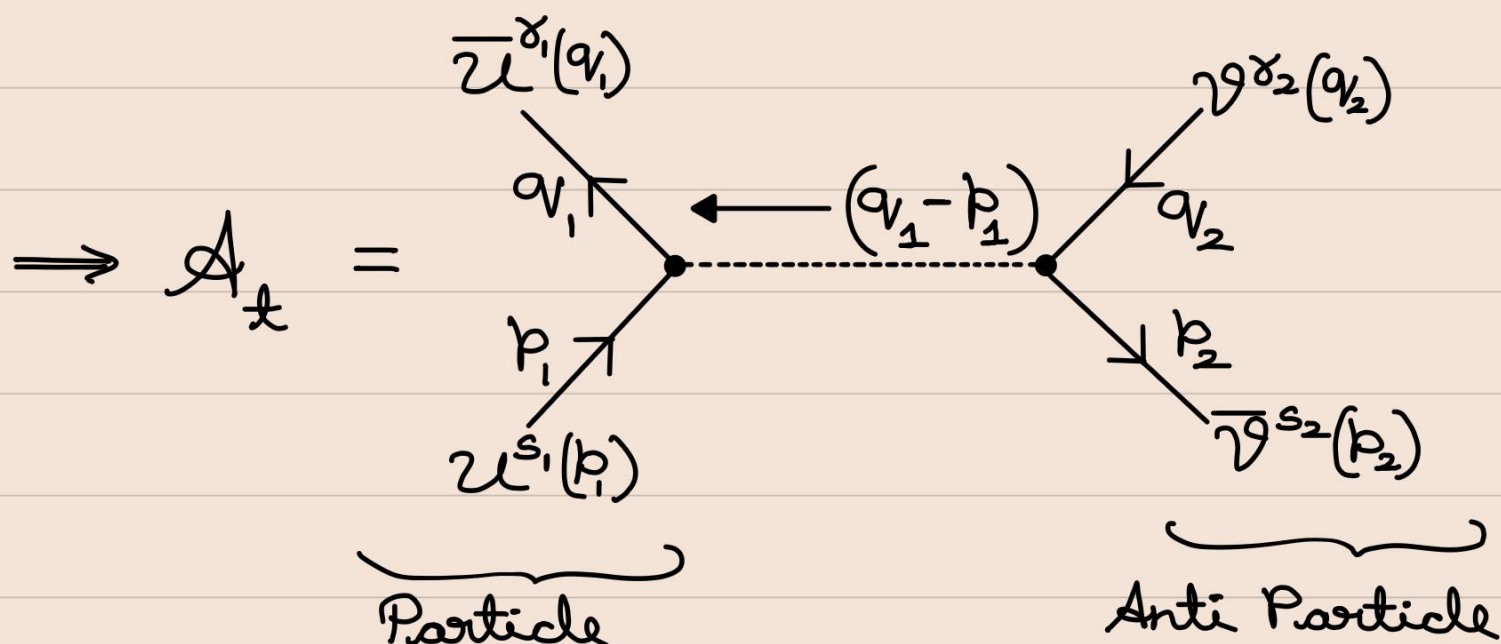
$$= -(-ig)^2 \frac{i}{(q_1 - p_1)^2 - \mu^2 + i\varepsilon} \bar{u}^{\gamma_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\gamma_2}(q_2) \times \underbrace{\int d^4x e^{i(q_1 + q_2 - p_1 - p_2)x}}_{(2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)}$$

We can denote $t = (q_1 - p_1)^2$
 (' $i\varepsilon$ ' can be neglected as it is not integrated)

$$\rightarrow \mathcal{A}_t = -(-ig)^2 \frac{i}{t - \mu^2} \bar{u}^{\gamma_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\gamma_2}(q_2) \times (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)$$

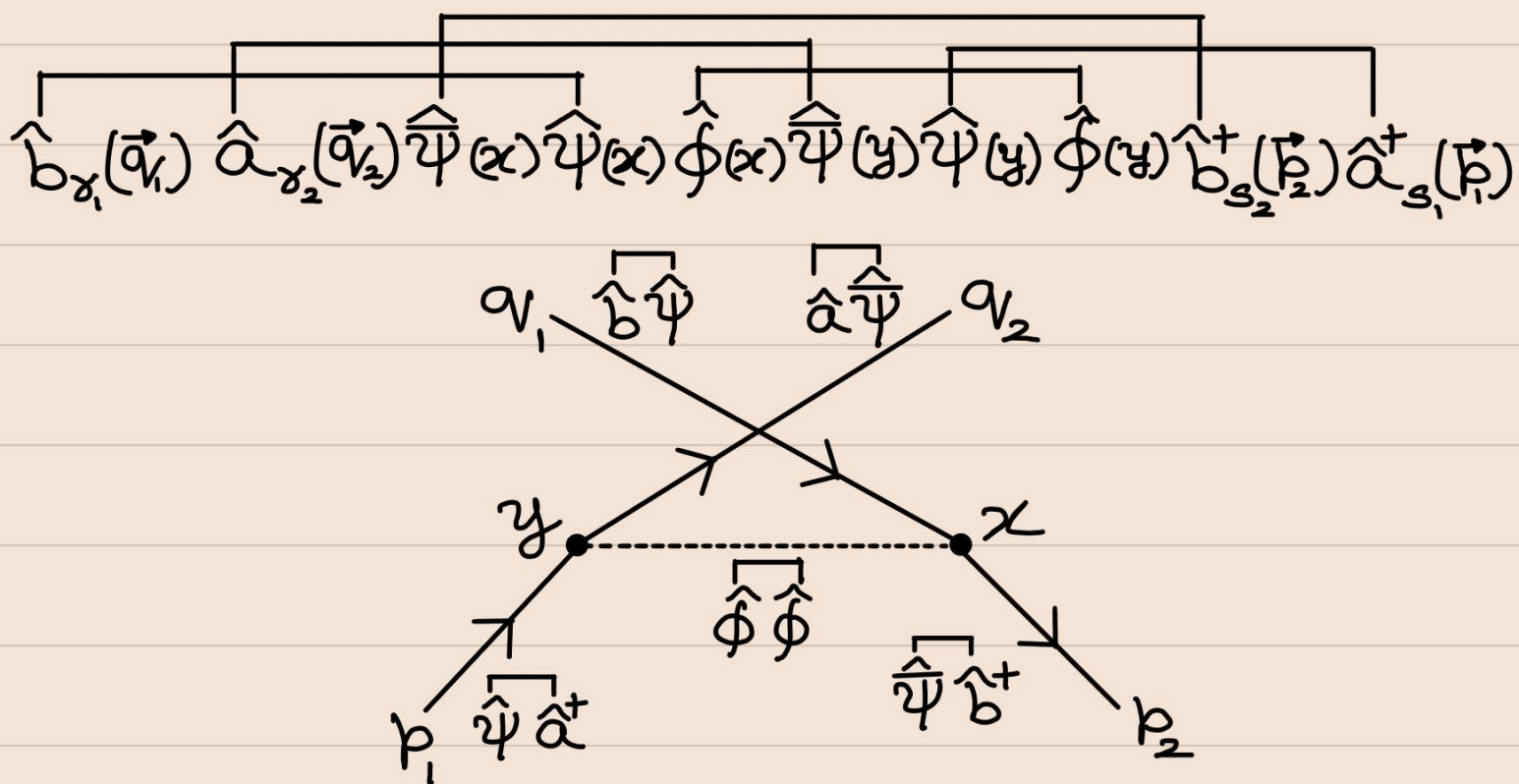
Then, the invariant matrix element is,

$$i\mathcal{M}_t = g^2 \frac{i}{t - \mu^2} \bar{u}^{\gamma_1}(q_1) u^{s_1}(p_1) \bar{v}^{s_2}(p_2) v^{\gamma_2}(q_2)$$



2-channel Scattering

It appears in the 2nd order terms in the expansion of $\langle \bar{q}_1, q_2 | \hat{S} | p_1, \bar{p}_2 \rangle$, where the anti-fermions have momenta $\bar{p}_2$ & $\bar{q}_1$. This process appears in the following contraction.



The amplitude for t-channel scattering is given by,

$$\mathcal{A}_t = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} (-ig)^2$$

$$\times \int d^4x d^4y \hat{b}_{s_1}(\vec{q}_1) \hat{a}_{s_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{b}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1)$$

(there are even(4) number of total exchanges)

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. So, we get,

$$A_u = (-ig)^2 \int d^4x d^4y \bar{u}^{\alpha_2}(q_2) u^{\alpha_1}(p_1) \bar{v}^{\alpha_1}(q_1) v^{\alpha_2}(p_2) \\ \times e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x} \Delta_\phi(x, y)$$

On going to momentum space, we get

$$A_u = (-ig)^2 \bar{u}^{\alpha_2}(q_2) u^{\alpha_1}(p_1) \bar{v}^{\alpha_1}(q_1) v^{\alpha_2}(p_2) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x}$$

$$= (-ig)^2 \bar{u}^{\alpha_2}(q_2) u^{\alpha_1}(p_1) \bar{v}^{\alpha_1}(q_1) v^{\alpha_2}(p_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} \int d^4x e^{-i(k+p_2-q_1)x} \\ \times \int d^4y e^{-i(p_1-k-q_2)y}$$

(Perform 'x' integral)

$$= (-ig)^2 \bar{u}^{\gamma_2}(q_2) u^{s_1}(p_1) \bar{v}^{\gamma_1}(q_1) v^{s_2}(p_2)$$

$$\times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} (2\pi)^4 \underbrace{\delta^{(4)}(k + p_2 - q_1)}_{k = (q_1 - p_2)} \\ \times \int d^4 y e^{-i(p_1 - k - q_2)y}$$

(Perform k -integral & then y -integral)

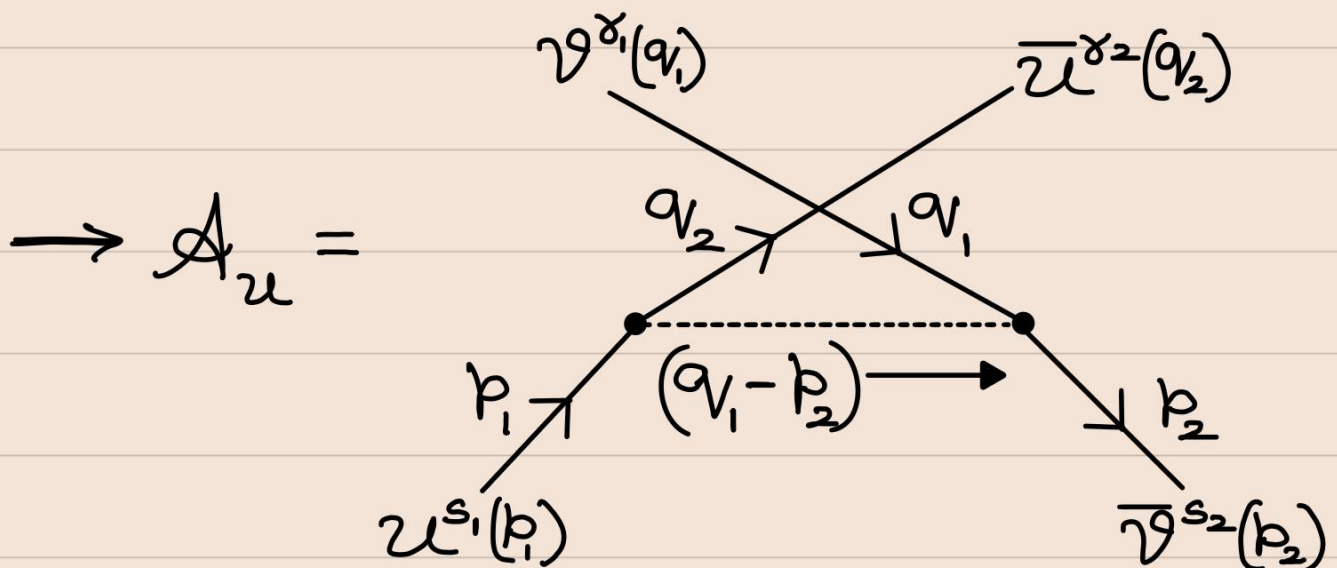
We can denote $u = (q_1 - p_2)^2$

('iε' can be neglected as it is not integrated)

$$\rightarrow \mathcal{A}_u = (-ig)^2 \frac{i}{u - \mu^2} \bar{u}^{\gamma_2}(q_2) u^{s_1}(p_1) \bar{v}^{\gamma_1}(q_1) v^{s_2}(p_2) \\ \times (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)$$

Then, the invariant matrix element is,

$$i\mathcal{M}_u = -g^2 \frac{i}{u - \mu^2} \bar{u}^{\gamma_2}(q_2) u^{s_1}(p_1) \bar{v}^{\gamma_1}(q_1) v^{s_2}(p_2)$$



Scattering of two anti-fermions

Here we shall consider 2 anti-fermions entering & 2 anti-fermions leaving. So the S-matrix expansion is $\langle \bar{q}_1, \bar{q}_2 | \hat{S} | \bar{p}_1, \bar{p}_2 \rangle$

The leading order non-trivial terms which actually induces scattering are given by the second order expansion,

$$\begin{aligned} \overline{A}^{(2)}(\bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2) &= (2\pi)^6 4\sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \\ &\times \frac{(-ig)^2}{2!} \int d^4x d^4y \langle 0 | T \left[\hat{b}_{x_1}(\vec{q}_1) \hat{b}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \right. \\ &\quad \left. \times \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{b}_{s_2}^\dagger(\vec{p}_2) \hat{b}_{s_1}^\dagger(\vec{p}_1) \right] | 0 \rangle \end{aligned}$$

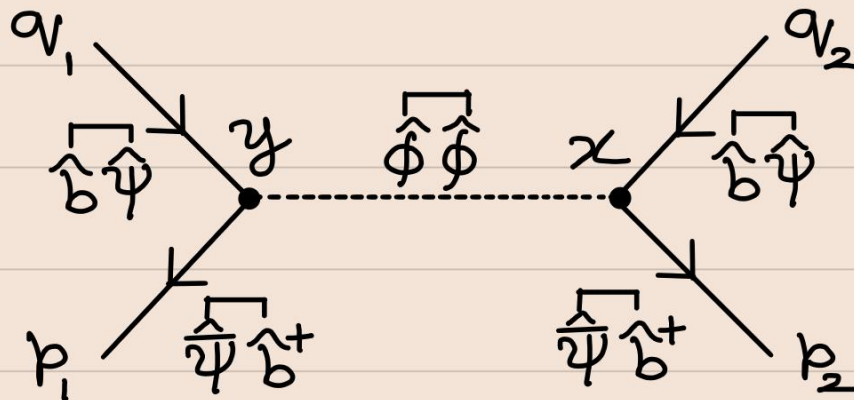
Here we shall look into the interacting diagrams where two fermions interact using a boson.

This gives the similar t-channel and u-channel processes that we saw in the scalar Yukawa theory.

t-channel Scattering

It is given by the following contraction

$$\hat{b}_{s_1}(\vec{q}_1) \hat{b}_{s_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{b}_{s_2}^+(\vec{p}_2) \hat{b}_{s_1}^+(\vec{p}_1)$$



The amplitude for t-channel scattering is given by,

$$\overline{\mathcal{A}}_t = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \hat{b}_{s_1}(\vec{q}_1) \hat{b}_{s_2}(\vec{q}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \hat{b}_{s_2}^+(\vec{p}_2) \hat{b}_{s_1}^+(\vec{p}_1)$$

(there are even (4) number of total exchanges)
(So no overall -ve sign appears)

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. So, we get,

$$\overline{\mathcal{A}}_t = (-ig)^2 \int d^4x d^4y \overline{v}^{s_1}(p_1) e^{-ip_1 y} v^{\delta_1}(q_1) e^{iq_1 y} \\ \times \overline{v}^{s_2}(p_2) e^{-ip_2 x} v^{\delta_2}(q_2) e^{iq_2 x} \Delta_\phi(x, y)$$

On going to momentum space, we get

$$\overline{\mathcal{A}}_t = (-ig)^2 \overline{v}^{s_1}(p_1) v^{\delta_1}(q_1) \overline{v}^{s_2}(p_2) v^{\delta_2}(q_2) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\varepsilon} e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} e^{-ip_2 x}$$

$$= (-ig)^2 \overline{v}^{s_1}(p_1) v^{\delta_1}(q_1) \overline{v}^{s_2}(p_2) v^{\delta_2}(q_2) \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4x e^{-i(k+p_2-q_2)x} \\ \times \int d^4y e^{-i(p_1-k-q_1)y}$$

(Perform 'y' integral)

$$\begin{aligned}
&= (-ig)^2 \bar{v}^{s_1}(p_1) v^{\delta_1}(q_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \\
&\quad \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4 x e^{-i(k+p_2-q_2)x} \\
&\quad \times (2\pi)^4 \underbrace{\delta^{(4)}(p_1 - k - q_1)}_{k = -(q_1 - p_1)}
\end{aligned}$$

(Perform 'k' integral)

$$\begin{aligned}
&= (-ig)^2 \bar{v}^{s_1}(p_1) v^{\delta_1}(q_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \\
&\quad \times \frac{i}{(q_1 - p_1)^2 - \mu^2} \underbrace{\int d^4 x e^{-i(p_1 - q_1 + p_2 - q_2)x}}_{(2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)}
\end{aligned}$$

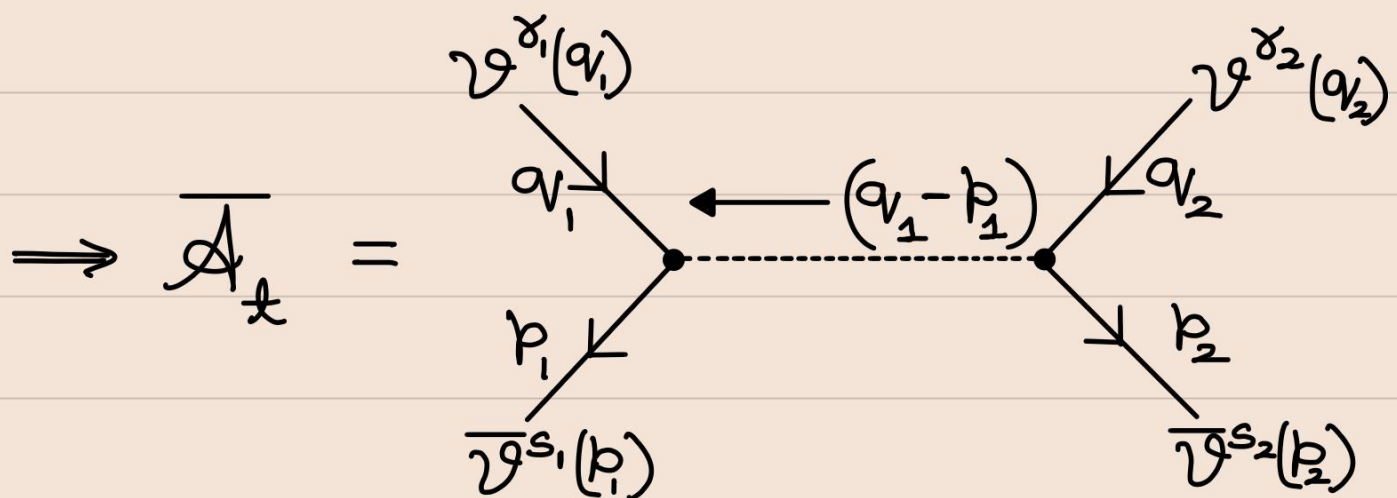
('i\varepsilon' can be neglected as it is not integrated)

If we denote $t = (q_1 - p_1)^2$, then we get the amplitude of t-channel process as,

$$\begin{aligned}
\bar{A}_t &= (-ig)^2 \bar{v}^{s_1}(p_1) v^{\delta_1}(q_1) \bar{v}^{s_2}(p_2) v^{\delta_2}(q_2) \frac{i}{t - \mu^2} \\
&\quad \times (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)
\end{aligned}$$

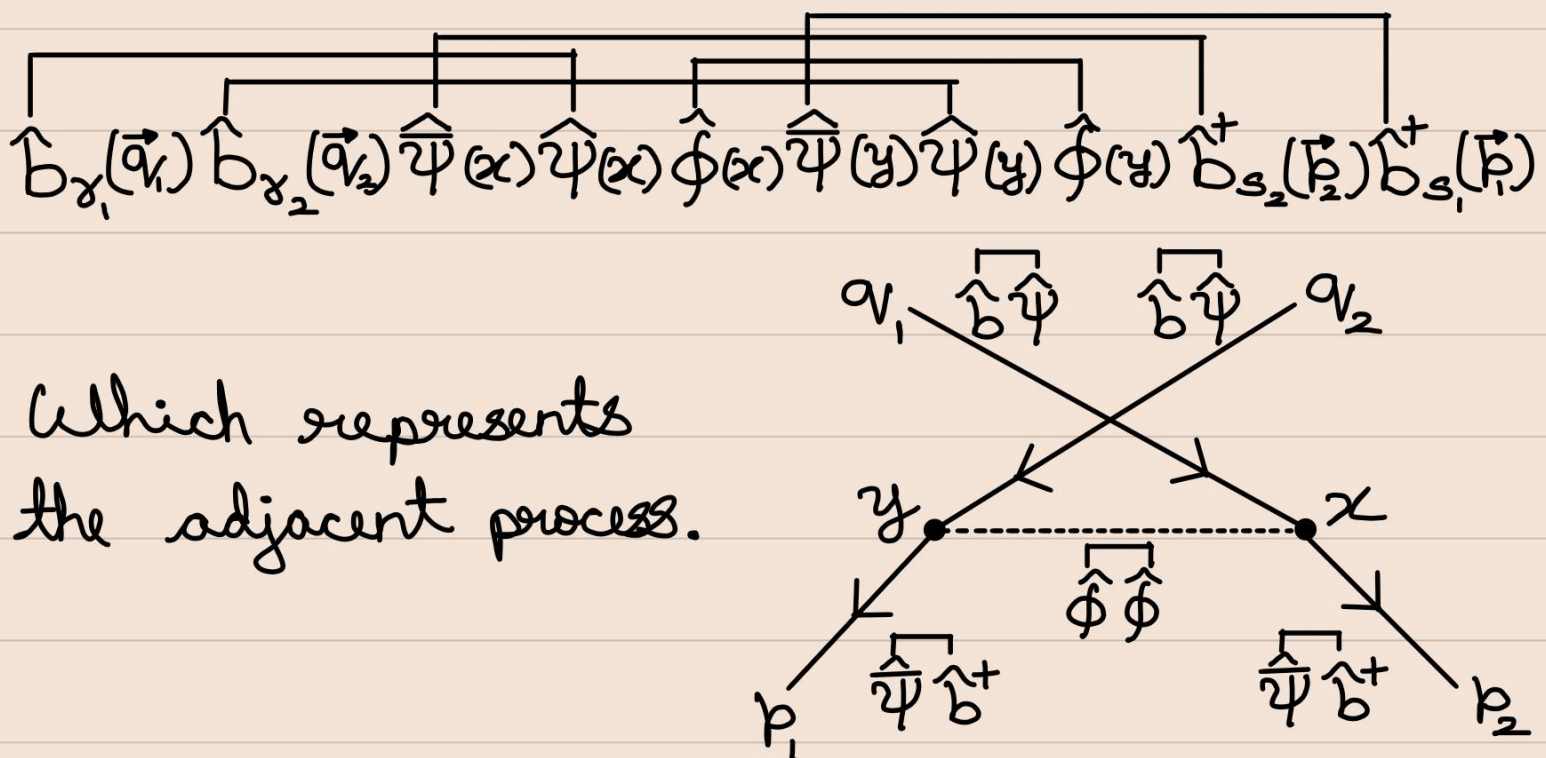
In terms of invariant matrix element we get it to be,

$$i\overline{\mathcal{M}}_t = -g^2 \frac{i}{t - \mu^2} \overline{\psi}^{s_1}(p_1) \psi^{\delta_1}(q_1) \overline{\psi}^{s_2}(p_2) \psi^{\delta_2}(q_2)$$



u-channel Scattering

It is given by the following contraction



The amplitude for u-channel scattering is given by,

$$\overline{\mathcal{A}}_u = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-ig)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \overline{\psi}_{s_1}(\vec{p}_1) \overline{\psi}_{s_2}(\vec{p}_2) \hat{\psi}(x) \hat{\psi}(x) \hat{\phi}(x) \hat{\psi}(y) \hat{\psi}(y) \hat{\phi}(y) \psi_{s_2}(\vec{p}_2) \psi_{s_1}(\vec{p}_1)$$

(there are odd(5) number of total exchanges)

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. So, we get,

$$\overline{\mathcal{A}}_u = -(-ig)^2 \int d^4x d^4y \overline{\psi}^{s_1}(p_1) e^{-ip_1 y} \psi^{s_2}(q_2) e^{iq_2 y} \\ \times \overline{\psi}^{s_2}(p_2) e^{-ip_2 x} \psi^{s_1}(q_1) e^{iq_1 x} \Delta_{\phi}(x, y)$$

On going to momentum space, we get

$$\overline{\mathcal{A}}_u = -(-ig)^2 \overline{\psi}^{s_1}(p_1) \psi^{s_2}(q_2) \overline{\psi}^{s_2}(p_2) \psi^{s_1}(q_1) \int d^4x d^4y \\ \times \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x}$$

$$\begin{aligned}
&= -(-ig)^2 \bar{\psi}^{s_1}(p_1) \psi^{\delta_2}(q_2) \bar{\psi}^{s_2}(p_2) \psi^{\delta_1}(q_1) \\
&\quad \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} \int d^4 x \, e^{-i(k+p_2-q_1)x} \\
&\quad \times \int d^4 y \, e^{-i(p_1-k-q_2)y}
\end{aligned}$$

(Perform 'x' integral)

$$\begin{aligned}
&= -(-ig)^2 \bar{\psi}^{s_1}(p_1) \psi^{\delta_2}(q_2) \bar{\psi}^{s_2}(p_2) \psi^{\delta_1}(q_1) \\
&\quad \times \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} (2\pi)^4 \delta^{(4)}(k+p_2-q_1) \\
&\quad \times \int d^4 y \, e^{-i(p_1-k-q_2)y}
\end{aligned}$$

(Perform 'k' integral)

$$\begin{aligned}
&= -(-ig)^2 \bar{\psi}^{s_1}(p_1) \psi^{\delta_2}(q_2) \bar{\psi}^{s_2}(p_2) \psi^{\delta_1}(q_1) \\
&\quad \times \frac{i}{(q_1-p_2)^2 - \mu^2 + i\varepsilon} \underbrace{\int d^4 y \, e^{-i(p_1-q_1+p_2-q_2)y}}_{(2\pi)^4 \delta^{(4)}(q_1+q_2-p_1-p_2)}
\end{aligned}$$

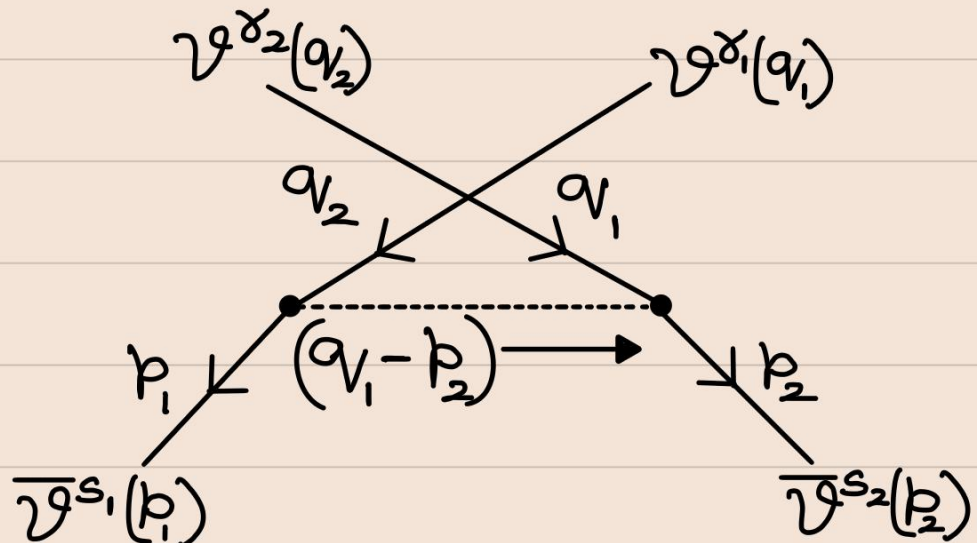
We can denote $u = (q_1-p_2)^2$.

('i\varepsilon' can be neglected as it is not integrated)

$$\Rightarrow \overline{\mathcal{A}}_u = -(-ig)^2 \overline{\psi}^{s_1}(p_1) \psi^{\delta_2}(q_2) \overline{\psi}^{s_2}(p_2) \psi^{\delta_1}(q_1) \\ \times \frac{i}{u - \mu^2} (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)$$

In terms of invariant matrix element we get it to be,

$$i\overline{\mathcal{M}}_u = g^2 \frac{i}{u - \mu^2} \overline{\psi}^{s_1}(p_1) \psi^{\delta_2}(q_2) \overline{\psi}^{s_2}(p_2) \psi^{\delta_1}(q_1)$$

$$\Rightarrow \overline{\mathcal{A}}_u =$$


Thus the full result to the total $\overline{\mathcal{M}}^{(2)}$ in the second order scattering of two anti fermions in the Yukawa theory is,

$$i\overline{\mathcal{M}}^{(2)} = i\overline{\mathcal{M}}_t + i\overline{\mathcal{M}}_u$$

$$\rightarrow i\overline{\mathcal{M}}^{(2)}(\overline{p}_1, \overline{p}_2 \rightarrow \overline{q}_1, \overline{q}_2)$$

$$= (-ig^2) \left[\frac{1}{(q_1 - p_1)^2 - \mu^2} \overline{\psi}^{s_1}(p_1) \gamma^{\delta_1}(q_1) \overline{\psi}^{s_2}(p_2) \gamma^{\delta_2}(q_2) \right. \\ \left. - \frac{1}{(q_1 - p_2)^2 - \mu^2} \overline{\psi}^{s_1}(p_1) \gamma^{\delta_2}(q_2) \overline{\psi}^{s_2}(p_2) \gamma^{\delta_1}(q_1) \right]$$

The -ve sign arises due to the interchange of two anti-fermions after scattering b/w the t & u processes. This result is the reflection of the fermi statistics of spin 1/2 fermions.

Yukawa Potential

In this theory the scattering interactions between two fermions involve the force carrying particle to have spin zero (Scalar), no any charge, but has a non-zero mass ' μ '. This type of particle is called as Yukawa's Force carrying Particle.

This theory was formulated by Yukawa to explain how nucleons (fermions) are held together by strong nuclear force by exchange of massive bosons (Pions).

To derive the potential b/w the fermions we shall consider the scattering of two distinguishable fermions in non-relativistic limit, which implies only the t -channel process contributes. Thus upto leading order of ' 2 ', the scattering amplitude we should consider is given below.

$$i\mathcal{M}_t = -g^2 \frac{i}{t - \mu^2} \bar{u}^{\alpha_1}(q_1) u^{\alpha_1}(p_1) \bar{u}^{\alpha_2}(q_2) u^{\alpha_2}(p_2)$$

$t = (q_1 - p_1)^2$. In non-relativistic limit,
 $q_i \sim (m, \vec{q}_i)$, $p_i \sim (m, \vec{p}_i)$, $m \gg |\vec{q}_i|, |\vec{p}_i|$

$$\begin{aligned} \rightarrow t &= (m - m, \vec{q}_1 - \vec{p}_1)^2, \quad \vec{k} = \vec{q}_1 - \vec{p}_1 \\ &= (0, \vec{k})^2 = 0 - |\vec{k}|^2 \end{aligned}$$

$$\rightarrow \underline{\underline{t = -|\vec{k}|^2}}$$

In non-relativistic limit, the spinors in momentum space are as follows,

$$u^s(p) = \sqrt{m} \begin{bmatrix} \xi^s \\ \xi^s \end{bmatrix}, \quad \xi^{s\dagger} \xi^{\alpha} = \delta_{s\alpha}$$

$$v^s(p) = \sqrt{m} \begin{bmatrix} \eta^s \\ -\eta^s \end{bmatrix}, \quad \eta^{s\dagger} \eta^{\alpha} = \delta_{s\alpha}$$

Then the Spinor inner products are

$$\begin{aligned}
 \bar{u}^{\sigma_1}(q_1) u^{s_1}(p_1) \bar{u}^{\sigma_2}(q_2) u^{s_2}(p_2) &= [\bar{u}^{\sigma_1}(q_1) u^{s_1}(p_1)] \\
 &\quad \times [\bar{u}^{\sigma_2}(q_2) u^{s_2}(p_2)] \\
 &= [\bar{u}^{\sigma_1 \dagger}(q_1) \gamma^0 u^{s_1}(p_1)] \times [\bar{u}^{\sigma_2 \dagger}(q_2) \gamma^0 u^{s_2}(p_2)] \\
 &= m^2 \begin{bmatrix} \xi^{\sigma_1 \dagger} & \xi^{\sigma_1 \dagger} \end{bmatrix} \begin{bmatrix} \xi^{s_1} \\ \xi^{s_1} \end{bmatrix} \begin{bmatrix} \xi^{\sigma_2 \dagger} & \xi^{\sigma_2 \dagger} \end{bmatrix} \begin{bmatrix} \xi^{s_2} \\ \xi^{s_2} \end{bmatrix} \\
 &= (2m \delta_{\sigma_1 s_1}) (2m \delta_{\sigma_2 s_2})
 \end{aligned}$$

In non-relativistic limit for distinguishable fermions only t-channel process contributes to the 2nd order scattering, so we get upto 2nd order,

$$i\mathcal{M}^{(2)} = g^2 \frac{i}{|\vec{k}|^2 + \mu^2} (4m^2 \delta_{\sigma_1 s_1} \delta_{\sigma_2 s_2})$$

This implies that the spins of the two fermions are conserved independently.

According to Born's approx. in single particle scattering in q. mech., The scattering amplitude is given by the transition matrix as,

$$\langle q_1 | i\hat{T} | p_1 \rangle = -i\tilde{U}(\vec{k}) (2\pi) \delta(E_{q_1} - E_{p_1})$$

where $\tilde{U}(\vec{k}) = \langle \vec{q}_1 | U(\vec{r}) | \vec{p}_1 \rangle$

This tells that the first particle is scattered under the influence of some potential (in its fourier transform), which we know through QFT that it is caused by the presence of second particle & this second particle mediates force to the 1st particle through ϕ -bosons.

Thus by comparing the two results, we get, $-i\tilde{U}(\vec{k}) = iM^{(2)}$

$$\Rightarrow \tilde{U}(\vec{k}) = - \frac{g^2 (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})}{|\vec{k}|^2 + \mu^2}$$

$$\rightarrow \langle \vec{v}_1 | \mathcal{U}(\vec{r}) | \vec{p}_1 \rangle = \frac{-g^2 (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})}{|\vec{v}_1 - \vec{p}_1|^2 + \mu^2}$$

$$\Rightarrow \underbrace{\int d^3x e^{-i(\vec{v}_1 - \vec{p}_1) \cdot \vec{r}} \mathcal{U}(\vec{r})}_{\text{fourier transform relation}} = \frac{-g^2 (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})}{|\vec{v}_1 - \vec{p}_1|^2 + \mu^2}$$

$$\Rightarrow \mathcal{U}(\vec{r}) = -4m^2 g^2 \delta_{s_1 s_1} \delta_{s_2 s_2} \underbrace{\int d^3k \frac{e^{-i\vec{k} \cdot \vec{r}}}{|\vec{k}|^2 + \mu^2}}_{\frac{e^{-\mu|\vec{r}|}}{4\pi|\vec{r}|}}$$

$$\Rightarrow \mathcal{U}(\vec{r}) = (4m^2 g^2 \delta_{s_1 s_1} \delta_{s_2 s_2}) \left[-\frac{e^{-\mu|\vec{r}|}}{4\pi|\vec{r}|} \right]$$

Thus the potential acting b/w two scattering fermions is the attractive Yukawa Potential (because of the -ve sign).

The range of this potential is in the order of $\frac{1}{\mu}$, i.e. for $|\vec{r}| < \frac{1}{\mu}$,

we see influence of particles & for $|\vec{r}| > \frac{1}{\mu}$, the strength of potential falls extremely fast.

For Pions whose mass is around $135 \text{ MeV}/c^2$, $\frac{1}{\mu} \sim 1.5 \text{ fm}$, which is roughly the interaction range of the nucleons inside the nucleons.

But what if we consider the scattering b/w two distinguishable anti-fermions. Then we need to compare the following two amplitudes in non-relativistic limit.

$$i\overline{\mathcal{M}}_t = -g^2 \frac{i}{t - \mu^2} \overline{v}^{s_1}(p_1) \gamma^{\delta_1}(q_1) \overline{v}^{s_2}(p_2) \gamma^{\delta_2}(q_2)$$

&

$$\langle q_2 | i\hat{T} | p_1 \rangle = -i\tilde{U}(\vec{k}) (2\pi) \delta(E_{q_1} - E_{p_1})$$

In non relativistic limit, $t = -|\vec{k}|^2$
 with $\vec{k} = \vec{q}_1 - \vec{p}_1$. Thus

$$i\overline{\mathcal{M}}_t = g^2 \frac{i}{|\vec{k}|^2 - \mu^2} \overline{\psi}^{s_1}(p_1) \psi^{\delta_1}(q_1) \overline{\psi}^{s_2}(p_2) \psi^{\delta_2}(q_2)$$

$$\overline{\psi}^{s_1}(p_1) \psi^{\delta_1}(q_1) \overline{\psi}^{s_2}(p_2) \psi^{\delta_2}(q_2) = [\overline{\psi}^{s_1}(p_1) \psi^{\delta_1}(q_1)] \times [\overline{\psi}^{s_2}(p_2) \psi^{\delta_2}(q_2)]$$

$$= [\psi^{\delta_1\dagger}(q_1) \gamma^0 \psi^{s_1}(p_1)] \times [\psi^{\delta_2\dagger}(q_2) \gamma^0 \psi^{s_2}(p_2)]$$

$$= m^2 \begin{bmatrix} \eta^{\delta_1\dagger} & -\eta^{\delta_1\dagger} \end{bmatrix} \begin{bmatrix} \eta^{s_1} \\ -\eta^{s_1} \end{bmatrix} \begin{bmatrix} \eta^{\delta_2\dagger} & -\eta^{\delta_2\dagger} \end{bmatrix} \begin{bmatrix} \eta^{s_2} \\ -\eta^{s_2} \end{bmatrix}$$

$$= (-2m\delta_{\delta_1 s_1})(-2m\delta_{\delta_2 s_2})$$

(only t-channel contributes)

$$\rightarrow i\overline{\mathcal{M}}^{(2)} = g^2 \frac{i}{|\vec{k}|^2 + \mu^2} (4m^2 \delta_{\delta_1 s_1} \delta_{\delta_2 s_2})$$

$$\rightarrow \tilde{\mathcal{U}}(\vec{k}) = -\frac{g^2}{|\vec{k}|^2 + \mu^2} (4m^2 \delta_{\delta_1 s_1} \delta_{\delta_2 s_2})$$

$$\rightarrow \mathcal{U}(\vec{r}) = (4m^2 g^2 \delta_{s_1 s_1} \delta_{s_2 s_2}) \left[-\frac{e^{-\mu|\vec{r}|}}{4\pi|\vec{r}|} \right]$$

Thus, even here, potential acting b/w two scattering anti-fermions is the attractive Yukawa Potential (because of the -ve sign).

Now, if we consider scattering between a fermion & anti-fermion, then, even here only t -channel contributes. So,

$$i\mathcal{M}_t = g^2 \frac{i}{t - \mu^2} [\bar{u}^{\alpha_1}(q_1) u^{s_1}(p_1)] [\bar{v}^{s_2}(p_2) v^{\alpha_2}(q_2)]$$

In non relativistic limit, we get

$$\begin{aligned} & [\bar{u}^{\alpha_1}(q_1) u^{s_1}(p_1)] [\bar{v}^{s_2}(p_2) v^{\alpha_2}(q_2)] \\ &= (2m \delta_{s_1 s_1}) (-2m \delta_{s_2 s_2}) \end{aligned}$$

$$\& \quad t \sim -|\vec{k}|^2 \quad \text{with} \quad \vec{k} = \vec{q}_1 - \vec{p}_1$$

Thus,

$$\rightarrow i\mathcal{M}^{(2)} = g^2 \frac{i}{|\vec{k}|^2 + \mu^2} (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})$$

$$\rightarrow \tilde{\mathcal{U}}(\vec{k}) = -\frac{g^2}{|\vec{k}|^2 + \mu^2} (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})$$

$$\rightarrow \underline{\mathcal{U}(\vec{r}) = (4m^2 g^2 \delta_{s_1 s_1} \delta_{s_2 s_2}) \left[-\frac{e^{-\mu|\vec{r}|}}{4\pi|\vec{r}|} \right]}$$

Thus, even here, potential acting b/w the scattering fermions & anti-fermions is the attractive Yukawa Potential.

Hence, we can conclude that the nature of potential acting between any two fermions / anti-fermions in Yukawa theory is universally attractive.