

Local Symmetry & Gauge theory for Fermions : QED Lagrangian

Here we try to build a Gauge theory of fermions described by the Dirac Lagrangian where the EM field acts as the Gauge field. This will lead to the minimal coupling interaction between the fermion (electron) field and the EM field.

The Dirac Lagrangian density is,

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\not{\partial} - m)\Psi$$

The Dirac Lagrangian is invariant under global phase rotation.

The Dirac Lagrangian is invariant under the global phase rotation of $\Psi(x)$ spinor. So it has $U(1)$ Symmetry.

If we perform an infinitesimal rot.

$$\psi(x) \longrightarrow \psi(x) e^{i(\delta\alpha)}$$

$$\text{where } e^{i(\delta\alpha)} = 1 + i(\delta\alpha)$$

$$\rightarrow \delta\psi = i(\delta\alpha)\psi \quad \& \quad \delta\bar{\psi} = i(\delta\alpha)\bar{\psi}$$

Then, the corresponding conserved Noether current is given by,

$$\Rightarrow J_N^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (\delta\psi) + \frac{\partial \mathcal{L}}{\partial(\underbrace{\partial_\mu \bar{\psi}}_0)} (\delta\bar{\psi})$$

$$\rightarrow \underline{J_N^\mu = -(\delta\alpha) \bar{\psi} \gamma^\mu \psi}$$

But $\bar{\psi} \gamma^\mu \psi = J_{\text{prob. density}}^\mu$ is the probability density current. So the $U(1)$ symmetry of Dirac field results in the conservation of Probability of fermions, provided that dirac fields vanish at the boundary (no flux of fermions in boundary).

But the dirac lagrangian has no local $U(1)$ symmetry. If the phase is position dependent, then,

$$\partial_\mu \psi \rightarrow \partial_\mu (\psi e^{i\alpha}) = e^{i\alpha} [\partial_\mu + i(\partial_\mu \alpha)] \psi$$

$$\rightarrow \mathcal{L}_{\text{dirac}} = \bar{\psi} [i \gamma^\mu \partial_\mu \psi - m \psi]$$

$\downarrow$ local $U(1)$

$$\bar{\psi} e^{-i\alpha} (i \gamma^\mu e^{i\alpha} [\partial_\mu + i(\partial_\mu \alpha)] \psi - m \psi e^{i\alpha})$$

$$= \bar{\psi} (i \gamma^\mu [\partial_\mu + i(\partial_\mu \alpha)] \psi - m \psi)$$

$$\rightarrow \delta \mathcal{L} = - \bar{\psi} (\not{\partial} \alpha) \psi \neq 0.$$

(Thus dirac lagrangian has no local symmetry)

But if we replace the normal derivative ∂_μ with covariant derivative $D_\mu = (\partial_\mu + iqA_\mu)$ in the dirac lagrangian, & also when $\psi \rightarrow \psi e^{i\alpha(x)}$, if we also do a gauge transformation of A_μ

Such that $A_\mu \rightarrow A_\mu - \frac{1}{q} \partial_\mu \alpha(x)$, then the modified dirac lagrangian has even the local $U(1)$ symmetry.

$$\begin{aligned}
 D_\mu \psi &\xrightarrow[U(1)]{\text{local}} D_\mu (\psi e^{i\alpha(x)}) \\
 &= (\partial_\mu + iqA_\mu)(\psi e^{i\alpha(x)}) \\
 &= e^{i\alpha(x)} \left[\partial_\mu + i(\partial_\mu \alpha(x)) \right] \psi + iqA_\mu \psi e^{i\alpha(x)} \\
 &= e^{i\alpha(x)} \left(\partial_\mu \psi + i\psi [\partial_\mu \alpha(x) + qA_\mu] \right)
 \end{aligned}$$

$$\text{Gauge transform} \downarrow A_\mu \rightarrow A_\mu - \frac{1}{q} \partial_\mu \alpha(x)$$

$$= e^{i\alpha(x)} (\partial_\mu + iqA_\mu) \psi = e^{i\alpha(x)} (D_\mu \psi)$$

Thus if $\mathcal{L}_{\text{modified dirac}} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi$

Then $\mathcal{L}_{\text{modified dirac}} \xrightarrow[\text{Gauge transform}]{\text{local } U(1)} \mathcal{L}_{\text{modified dirac}}$

So, local $U(1)$ symmetry always has an associated gauge symmetry and requires a gauge field A_μ to exist.

If the gauge field is EM field then the e.m. lagrangian in the absence of external source, we have $\mathcal{L}_{\text{e.m.}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$F_{\mu\nu}$ can also be got by evaluating

$$\begin{aligned} [D_\mu, D_\nu] &= [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] \\ &= [\partial_\mu, iqA_\nu] - [\partial_\nu, iqA_\mu] \\ &= iq(\partial_\mu A_\nu - \partial_\nu A_\mu) \end{aligned}$$

$$\Rightarrow \underline{[D_\mu, D_\nu] = iq F_{\mu\nu}}$$

Thus in the presence of EM field, through minimal coupling procedure we can write the total lagrangian for fermions & EM fields as

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

Minimal coupling is just modifying $\partial_\mu \longrightarrow \partial_\mu + iqA_\mu = D_\mu$.

Thus the total lagrangian can be written more intuitively as,

$$\begin{aligned}\mathcal{L} &= \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= \bar{\Psi} [i\gamma^\mu (\partial_\mu + iqA_\mu) - m] \Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= \underbrace{\bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi}_{\mathcal{L}_{\text{Dirac}}} - \underbrace{q \bar{\Psi} \gamma^\mu \Psi A_\mu}_{\mathcal{L}_{\text{int}}} - \underbrace{\frac{1}{4} F^{\mu\nu} F_{\mu\nu}}_{\mathcal{L}_{\text{EM}}}\end{aligned}$$

Thus we get the QED lagrangian as sum of Dirac lagrangian, EM lagrangian without source and a simple minimal coupling interaction term b/w Ψ field & EM field with 'q' as the coupling constant which turns out to be the charge of fermion (Dirac) particles.

Note that $\bar{\Psi}\gamma^\mu\Psi = J_{\text{prob.}}^\mu$ is the conserved probability density current of fermions. Then, the charge q multiplied with probability density ' $\Psi\gamma^\mu\Psi$ ' acts as current density. Thus,

$$J_{\text{current}}^\mu = q\bar{\Psi}\gamma^\mu\Psi = qJ_{\text{prob.}}^\mu$$

Then, $\mathcal{L}_{\text{int}} = -J_{\text{current}}^\mu A_\mu$ acts like the source term for EM fields. So, the fermions act as source to EM field. This implies that in the presence of many fermions, the charge of fermion is the main cause of interaction which is mediated through the EM field.

Thus the QED lagrangian is written as,

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_{\text{dirac}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int. (minimal coupling)}}$$

$$\Rightarrow \boxed{\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\not{D} - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\bar{\Psi}\gamma^\mu A_\mu\Psi}$$

Introduction to Quantum Electrodynamics

Quantum Electrodynamics is the quantum field theory that describes the interaction of Light (Electromagnetic field) & Matter (Fermions) at the fundamental level of Elementary Particles.

The QED lagrangian is written as,

$$\mathcal{L}_{\text{QED}} = \bar{\Psi}(i\not{\partial} - m)\Psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - q\bar{\Psi}\gamma^{\mu}A_{\mu}\Psi$$

Where the interaction term is

$$\mathcal{L}_{\text{int}} = -q\bar{\Psi}\gamma^{\mu}A_{\mu}\Psi$$

The hamiltonian density is given by,

$$\mathcal{H}_{\text{QED}} = \mathcal{H}_{\text{dirac}} + \mathcal{H}_{\text{EM}} + \mathcal{H}_{\text{int.}}, \text{ where}$$

$\mathcal{H}_{\text{int.}} = q\bar{\Psi}\gamma^{\mu}A_{\mu}\Psi$ is the Interaction part.

But $H_0 = H_{\text{dirac}} + H_{\text{EM}}$ is the canonically second quantizable part. So, in the interaction picture, the field operators are governed only by this H_0 part. So Ψ_I solves the dirac eqⁿ & A_I^μ will solve the maxwell's eqⁿ. Thus from now we will always work in the interaction picture & thus drop the subscript 'I' from the field operators.

Free photon Propagator

Now we choose the photon propagator in the Lorenz gauge. Thus the 4-vector field A^μ obeys massless klein-gordon eqⁿ. (In interaction picture). So, $\square^2 A^\mu = 0$. Thus the free photon propagator $D_{\mu\nu}(x, y)$ in Lorenz Gauge satisfies,

$$\square^2 \eta^{\mu\sigma} D_{\mu\nu}(x, y) = i \delta_{\nu}^{\sigma} \delta(x - y)$$

And as a fourier integral, the photon propagator can be written as,

$$D_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4} \tilde{D}_{\mu\nu}(k) e^{-ik(x-y)}$$

where $\tilde{D}_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}$ is the

photon propagator in momentum space with the four momentum 'k'. In position space it is proportional to the massless Klein-Gordon propagator. Thus the on-shell condition for photons is,

$$p^0 = E_p = |\vec{p}| \quad \& \quad p^2 = 0, \text{ as } m=0.$$

$$\rightarrow \underbrace{D_{\mu\nu}^F(x,y)}_{\text{Feynmann propagator for free photons}} = -\eta_{\mu\nu} \underbrace{\Delta_F(x,y)}_{\text{K.G propagator with } m=0.}$$

Feynmann propagator for free photons

K.G propagator with $m=0$.

$$\Delta_F(x,y) = \int \frac{d^3p}{(2\pi)^3 (2E_p)} \left[\Theta(t_x - t_y) e^{-ip(x-y)} + \Theta(t_y - t_x) e^{ip(x-y)} \right]$$

This can be related to the EM field operators A^μ as follows.

$$\underline{D_{\mu\nu}^F(x,y) = \langle 0 | T [\hat{A}_\mu(x) \hat{A}_\nu(y)] | 0 \rangle}$$

$$\hat{A}^\mu(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \left[\sum_{\lambda=0}^3 \left[\varepsilon_\lambda^\mu(\vec{p}) \hat{c}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} + \varepsilon_\lambda^{\mu*}(\vec{p}) \hat{c}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right] \right]$$

where $\lambda \rightarrow$ Polarization states. $\hat{c}_\lambda \neq \hat{c}_\lambda^\dagger$ are annihilation & creation operators for photons in a particular λ polarization state.

$\varepsilon_\lambda^\mu(\vec{p})$ are the polarization 4-vectors. $\{\varepsilon_0^\mu(\vec{p}), \varepsilon_1^\mu(\vec{p}), \varepsilon_2^\mu(\vec{p}), \varepsilon_3^\mu(\vec{p})\}$ form a basis for polarization vectors which follow the Minkowski orthogonality condition given by $\underline{\eta_{\mu\nu} \varepsilon_\lambda^{\mu*} \varepsilon_{\lambda'}^\nu = \eta_{\lambda\lambda'}}$ & satisfy Polarization

sum relation $\underline{\sum_{\lambda, \lambda'} \varepsilon_{\lambda'}^\mu(\vec{p}) \varepsilon_\lambda^{\nu*}(\vec{p}) \eta_{\lambda\lambda'} = -\eta^{\mu\nu}}$

And are Minkowski orthogonal to the four momentum $\rightarrow \underline{p_\mu \varepsilon_\gamma^\mu(p) = p_\mu \varepsilon_\gamma^{\mu*}(p) = 0}$

Even though we have allowed four basis vectors for polarization, not all of them are physical. There are two gauge degrees of freedom for any general A_μ . One of them is removed by Lorenz gauge condition. But still one more degree of freedom persists as seen in the chapter of Quantization of EM field. Thus we generally choose $A^0 = 0$, which makes the photons to be in a transverse polarized state.

So, for incoming & outgoing on-mass shell physically observable photons we need to set transverse polarization condition

$$\vec{p} \cdot \vec{\epsilon}_\alpha(\vec{p}) = 0, \quad \vec{p} \cdot \vec{\epsilon}_\alpha^*(\vec{p}) = 0,$$

Eg: For $\vec{p}$ along z -axis, the right & left handed polarization vectors are

$$\epsilon^\mu = \frac{1}{\sqrt{2}} (0, 1, \pm i, 0) \quad \begin{array}{l} + \rightarrow \text{right} \\ - \rightarrow \text{left} \end{array}$$

Thus we can write the Contractions of EM gauge field as,

$$\overbrace{\hat{A}_\mu(x) \hat{A}_\nu(y)} = D_{\mu\nu}^F(x, y) = \int \frac{d^4 k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon} e^{-ik(x-y)}$$

But in QED we come across the contraction of the form

$$\overbrace{\cancel{\hat{A}}(x) \cancel{\hat{A}}(y)} = \langle 0 | T [\gamma^\mu \hat{A}_\mu(x) \gamma^\nu \hat{A}_\nu(y)] | 0 \rangle$$

$$= \langle 0 | T [\gamma^\mu \hat{A}_\mu(x) \gamma^\nu \hat{A}_\nu(y)] | 0 \rangle$$

$$= \gamma^\mu \overbrace{\hat{A}_\mu(x) \hat{A}_\nu(y)} \gamma^\nu$$

$$\Rightarrow \overbrace{\cancel{\hat{A}}(x) \cancel{\hat{A}}(y)} = \gamma^\mu D_{\mu\nu}^F(x, y) \gamma^\nu$$

$$D_{\mu\nu}^F(x, y) = \int \frac{d^4 k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon}$$

Free dirac field Propagator

We have already seen the Spin- $1/2$ Fermions (dirac particles) described by the Dirac Spinor field $\psi(x)$ solves Dirac eqⁿ

$$(i\not{\partial} - m)\psi(x) = 0$$

$$\rightarrow \psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}$$

, ψ_A is A^{th} component of the spinor

$\not{\partial} = \gamma^\mu \partial_\mu$, and in chiral basis,

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$$

$$\sigma^\mu = (\mathbb{I}_2, \vec{\sigma})$$

$$\bar{\sigma}^\mu = (\mathbb{I}_2, -\vec{\sigma})$$

Then the dirac conjugate field $\bar{\psi}(x)$ is defined using the adjoint dirac eqⁿ

$$\bar{\psi}(x)(i\overleftarrow{\not{\partial}} - m) = 0$$

The dispersion relation that the Dirac field satisfies is given by,

$$\underline{E_p^2 = |\vec{p}|^2 + m^2} \rightarrow \text{mass of fermion}$$

The mode expansion of $\psi(x)$ & $\bar{\psi}(x)$ is,

$$\hat{\psi}(x) = \int \frac{d^3p \sum_{s=1}^2 [\psi^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} + \psi^s(p) \hat{b}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\bar{\psi}}(x) = \int \frac{d^3p \sum_{s=1}^2 [\bar{\psi}^s(p) \hat{a}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu} + \bar{\psi}^s(p) \hat{b}_s(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

where $\psi^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi^s \end{bmatrix} \rightarrow \text{Dirac Particle}$

$\psi^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \eta^s \end{bmatrix} \rightarrow \text{Dirac anti-particle}$

$$\epsilon^1 = \eta^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \text{gives physical spin up state } S_z = +\frac{1}{2} \text{ (+ve helicity)}$$

$$\epsilon^2 = \eta^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \text{gives physical spin down state } S_z = -\frac{1}{2} \text{ (-ve helicity)}$$

The creation and annihilation operators are given as follows,

$\hat{a}_1^+(\vec{p}) \rightarrow$ creates a spin up particle along $\vec{p}$ axis & +ve helicity.

$\hat{a}_2^+(\vec{p}) \rightarrow$ creates a spin down particle along $\vec{p}$ axis & -ve helicity.

$\hat{b}_1^+(\vec{p}) \rightarrow$ creates a spin up anti-particle along $\vec{p}$ axis & +ve helicity.

$\hat{b}_2^+(\vec{p}) \rightarrow$ creates a spin down anti-particle along $\vec{p}$ axis & +ve helicity.

The Feynman propagator $\mathcal{S}_F(x, y)$ for Dirac eqⁿ is defined by,

$$[i\gamma^\mu \partial_{x^\mu} - m] \mathcal{S}_F(x, y) = i\delta(x-y) \mathbb{I}_{4 \times 4}$$

The feynman propagator is given by the following fourier integral,

$$\mathcal{S}_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \right] e^{-ip(x-y)}$$

Where we perform the p^0 integral with feynman's prescription of poles. This can be related to the Spinor field as:

$$\mathcal{S}_F^{AB}(x, y) = \langle 0 | \mathcal{T} [\hat{\psi}_A(x) \hat{\psi}_B(y)] | 0 \rangle$$

where,

$$\underbrace{\mathcal{T} [\hat{\psi}(x) \hat{\psi}(y)]}_{4 \times 4 \text{ matrix}} = \begin{cases} \hat{\psi}(x) \hat{\psi}(y), & x^0 > y^0 \\ - \underbrace{[\hat{\psi}(y)]^T}_{\text{column Spinor}} \underbrace{[\hat{\psi}(x)]^T}_{\text{row Spinor}}, & y^0 > x^0 \end{cases}$$

$$\rightarrow \overbrace{\hat{\psi}(x)\hat{\psi}(y)} = \langle 0 | T [\hat{\psi}(x)\hat{\psi}(y)] | 0 \rangle = \mathcal{S}_F(x, y)$$

Note that $\{\hat{\psi}_A(x)\} \neq \{\hat{\psi}_B(y)\}$ are set of fermionic operators that follow the anti-commutation relations, thus whenever we exchange two fermionic operators, we get a -ve sign.

The only surviving contractions are:

$$\bullet \overbrace{\hat{\psi}(x)\hat{a}_s^\dagger(\vec{p})} = \frac{u^s(p)e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\bullet \overbrace{\hat{a}_s(\vec{p})\hat{\psi}(x)} = \frac{\bar{u}^s(p)e^{ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\bullet \overbrace{\hat{b}_s(\vec{p})\hat{\psi}(x)} = \frac{v^s(p)e^{ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\bullet \overbrace{\hat{\psi}(x)\hat{b}_s^\dagger(\vec{p})} = \frac{\bar{v}^s(p)e^{-ip_\mu x^\mu}}{\sqrt{(2\pi)^3(2E_p)}}$$

- $\underbrace{\hat{\psi}(x)\hat{\psi}(y)}_{\text{fermion line}} = S_F(x, y) = - \underbrace{\hat{\bar{\psi}}(y)\hat{\psi}(x)}_{\text{anti-fermion line}}$

Thus the feynman rules for fermions in momentum space is,

- An incoming fermion line with a four momentum ' p ' & a spin state ' s ', has a factor $u^s(p)$.
- An outgoing fermion line with a four momentum ' q ' & a spin state ' r ', has a factor $\bar{u}^r(q)$.
- An incoming antifermion line with a four momentum ' p ' & a spin state ' s ', has a factor $\bar{v}^s(p)$.
- An outgoing antifermion line with a four momentum ' q ' & a spin state ' r ', has a factor $v^r(q)$.

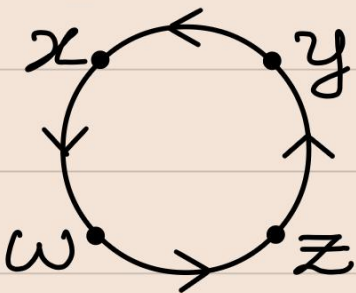
- The spinors should have an appropriate inner product based on the line's path.
- Every internal fermion line has a factor of $\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}$ where 'k' is the four momentum carried by the propagator.
- The fermion lines have arrows along the direction of time, and opposite for anti-fermions.
- Trace over the matrix products arising from fermion loop, every fermion loop carries a factor of -1 .
- Multiply '-1' factors appropriately to reflect fermion statistics.

Eg:

$$\begin{aligned}
 \text{a) } & [\hat{\psi}(x)\hat{\psi}(x)][\hat{\psi}(y)\hat{\psi}(y)][\hat{\psi}(z)\hat{\psi}(z)][\hat{\psi}(w)\hat{\psi}(w)] \\
 &= + [\hat{\psi}(x)\hat{\psi}(x)][hat{\psi}(z)\hat{\psi}(z)][\hat{\psi}(y)\hat{\psi}(y)][\hat{\psi}(w)\hat{\psi}(w)] \\
 &= \hat{\psi}(x) \mathcal{S}_F(x, z) \mathcal{S}_F(z, y) \mathcal{S}_F(y, w) \hat{\psi}(w)
 \end{aligned}$$

b) But in closed loop of any fermion propagator, we get a factor of (-1) & trace of a product of Propagators

$$\begin{aligned}
 & [\hat{\psi}(x)\hat{\psi}(x)][\hat{\psi}(y)\hat{\psi}(y)][\hat{\psi}(z)\hat{\psi}(z)][\hat{\psi}(w)\hat{\psi}(w)] \\
 &= - \text{Tr} [\hat{\psi}(w)\hat{\psi}(x)\hat{\psi}(x)\hat{\psi}(y)\hat{\psi}(y)\hat{\psi}(z)\hat{\psi}(z)\hat{\psi}(w)] \\
 &= - \text{Tr} [\mathcal{S}_F(w, x) \mathcal{S}_F(x, y) \mathcal{S}_F(y, z) \mathcal{S}_F(z, w)] \\
 &\equiv \text{Diagram} \rightarrow \text{Fermion loop. } (-ve \text{ sign})
 \end{aligned}$$



Scattering Amplitudes in QED

S-matrix expansion

The interactive Hamiltonian density part in interaction picture is given by,

$$\hat{H}_I(x) = q \hat{\bar{\psi}}(x) \hat{A}(x) \hat{\psi}(x), \quad \hat{A}(x) = \gamma^\mu A_\mu(x)$$

where all the field operators are in the interaction picture. Thus the S-matrix is given by,

$$\Rightarrow \hat{S} = T \left[\exp \left(-iq \int d^4x \hat{\bar{\psi}}(x) \hat{A}(x) \hat{\psi}(x) \right) \right]$$

Then, on expanding $\hat{S}$ above, we get,

$$\begin{aligned} \hat{S} = & \hat{I} - iq \int d^4x T \left[\hat{\bar{\psi}}(x) \hat{A}(x) \hat{\psi}(x) \right] \\ & + \frac{(-iq)^2}{2!} \int d^4x d^4y T \left[\hat{\bar{\psi}}(x) \hat{A}(x) \hat{\psi}(x) \right. \\ & \quad \left. \times \hat{\bar{\psi}}(y) \hat{A}(y) \hat{\psi}(y) \right] \\ & + \dots \end{aligned}$$

Scattering of two fermions

Here we shall find the S -matrix element given by $\langle q_1, q_2 | \hat{S} | p_1, p_2 \rangle$ which represents the amplitude of two fermions entering with four momentum states p_1 & p_2 , getting scattered, & two fermions leaving with four momentum q_1 & q_2 .

The leading order non-trivial terms which actually induces scattering are given by the second order expansion,

$$\begin{aligned} \mathcal{A}^{(2)}(p_1, p_2, q_1, q_2) &= (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \\ &\times \frac{(-iq)^2}{2!} \int d^4x d^4y \langle 0 | T \left[\hat{a}_{s_1}(\vec{p}_1) \hat{a}_{s_2}(\vec{p}_2) \hat{\psi}(x) \hat{A}(x) \hat{\psi}(x) \right. \\ &\quad \left. \times \hat{\psi}(y) \hat{A}(y) \hat{\psi}(y) \hat{a}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1) \right] \end{aligned}$$

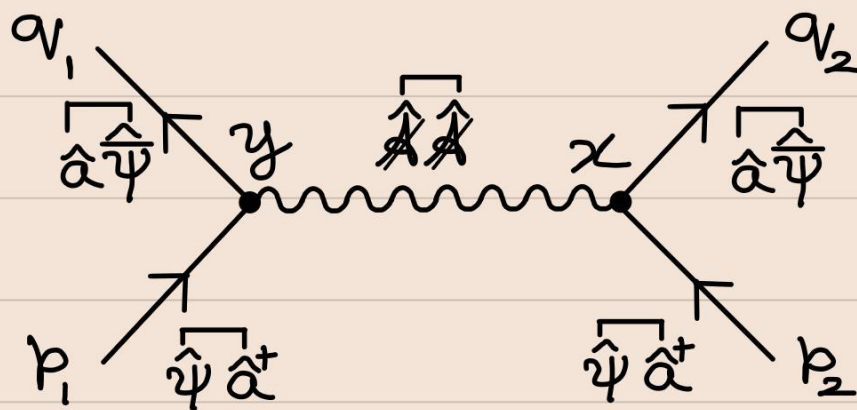
s & $\bar{s}$ are the spin states before and after the scattering respectively.

t-channel Scattering

It is given by the following contraction

$$\hat{a}_{x_1}(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{A}(x) \hat{\psi}(x) \hat{\psi}(y) \hat{A}(y) \hat{\psi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1)$$

This represents the following process



The amplitude for t-channel scattering is given by,

$$\mathcal{A}_t = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(iq)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \hat{a}_{x_1}(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{A}(x) \hat{\psi}(x) \hat{\psi}(y) \hat{A}(y) \hat{\psi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1)$$

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. To untangle,

$$\begin{aligned}
 & \hat{a}_{x_1}(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \cancel{\hat{\psi}(x)} \hat{\psi}(x) \hat{\psi}(y) \cancel{\hat{\psi}(y)} \hat{\psi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1) \\
 &= - \hat{a}_{x_1}(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \cancel{\hat{\psi}(x)} \hat{\psi}(y) \hat{\psi}(x) \cancel{\hat{\psi}(y)} \hat{\psi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1) \\
 &= - \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{a}_{x_1}(\vec{q}_1) \cancel{\hat{\psi}(x)} \hat{\psi}(y) \hat{\psi}(x) \cancel{\hat{\psi}(y)} \hat{\psi}(y) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{a}_{s_1}^\dagger(\vec{p}_1) \\
 &= (-)^2 \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{a}_{x_1}(\vec{q}_1) \cancel{\hat{\psi}(x)} \hat{\psi}(y) \hat{\psi}(x) \cancel{\hat{\psi}(y)} \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{\psi}(y) \hat{a}_{s_1}^\dagger(\vec{p}_1) \\
 &= (-)^2 \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{a}_{x_1}(\vec{q}_1) \hat{\psi}(y) \cancel{\hat{\psi}(x)} \cancel{\hat{\psi}(y)} \hat{\psi}(x) \hat{a}_{s_2}^\dagger(\vec{p}_2) \hat{\psi}(y) \hat{a}_{s_1}^\dagger(\vec{p}_1) \\
 & \quad (\text{there are even number of total exchanges})
 \end{aligned}$$

Thus by preserving the inner product relation,

$$\begin{aligned}
 \mathcal{A}_t &= (-iq)^2 \int d^4x d^4y \bar{u}^{s_2}(q_2) \gamma^\mu u^{s_2}(p_2) D_{\mu\nu}^F(x, y) \\
 & \quad \times \bar{u}^{s_1}(q_1) \gamma^\nu u^{s_1}(p_1) e^{iq_2 x} e^{iq_1 y} e^{-ip_2 x} e^{-ip_1 y}
 \end{aligned}$$

On going to momentum space, we get

$$\begin{aligned}
 \mathcal{A}_t &= (-iq)^2 \bar{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \int \frac{d^4 k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon} \\
 &\times \bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1) \int d^4 x d^4 y e^{-ik(x-y)} e^{iq_1 y} e^{iq_2 x} e^{-ip_1 y} \\
 &\times e^{-ip_2 x} \\
 &= (-iq)^2 \bar{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \int \frac{d^4 k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon} \\
 &\times \bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1) \int d^4 x e^{-i(k+p_2-q_2)x} \\
 &\times \int d^4 y e^{-i(p_1-k-q_1)y}
 \end{aligned}$$

(Perform 'y' integral)

$$\begin{aligned}
 &= (-iq)^2 \bar{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \int \frac{d^4 k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon} \\
 &\times \bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1) \int d^4 x e^{-i(k+p_2-q_2)x} \\
 &\times (2\pi)^4 \delta^{(4)}(\underbrace{p_1 - k - q_1}_{k = -(q_1 - p_1)})
 \end{aligned}$$

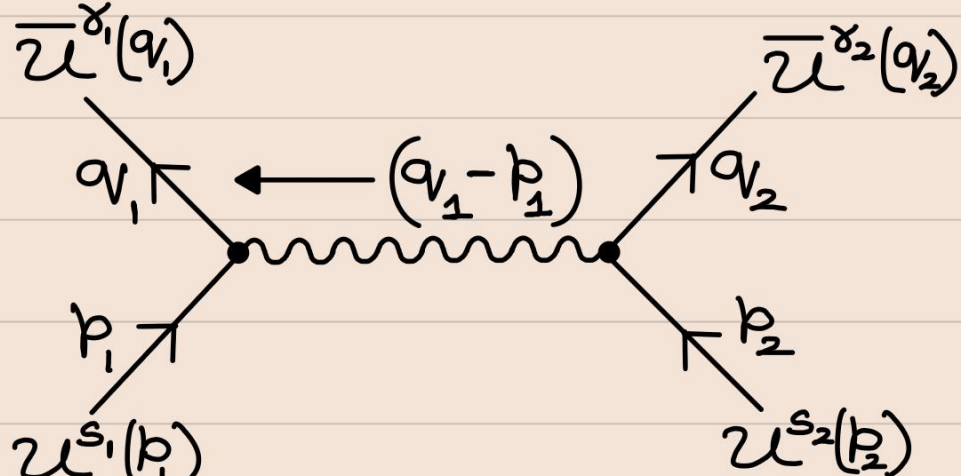
(Perform 'k' integral & then x-integral)

' $i\varepsilon$ ' can be neglected, denote $t = (q_1 - p_1)^2$, then we get the amplitude as,

$$\mathcal{A}_t = (-iq)^2 \bar{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \frac{(-i\eta_{\mu\nu})}{t} \bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1) \\ \times (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)$$

In terms of invariant matrix element we get it to be,

$$i\mathcal{M}_t = (-iq)^2 \bar{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_1)^2} \bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1)$$

$$\Rightarrow \mathcal{A}_t =$$


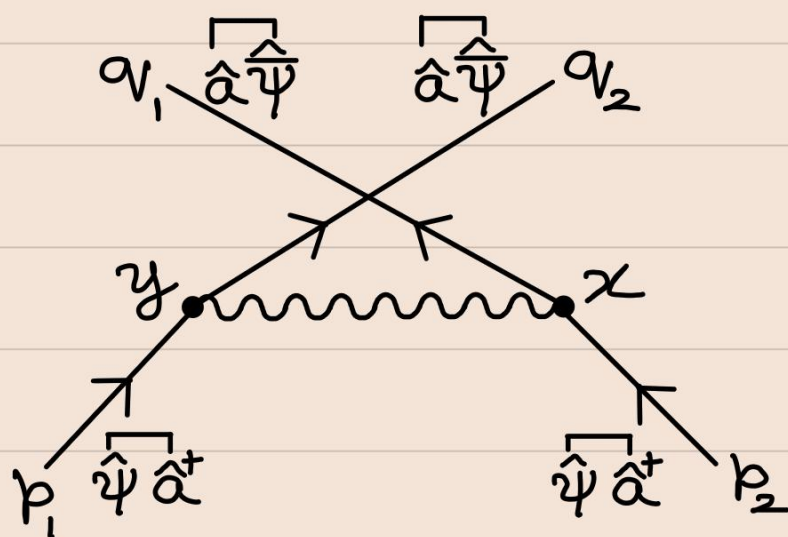
The diagram shows a t-channel exchange process. On the left, an incoming electron with momentum p_1 and spin s_1 meets a vertex. From this vertex, an outgoing positron with momentum q_1 and spin δ_1 goes upwards, and a fermion line continues downwards to another vertex. On the right, an incoming electron with momentum p_2 and spin s_2 meets a vertex. From this vertex, an outgoing positron with momentum q_2 and spin δ_2 goes upwards, and a fermion line continues downwards to another vertex. A wavy line (photon) connects the two vertices, with momentum $(q_1 - p_1)$ indicated by an arrow pointing from left to right.

u-channel Scattering

It is given by the following contraction

$$\hat{a}_{x_1}(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{A}(x) \hat{\psi}(x) \hat{\psi}(y) \hat{A}(y) \hat{\psi}(y) \hat{a}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1)$$

Which represents the adjacent process.



The amplitude for u-channel scattering is given by,

$$\mathcal{A}_u = (2\pi)^6 4 \sqrt{E_{p_1} E_{p_2} E_{q_1} E_{q_2}} \frac{(-iq)^2}{2!} \times (2)$$

$$\times \int d^4x d^4y \hat{a}_{x_1}(\vec{q}_1) \hat{a}_{x_2}(\vec{q}_2) \hat{\psi}(x) \hat{A}(x) \hat{\psi}(x) \hat{\psi}(y) \hat{A}(y) \hat{\psi}(y) \hat{a}_{s_2}^+(\vec{p}_2) \hat{a}_{s_1}^+(\vec{p}_1)$$

(there are odd number of total exchanges)

Also there are two ways of obtaining this amplitude (Interchange $x \leftrightarrow y$). Thus $2!$ in the denominator gets cancelled. So, we get,

$$A_u = -(-iq)^2 \int d^4x d^4y \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{s_2}(p_2) D_{\mu\nu}^F(x, y) \\ \times \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{s_1}(p_1) e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x}$$

On going to momentum space, we get

$$A_u = -(-iq)^2 \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{s_2}(p_2) \frac{\int d^4k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon} \\ \times \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{s_1}(p_1) \int d^4x d^4y e^{iq_1 x} e^{iq_2 y} e^{-ip_1 y} e^{-ip_2 x} \\ \times e^{-ik(x-y)} \\ = -(-iq)^2 \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{s_2}(p_2) \frac{\int d^4k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon} \\ \times \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{s_1}(p_1) \int d^4x e^{-i(k+p_2-q_1)x} \\ \times \int d^4y e^{-i(p_1-k-q_2)y}$$

(Perform 'x' integral)

$$\begin{aligned}
&= - (iq)^2 \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{\delta_2}(p_2) \int \frac{d^4 k}{(2\pi)^4} \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon} \\
&\quad \times \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{\delta_1}(p_1) (2\pi)^4 \delta^{(4)}(k + p_2 - q_1) \\
&\quad \times \int d^4 y e^{-i(p_1 - k - q_2)y}
\end{aligned}$$

(Perform 'k' integral)

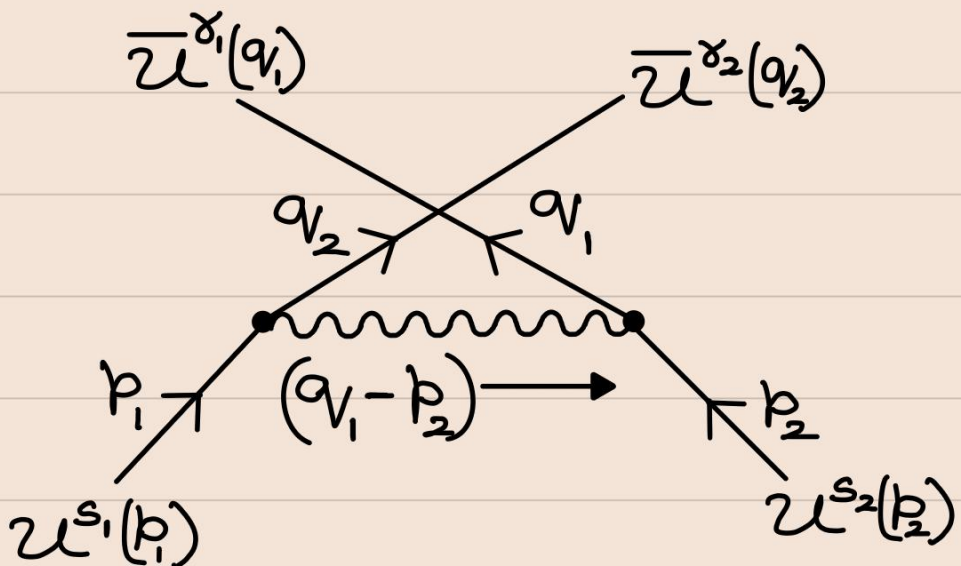
$$\begin{aligned}
&= - (iq)^2 \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{\delta_2}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_2)^2 + i\epsilon} \\
&\quad \times \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{\delta_1}(p_1) \underbrace{\int d^4 y e^{-i(p_1 - k - q_2)y}}_{(2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)}
\end{aligned}$$

We can denote $u = (q_1 - p_2)^2$.
('iε' can be neglected as it is not integrated)

$$\begin{aligned}
\Rightarrow \mathcal{A}_u &= - (iq)^2 \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{\delta_2}(p_2) \frac{(-i\eta_{\mu\nu})}{u} \\
&\quad \times \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{\delta_1}(p_1) (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p_1 - p_2)
\end{aligned}$$

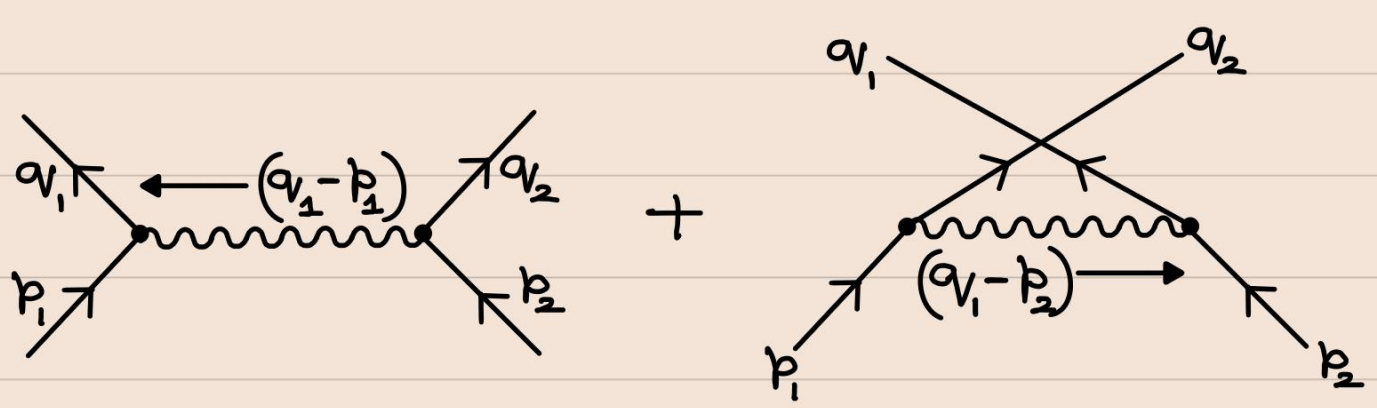
In terms of invariant matrix element we get it to be,

$$i\mathcal{M}_u = -(iq)^2 \bar{u}^{\delta_1}(q_1) \gamma^\mu u^{s_2}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_2)^2} \bar{u}^{\delta_2}(q_2) \gamma^\nu u^{s_1}(p_1)$$

$$\Rightarrow \mathcal{A}_u =$$


Thus the full result to the total $\mathcal{M}^{(2)}$ in the second order scattering of two fermions in QED is,

$$i\mathcal{M}^{(2)} = i\mathcal{M}_t + i\mathcal{M}_u$$



$$\rightarrow i\mathcal{M}^{(2)}(p_1, p_2 \rightarrow q_1, q_2)$$

$$= (-iq)^2 \left[\overline{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_1)^2} \overline{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1) \right. \\ \left. - (-iq)^2 \overline{u}^{\delta_1}(q_1) \gamma^\mu u^{s_2}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_2)^2} \overline{u}^{\delta_2}(q_2) \gamma^\nu u^{s_1}(p_1) \right]$$

The -ve sign arises due to the interchange of two fermions after the scattering b/w the t & u processes.

This result is the reflection of the fermi statistics of spin $1/2$ fermions.

But if in case we have two distinguishable fermions, then the u -channel doesn't exist. So then in that case only the t -channel contributes to the second order scattering process.

$$i\mathcal{M}^{(2)}_{(\text{distinguishable})} = (-iq)^2 \overline{u}^{\delta_2}(q_2) \gamma^\mu u^{s_2}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_1)^2} \overline{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1)$$

The Coulomb Potential

To derive the potential b/w the fermions we shall consider the scattering of two distinguishable fermions in non-relativistic limit, which implies only the t -channel process contributes. Thus upto leading order of '2', the scattering amplitude we should consider is given below.

$$i\mathcal{M}^{(2)}_{(\text{distinguishable})} = (-iq)^2 \bar{u}^{\alpha_2}(q_2) \gamma^\mu u^{\alpha_1}(p_2) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_1)^2} \bar{u}^{\beta_1}(q_1) \gamma^\nu u^{\beta_2}(p_1)$$

In non-relativistic limit,

$$q_i \sim (m, \vec{q}_i), p_i \sim (m, \vec{p}_i), m \gg |\vec{q}_i|, |\vec{p}_i|$$

$$\rightarrow (q_1 - p_1)^2 = (m - m, \vec{q}_1 - \vec{p}_1)^2$$

$$\text{Let, } \vec{k} = \vec{q}_1 - \vec{p}_1 \rightarrow (q_1 - p_1)^2 = -|\vec{k}|^2$$

In non-relativistic limit, the spinors in momentum space are as follows,

$$u^s(p) = \sqrt{m} \begin{bmatrix} \xi^s \\ \xi^s \end{bmatrix}, \quad \xi^{s\dagger} \xi^x = \delta_{sx}$$

$$v^s(p) = \sqrt{m} \begin{bmatrix} \eta^s \\ -\eta^s \end{bmatrix}, \quad \eta^{s\dagger} \eta^x = \delta_{sx}$$

$$\rightarrow \bar{u}^{x_1}(q_1) \gamma^\nu u^{s_1}(p_1) = [u^{x_1}(q_1)]^\dagger \gamma^0 \gamma^\nu u^{s_1}(p_1)$$

$$\gamma^0 \gamma^\nu = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{bmatrix} = \begin{bmatrix} \bar{\sigma}^\nu & 0 \\ 0 & \sigma^\nu \end{bmatrix}$$

$$\rightarrow \bar{u}^{x_1}(q_1) \gamma^\nu u^{s_1}(p_1)$$

$$= m \begin{bmatrix} \xi^{x_1\dagger} & \xi^{x_1\dagger} \end{bmatrix} \begin{bmatrix} \bar{\sigma}^\nu & 0 \\ 0 & \sigma^\nu \end{bmatrix} \begin{bmatrix} \xi^{s_1} \\ \xi^{s_1} \end{bmatrix}$$

$$= m \left[(\xi^{x_1\dagger} \bar{\sigma}^\nu \xi^{s_1}) + (\xi^{x_1\dagger} \sigma^\nu \xi^{s_1}) \right]$$

$$\overline{\sigma}^\nu = (\mathbb{I}, -\vec{\sigma}) \quad \& \quad \sigma^\nu = (\mathbb{I}, \vec{\sigma})$$

$$\rightarrow \bar{u}^{s_1}(q_1) \gamma^0 u^{s_1}(p_1) = 2m \xi^{s_1\dagger} \xi^{s_1} = 2m \delta_{s_1 s_1}$$

$$\bar{u}^{s_1}(q_1) \vec{\gamma} u^{s_1}(p_1) = m \left[\underbrace{-\xi^{s_1\dagger} \vec{\sigma} \xi^{s_1} + \xi^{s_1\dagger} \vec{\sigma} \xi^{s_1}}_0 \right]$$

$$\rightarrow \bar{u}^{s_1}(q_1) \gamma^\nu u^{s_1}(p_1) = 2m \delta_{s_1 s_1} \delta^\nu_0$$

So, only γ^0 component term survives.
Thus we get,

$$i\mathcal{M}^{(2)} = \underbrace{(-iq)^2}_{\substack{\text{(distinguishable)} \\ \text{(Non-relativistic)}}} (2m \delta_{s_2 s_5} \delta^\mu_0) \frac{(-i\eta_{\mu\nu})}{(-|\vec{k}|^2)} (2m \delta_{s_1 s_4} \delta^\nu_0)$$

$$\rightarrow i\mathcal{M}^{(2)} = \frac{-q^2 i\eta_{00}}{|\vec{k}|^2} [4m^2 \delta_{s_1 s_4} \delta_{s_2 s_5}]$$

(distinguishable)
(Non-relativistic)

This implies that the spins of the two fermions are conserved independently.

According to Born's approx. in single particle scattering in q. mech., The scattering amplitude is given by the transition matrix as,

$$\langle q_1 | i\hat{T} | p_1 \rangle = -i\tilde{U}(\vec{k}) (2\pi) \delta(E_{q_1} - E_{p_1})$$

where $\tilde{U}(\vec{k}) = \langle \vec{q}_1 | U(\vec{r}) | \vec{p}_1 \rangle$

This tells that the first particle is scattered under the influence of some potential (in its fourier transform), which we know through QFT that it is caused by the presence of second particle & this second particle mediates force to the 1st particle through the photons.

Thus by comparing the two results, we get, $-i\tilde{U}(\vec{k}) = i\mathcal{M}^{(2)}$

$$\Rightarrow \tilde{U}(\vec{k}) = \frac{g^2 \eta_{00}}{|\vec{k}|^2} [4m^2 \delta_{s_1 s_1'} \delta_{s_2 s_2'}]$$

But $\eta_{00} = +1$. Thus we get,

$$\langle \vec{v}_1 | \mathcal{U}(\vec{r}) | \vec{p}_1 \rangle = \frac{q^2 (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})}{|\vec{v}_1 - \vec{p}_1|^2}$$

$$\rightarrow \underbrace{\int d^3 r e^{-i(\vec{v}_1 - \vec{p}_1) \cdot \vec{r}}}_{\text{fourier transform relation}} \mathcal{U}(\vec{r}) = \frac{q^2 (4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2})}{|\vec{v}_1 - \vec{p}_1|^2}$$

$$\rightarrow \mathcal{U}(\vec{r}) = 4m^2 q^2 \delta_{s_1 s_1} \delta_{s_2 s_2} \underbrace{\int \frac{d^3 k e^{-i\vec{k} \cdot \vec{r}}}{|\vec{k}|^2}}_{\frac{1}{4\pi |\vec{r}|}}$$

(Let, $\vec{k} = \vec{v}_1 - \vec{p}_1$)

$$\rightarrow \boxed{\mathcal{U}(\vec{r}) = (4m^2 q^2 \delta_{s_1 s_1} \delta_{s_2 s_2}) \left[\frac{1}{4\pi |\vec{r}|} \right]}$$

repulsive Coulomb

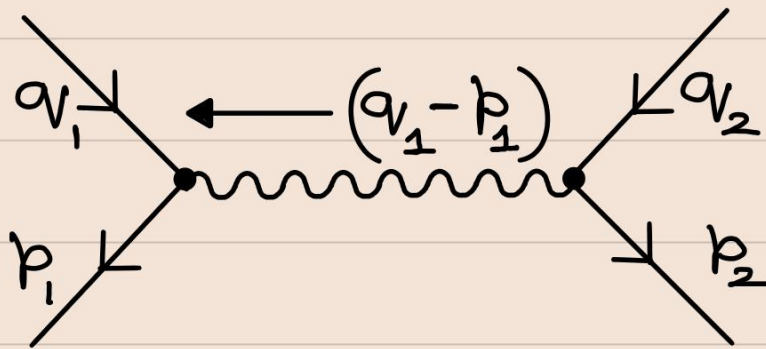
Thus the potential acting b/w two like fermions scattering is repulsive Coulomb Potential (because of the +ve sign) & since the photons are massless, exponential factor is 1. So Coulomb potential has long range.

Feynman Rules for QED

- Use all the fermion feynman rules as stated previously.
- Every interaction vertex contributes to a factor of $-iq\gamma^\mu$ where $q = Q|e|$, e is the charge of electron.
- The γ^μ of the interaction vertex has to be sandwiched between the spinors that represent the exit & entry of the fermions out of & into the vertex.
- Every internal photon line b/w the vertices with gamma matrices γ^μ & γ^ν represents $D_{\mu\nu}^F(k) = \frac{(-i\eta_{\mu\nu})}{k^2 + i\epsilon}$
- Incoming photon lines contribute to the Polarization $\epsilon_\lambda^\mu(p)$ & Outgoing photon lines contribute to Polarization $[\epsilon_\lambda^\mu(p)]^*$.

Scattering b/w two Anti-fermions

If we consider the scattering between two distinguishable anti-fermions, the process looks like,



Then we need get the following amplitude.

$$i\overline{M}^{(2)}_{(\text{distinguishable})} = (-iq)^2 \overline{v}^{\delta_2}(q_2) \gamma^\mu v^{\delta_1}(p_1) \frac{(-i\eta_{\mu\nu})}{(q_1 - p_1)^2} \overline{v}^{\delta_1}(q_1) \gamma^\nu v^{\delta_2}(p_2)$$

In non-relativistic limit with the Born's approx. $\langle q_2 | i\hat{T} | p_1 \rangle = -i\tilde{U}(\vec{k})(2\pi) \delta(E_{q_1} - E_{p_1})$

$$q_i \sim (m, \vec{q}_i), p_i \sim (m, \vec{p}_i), m \gg |\vec{q}_i|, |\vec{p}_i|$$

$$\rightarrow (q_1 - p_1)^2 = (m - m, \vec{q}_1 - \vec{p}_1)^2$$

$$\text{Let, } \vec{k} = \vec{q}_1 - \vec{p}_1 \rightarrow (q_1 - p_1)^2 = -|\vec{k}|^2$$

In non-relativistic limit, the spinors in momentum space are as follows,

$$v^s(p) = \sqrt{m} \begin{bmatrix} \eta^s \\ -\eta^s \end{bmatrix}, \quad \eta^{s\dagger} \eta^r = \delta_{sr}$$

$$\rightarrow \bar{v}^{r_1}(q_1) \gamma^\nu v^{s_1}(p_1) = -2m \delta_{r_1 s_1} \delta_0^\nu$$

$$\& \bar{v}^{r_2}(q_2) \gamma^\mu v^{s_2}(p_2) = -2m \delta_{r_2 s_2} \delta_0^\mu$$

Thus we get,

$$i\overline{\mathcal{M}}^{(2)}_{\text{(distinguishable)}}^{\text{(Non-relativistic)}} = (-iq)^2 (-2m \delta_{r_2 s_2} \delta_0^\mu) \frac{(-i\eta_{\mu\nu})}{(-|\vec{k}|^2)} (-2m \delta_{r_1 s_1} \delta_0^\nu)$$

$$\rightarrow i\overline{\mathcal{M}}^{(2)}_{\text{(distinguishable)}}^{\text{(Non-relativistic)}} = \frac{-q^2 i \eta_{00}}{|\vec{k}|^2} [4m^2 \delta_{r_1 s_1} \delta_{r_2 s_2}]$$

This implies that the spins of the two anti-fermions are conserved independently.

According to Born's approx.,

$$\langle q_1 | i\hat{T} | p_1 \rangle = -i\tilde{U}(\vec{k}) (2\pi) \delta(E_{q_1} - E_{p_1})$$

$$\text{where } \tilde{U}(\vec{k}) = \langle \vec{q}_1 | U(\vec{r}) | \vec{p}_1 \rangle$$

Thus by comparing the two results,
we get, $-i\tilde{U}(\vec{k}) = iM^{(2)}$

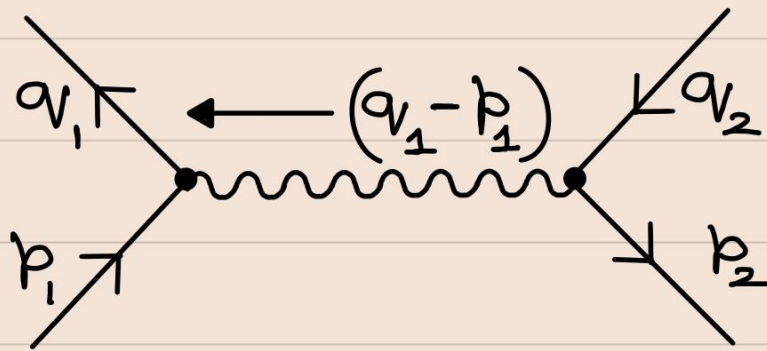
$$\rightarrow \tilde{U}(\vec{k}) = \frac{q^2 \eta_{00}}{|\vec{k}|^2} [4m^2 \delta_{s_1 s_1} \delta_{s_2 s_2}]$$

$$\rightarrow U(\vec{r}) = (4m^2 q^2 \delta_{s_1 s_1} \delta_{s_2 s_2}) \underbrace{\left[\frac{1}{4\pi |\vec{r}|} \right]}_{\text{repulsive coulomb}}$$

Thus the potential acting b/w two anti-fermions scattering is repulsive coulomb Potential (because of the +ve sign).

Scattering b/w fermion & Anti-fermion

If we consider the scattering between two distinguishable anti-fermions, the process looks like,



Then we need get the following amplitude during scattering (not annihilation/creation)

$$i\vec{M}^{(2)} = \underbrace{(-iq)^2 \bar{v}^{\delta_2}(q_2) \gamma^\mu v^{s_2}(p_2)}_{\text{Anti fermion}} \underbrace{\frac{(-i\eta_{\mu\nu})}{(q_1 - p_1)^2}}_{\text{Photon}} \underbrace{\bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1)}_{\text{fermion}}$$

In non-relativistic limit,

$$\bar{u}^{\delta_1}(q_1) \gamma^\nu u^{s_1}(p_1) = 2m \delta_{\delta_1 s_1} \delta_0^\nu$$

$$\& \bar{v}^{\delta_2}(q_2) \gamma^\mu v^{s_2}(p_2) = -2m \delta_{\delta_2 s_2} \delta_0^\mu$$

$$\& (q_1 - p_1)^2 = -|\vec{q}_1 - \vec{p}_1|^2 = -|\vec{k}|^2$$

Thus we get,

$$i\tilde{M}^{(2)} = (-iq)^2 (-2m\delta_{x_2 s_2} \delta_0^\mu) \frac{(-i\eta_{\mu\nu})}{(-|\vec{k}|^2)} (2m\delta_{x_1 s_1} \delta_0^\nu)$$

(Non-relativistic)

$$\rightarrow i\tilde{M}^{(2)}_{\text{(Non-relativistic)}} = \frac{q^2 i\eta_{00}}{|\vec{k}|^2} [4m^2 \delta_{x_1 s_1} \delta_{x_2 s_2}]$$

According to born's approx. we get,

$$-i\tilde{U}(\vec{k}) = i\tilde{M}^{(2)}$$

$$\rightarrow \tilde{U}(\vec{k}) = -\frac{q^2 \eta_{00}}{|\vec{k}|^2} [4m^2 \delta_{x_1 s_1} \delta_{x_2 s_2}]$$

$$\rightarrow \boxed{U(\vec{r}) = (4m^2 q^2 \delta_{x_1 s_1} \delta_{x_2 s_2}) \underbrace{\left[\frac{-1}{4\pi |\vec{r}|} \right]}_{\text{attractive}}}$$

Thus the potential acting b/w fermion & anti-fermion during scattering is attractive coulomb Potential (because of the -ve sign).