

Path Integrals in Quantum Mechanics

The path integral approach gives a description of quantum Mechanics that will generalize the idea of 'Principle of Least Action' in Classical mechanics.

Quantum Mechanics deals with probability amplitudes. If a particle has to go to a space-time point ' x ' from a space-time point ' y ', then the probability amplitude for this, is given by the below 'Propagator'.

$$G(x, y) = \langle \vec{x} | \hat{U}(t_x - t_y) | \vec{y} \rangle$$

Where ' $\hat{U}(t_x - t_y)$ ' is the time-evolution operator. Now, we split the trajectory into ' N ' number of infinitesimal steps of time intervals ' Δt '. This is called as 'Time Slicing'. $\rightarrow (\Delta t)N = t_x - t_y$.

It is possible because of the below property of the unitary operator.

$$" \hat{U}(t_x - t_y) = \hat{U}(t_x - t_1) \hat{U}(t_1 - t_y) "$$

We shall consider the simple time-independent hamiltonian of the form;

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

$$\rightarrow \hat{U}(t_x - t_y) = e^{-i\hat{H}(t_x - t_y)}$$

$$\rightarrow G(x, y) = \langle \vec{x} | e^{-i\hat{H}(t_x - t_y)} | \vec{y} \rangle$$

$$\rightarrow G(x, y) = \langle \vec{x} | e^{-i\hat{H}\Delta t} e^{-i\hat{H}\Delta t} \dots e^{-i\hat{H}\Delta t} | \vec{y} \rangle$$

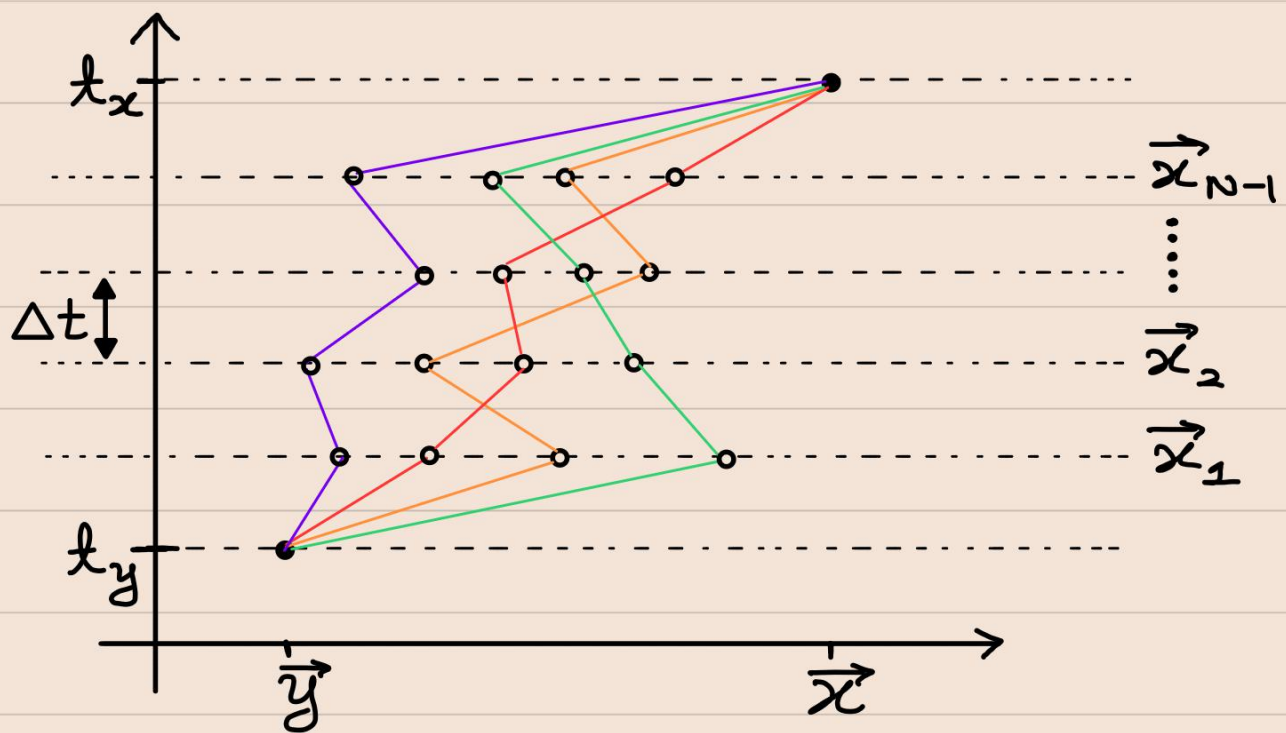
Here each n^{th} ' $e^{-i\hat{H}\Delta t}$ ' factor evolves the particles position from $|\vec{x}_n\rangle$ to $|\vec{x}_{n+1}\rangle$. Thus we can use the completeness relations

$$\hat{1} = \int d^3x_n |\vec{x}_n\rangle \langle \vec{x}_n|$$

We have to insert $N-1$ number of identities between two tiny time evolution operators. So,

$$\begin{aligned}
 G(x, y) = & \langle \vec{x} | e^{-i\hat{H}\Delta t} \int d^3x_{N-1} | \vec{x}_{N-1} \rangle \langle \vec{x}_{N-1} | \\
 & \times e^{-i\hat{H}\Delta t} \int d^3x_{N-2} | \vec{x}_{N-2} \rangle \langle \vec{x}_{N-2} | e^{-i\hat{H}\Delta t} \\
 & \times \dots \times \int d^3x_{n+1} | \vec{x}_{n+1} \rangle \langle \vec{x}_{n+1} | e^{-i\hat{H}\Delta t} \\
 & \times \int d^3x_n | \vec{x}_n \rangle \langle \vec{x}_n | e^{-i\hat{H}\Delta t} \int d^3x_{n-1} | \vec{x}_{n-1} \rangle \langle \vec{x}_{n-1} | \\
 & \times \dots \times \int d^3x_1 | \vec{x}_1 \rangle \langle \vec{x}_1 | e^{-i\hat{H}\Delta t} | y \rangle
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow G(x, y) = & \int \left[\prod_{n=1}^{N-1} d^3x_n \right] \langle \vec{x} | e^{-i\hat{H}\Delta t} | \vec{x}_{N-1} \rangle \\
 & \times \langle \vec{x}_{N-1} | e^{-i\hat{H}\Delta t} | \vec{x}_{N-2} \rangle \langle \vec{x}_{N-2} | e^{-i\hat{H}\Delta t} | \vec{x}_{N-3} \rangle \\
 & \times \dots \times \langle \vec{x}_{n+1} | e^{-i\hat{H}\Delta t} | \vec{x}_n \rangle \langle \vec{x}_n | e^{-i\hat{H}\Delta t} | \vec{x}_{n-1} \rangle \\
 & \times \dots \times \langle \vec{x}_2 | e^{-i\hat{H}\Delta t} | \vec{x}_1 \rangle \langle \vec{x}_1 | e^{-i\hat{H}\Delta t} | y \rangle
 \end{aligned}$$



Notice that here, in any n^{th} time step we are varying $\vec{x}_n$ independently in the entire space of allowed region. Hence, this procedure corresponds to the particle to go along all the possible trajectories.

Thus $G(\vec{x}, \vec{y})$ is an integration of all the mini-propagators ' G_n '.

$$G_n = \langle \vec{x}_{n+1} | e^{-i\hat{H}\Delta t} | \vec{x}_n \rangle = \langle \vec{x}_{n+1} | e^{-i\left[\frac{\hat{p}^2}{2m} + \hat{V}\right]\Delta t} | \vec{x}_n \rangle$$

(If Δt is infinitesimal, $(\Delta t)^2 \rightarrow 0$)

$$\rightarrow e^{-i\left[\frac{\hat{p}^2}{2m} + \hat{V}\right]\Delta t} = e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\hat{V}\Delta t} + e^{\mathcal{O}(\Delta t^2)}$$

$$\rightarrow G_n = \langle \vec{x}_{n+1} | e^{-i \frac{\hat{p}^2}{2m} \Delta t} e^{-i \hat{V} \Delta t} | \vec{x}_n \rangle$$

$$= \langle \vec{x}_{n+1} | e^{-i \frac{\hat{p}^2}{2m} \Delta t} | \vec{x}_n \rangle e^{-i \hat{V}(\vec{x}_n) \Delta t}$$

Insert $\int d^3p |\vec{p}\rangle \langle \vec{p}| = \hat{1}$

$$\rightarrow G_n = \int d^3p \langle \vec{x}_{n+1} | e^{-i \frac{\hat{p}^2}{2m} \Delta t} | \vec{p} \rangle \langle \vec{p} | \vec{x}_n \rangle \times e^{-i \hat{V}(\vec{x}_n) \Delta t}$$

$$= \int d^3p \langle \vec{x}_{n+1} | \vec{p} \rangle \langle \vec{p} | \vec{x}_n \rangle e^{-i \frac{|\vec{p}|^2}{2m} \Delta t} e^{-i \hat{V}(\vec{x}_n) \Delta t}$$

$$\langle \vec{p} | \vec{x}_n \rangle = \frac{e^{-i \vec{p} \cdot \vec{x}_n}}{(2\pi)^{3/2}}, \quad \langle \vec{x}_{n+1} | \vec{p} \rangle = \frac{e^{i \vec{p} \cdot \vec{x}_{n+1}}}{(2\pi)^{3/2}}$$

$$\rightarrow G_n = \int \frac{d^3p}{(2\pi)^3} e^{i \vec{p} \cdot (\vec{x}_{n+1} - \vec{x}_n) - i \left[\frac{|\vec{p}|^2}{2m} + \hat{V}(\vec{x}_n) \right] \Delta t}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{\left[-\frac{i(\Delta t)}{2m} |\vec{p}|^2 + i \vec{p} \cdot (\vec{x}_{n+1} - \vec{x}_n) - i(\Delta t) V(\vec{x}_n) \right]}$$

(using $\int_{-\infty}^{\infty} dz e^{(-\frac{a}{2} z^2 + bz + c)} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a} + c}$)

$$\rightarrow G_n = \left[\frac{m}{(2\pi)i(\Delta t)} \right]^{3/2} e^{i \left[\frac{m |\vec{x}_{n+1} - \vec{x}_n|^2}{2 (\Delta t)^2} - V(\vec{x}_n) \right] (\Delta t)}$$

Now, $\lim_{(\Delta t) \rightarrow 0} \frac{|\vec{x}_{n+1} - \vec{x}_n|^2}{(\Delta t)^2} = |\dot{\vec{x}}_n|^2$

Let $\mathcal{E}_S = \left[\frac{(2\pi)i(\Delta t)}{m} \right]^{1/2}$

$$\rightarrow G_n = \frac{1}{\mathcal{E}_S^3} e^{i \left[\frac{1}{2} m |\dot{\vec{x}}_n|^2 - V(\vec{x}_n) \right] (\Delta t)}$$

But the limit $\Delta t \rightarrow 0$ is valid only when $N \rightarrow \infty$.

$$\Rightarrow G(\mathbf{x}, \mathbf{y}) = \lim_{N \rightarrow \infty} \int \prod_{n=1}^{N-1} d^3 \mathbf{x}_n G_n$$

Let $\lim_{N \rightarrow \infty} \int \prod_{n=1}^{N-1} \left[\frac{d^3 \mathbf{x}_n}{\mathcal{E}_S^3} \right] = \underbrace{\int \mathcal{D}[\vec{x}(t)]}_{\text{Functional measure}}$

now the product of all exponentials becomes the exponential of the sum.

$$\begin{aligned} \Rightarrow \prod_n e^{i \left[\frac{1}{2} m |\dot{\vec{x}}_n|^2 - V(\vec{x}_n) \right] \Delta t} \\ = e^{\sum_n \Delta t \, i \left[\frac{1}{2} m |\dot{\vec{x}}_n|^2 - V(\vec{x}_n) \right]} \end{aligned}$$

Then in the lim $N \rightarrow \infty$, $\Delta t \rightarrow 0$,
we can write $\sum_n (\Delta t) = \int_{t_y}^{t_x} dt$

$$\Rightarrow G(\vec{x}, \vec{y}) = \int D[\vec{x}(t)] e^{i \int_{t_y}^{t_x} dt \left[\frac{1}{2} m |\dot{\vec{x}}(t)|^2 - V(\vec{x}(t)) \right]}$$

where $\vec{x}(t_y) = \vec{y}$ (starting point)
& $\vec{x}(t_x) = \vec{x}$ (ending point)

$\int D[\vec{x}(t)]$ tells us to integrate over all the possible paths.

Also, $\mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] = \frac{1}{2} m |\dot{\vec{x}}(t)|^2 - V(\vec{x}(t))$
is the lagrangian of the system.

$$\rightarrow G(x, y) = \int D[\vec{x}(t)] e^{i \int_{t_y}^{t_x} dt \mathcal{L}(\vec{x}(t), \dot{\vec{x}}(t))}$$

$$\text{But } \int_{t_y}^{t_x} dt \mathcal{L}(\vec{x}(t), \dot{\vec{x}}(t)) = S[\vec{x}(t), t_x, t_y]$$

is the action for a given path $\vec{x}(t)$.

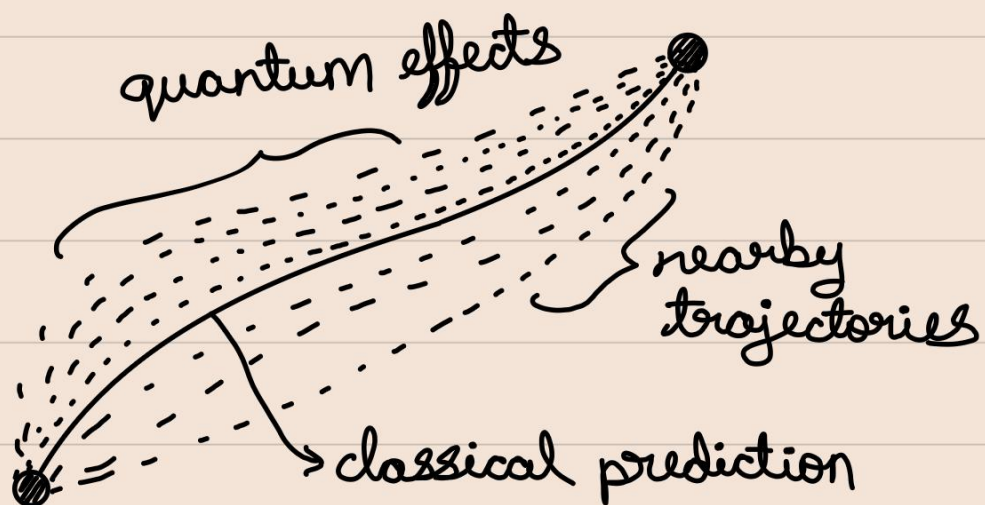
Thus we finally get,

$$G(x, y) = \int D[\vec{x}(t)] e^{i S[\vec{x}(t), t_x, t_y]}$$

So, to get the probability amplitude for a particle to go from $(t_y, \vec{y})$ to $(t_x, \vec{x})$. We allow the particle to take every possible trajectory, where for each path there is a corresponding action 'S' & a phase factor contribution e^{iS} , & then we sum all the phase factors.

This closely relates to Least action Principle. If the trajectories are random, those phase factors cancel each other.

But those trajectories very close to the one predicted by 'Least action principle' will have similar actions & thus the phase factors align giving an 'In-phase' contribution. Thus, the sum is dominated by the result predicted by Euler-eqⁿ of motion. But still some quantum fluctuations are possible due to the contributions of nearby non-classical trajectories.



Gaussian type Functional Integral

Saddle point method to find

$$I = \int_{-\infty}^{\infty} dx e^{-\frac{ax^2}{2} + bx}, \quad a > 0$$

$$\text{Let } P(x) = -\frac{ax^2}{2} + bx$$

$$\Rightarrow P'(x) = -ax + b$$

$$\text{Let } P'(x_m) = 0 \Rightarrow x_m = \frac{b}{a}$$

$$P''(x_m) = -a \Rightarrow x_m \rightarrow \text{maxima}$$

$$\& P(x_m) = \frac{b^2}{2a}$$

$$\begin{aligned} \rightarrow P(x) - P(x_m) &= -\frac{ax^2}{2} + bx - \frac{b^2}{2a} \\ &= -\frac{a}{2} \left(x - \frac{b}{a}\right)^2 \end{aligned}$$

$$\rightarrow I = e^{P(x_m)} \int_{-\infty}^{\infty} e^{P(x) - P(x_m)} dx$$

$$\rightarrow I = e^{\frac{b^2}{2a}} \underbrace{\int_{-\infty}^{\infty} dx e^{-\frac{a}{2} \left(x - \frac{b}{a}\right)^2}}_{\sqrt{\frac{2\pi}{a}}}$$

$$\rightarrow \int_{-\infty}^{\infty} dx e^{-\frac{ax^2}{2} + bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{b^2}{2a}}$$

But now consider a multivariable vector gaussian integral of the form,

$$I_v = \int d^N x e^{\left[-\frac{1}{2} (\vec{x})^T \hat{A} (\vec{x}) + (\vec{b})^T \vec{x} \right]}$$

where,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_N \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$$

Symmetric &
Invertible matrix

Now,

$$P(\vec{x}) = -\frac{1}{2} (\vec{x})^T \hat{A} (\vec{x}) + (\vec{b})^T \vec{x}$$

$$\rightarrow P(\vec{x}) = -\frac{1}{2} x_i A_{ij} x_j + b_j x_j$$

$$\frac{\partial P}{\partial x_k} = -\frac{1}{2} [x_i A_{ij} \delta_{jk} + \delta_{ik} A_{ij} x_j] + b_j \delta_{jk}$$

$$= -\frac{1}{2} [x_i A_{ik} + A_{kj} x_j] + b_k$$

Since $A_{ik} = A_{ki}$

$$\rightarrow \frac{\partial P}{\partial x_k} = -A_{kj} x_j + b_k$$

$$\text{or } \underline{\vec{\nabla} P(\vec{x}) = -\hat{A} \vec{x} + \vec{b}}$$

To get the maxima,

$$[\vec{\nabla} P(\vec{x})]_{\vec{x} = \vec{x}_m} = \vec{0}$$

$$\rightarrow -\hat{A} \vec{x}_m + \vec{b} = 0 \Rightarrow \underline{\vec{x}_m = \hat{A}^{-1} \vec{b}}$$

$$\rightarrow P(\vec{x}_m) = \frac{-1}{2} (\vec{b})^T (\hat{A}^{-1})^T \hat{A} \hat{A}^{-1} \vec{b} + (\vec{b})^T \hat{A}^{-1} \vec{b}$$

$$\Rightarrow P(\vec{x}_m) = \frac{1}{2} (\vec{b})^T \hat{A}^{-1} \vec{b}$$

Then,

$$P(\vec{x}) - P(\vec{x}_m) = -\frac{1}{2} (\vec{x})^T \hat{A} \vec{x} + (\vec{b})^T \vec{x} - \frac{1}{2} (\vec{b})^T \hat{A}^{-1} \vec{b}$$

$$P(\vec{x}) - P(\vec{x}_m) = -\frac{1}{2} [\vec{x} - \hat{A}^{-1}(\vec{b})]^T \hat{A} [\vec{x} - \hat{A}^{-1} \vec{b}]$$

$$\text{Let } \vec{y} = \vec{x} - \hat{A}^{-1} \vec{b}$$

$$\rightarrow P(\vec{x}) - P(\vec{x}_m) = -\frac{1}{2} (\vec{y})^T \hat{A} \vec{y}$$

$$\& d^N x = d^N y$$

$$\& e^{P(\vec{x})} = e^{(-\frac{1}{2} (\vec{y})^T \hat{A} \vec{y})} e^{\frac{1}{2} (\vec{b})^T \hat{A}^{-1} \vec{b}}$$

Thus the integral finally becomes,

$$I_v = e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}} \int d^N y e^{\left[-\frac{1}{2}(\vec{y})^T \hat{A} \vec{y}\right]}$$

Now we perform an orthogonal transformation on $\hat{A}$, where we write

$\hat{A} = \hat{O}^T \hat{D} \hat{O}$, with $\hat{O}$ as the orthogonal matrix that diagonalizes $\hat{A}$.
& $\hat{D}$ is diagonal matrix of eigen values of $\hat{A}$.

$$\rightarrow \int d^N y e^{\left[-\frac{1}{2}(\vec{y})^T \hat{A} \vec{y}\right]}$$

$$= \int d^N y e^{\left[-\frac{1}{2}(\vec{y})^T \hat{O}^T \hat{D} \hat{O} \vec{y}\right]}$$

Let $\vec{u} = \hat{O} \vec{y}$ be the orthogonal transformation. Thus $d^N y = d^N u$.
Since the Jacobian $= |\det(\hat{O})| = 1$.

$$\Rightarrow I_v = e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}} \int d\vec{u} e^{\left[-\frac{1}{2}(\vec{u})^T \hat{D} \vec{u}\right]}$$

Now $\hat{D} = \text{diag}(D_{11}, D_{22}, \dots, D_{NN})$
 where none of the eigen value is
 Zero. Since $\hat{A}$ is invertible.

$$\Rightarrow I_v = e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}} \int d\vec{u} e^{-\frac{1}{2} \sum_j u_j^2 D_{jj}}$$

$$= e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}} \prod_j \underbrace{\int du_j e^{-\frac{1}{2} D_{jj} u_j^2}}_{\sqrt{\frac{2\pi}{D_{jj}}}}$$

$$= e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}} \prod_j \sqrt{\frac{2\pi}{D_{jj}}}$$

$$= \sqrt{\frac{(2\pi)^N}{\left(\prod_j D_{jj}\right)}} e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}}$$

But the product of all the eigen values of the matrix is just its determinant. Hence we can write,

$$\int d^N x e^{\left[-\frac{1}{2}(\vec{x})^T \hat{A} \vec{x} + (\vec{b})^T \vec{x}\right]}$$

$$= \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(\hat{A})}} e^{\frac{1}{2}(\vec{b})^T \hat{A}^{-1} \vec{b}}$$

Now, we can generalize the notion of the vector $\vec{x}$ to infinite dimensions as a function. And the matrix $\hat{A}$ as an operator acting on the function.

$$\Rightarrow \vec{x} \longrightarrow f(x), \quad \vec{b} \longrightarrow b(x)$$

$$\hat{A} \longrightarrow \hat{A}(x, y),$$

$$\rightarrow (\vec{x})^T \hat{A} \vec{x} \longrightarrow \int dx dy f(y) \hat{A}(x, y) f(x)$$

$$d^N x \longrightarrow \int D[f(x)], \quad \text{for } N \rightarrow \infty$$

Thus we get,

$$\int D[f(x)] e^{\left[-\frac{1}{2} \int dx dy f(y) \hat{A}(x, y) f(x) + \int dx b(x) f(x) \right]}$$
$$= \frac{B}{\sqrt{\det(\hat{A})}} e^{\frac{1}{2} \int dx dy b(y) \hat{A}^{-1}(x, y) b(x)}$$

where $B = \lim_{N \rightarrow \infty} (2\pi)^{\frac{N}{2}}$

& $\hat{A}^{-1}(x, y)$ is defined as,

$$\int dz \hat{A}(x, z) \hat{A}^{-1}(z, y) = \delta(x - y)$$

$\rightarrow \hat{A}^{-1}(z, y)$ is just the Green's function of operator $\hat{A}(x, y)$.

Propagator for Harmonic Oscillator

Here we shall derive the propagator for quantum harmonic oscillator for a special case as follows. If we put a particle in the minimum of the oscillator at time $t_y=0$ and $y=0$, then what is the probability of the particle to still be at $x=0$ at a time $t_x=\tau$. Thus we need to find $G(x=0, t_x=\tau, y=0, t_y=0)$

$$\Rightarrow G(0, \tau, 0, 0) = \int_{x(0)=0}^{x(\tau)=0} D[x(t)] e^{i \int_0^\tau dt L(x, \dot{x})}$$

$$\text{Lagrangian} \rightarrow \mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$\text{Action} \rightarrow S = \int_0^\tau dt \left[\frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right]$$

(integrate the first term by parts)

$$\frac{m}{2} \int_0^{\tau} dt \left(\frac{dx}{dt} \right)^2 = \left[\frac{m}{2} \left(\frac{dx}{dt} \right) \int dt \left(\frac{dx}{dt} \right) \right]_{t=0}^{t=\tau}$$

$$- \frac{m}{2} \int_0^{\tau} \left[\int dt \left(\frac{dx}{dt} \right) \right] \frac{d}{dt} \left(\frac{dx}{dt} \right)$$

$$= \underbrace{\left[\frac{m}{2} \dot{x}(t) x(t) \right]_{t=0}^{t=\tau}}_{\text{Zero at both limits}} - \frac{m}{2} \int_0^{\tau} dt \, x(t) \frac{d^2 x(t)}{dt^2}$$

$$\rightarrow \frac{m}{2} \int_0^{\tau} dt \left(\frac{dx}{dt} \right)^2 = - \frac{m}{2} \int_0^{\tau} dt \, x(t) \frac{d^2 x(t)}{dt^2}$$

Thus we get the action as,

$$S = \int_0^{\tau} dt \left[- \frac{m}{2} x(t) \frac{d^2 x(t)}{dt^2} - \frac{1}{2} m \omega^2 [x(t)]^2 \right]$$

$$\rightarrow S = \int_0^{\tau} dt \left[- \frac{1}{2} x(t) m \left(\frac{d^2}{dt^2} + \omega^2 \right) x(t) \right]$$

$\Rightarrow G(0, T, 0, 0)$ is given by,

$$= \int_{x(0)=0}^{x(T)=0} D[x(t)] e^{\int_0^T dt \left[-\frac{1}{2} x(t) \operatorname{im} \left(\frac{d^2}{dt^2} + \omega^2 \right) x(t) \right]}$$

The above functional integral resembles the functional gaussian integral

$$\int D[x(t)] e^{\left[-\frac{1}{2} \int dt x(t) \hat{A}(t) x(t) \right]} = \frac{B}{\sqrt{\det(\hat{A})}}$$

where $\hat{A} = \operatorname{im} \left(\frac{d^2}{dt^2} + \omega^2 \right)$, & B is

a constant given by $\lim_{N \rightarrow \infty} (2\pi)^{N/2}$.

Now, the eigen functions of the operator $\hat{A}$ should satisfy

$$\hat{A} u_n(t) = \operatorname{im} \left(\frac{d^2}{dt^2} + \omega^2 \right) u_n(t) = \underbrace{i\lambda_n}_{\text{Eigen value}} u_n(t)$$

$$\Rightarrow u_n''(t) = -k_n^2 u_n(t)$$

$$\text{where, } k_n^2 = \left[\omega^2 - \frac{\lambda_n}{m} \right]$$

$$\rightarrow u_n(t) = C_1 e^{ik_n t} + C_2 e^{-ik_n t}$$

$$\text{But } u_n(t=0) = 0 \Rightarrow C_1 + C_2 = 0$$

$$\rightarrow u_n(t) = C_1 (2i) \sin(k_n t)$$

$$\& u_n(\tau) = 2iC_1 \sin(k_n \tau) = 0$$

$$\rightarrow \underline{k_n = \frac{n\pi}{\tau}} \Rightarrow \omega^2 - \frac{\lambda_n}{m} = \left(\frac{n\pi}{\tau} \right)^2$$

$$\rightarrow \underline{\lambda_n = m \left[\omega^2 - \left(\frac{n\pi}{\tau} \right)^2 \right]}$$

Thus the eigen values of $\hat{A}$ are given by

$$\underline{i\lambda_n = im \left[\omega^2 - \left(\frac{n\pi}{\tau} \right)^2 \right]}$$

$$\rightarrow \det(A) = \lim_{N \rightarrow \infty} \prod_{n=1}^N (i\lambda_n)$$

$$\Rightarrow G_1(0, \tau, 0, 0) = \frac{B}{\sqrt{\det(\hat{A})}}$$

$$= \frac{\lim_{N \rightarrow \infty} (2\pi)^{N/2}}{\sqrt{\lim_{N \rightarrow \infty} \prod_{n=1}^N \left(im \left[\omega^2 - \left(\frac{n\pi}{\tau} \right)^2 \right] \right)}}$$

$$= \lim_{N \rightarrow \infty} \frac{(2\pi)^{N/2}}{(-i)^{N/2} \sqrt{\prod_{n=1}^N m \left[\left(\frac{n\pi}{\tau} \right)^2 - \omega^2 \right]}}$$

$$\Rightarrow G_1(0, \tau, 0, 0) = \lim_{N \rightarrow \infty} \frac{(2\pi i)^{N/2}}{\sqrt{\prod_{n=1}^N m \left[\left(\frac{n\pi}{\tau} \right)^2 - \omega^2 \right]}}$$

Now when there is no harmonic pot.
i.e. when $\omega = 0$, the result is for a
free particle.

$$\rightarrow G_1(0, \tau, 0, 0)_{(\omega=0)} = \lim_{N \rightarrow \infty} \frac{(2\pi i)^{N/2}}{\sqrt{\prod_{n=1}^N m \left[\left(\frac{n\pi}{\tau} \right)^2 \right]}}$$

But the forward propagator for a free particle as seen before is given by,

$$\Theta(t_x - t_y) \sqrt{\frac{m}{2\pi i(t_x - t_y)}} \exp\left[\frac{im(\vec{x} - \vec{y})^2}{2(t_x - t_y)}\right]$$

Substituting $t_x = \tau$, $t_y = 0$, $\vec{x} = 0$, $\vec{y} = 0$, we get,

$$G_{\text{free}}(0, \tau, 0, 0) = \sqrt{\frac{m}{2\pi i\tau}}$$

Thus by equating $G_{\omega=0} \neq G_{\text{free}}$ we can get rid of the factor $\lim_{N \rightarrow \infty} (2\pi i)^{N/2}$

$$\rightarrow G(0, \tau, 0, 0) = G(0, \tau, 0, 0) \frac{G_{\text{free}}(0, \tau, 0, 0)}{G_{\omega=0}(0, \tau, 0, 0)}$$

$$\rightarrow G(0, \tau, 0, 0) = \lim_{N \rightarrow \infty} \left[\frac{(2\pi i)^{N/2} \sqrt{\frac{m}{2\pi i\tau}}}{\sqrt{\prod_{n=1}^N m \left[\left(\frac{n\pi}{\tau} \right)^2 - \omega^2 \right]}} \cdot \frac{(2\pi i)^{N/2}}{\sqrt{\prod_{n=1}^N m \left[\left(\frac{n\pi}{\tau} \right)^2 \right]}} \right]$$

$$\Rightarrow G_1(0, \tau, 0, 0) = \sqrt{\frac{m}{2\pi i \tau}} \times \lim_{N \rightarrow \infty} \prod_{n=1}^N \sqrt{\frac{\left[\left(\frac{n\pi}{\tau}\right)^2\right]}{\left[\left(\frac{n\pi}{\tau}\right)^2 - \omega^2\right]}}$$

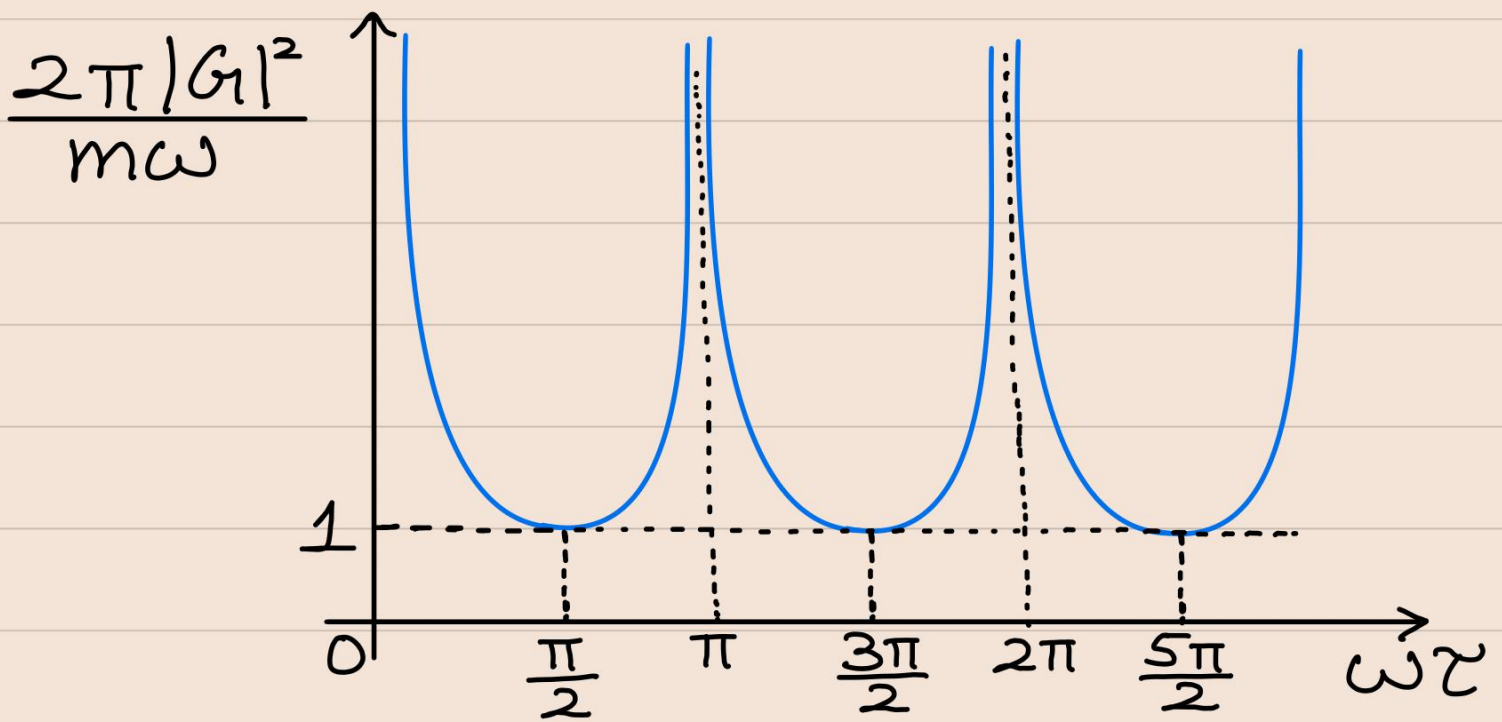
$$\Rightarrow G_1(0, \tau, 0, 0) = \sqrt{\frac{m}{2\pi i \tau}} \prod_{n=1}^{\infty} \left[1 - \left(\frac{\omega \tau}{n\pi}\right)^2\right]^{-1/2}$$

Using the identity $\prod_{n=1}^{\infty} \left[1 - \left(\frac{x}{n\pi}\right)^2\right]^{-1/2} = \sqrt{\frac{x}{\sin(x)}}$
we get,

$$G_1(0, \tau, 0, 0) = \sqrt{\frac{m}{2\pi i \tau}} \sqrt{\frac{\omega \tau}{\sin(\omega \tau)}}$$

$$\Rightarrow G_1(0, \tau, 0, 0) = \sqrt{\frac{-im\omega}{2\pi \sin(\omega \tau)}}$$

$$\Rightarrow |G_1|^2 = \frac{m\omega}{2\pi} |\operatorname{cosec}(\omega \tau)|$$



Thus we see that the probability of finding the particle to be in the potential minima at a later time τ peaks at $\tau = \frac{n\pi}{\omega} = \left\{0, \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \frac{3\pi}{\omega}, \dots\right\}$.

This is exactly what is found in the classical result. At every half-a-period $\left(\frac{\pi}{\omega}\right)$ time interval, the particle crosses the minima.

From the path-integral point of view, the peaks are due to constructive interference of all paths that start & end at origin.

Analogue of Wick's theorem using the Gaussian Integral

We have already seen that,

$$\int d^N x e^{\left[-\frac{1}{2} x_j A_{jk} x_k + b_j x_j\right]} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(\hat{A})}} e^{\frac{1}{2} b_j (\hat{A}^{-1})_{jk} b_k}$$

We shall define $\langle x_i \rangle$ as

$$\langle x_i \rangle = \frac{\int d^N x x_i e^{\left[-\frac{1}{2} x_j A_{jk} x_k\right]}}{\int d^N x e^{\left[-\frac{1}{2} x_j A_{jk} x_k\right]}}$$

Now, we want to get an expression for $\langle x_i \rangle$, $\langle x_i x_j \rangle$, $\langle x_i x_j x_k \rangle$, $\langle x_i x_j x_k x_l \rangle$, ... so on.

$$\text{Let, } \mathcal{K}(\vec{b}) = \int d^N x \, e^{\left[-\frac{1}{2} x_j A_{jk} x_k + b_j x_j\right]}$$

$$= \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(\hat{A})}} e^{\frac{1}{2} b_j (\hat{A}^{-1})_{jk} b_k}$$

$$\rightarrow \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = \int d^N x \, x_i e^{\left[-\frac{1}{2} x_j A_{jk} x_k + b_j x_j\right]}$$

$$\rightarrow \lim_{b_i \rightarrow 0} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = \int d^N x \, x_i e^{\left[-\frac{1}{2} x_j A_{jk} x_k\right]}$$

$$\rightarrow \frac{\lim_{b_i \rightarrow 0} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})}{\mathcal{K}(\vec{b}=0)} = \frac{\int d^N x \, x_i e^{\left[-\frac{1}{2} x_j A_{jk} x_k\right]}}{\int d^N x \, e^{\left[-\frac{1}{2} x_j A_{jk} x_k\right]}}$$

$$\Rightarrow \langle x_i \rangle = \frac{\lim_{\vec{b} \rightarrow 0} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})}{\mathcal{K}(\vec{b}=0)}$$

$$\langle x_i \rangle = \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

$$\text{Now, } \mathcal{K}(\vec{b}) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(\hat{A})}} e^{\frac{1}{2} b_j (\hat{A}^{-1})_{jk} b_k}$$

$$\rightarrow \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = \mathcal{K}(\vec{b}) \frac{1}{2} \left[b_j (\hat{A}^{-1})_{jk} \delta_{ki} + \delta_{ji} (\hat{A}^{-1})_{jk} b_k \right]$$

$$\rightarrow \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = \mathcal{K}(\vec{b}) (\hat{A}^{-1})_{ik} b_k$$

$$\rightarrow \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = (\hat{A}^{-1})_{ik} b_k$$

$$\rightarrow \langle x_i \rangle = \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

$$\rightarrow \langle x_i \rangle = \lim_{\vec{b} \rightarrow 0} (\hat{A}^{-1})_{ik} b_k = 0$$

$$\rightarrow \underline{\langle x_i \rangle = 0}$$

Similarly,

$$\langle x_i x_j \rangle = \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

Now,

$$\frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = \frac{\partial}{\partial b_j} \left[\mathcal{K}(\vec{b}) (\hat{A}^{-1})_{ik} b_k \right]$$

$$= \mathcal{K}(\vec{b}) \delta_{jk} (\hat{A}^{-1})_{ik} + (\hat{A}^{-1})_{ik} b_k \frac{\partial}{\partial b_j} \mathcal{K}(\vec{b})$$

$$= \mathcal{K}(\vec{b}) \left[(\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jl} b_k b_l \right]$$

$$\frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = \mathcal{K}(\vec{b}) \left[(\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jl} b_k b_l \right]$$

$$\rightarrow \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b}) = (\hat{A}^{-1})_{ij}$$

$$\rightarrow \boxed{\langle x_i x_j \rangle = (\hat{A}^{-1})_{ij}}$$

$$\text{Now, } \langle x_i x_j x_k \rangle =$$

$$= \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

$$= \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_k} \mathcal{K}(\vec{b}) \left[(\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jl} b_m b_l \right]$$

$$= \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_k} \mathcal{K}(\vec{b}) \left[(\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jl} b_m b_l \right]$$

$$= \lim_{\vec{b} \rightarrow 0} \left[(\hat{A}^{-1})_{kn} b_n \left[(\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jl} b_m b_l \right] + \left[(\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jl} (2_{mk} b_l + 2_{lk} b_m) \right] \right]$$

$$= \lim_{\vec{b} \rightarrow 0} \left[\left[(\hat{A}^{-1})_{kl} (\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jl} + (\hat{A}^{-1})_{il} (\hat{A}^{-1})_{jk} \right] b_l + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jl} (\hat{A}^{-1})_{kn} b_m b_l b_n \right]$$

$$\rightarrow \underline{\langle x_i x_j x_k \rangle = 0}$$

Let's find $\langle x_i x_j x_k x_l \rangle$

$$= \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

$\frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$ is given by,

$$= \frac{\partial}{\partial b_l} \mathcal{K}(\vec{b}) \left[\begin{aligned} & \left[(\hat{A}^{-1})_{k\gamma} (\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{j\gamma} \right. \\ & \quad \left. + (\hat{A}^{-1})_{i\gamma} (\hat{A}^{-1})_{jk} \right] b_\gamma \\ & \quad + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{j\gamma} (\hat{A}^{-1})_{kn} b_m b_\gamma b_n \end{aligned} \right]$$

$$\Rightarrow \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

$$= \left[\begin{aligned} & (\hat{A}^{-1})_{ij} (\hat{A}^{-1})_{kl} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jl} + (\hat{A}^{-1})_{il} (\hat{A}^{-1})_{jk} \\ & + \left[\begin{aligned} & (\hat{A}^{-1})_{il} (\hat{A}^{-1})_{jm} (\hat{A}^{-1})_{kn} + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jl} (\hat{A}^{-1})_{kn} \\ & + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jn} (\hat{A}^{-1})_{kl} + (\hat{A}^{-1})_{ln} (\hat{A}^{-1})_{km} (\hat{A}^{-1})_{ij} \\ & + (\hat{A}^{-1})_{ln} (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jm} + (\hat{A}^{-1})_{ln} (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{jk} \end{aligned} \right] b_m b_n \\ & + (\hat{A}^{-1})_{im} (\hat{A}^{-1})_{j\gamma} (\hat{A}^{-1})_{kn} (\hat{A}^{-1})_{lq} b_m b_\gamma b_n b_q \end{aligned} \right]$$

$$\Rightarrow \lim_{\vec{b} \rightarrow 0} \frac{1}{\mathcal{K}(\vec{b})} \frac{\partial}{\partial b_l} \frac{\partial}{\partial b_k} \frac{\partial}{\partial b_j} \frac{\partial}{\partial b_i} \mathcal{K}(\vec{b})$$

$$= (\hat{A}^{-1})_{kl} (\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jl} + (\hat{A}^{-1})_{il} (\hat{A}^{-1})_{jk}$$

$$\Rightarrow \boxed{\langle x_i x_j x_k x_l \rangle = (\hat{A}^{-1})_{kl} (\hat{A}^{-1})_{ij} + (\hat{A}^{-1})_{ik} (\hat{A}^{-1})_{jl} + (\hat{A}^{-1})_{il} (\hat{A}^{-1})_{jk}}$$

We observe that, average of odd number of x_i 's gives zero.

$$\Rightarrow \underline{\langle x_i x_j x_k x_l x_m \rangle = 0}$$

This resembles the Wick's theorem for vacuum expectation values of correlation functions.

Thus the vector ' $\vec{b}$ ' acts as a source using which can find average of any function of x_i 's. Thus $\mathcal{K}(\vec{b})$ is called as the Generating functional.