Path integral for Scalar field theory

In Single particle quantum mechanics, to find the particle's propagator we dealt with the classical lagrangian '$L(\vec{x}(t), \dot{\vec{x}}(t))$' & used the path integral over all the possible paths.

But for fields we shall try to find the analogue by writing a functional integral which deals with the Lagrangian density $\mathcal{L}[\phi(x), \partial_\mu \phi(x)]$ of the corresponding classical field and sum over all the possible field Configurations that can exist ($x \equiv x^\mu$)

Thus a generating functional integral over fields can be written as

$$Z[J] = \int D[\phi(x)] e^{i\int d^4x \left(\mathcal{L}[\phi(x), \partial_\mu \phi(x)] + J(x)\phi(x)\right)}$$

(x^μ varies throughout Space-time)

where $J(x)$ is a Source field.

$Z[J]$ gives the probability amplitude

$$\left\langle \begin{matrix} \text{no particles} \\ \text{at } x^0 = \infty \end{matrix} \middle| \begin{matrix} \text{no particles} \\ \text{at } y^0 = -\infty \end{matrix} \right\rangle_{J \neq 0}$$

Note that $\phi(x)$ can also represent a set of fields denoted by $\phi_A(x)$, $A = \{1, 2, \cdots\}$ is the field index. Then $J \to J_A$ and the source terms becomes '$J_A(x)\phi_A(x)$', one term for each field.

The source field $J(x)$ has to obey the usual rules of functional derivatives.

- $\dfrac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y)$
- $\dfrac{\delta}{\delta J(x)} \int d^4y\, J(y)\phi(y) = \phi(x)$
- $\dfrac{\delta}{\delta J(y)} \int d^4x \left[\partial_\mu J(x)\right] f(x) = -\partial_\mu f(y)$

we first apply integration by parts, & then perform functional derivative.

Now we can find correlation functions using the generating functional.

$$Z[J] = \int \mathcal{D}[\phi(x)] e^{i\int d^4x \left(\mathcal{L}[\phi(x), \partial_\mu \phi(x)] + J(x)\phi(x)\right)}$$

$$\longrightarrow \lim_{J(x)\to 0} \frac{-i}{Z[J]} \left(\frac{\delta Z[J]}{\delta J(x)}\right)$$

$$= \frac{\int \mathcal{D}[\phi(x)]\, \phi(x)\, e^{i\int d^4x\, \mathcal{L}[\phi(x), \partial_\mu \phi(x)]}}{\int \mathcal{D}[\phi(x)] e^{i\int d^4x\, \mathcal{L}[\phi(x), \partial_\mu \phi(x)]}}$$

We call this quantity as $\langle \phi(x) \rangle$.
Similarly, $\langle \phi(x)\phi(y) \rangle$ is given by,

$$\langle \phi(x)\phi(y) \rangle = \lim_{J(x)\to 0} \frac{(-i)^2}{Z[J]} \left(\frac{\delta^2 Z[J]}{\delta J(x)\, \delta J(y)}\right)$$

Later we show that $\langle \ldots\ldots \rangle$ gives the ground state expectation values.

Generating functional for free Scalar field

The lagrangian density for free Scalar field theory is given by,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow$$

$$Z_0[J] = \int \mathcal{D}[\phi(x)] e^{i\int d^4x \left(\frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] + J(x)\phi(x)\right)}$$

(Subscript '0' represents free field functional)

Consider the factor $\int d^4x\, (\partial_\mu \phi)^2$ done using integration by parts.

$$\rightarrow \int d^4x\, (\partial_\mu \phi)^2 = \underbrace{\left[\partial_\mu \phi\ \phi(x)\right]}_{\text{Surface}} - \int d^4x\, \phi\, \partial_\mu \partial^\mu \phi$$

$$\underbrace{\qquad\qquad}_{0}$$

$$\rightarrow \int d^4x\, (\partial_\mu \phi)^2 = -\int d^4x\, \phi(x) \Box_x^2 \phi(x)$$

$$\rightarrow Z_0[J] = \int D[\phi(x)]\, e^{i\int d^4x \left(\frac{1}{2}\phi\left[-\Box_x^2 - m^2\right]\phi + J\phi\right)}$$

$$= \int D[\phi(x)]\, e^{-\frac{1}{2}\int d^4x\, \phi\, i\left[\Box_x^2 + m^2\right]\phi + i\int d^4x\, J\phi}$$

Consider the exponent factor which has the Klein Gordon operator

$$-\frac{1}{2}\int d^4x\, \phi(x)\, i\left[\Box_x^2 + m^2\right]\phi(x)$$

$$= -\frac{1}{2}\int d^4x\, d^4y\, \phi(y)\left(\delta^{(4)}(x-y)\, i\left[\Box_x^2 + m^2\right]\right)\phi(x)$$

Let $\hat{K}(x,y) = i\delta^{(4)}(x-y)\left[\Box_x^2 + m^2\right]$

$$\rightarrow Z_0[J] = \int D[\phi(x)]\, e^{-\frac{1}{2}\int d^4x\, d^4y\, \phi(y)\hat{K}(x,y)\phi(x)} \times e^{i\int d^4x\, J(x)\phi(x)}$$

Now, we are in the stage to use the Gaussian integral Identity.

We know that

$$\int \mathcal{D}[f(x)]\, e^{\left[\begin{array}{c} -\frac{1}{2}\int dx\,dy\, f(y)\,\hat{A}(x,y)\,f(x) \\ +\int dx\, b(x) f(x) \end{array}\right]}$$

$$= \frac{B}{\sqrt{\det(\hat{A})}}\, e^{\frac{1}{2}\int dx\,dy\, b(x)\hat{A}^{-1}(x,y)b(y)}$$

$B = \lim_{N\to\infty} (2\pi)^{\frac{N}{2}}$ & $\hat{A}^{-1}(x,y)$ satisfies,

$$\int dz\, \hat{A}(x,z)\hat{A}^{-1}(z,y) = \delta(x-y)$$

So we need to find the inverse of

$\hat{K}(x,z) = \delta^{(4)}(x-z)\, i\left[\Box_x^2 + m^2\right]$ which satisfies

$$\int d^4z\, \delta^{(4)}(x-z)\, i\left[\Box_z^2+m^2\right]\hat{K}^{-1}(z,y) = \delta^{(4)}(x-y)$$

$\longrightarrow \hat{K}^{-1}(z,y)$ should satisfy,

$$i\left[\Box_z^2+m^2\right]\hat{K}^{-1}(z,y) = \delta^{(4)}(z-y)$$

But we defined the feynman propagator for K.G field as $\Delta_F(x,y)$ which satisfies,

$$(\Box_x^2 + m^2)\Delta_F(x,y) = -i\delta^{(4)}(x-y)$$

$$\rightarrow i(\Box_x^2 + m^2)\Delta_F(x,y) = \delta^{(4)}(x-y)$$

Thus by comparision, $\hat{K}^{-1}(x,y) = \Delta_F(x,y)$

$$\rightarrow Z_0[J] = \frac{B\, e^{\frac{-1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x,y)J(y)}}{\sqrt{\det(\hat{K})}}$$

---

where,

$$\frac{B}{\sqrt{\det(\hat{K})}} = \frac{\lim_{N\to\infty}(2\pi i)^{\frac{N}{2}}}{\sqrt{\det(-\Box^2 - m^2)}}$$

But observe that

$$Z_0[0] = \int D[\phi(x)]\, e^{\frac{-1}{2}\int d^4x\, \phi\, i[\Box^2 + m^2]\phi} = \frac{B}{\sqrt{\det(\hat{K})}}$$

$$\Rightarrow \frac{Z_0[J]}{Z_0[0]} \text{ is given by}$$

$$= \frac{\int \mathcal{D}[\phi(x)]\, e^{i\int d^4x \left(\frac{1}{2}\phi[-\Box^2 - m^2]\phi + J\phi\right)}}{\int \mathcal{D}[\phi(x)]\, e^{\frac{-1}{2}\int d^4x\, \phi\, i[\Box^2 + m^2]\phi}}$$

$$= e^{\frac{-1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x,y)J(y)}$$

Thus we define the normalized generating functional as $Z_0[J] = \frac{Z_0[J]}{Z_0[0]}$ $\left(Z_0[0] = 1\right)$

$$\Rightarrow \boxed{Z_0[J] = e^{\frac{-1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x,y)J(y)}}$$

We can use the above normalized generating functional to get the free field correlation functions.

Free field Correlation functions

Consider $$\frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(y)} = \frac{\delta}{\delta J(x)}\left(\frac{\delta\, Z_0[J]}{\delta J(y)}\right)$$

$$= \frac{\delta}{\delta J(x)}\left[\left(-\frac{1}{2}\int d^4z\, J(z)\Delta_F(z,y) - \frac{1}{2}\int d^4\omega\, \Delta_F(y,\omega)J(\omega)\right) \times e^{\frac{-1}{2}\int d^4z\, d^4\omega J(z)\Delta_F(z,\omega)J(\omega)}\right]$$

Using $\Delta_F(z,y) = \Delta_F(y,z)$, we get,

$$\frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(y)} = \frac{\delta}{\delta J(x)}\left[-Z_0[J]\int d^4z\, J(z)\Delta_F(z,y)\right]$$

$$= Z_0[J]\left(-\Delta_F(x,y) + \int d^4z\, J(z)\Delta_F(z,y) \times \int d^4\omega J(\omega)\Delta_F(\omega,x)\right)$$

$$\Longrightarrow \boxed{\lim_{J(x)\to 0}(-i)^2\frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(y)} = \Delta_F(x,y)}$$

But for the free Scalar field theory,

$$\Delta_F(x,y) = \langle 0|\, T[\phi(x)\phi(y)]\,|0\rangle$$

Thus we prooved that,

$$\langle 0|\, T[\phi(x)\phi(y)]\,|0\rangle = (-i)^2 \frac{\delta^2 Z_0[J]}{\delta J(x)\,\delta J(y)}\Bigg|_{J(x)=0}$$

A useful relation is

$$\frac{\delta Z_0[J]}{\delta J(x)} = -Z_0[J]\int d^4z\, J(z)\,\Delta_F(z,x)$$

Now similarly we get

$$\frac{\delta^3 Z_0[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)} =$$

$$= \frac{\delta}{\delta J(x)} Z_0[J]\Big(-\Delta_F(y,z) + \int d^4u\, J(u)\,\Delta_F(u,z) \times \int d^4w\, J(w)\,\Delta_F(w,y)\Big)$$

$$= -Z_0[J]\int d^4v\, J(v)\,\Delta_F(v,x)$$

$$\times\left(-\Delta_F(y,z)+\int d^4u\, J(u)\,\Delta_F(u,z)\times\int d^4w\, J(w)\,\Delta_F(w,y)\right)$$

$$+Z_0[J]\left[\Delta_F(x,y)\int d^4u\, J(u)\,\Delta_F(u,z)+\Delta_F(x,z)\int d^4w\, J(w)\,\Delta_F(w,y)\right]$$

$$= Z_0[J]\left[\begin{array}{l}\Delta_F(x,y)\int d^4w\, J(w)\,\Delta_F(w,z)\\ +\Delta_F(x,z)\int d^4w\, J(w)\,\Delta_F(w,y)\\ +\Delta_F(y,z)\int d^4w\, J(w)\,\Delta_F(w,x)\\ -\int d^4v\, J(v)\,\Delta_F(v,x)\int d^4w\, J(w)\,\Delta_F(w,y)\\ \qquad\times\int d^4u\, J(u)\,\Delta_F(u,z)\end{array}\right]$$

$$\Longrightarrow (-i)^3\left.\frac{\delta^3 Z_0[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)}\right|_{J(x)=0}=0$$

$$\Longrightarrow \langle 0|\,T[\phi(x)\phi(y)\phi(z)]\,|0\rangle = 0$$

Similarly $\dfrac{\delta^4 Z_0[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)\,\delta J(w)}$ is

$$= \frac{\delta}{\delta J(x)} Z_0[J] \Big[\Delta_F(y,z) \int d^4u\, J(u) \Delta_F(u,w) + \Delta_F(y,w) \int d^4u\, J(u) \Delta_F(u,z) + \Delta_F(z,w) \int d^4u\, J(u) \Delta_F(u,y) - \int d^4v\, J(v) \Delta_F(v,y) \int d^4s\, J(s) \Delta_F(s,z) \times \int d^4u\, J(u) \Delta_F(u,w) \Big]$$

In the limit of $J(x) \longrightarrow 0$, it reduces to

$$= \Delta_F(x,y)\Delta_F(z,w) + \Delta_F(x,z)\Delta_F(y,w) + \Delta_F(y,z)\Delta_F(x,w)$$

$$= \langle 0 | T[\phi(x)\phi(y)\phi(z)\phi(w)] | 0 \rangle$$

$$\Rightarrow \boxed{(-i)^4 \frac{\delta^4 Z_0[J]}{\delta J(x)\,\delta J(y)\,\delta J(z)\,\delta J(w)} = \langle 0 | T[\phi(x)\phi(y)\phi(z)\phi(w)] | 0 \rangle}$$

Thus we can derive wick's theorem & also get the n-point correlation function using the Generating functional.

$$\rightarrow \left.\frac{(-i)^n \, \delta^n Z_0[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\right|_{J(x)=0} = \langle 0|T[\phi(x_1)\phi(x_2)\cdots\phi(x_n)]|0\rangle$$

Remember that we have created the Generating functional for the free Scalar field. That is why we get the free field propogator upon functional differentiation.

But in the presence of a perturbation or interactions, this will not be same. There we would like to obtain the quantities like $\langle\Omega|T[\phi(x_1)\phi(x_2)\cdots\phi(x_n)]|\Omega\rangle$ which are the n point correlation functions where $|\Omega\rangle$ is the ground state of the perturbed (full) hamiltonian.

Generating functional for Interacting Theory

Here we consider the self-interactions for a field whose interaction lagrangian is given by $\mathcal{L}_{int.}(\phi)$. Thus the total lagrangian is given by:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int.}$$

Thus the generating functional for the complete theory is given by,

$$Z[J] = \int \mathcal{D}[\phi]\, e^{i\int d^4x \left(\mathcal{L}_0 + \mathcal{L}_{int.}(\phi) + J\phi\right)}$$

(Note that ϕ is not an operator here)

$$\rightarrow Z[J] = \int \mathcal{D}[\phi]\, e^{i\int d^4x\, \mathcal{L}_{int.}(\phi)}\, e^{i\int d^4x \left(\mathcal{L}_0 + J\phi\right)}$$

Now by functional differentiation,

$$\frac{-i\partial}{\partial J(x)}\left(e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}\right)=\phi(x)\,e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}$$

$$\Rightarrow \mathcal{L}_{int.}\left[\frac{-i\partial}{\partial J(x)}\right] \text{ acting on } e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}$$

is given by,

$$\mathcal{L}_{int.}\left[\frac{-i\partial}{\partial J(x)}\right]e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}$$

$$=\mathcal{L}_{int.}(\phi(x))\,e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}$$

$$\Rightarrow \underbrace{\int d^4x\,\mathcal{L}_{int.}\left[\frac{-i\partial}{\partial J(x)}\right]}_{\text{functional derivative operator}}\,e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}$$

$$=\int d^4x\,\mathcal{L}_{int.}(\phi(x))\,e^{i\int d^4z\,(\mathcal{L}_0+J\phi)}$$

We can expand $e^{i\int d^4x\,\mathcal{L}_{int.}(\phi)}$ as a power series of $i\int d^4x\,\mathcal{L}_{int.}(\phi(x))$

$$\rightarrow e^{i\int d^4x\, \mathcal{L}_{int.}(\phi)} = 1 + i\int d^4x\, \mathcal{L}_{int.}(\phi(x)) + i^2 \int d^4x\, d^4y\, \mathcal{L}_{int.}(\phi(x))\mathcal{L}_{int.}(\phi(y)) + \cdots$$

$$\Rightarrow e^{i\int d^4x\, \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right]} e^{i\int d^4z (\mathcal{L}_0 + J\phi)}$$

$$= e^{i\int d^4x\, \mathcal{L}_{int.}(\phi)} e^{i\int d^4z (\mathcal{L}_0 + J\phi)}$$

Thus,

$$Z[J] = \int \mathcal{D}[\phi]\, e^{i\int d^4x\, \mathcal{L}_{int.}(\phi)} e^{i\int d^4z (\mathcal{L}_0 + J\phi)}$$

can be written as a functional derivative

$$Z[J] = \int \mathcal{D}[\phi]\, \underbrace{e^{i\int d^4x\, \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right]}}_{\text{Independent of } \phi} e^{i\int d^4z (\mathcal{L}_0 + J\phi)}$$

Thus the functional derivative operator can be taken outside the functional integral.

Here we get,

$$Z[J] = e^{i\int d^4x\, \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right]} \underbrace{\int D[\phi]\, e^{i\int d^4z (\mathcal{L}_0 + J\phi)}}_{\text{Generating function of free field theory, } Z_0[J]}$$

$$\rightarrow \boxed{Z[J] = \left(e^{i\int d^4x\, \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right]}\right) Z_0[J]}$$

In analogy with normalized generating functional for free field we can define the 'free field normalized' generating functional for the interacting theory as.

$$Z[J] = \frac{Z[J]}{Z_0[0]} \qquad (\text{observe } Z[0] \neq 1)$$

$$\rightarrow Z[J] = \left(e^{i\int d^4x\, \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right]}\right) Z_0[J]$$

Correlation functions of the interacting theory

We know that the correlation functions are got by the functional differentiation of the generating functional.

Let $\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle$ be the n-point correlation function of the interacting theory which is calculated with respect to the free field vacuum. i.e.

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle =$$

$$\frac{\int D[\phi(x)]\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\,e^{i\int d^4x\,\mathcal{L}[\phi(x),\partial_\mu\phi(x)]}}{\int D[\phi(x)]\,e^{i\int d^4x\,\mathcal{L}_0[\phi(x),\partial_\mu\phi(x)]}}$$

Denominator is with respect to the free field theory.

Thus if,

$$Z[J]=\int \mathcal{D}[\phi(x)]\, e^{i\int d^4x\left(\mathcal{L}[\phi(x),\partial_\mu\phi(x)]+J(x)\phi(x)\right)}$$

&

$$Z_0[J]=\int \mathcal{D}[\phi(x)]\, e^{i\int d^4x\left(\mathcal{L}_0[\phi(x),\partial_\mu\phi(x)]+J(x)\phi(x)\right)}$$

$$\longrightarrow \lim_{J(x)\to 0} \frac{(-i)^n}{Z_0[0]}\,\frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}$$

$$=\frac{\int \mathcal{D}[\phi(x)]\,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\, e^{i\int d^4x\,\mathcal{L}[\phi(x),\partial_\mu\phi(x)]}}{\int \mathcal{D}[\phi(x)]\, e^{i\int d^4x\,\mathcal{L}_0[\phi(x),\partial_\mu\phi(x)]}}$$

But we defined $Z[J]=\dfrac{Z[J]}{Z_0[0]}$

$\Rightarrow$

$$\boxed{\langle\phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle=\frac{(-i)^n\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\Bigg|_{J\to 0}}$$

ϕ^4 Theory using Path integrals

Here $\mathcal{L}_{int.}(\phi(x)) = -\frac{\lambda}{4!}\phi^4(x)$

$$\Rightarrow \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right] = -\frac{\lambda}{4!}\left(\frac{-i\delta}{\delta J(x)}\right)^4 =$$

And we previously saw that for free Scalar field,

$$Z_0[J] = e^{\frac{-1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x,y)J(y)}$$

where $\Delta_F(x,y)$ is the free field propagator

Thus the generating functional for $\phi^4(x)$ interaction of free scalar field is $Z[J]$

$$\boxed{Z[J] = \left(e^{\frac{-i\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J(x)^4}}\right) e^{\frac{-1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x,y)J(y)}}$$

Perturbative expansion of generating functional for $\phi^4(x)$ interaction

Here we shall expand the exponential of the functional derivative in power series

$$\rightarrow e^{\frac{-i\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J(x)^4}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-i\lambda}{4!}\int d^4x \frac{\delta^4}{\delta J(x)^4}\right)^n$$

$$= 1 + \frac{(-i\lambda)}{4!}\int d^4x \frac{\delta^4}{\delta J(x)^4} + \frac{1}{2!}\frac{(-i\lambda)^2}{4!}\int d^4x \frac{\delta^4}{\delta J(x)^4}\int d^4y \frac{\delta^4}{\delta J(y)^4} + \cdots$$

$$\rightarrow Z[J] = Z_0[J] + \frac{(-i\lambda)}{4!}\int d^4x \frac{\delta^4 Z_0[J]}{\delta J(x)^4} + \frac{1}{2!}\frac{(-i\lambda)^2}{4!}\int d^4x \int d^4y \frac{\delta^4}{\delta J(x)^4}\frac{\delta^4}{\delta J(y)^4} Z_0[J] + \cdots$$

We shall consider these terms order by order of 'λ'.

0^{th} order term is the free field contribution where there are no interactions.

1^{st} order term

$$Z^{(1)}[J] = \frac{(-i\lambda)}{4!}\int d^4x \, \frac{\delta^4 Z_0[J]}{\delta J(x)^4}$$

$$Z_0[J] = e^{\frac{-1}{2}\int d^4z\, d^4w\, J(z)\Delta_F(z,w)J(w)}$$

$$\Longrightarrow \frac{\delta Z_0[J]}{\delta J(x)} = -Z_0[J]\int d^4z\, J(z)\Delta_F(z,x)$$

$$\Longrightarrow \frac{\delta^2 Z_0[J]}{\delta J(x)^2} = \frac{\delta}{\delta J(x)}\left[-Z_0[J]\int d^4z\, J(z)\Delta_F(z,x)\right]$$

$$= Z_0[J]\left(-\Delta_F(x,x) + \int d^4z\, J(z)\Delta_F(z,x) \times \int d^4w\, J(w)\Delta_F(w,x)\right)$$

$$\frac{\delta^2 Z_0[J]}{\delta J(x)^2} = Z_0[J]\left(-\Delta_F(x,x) + \left[\int d^4z\, J(z)\Delta_F(z,x)\right]^2\right)$$

$$\Rightarrow \frac{\delta^3 Z_0[J]}{\delta J(x)^3} = \frac{\delta}{\delta J(x)} Z_0[J] \left(-\Delta_F(x,x) + \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^2 \right)$$

$$\Rightarrow$$

$$\frac{\delta^3 Z_0[J]}{\delta J(x)^3} = Z_0[J] \left(3\Delta_F(x,x) \int d^4z \, J(z) \Delta_F(z,x) - \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^3 \right)$$

Similarly,

$$\frac{\delta^4 Z_0[J]}{\delta J(x)^4} = Z_0[J] \left(3\left[\Delta_F(x,x)\right]^2 - 6\Delta_F(x,x) \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^2 + \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^4 \right)$$

Now we get the 1st order term as,

$$\frac{Z^{(1)}[J]}{Z_0[J]} = \frac{(-i\lambda)}{4!} \int d^4x \left(3\left[\Delta_F(x,x)\right]^2 - 6\Delta_F(x,x) \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^2 + \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^4 \right)$$

$$= \frac{(-i\lambda)}{4!}\left(\begin{array}{l} 3\int d^4x \left[\Delta_F(x,x)\right]^2 \\ -6\int d^4x\, \Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,x)\right]^2 \\ \qquad +\int d^4x \left[\int d^4z\, J(z)\Delta_F(z,x)\right]^4 \end{array}\right)$$

$$= (-i\lambda)\left(\begin{array}{l} \frac{1}{8}\int d^4x \left[\Delta_F(x,x)\right]^2 \\ -\frac{1}{4}\int d^4x\, \Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,x)\right]^2 \\ \qquad +\frac{1}{24}\int d^4x \left[\int d^4z\, J(z)\Delta_F(z,x)\right]^4 \end{array}\right)$$

To interpret this we need to expand the powers of $\left[\int d^4z\, J(z)\Delta_F(z,x)\right]$.

$$\int d^4x\, \Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,x)\right]^2$$

$$= \int d^4x\, d^4y\, d^4z\, J(y)\Delta_F(y,x)\Delta_F(x,x)\Delta_F(x,z)J(z)$$

$$\int d^4x \left[\int d^4z\, J(z)\Delta_F(z,x)\right]^4$$

$$= \int d^4x\, d^4z_1 d^4z_2 d^4z_3 d^4z_4 J(z_1)J(z_2)J(z_3)J(z_4) \times \Delta_F(z_1,x)\Delta_F(z_2,x)\Delta_F(z_3,x)\Delta_F(z_4,x)$$

Thus we arrive at,

$$\frac{Z^{(1)}[J]}{Z_0[J]} = \frac{(-i\lambda)}{8}\int d^4x \left[\Delta_F(x,x)\right]^2$$

$$- \frac{(-i\lambda)}{4}\int d^4x\, d^4y\, d^4z\, J(y)\Delta_F(y,x)\Delta_F(x,x)\Delta_F(x,z)J(z)$$

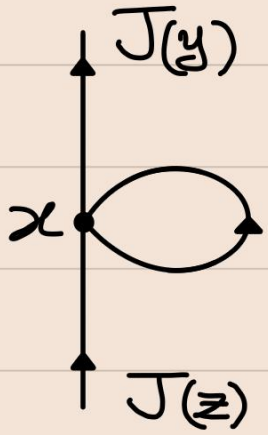
$$+ \frac{(-i\lambda)}{24}\int d^4x\, d^4z_1 d^4z_2 d^4z_3 d^4z_4\, J(z_1)J(z_2)J(z_3)J(z_4) \times \Delta_F(z_1,x)\Delta_F(z_2,x)\Delta_F(z_3,x)\Delta_F(z_4,x)$$

These 1st order terms actually gives rise to the following feynman diagrams.

$$\frac{(-i\lambda)}{8}\int d^4x \left[\Delta_F(x,x)\right]^2 \sim$$

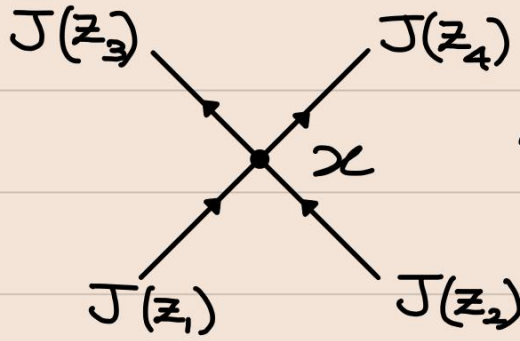


→ Vacuum bubble (no source term)



$$\sim -\frac{(-i\lambda)}{4}\int d^4x\, d^4y\, d^4z\, J(y)\,\Delta_F(y,x)\,\Delta_F(x,x)\times \Delta_F(x,z)\,J(z)$$

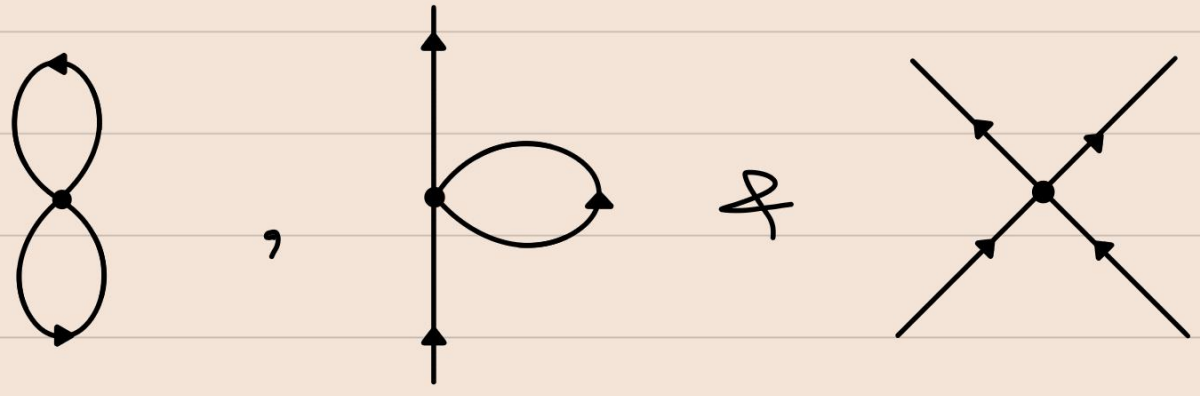
(It has two sources z & y. & a loop '$\Delta_F(x,x)$' at x.)



$$\sim \frac{(-i\lambda)}{24}\int d^4x\, d^4z_1 d^4z_2 d^4z_3 d^4z_4 \times J(z_1)J(z_2)J(z_3)J(z_4) \times \Delta_F(x,z_1)\Delta_F(x,z_2) \times \Delta_F(z_3,x)\Delta_F(z_4,x)$$

(4 Sources and 1 vertex is present)

Thus $Z^{(1)}[J]$ can give rise to the following diagrams



But consider the sourceless function given by 'Z[0]' which is,

$$Z[0] = 1 + \frac{(-i\lambda)}{8}\int d^4x \left[\Delta_F(x,x)\right]^2 + \Theta(\lambda^2)$$

Remember that,

$$Z[0] = \frac{\int D[\phi(x)]\, e^{i\int d^4x\, \mathcal{L}[\phi(x), \partial_\mu\phi(x)]}}{\int D[\phi(x)]\, e^{i\int d^4x\, \mathcal{L}_0[\phi(x), \partial_\mu\phi(x)]}}$$

$$= \frac{\int D[\phi(x)]\, e^{\frac{-i\lambda}{4!}\int d^4x\, \phi(x)^4}\, e^{i\int d^4x\, \mathcal{L}_0}}{\int D[\phi(x)]\, e^{i\int d^4x\, \mathcal{L}_0}}$$

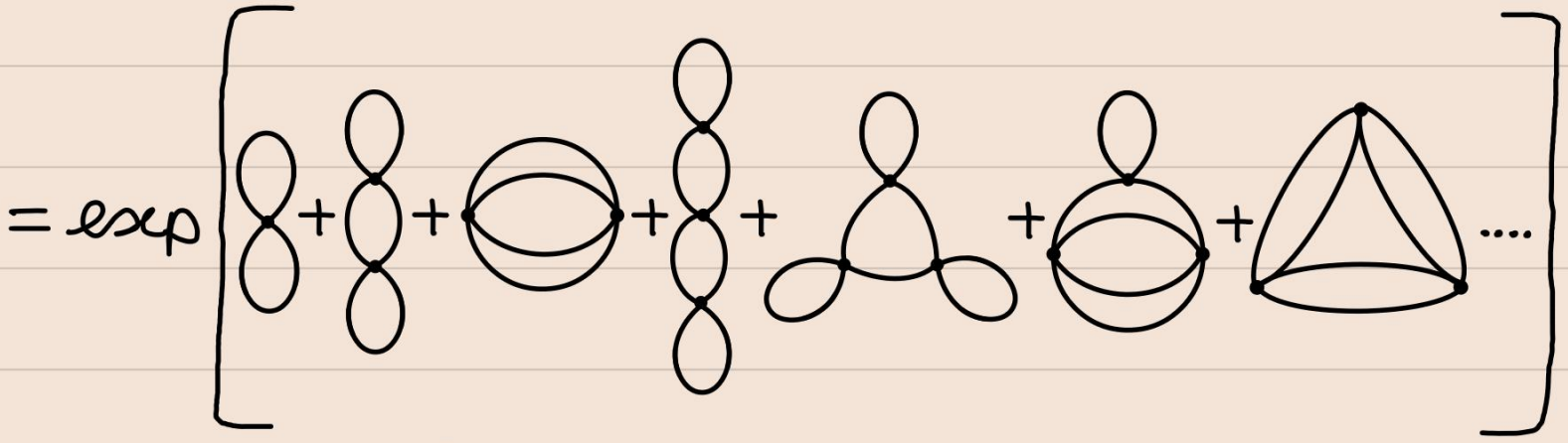
$$\rightarrow Z[0] = \left\langle e^{\frac{-i\lambda}{4!}\int d^4x\, \phi(x)^4} \right\rangle$$

with respect to free field

Thus this resembles the expression of $\langle 0 | e^{\frac{-i\lambda}{4!}\int d^4x\, \hat{\phi}_I(x)^4} | 0 \rangle$ of the canonical quantization. Thus we can write,

$$Z[0] = \exp\left[\sum \left(\begin{array}{c}\text{Sum of all connected}\\ \text{vacuum diagrams with}\\ \text{no sources/external legs}\end{array}\right)\right]$$

$$= \exp\Big[\quad + \quad + \quad + \quad + \quad + \quad + \quad \ldots \Big]$$



where  $= \frac{(-i\lambda)}{8}\int d^4x \left[\Delta_F(x,x)\right]^2$

 $= \frac{(-i\lambda)^2}{16}\int d^4x\, d^4y\, \Delta_F(x,x)\Delta_F(x,y)^2\Delta_F(y,y)$

Connected vacuum diagram means all terms have one piece without any source. The diagrams with disconnected pieces aren't kept.

Perturbative expansion of n-Point Correlation for $\phi^4(x)$ interaction

We have already seen that the n-Point Correlation function is given by,

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle = \left.\frac{(-i)^n \delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\right|_{J\to 0}$$

(The above is free-field normalized.)

Using first order expansion of generating functional we can only get the first order contributions to the 2-point and 4-point correlation functions of the ϕ^4 theory.

Upto 1st order we get

$$Z[J] = Z_0[J] + Z^{(1)}[J]$$

where $$\frac{\delta Z_0[J]}{\delta J(y)} = -Z_0[J]\int d^4z\, J(z)\,\Delta_F(z,y)$$

2 point correlation function : ϕ^4 theory

The Zero-th order contribution is just a free field propagator.

$$\frac{\delta^2 Z_0[J]}{\delta J(x)\delta J(y)} = Z_0[J]\left(-\Delta_F(x,y) + \int d^4z\, J(z)\Delta_F(z,y) \times \int d^4w\, J(w)\Delta_F(w,x)\right)$$

$$\rightarrow$$

$$\langle\phi(x)\phi(y)\rangle^{(0)} = \left.\frac{(-i)^n \delta^2 Z_0[J]}{\delta J(x)\delta J(y)}\right|_{J\to 0} = \Delta_F(x,y)$$

As a feynman diagram it is 

First order contribution

1st order contribution to the 2-point correlation is $\langle\phi(x)\phi(y)\rangle^{(1)}$ is got by

$$\langle\phi(x)\phi(y)\rangle^{(1)} = \lim_{J(x)\to 0} (-i)^2 \left(\frac{\delta^2 Z^{(1)}[J]}{\delta J(x)\,\delta J(y)}\right)$$

$$Z^{(1)}[J] = (-i\lambda) Z_0[J] \left(\frac{1}{8} \int d^4w \left[\Delta_F(w,w) \right]^2 \right.$$

$$- \frac{1}{4} \int d^4w \, \Delta_F(w,w) \left[\int d^4z \, J(z) \Delta_F(z,w) \right]^2$$

$$\left. + \frac{1}{24} \int d^4x \left[\int d^4z \, J(z) \Delta_F(z,x) \right]^4 \right)$$

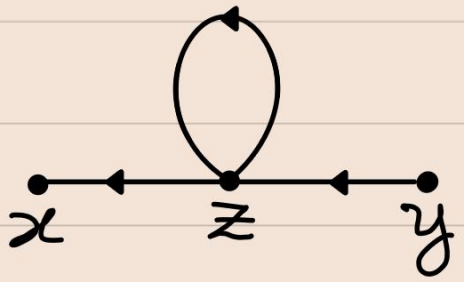
Now, after differentiating twice and then taking $J \to 0$, only the 2nd order term of 'J' in the above expression will contribute. So, focusing on this term,

$$\left. \frac{(-i)^2 \delta^2 Z^{(1)}[J]}{\delta J(x) \, \delta J(y)} \right|_{J \to 0} = -\frac{(-i\lambda)}{2} \int d^4z \, \Delta_F(x,z) \Delta_F(z,z) \times \Delta_F(z,y)$$

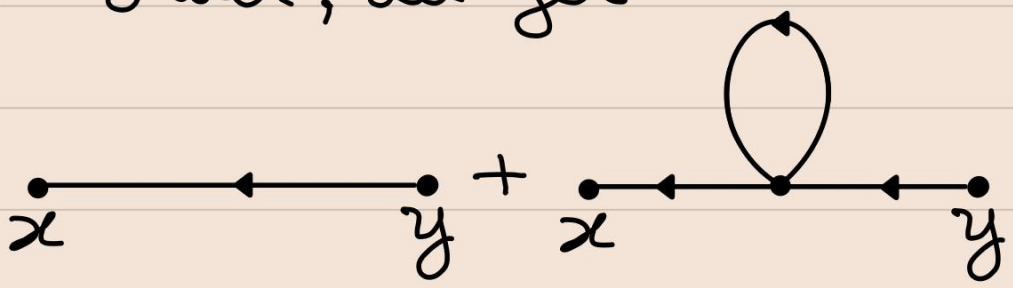
$$\to \langle \phi(x) \phi(y) \rangle^{(1)} = \frac{(-i\lambda)}{2} \int d^4z \, \Delta_F(x,z) \Delta_F(z,z) \Delta_F(z,y)$$

This is written in terms of feynmann diagram as

$$\langle \phi(x) \phi(y) \rangle^{(1)} =$$



Thus upto 1st order, we get

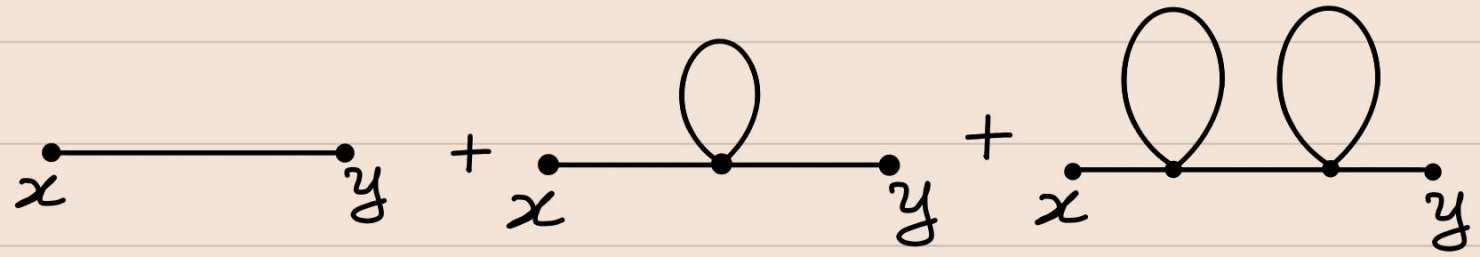
$\langle \phi(x)\phi(y) \rangle =$ 

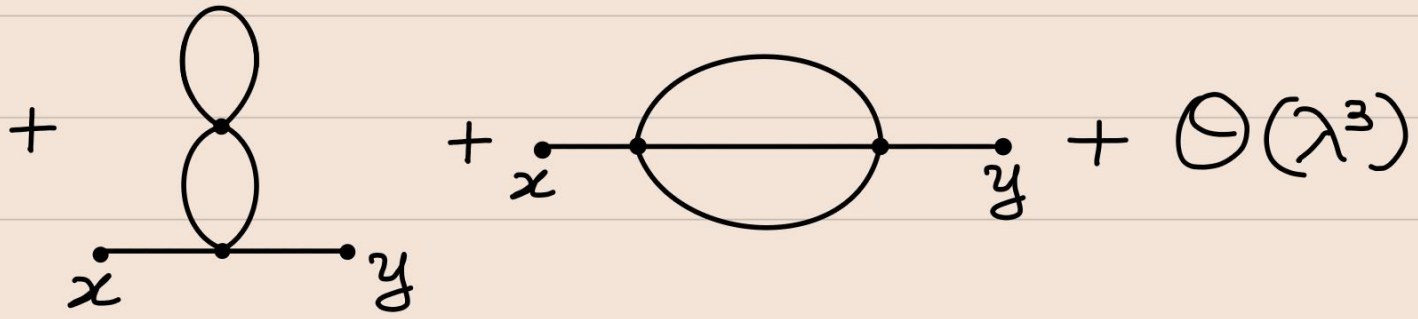
We can generalize this to any order

$$\langle \phi(x)\phi(y) \rangle = \sum \left(\begin{array}{l} \text{Sum of all possible} \\ \text{Connected pieces with two} \\ \text{external points} \end{array} \right)$$

Thus vacuum bubbles do not contribute for any n-point correlation functions.

Thus diagrammatically $\langle \phi(x)\phi(y) \rangle$ is,





4 point correlation function : ϕ^4 theory

It is given by

$$\langle \phi(x)\phi(y)\phi(z)\phi(w)\rangle = \left.\frac{(-i)^4 \delta^4 Z[J]}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0}$$

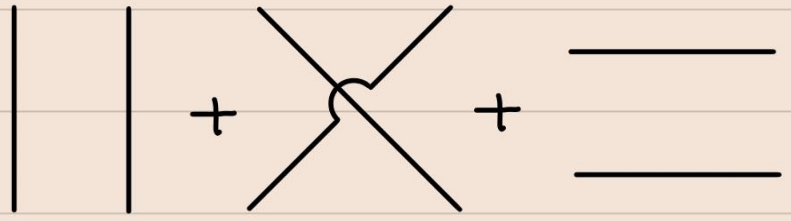
Upto 1st order we get

$$Z[J] = Z_0[J] + Z^{(1)}[J]$$

Now, $\left.\frac{(-i)^4 \delta^4 Z_0[J]}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0}$ which gives

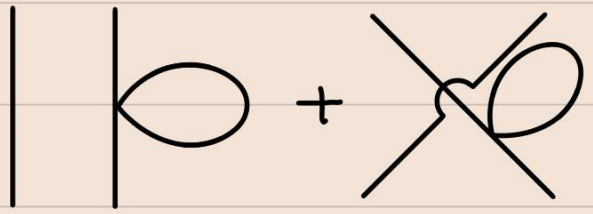
the Zeroth order contribution is just

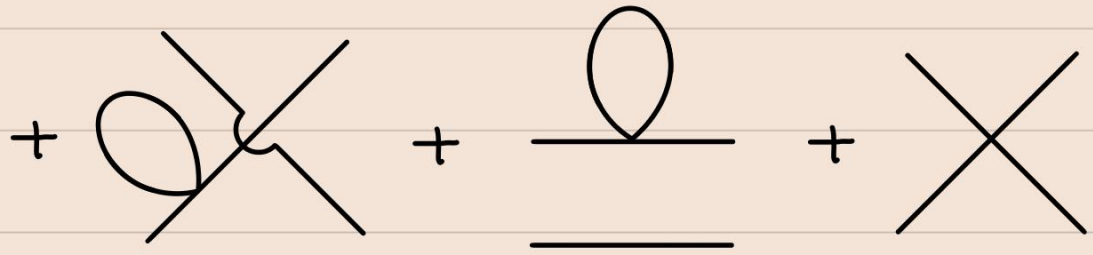
$$\langle 0|T[\phi(x)\phi(y)\phi(z)\phi(w)]|0\rangle = \Delta_F(x,y)\Delta_F(z,w) + \Delta_F(x,z)\Delta_F(y,w) + \Delta_F(y,z)\Delta_F(x,w)$$

$=$  $+$ $+$

Then the 1st order contributions are given by the following sum of diagrams.

$$\left.\frac{(-i)^4 \delta^4 Z^{(1)}[J]}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0} =$$

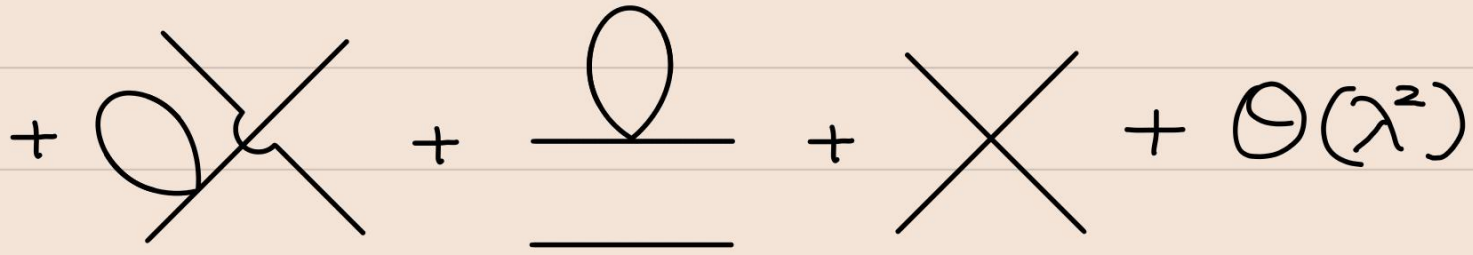




Thus, we can write,

$$\langle \phi(x)\phi(y)\phi(z)\phi(w)\rangle = \sum \left(\begin{array}{c}\text{All Connected pieces}\\ \text{with four external}\\ \text{points}\end{array}\right)$$

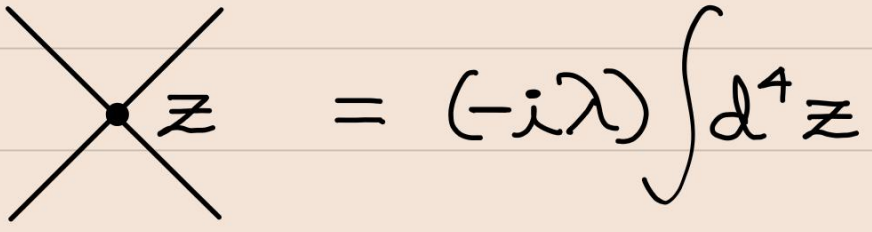




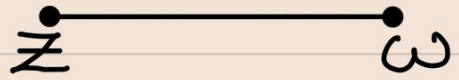
We get the feynman rules exactly same as before when we used the method of canonical quantization in ϕ^4 theory.

The position Space feynman rules

- Every internal vertex is represented by,



- Each propagator b/w two vertices is a line,

$\Delta_F(z,w) =$ 

$\Delta_F(z,z) =$ 

→ loop

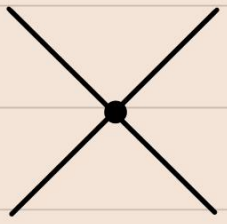
- Each external point, 'x':



- Divide by the symmetry factor

The momentum Space feynman rules

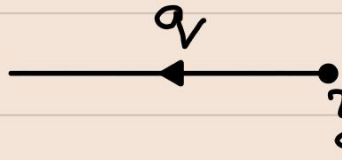
- Propagator:  $= \frac{i}{k^2 - m^2 + i\varepsilon}$

- Vertex:  $= -i\lambda$ [Enforce momentum Conservation]

- External points carry a plane wave factor



$= e^{-ipx}$



$= e^{iqy}$

- Integrate over all undetermined momenta 'k', $\int \frac{d^4k}{(2\pi)^4}$

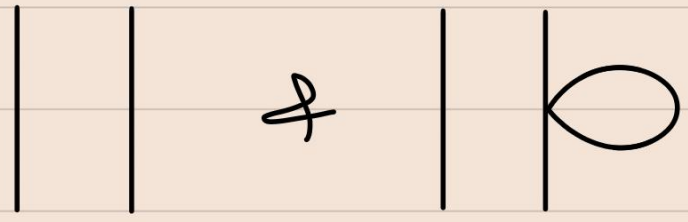
- Divide by the symmetry factor

- Every vacuum bubble, which is a disconnected piece of a diagram, that is not connected to external point will carry a factor of $(2\pi)^4 \delta(0)$

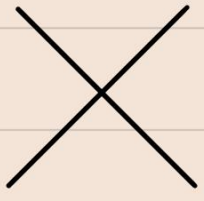
Connected Correlation functions

The correlation function is said to be Connected if all the external legs are connected to each other via some path.

Eg:- In the four-point correlation for ϕ^4 theory, we see that upto 1st order



types are the disconnected 4-point correlation functions

But  → It is the only Connected 4-point correlation function upto first order.

Now we need a Generating functional to obtain only Connected n-point Correlation functions denoted by 'G' as $G(x_1, x_2, x_3, \cdots x_n)$.

For ϕ^4 theory we saw that, upto 1st order,

$$Z[J] = Z_0[J] + Z^{(1)}[J]$$

where $Z_0[J] = e^{\frac{-1}{2}\int d^4z\, d^4w\, J(z)\Delta_F(z,w)J(w)}$

&

$$\frac{Z^{(1)}[J]}{Z_0[J]} = \frac{(-i\lambda)}{8}\int d^4x \left[\Delta_F(x,x)\right]^2$$

$$- \frac{(-i\lambda)}{4}\int d^4x\, d^4y\, d^4z\, J(y)\Delta_F(y,x)\Delta_F(x,x)\Delta_F(x,z)J(z)$$

$$+ \frac{(-i\lambda)}{24}\int d^4x\, d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\, J(z_1)J(z_2)J(z_3)J(z_4)$$
$$\times \Delta_F(z_1,x)\,\Delta_F(z_2,x)\,\Delta_F(z_3,x)\,\Delta_F(z_4,x)$$

$$\rightarrow \ln\left(Z[J]\right) = \ln\left(Z_0[J]\right)$$
$$+ \ln\left[1 + \frac{Z^{(1)}[J]}{Z_0[J]} + \frac{Z^{(2)}[J]}{Z_0[J]} + \cdots\right]$$

$$\ln(Z_0[J]) = \frac{-1}{2}\int d^4z\, d^4w\, J(z)\Delta_F(z,w)J(w)$$

& $(-i)^2 \dfrac{\delta^2 \ln(Z_0[J])}{\delta J(x)\delta J(y)}\Big|_{J\to 0}$ can be found.

$$\rightarrow (-i)^2 \frac{\delta^2 \ln(Z_0[J])}{\delta J(x)\delta J(y)}\Big|_{J\to 0} = \Delta_F(x,y) =$$



This is the connected 2-point correlation function upto 0^{th} order.

Let us consider $\ln(Z[J])$ only upto the 1^{st} order.

$$\rightarrow \ln(Z[J]) = \ln(Z_0[J]) + \ln\left[1 + \frac{Z^{(1)}[J]}{Z_0[J]}\right]$$

Upto 1^{st} order in 'λ',

$$\ln\left[1 + \frac{Z^{(1)}[J]}{Z_0[J]}\right] = \frac{Z^{(1)}[J]}{Z_0[J]}$$

Then,

$$\ln(Z[J]) = \ln(Z_0[J]) + \frac{Z^{(1)}[J]}{Z_0[J]} + \Theta(\lambda^3)$$

$$\Longrightarrow (-i)^2 \frac{\delta^2 \ln(Z[J])}{\delta J(x)\delta J(y)}\bigg|_{J\to 0} = (-i)^2 \frac{\delta^2 \ln(Z_0[J])}{\delta J(x)\delta J(y)}\bigg|_{J\to 0}$$

$$+ \frac{(-i)^2\delta^2}{\delta J(x)\delta J(y)}\left(\frac{Z^{(1)}[J]}{Z_0[J]}\right)\bigg|_{J\to 0}$$

$$= \Delta_F(x,y) + \frac{(-i)^2\delta^2}{\delta J(x)\delta J(y)}\left(\frac{Z^{(1)}[J]}{Z_0[J]}\right)\bigg|_{J\to 0}$$

Now, we find, $\dfrac{(-i)^2\delta^2}{\delta J(x)\delta J(y)}\left(\dfrac{Z^{(1)}[J]}{Z_0[J]}\right)\bigg|_{J\to 0}$

Note that only the term with 2nd order in 'J' will survive.

$$\Rightarrow \frac{(-i)^2}{\delta J(x)\delta J(y)}\left(\frac{Z^{(1)}[J]}{Z_0[J]}\right)\Bigg|_{J\to 0}$$

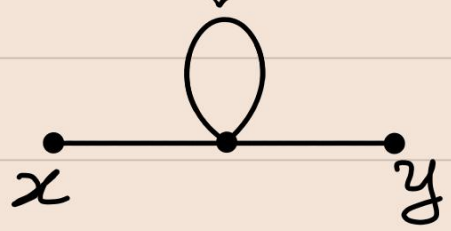
$$= \frac{(-i\lambda)}{2}\int d^4z\, \Delta_F(x,z)\Delta_F(z,z)\Delta_F(z,y)$$

Thus we get upto 1st order in 'λ',

$$(-i)^2 \frac{\delta^2 \ln(Z[J])}{\delta J(x)\delta J(y)}\Bigg|_{J\to 0}$$ to be equal to

$$= \underbrace{\Delta_F(x,y)} + \underbrace{\frac{(-i\lambda)}{2}\int d^4z\, \Delta_F(x,z)\Delta_F(z,z)\Delta_F(z,y)}$$





If we generalize it then,

$$(-i)^2 \frac{\delta^2 \ln(Z[J])}{\delta J(x)\delta J(y)}\Bigg|_{J\to 0} = \sum \left[\begin{array}{l}\text{Sum of all connected}\\ \text{diagrams with two}\\ \text{external legs}\end{array}\right]$$

But this is also the 2-point connected correlation function $G(x,y) = \langle \phi(x)\phi(y)\rangle$

Similarly, if we calculate,

$$\left.\frac{(-i)^4\delta^4 \ln(Z[J])}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0}$$ upto 1st order,

only the term with 4th order of 'J' will survive. Thus

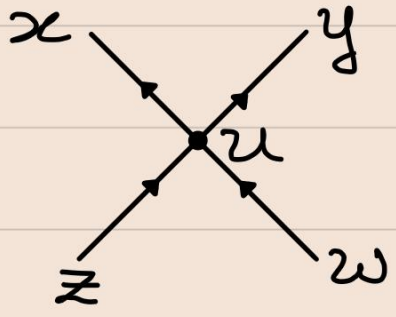
$$\left.\frac{(-i)^4\delta^4 \ln(Z[J])}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0}$$

$$= \frac{(-i\lambda)}{24}\int d^4u\, d^4z_1\, d^4z_2\, d^4z_3\, d^4z_4\; \Delta_F(z_1,u)\,\Delta_F(z_2,u)\,\Delta_F(z_3,u)$$

$$\times \Delta_F(z_4,u)\;\frac{(-i)^4\delta^4\, J(z_1)J(z_2)J(z_3)J(z_4)}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}$$

$$= (-i\lambda)\int d^4u\; \Delta_F(x,u)\,\Delta_F(y,u)\,\Delta_F(u,z)\,\Delta_F(u,w)$$

$$\Longrightarrow \left.\frac{(-i)^4\delta^4 \ln(Z[J])}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0} =$$

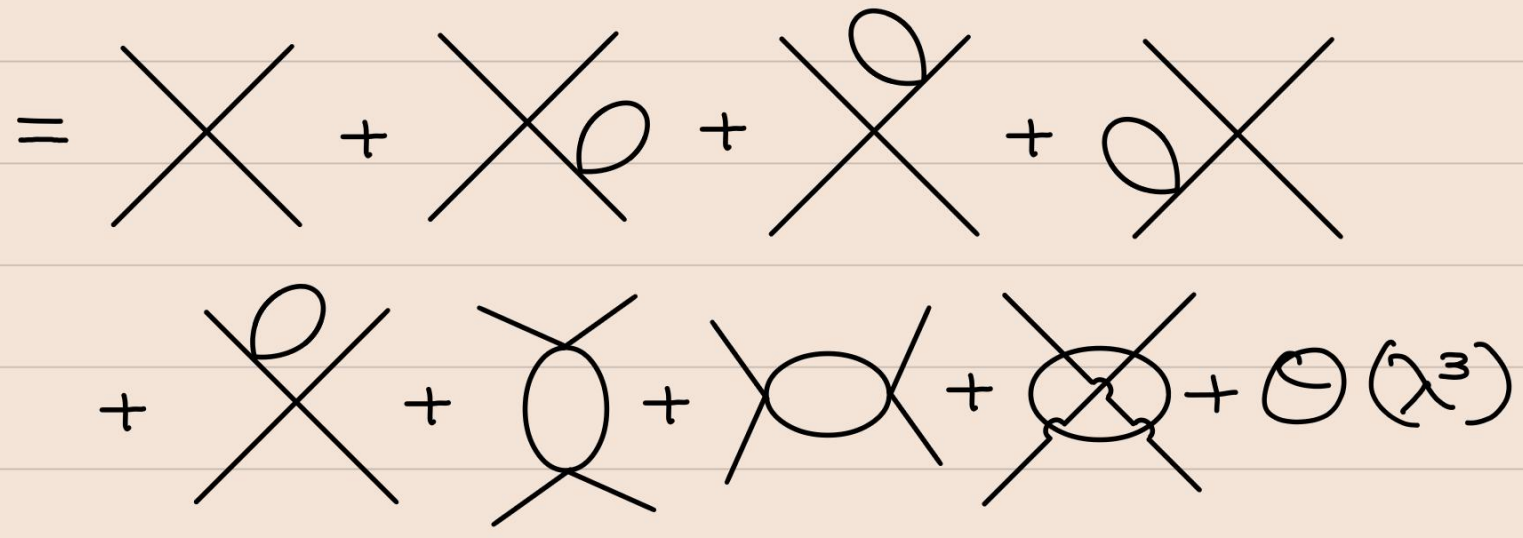


$+\Theta(\lambda^2)$

Thus we can generalize the result as

$$\left.\frac{(-i)^4 \delta^4 \ln(Z[J])}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)}\right|_{J\to 0}$$

$$= \sum \left[\begin{array}{l}\text{Sum of all fully connected} \\ \text{diagrams with 4-external legs}\end{array}\right]$$



Thus we observe that $\ln(Z[J])$ generates connected correlation functions $G(x_1, \cdots x_n)$ Hence, we define a new functional $W[J]$ as,

$$\underline{W[J] = \ln(Z[J])} \quad \text{where,}$$

$$\Rightarrow G(x_1, x_2 \cdots x_n) = \left.\frac{(-i)^n \delta^n W[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\right|_{J\to 0}$$

Now since $W[J]$ gives rise to only the connected diagrams, we can think of $Z[J]$ as a product of exponential of sum of vacuum diagram given by $Z[0]$ and exponential of sum of all fully connected source to source diagrams.

$$\Rightarrow Z[J] = Z[0]\, e^{\sum \left[\begin{array}{c}\text{fully connected source} \\ \text{to source diagrams}\end{array}\right]}$$

where $$Z[0] = e^{\sum \left[\begin{array}{c}\text{all connected vacuum} \\ \text{diagrams without} \\ \text{any source}\end{array}\right]}$$

Thus we can define interacting field normalized generating functional $Z[J]$ such that $Z[0] = 1$ as,

$$Z[J] = \frac{Z[J]}{Z[0]} = e^{\sum \left[\begin{array}{c}\text{fully connected source} \\ \text{to source diagrams}\end{array}\right]}$$

[Note: These results are valid for any interacting theory]

Then if $W[J] = \ln(Z[J])$

$$\Rightarrow Z[J] = e^{W[J]}$$

$$\Rightarrow \boxed{W[J] = \sum \left[\begin{array}{l}\text{Sum of all fully Connected} \\ \text{Source to Source and} \\ \text{Connected Vacuum diagrams}\end{array}\right]}$$

Therefore we have demonstrated the <u>linked-Cluster theorem</u> in ϕ^4 theory.

$$Z[J] = \sum \left[\begin{array}{l}\text{Sum of all possible disconnected} \\ + \text{Connected Vacuum \& Source} \\ \text{to Source diagrams}\end{array}\right]$$

Also we showed,

$$Z[J] = \exp\left(\sum \left[\begin{array}{l}\text{Sum of all fully Connected} \\ \text{Source to Source and} \\ \text{Connected Vacuum diagrams}\end{array}\right]\right)$$

$$\boxed{\sum (\text{all diagrams}) = e^{\sum (\text{Connected diagrams})}}$$

ϕ^3 Theory using Path integrals

Here $\mathcal{L}_{int.}(\phi(x)) = -\frac{g}{3!}\phi^3(x)$

$$\Rightarrow \mathcal{L}_{int.}\left[\frac{-i\delta}{\delta J(x)}\right] = -\frac{g}{3!}\left(\frac{-i\delta}{\delta J(x)}\right)^3 = \frac{ig}{3!}\frac{\delta^3}{\delta J(x)^3}$$

And we previously saw that for free Scalar field,

$$Z_0[J] = e^{\frac{-1}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x,y)J(y)}$$

where $\Delta_F(x,y)$ is the free field propagator

Thus the generating functional for $\phi^3(x)$ interaction of free scalar field is $Z[J]$

$$\boxed{Z[J] = \left(e^{\frac{-ig}{3!}\int d^4x \frac{\delta^3}{i^3\delta J(x)^3}}\right) e^{\frac{-1}{2}\int d^4x d^4y J(x)\Delta_F(x,y)J(y)}}$$

Perturbative expansion of generating functional for $\phi^3(x)$ interaction

Here we shall expand the exponential of the functional derivative in power series

$$\rightarrow e^{\frac{-ig}{3!}\int d^4x \frac{\delta^3}{i^3 \delta J(x)^3}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{(-ig)}{3!}\int d^4x \frac{\delta^3}{i^3\delta J(x)^3}\right)^n$$

$$= 1 + \frac{(-ig)}{3!}\int d^4x \frac{\delta^3}{i^3\delta J(x)^3}$$

$$+ \frac{1}{2!}\frac{(-ig)^2}{3!}\int d^4x \frac{\delta^3}{i^3\delta J(x)^3}\int d^4y \frac{\delta^3}{i^3\delta J(y)^3} + \cdots$$

$$\rightarrow Z[J] = Z_0[J] + \frac{(-ig)}{3!}\int d^4x \frac{\delta^3 Z_0[J]}{i^3\delta J(x)^3}$$

$$+ \frac{1}{2!}\frac{(-ig)^2}{(3!)^2}\int d^4x \int d^4y \frac{\delta^3}{i^3\delta J(x)^3}\frac{\delta^3}{i^3\delta J(y)^3} Z_0[J]$$

$$+ \cdots$$

We shall consider these terms order by order of 'g'.

0^{th} order term is the free field contribution where there are no interactions.

1^{st} order term

$$Z^{(1)}[J] = \frac{(-ig)}{3!}\int d^4x \quad \frac{\delta^3 Z_0[J]}{i^3 \delta J(x)^3}$$

$$Z_0[J] = e^{\frac{-1}{2}\int d^4z\, d^4w\, J(z)\Delta_F(z,w)J(w)}$$

$$\Longrightarrow \frac{\delta Z_0[J]}{\delta J(x)} = -Z_0[J]\int d^4z\, J(z)\Delta_F(z,x)$$

$$\Longrightarrow \frac{\delta^2 Z_0[J]}{\delta J(x)^2} = \frac{\delta}{\delta J(x)}\left[-Z_0[J]\int d^4z\, J(z)\Delta_F(z,x)\right]$$

$$= Z_0[J]\left(-\Delta_F(x,x) + \int d^4z\, J(z)\Delta_F(z,x) \times \int d^4w\, J(w)\Delta_F(w,x)\right)$$

$$\frac{\delta^2 Z_0[J]}{\delta J(x)^2} = Z_0[J]\left(-\Delta_F(x,x) + \left[\int d^4z\, J(z)\Delta_F(z,x)\right]^2\right)$$

$$\Rightarrow \frac{\delta^3 Z_0[J]}{\delta J(x)^3} = \frac{\delta}{\delta J(x)} Z_0[J] \left(-\Delta_F(x,x) + \left[\int d^4z\, J(z) \Delta_F(z,x)\right]^2 \right)$$

$$\Rightarrow$$

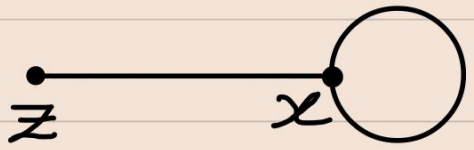
$$\frac{\delta^3 Z_0[J]}{\delta J(x)^3} = Z_0[J] \left(3\Delta_F(x,x) \int d^4z\, J(z) \Delta_F(z,x) - \left[\int d^4z\, J(z) \Delta_F(z,x)\right]^3 \right)$$

Now we get the 1st order term as,

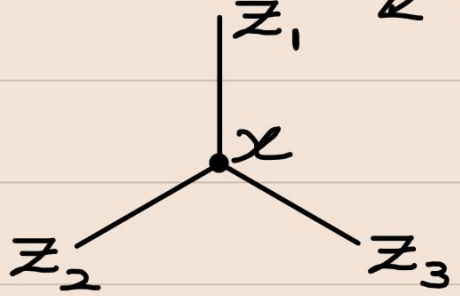
$$\frac{Z^{(1)}[J]}{Z_0[J]} = \frac{(-ig)}{3!} \int \frac{d^4x}{i^3} \left(3\Delta_F(x,x) \int d^4z\, J(z) \Delta_F(z,x) - \left[\int d^4z\, J(z) \Delta_F(z,x)\right]^3 \right)$$

$$= \frac{g}{2} \int d^4x\, d^4z\, \Delta_F(x,x) \Delta_F(x,z) J(z) - \frac{g}{3!} \int d^4x \left[\int d^4z\, J(z) \Delta_F(z,x)\right]^3$$

gives



gives



Now we shall try finding 2nd order term in 'g':

$$\frac{\delta^3 Z_0[J]}{\delta J(y)^3} = Z_0[J]\left(3\Delta_F(y,y)\int d^4z\, J(z)\Delta_F(z,y) - \left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3\right)$$

$$\frac{\delta}{\delta J(x)}\left(\frac{\delta^3 Z_0[J]}{\delta J(y)^3}\right)$$

$$= Z_0[J]\left(\begin{array}{l} 3\Delta_F(y,y)\Delta_F(x,y) \\ \quad -3\Delta_F(x,y)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \\ -3\Delta_F(y,y)\int d^4w\, J(w)\Delta_F(w,x) \\ \qquad\qquad \times \int d^4z\, J(z)\Delta_F(z,y) \\ +\int d^4w\, J(w)\Delta_F(w,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \end{array}\right)$$

$$\frac{\delta^2}{\delta J(x)^2}\left(\frac{\delta^3 Z_0[J]}{\delta J(y)^3}\right) =$$

$$= Z_0[J] \Bigg(\begin{aligned} & -6\Delta_F(x,y)^2 \int d^4z\, J(z)\Delta_F(z,y) \\ & -6\Delta_F(y,y)\Delta_F(x,y)\int d^4z\, J(z)\Delta_F(z,x) \\ & -3\Delta_F(y,y)\Delta_F(x,x)\int d^4z\, J(z)\Delta_F(z,y) \\ & +\Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \\ & +6\Delta_F(x,y)\int d^4w\, J(w)\Delta_F(w,x) \\ & \qquad \times\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \\ & +3\Delta_F(y,y)\int d^4z\, J(z)\Delta_F(z,y) \\ & \qquad \times\left[\int d^4w\, J(w)\Delta_F(w,x)\right]^2 \\ & -\left[\int d^4w\, J(w)\Delta_F(w,x)\right]^2\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \end{aligned} \Bigg)$$

Then finally $\frac{\delta^3}{\delta J(x)^3}\left(\frac{\delta^3 Z_0[J]}{\delta J(y)^3}\right)$ is,

$$
\begin{aligned}
= Z_0[J] \Bigg| & -6\Delta_F(x,y)^3 - 9\Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x) \\
& + 3\Delta_F(x,x)\Delta_F(x,y)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \\
& + 6\Delta_F(x,y)\Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \\
& + 18\left[\Delta_F(x,y)\right]^2 \int d^4w\, J(w)\Delta_F(w,x) \\
& \qquad \times \int d^4z\, J(z)\Delta_F(z,y) \\
& + 9\Delta_F(y,y)\Delta_F(x,y)\left[\int d^4w\, J(w)\Delta_F(w,x)\right]^2 \\
& + 9\Delta_F(y,y)\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x) \\
& \qquad \times \int d^4z\, J(z)\Delta_F(z,y) \\
& - 3\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x) \\
& \qquad \times \left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \\
& - 9\Delta_F(x,y)\left[\int d^4w\, J(w)\Delta_F(w,x)\right]^2 \\
& \qquad \times \left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \\
& - 3\Delta_F(y,y)\int d^4z\, J(z)\Delta_F(z,y) \\
& \qquad \times \left[\int d^4w\, J(w)\Delta_F(w,x)\right]^3 \\
& + \left[\int d^4w\, J(w)\Delta_F(w,x)\right]^3\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \Bigg|
\end{aligned}
$$

Thus, we get the 2^{nd} order term as,

$$\frac{Z^{(2)}[J]}{Z_0[J]} = \frac{1}{2!}\frac{(-ig)^2}{(3!)^2}\int\frac{d^4x\,d^4y}{i^6}$$

$$\times \Bigg|\; -6\Delta_F(x,y)^3 - 9\Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x)$$

$$+9\,\Delta_F(x,y)\Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \leftarrow \text{Same}$$

$$+18\left[\Delta_F(x,y)\right]^2\int d^4w\, J(w)\Delta_F(w,x)\int d^4z\, J(z)\Delta_F(z,y)$$

$$+9\Delta_F(y,y)\Delta_F(x,y)\left[\int d^4w\, J(w)\Delta_F(w,x)\right]^2 \leftarrow$$

$$+9\Delta_F(y,y)\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x)\int d^4z\, J(z)\Delta_F(z,y)$$

$$-3\,\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \leftarrow \text{Same}$$

$$-9\,\Delta_F(x,y)\left[\int d^4w\, J(w)\,\Delta_F(w,x)\right]^2\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2$$

$$-3\Delta_F(y,y)\int d^4z\, J(z)\Delta_F(z,y)\left[\int d^4w\, J(w)\Delta_F(w,x)\right]^3 \leftarrow$$

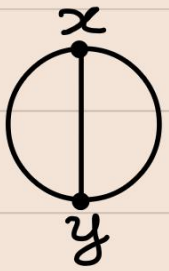
$$+\left[\int d^4w\, J(w)\,\Delta_F(w,x)\right]^3\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \;\Bigg|$$

$$\frac{Z^{(2)}[J]}{Z_0[J]} = \frac{1}{2!}\frac{(-ig)^2}{(3!)^2}\int \frac{d^4x\, d^4y}{i^6}$$

$$\times \Big| \begin{aligned}
&-6\Delta_F(x,y)^3 - 9\Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x) \\
&+18\,\Delta_F(x,y)\Delta_F(x,x)\Big[\int d^4z\, J(z)\Delta_F(z,y)\Big]^2 \\
&+18\big[\Delta_F(x,y)\big]^2 \int d^4w\, J(w)\Delta_F(w,x)\int d^4z\, J(z)\Delta_F(z,y) \\
&+9\Delta_F(y,y)\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x)\int d^4z\, J(z)\Delta_F(z,y) \\
&-6\,\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x)\Big[\int d^4z\, J(z)\Delta_F(z,y)\Big]^3 \\
&-9\,\Delta_F(x,y)\Big[\int d^4w\, J(w)\Delta_F(w,x)\Big]^2\Big[\int d^4z\, J(z)\Delta_F(z,y)\Big]^2 \\
&+\Big[\int d^4w\, J(w)\Delta_F(w,x)\Big]^3\Big[\int d^4z\, J(z)\Delta_F(z,y)\Big]^3
\end{aligned} \Big|$$

We can get the vacuum diagrams by finding $Z[0]$. There are no vacuum diagrams in 1^{st} order or any odd-orders.

Upto 2^{nd} order what we have calculated we can show the following.

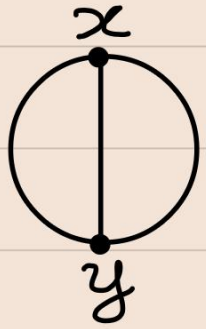
$$\frac{(-ig)^2}{12}\int d^4x\, d^4y\, \Delta_F(x,y)^3 =$$ 

$(S=12)$

$$\frac{(-ig)^2}{8}\int d^4x\, d^4y\, \Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x) =$$ 

$(S=8)$

$S \rightarrow$ Symmetry factors

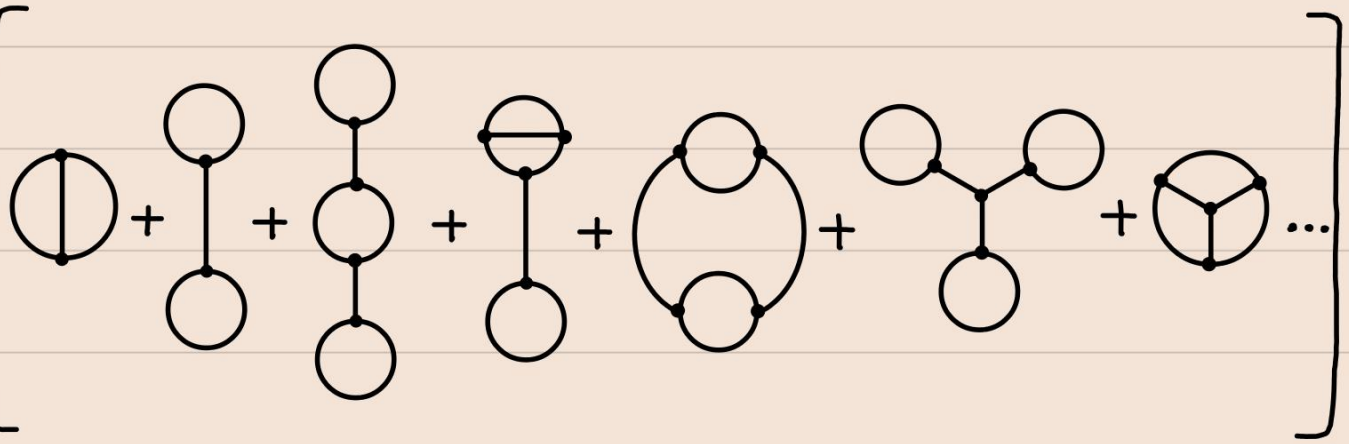
$$\rightarrow Z[0] = 1 + $$ 

$+$ 

$+ \mathcal{O}(\lambda^4)$

This can be generalized as usual

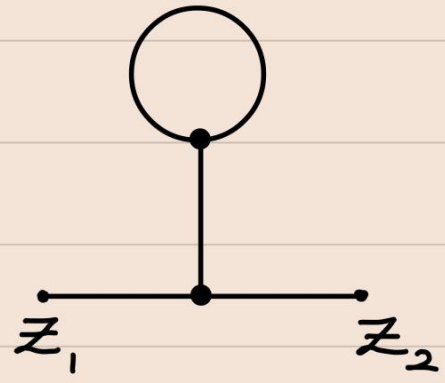
$$Z[0] = \text{esep}\left[\sum \left(\begin{array}{c}\text{Sum of all connected}\\ \text{vacuum diagrams with}\\ \text{no sources/external legs}\end{array}\right)\right]$$

$=$ esep $\Big[$  $\Big]$

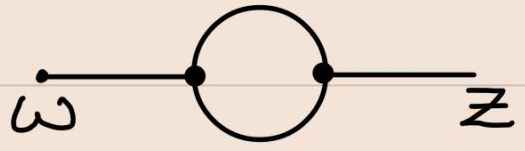
$S \longrightarrow 12\,,\ 8\,,\ 16\,,\ 8\,,\ 16\,,\ 48\,,\ 24$

The other terms in $\frac{Z^{(2)}[J]}{Z_0[J]}$, gives rise to the following diagrams.

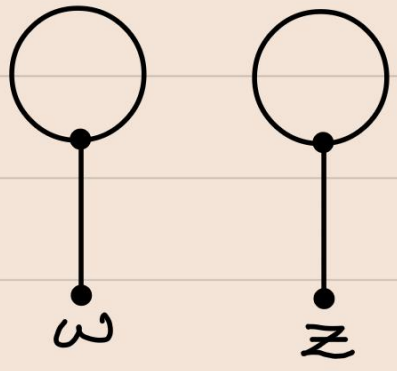
$$\Delta_F(x,y)\Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2 \sim$$



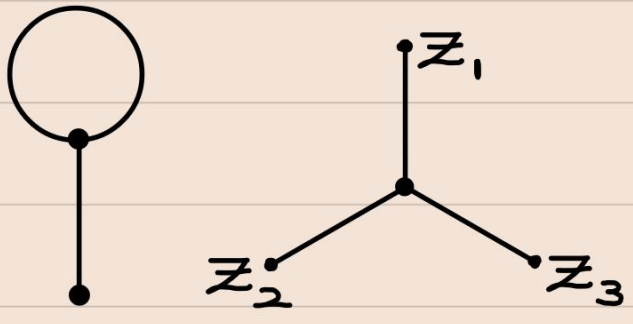
$$\left[\Delta_F(x,y)\right]^2 \int d^4w\, J(w)\Delta_F(w,x) \int d^4z\, J(z)\Delta_F(z,y) \sim$$



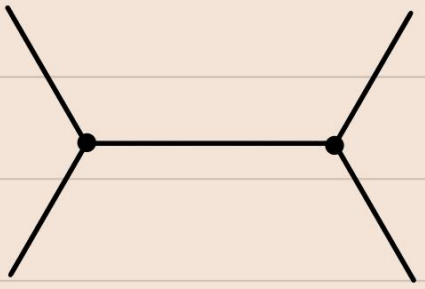
$$\left(\Delta_F(y,y)\int d^4z\, J(z)\Delta_F(z,y) \times \Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x)\right) \sim$$



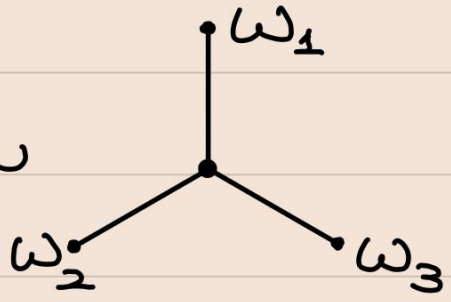
$$\Delta_F(x,x)\int d^4w\, J(w)\,\Delta_F(w,x) \times \left[\int d^4z\, J(z)\Delta_F(z,y)\right]^3 \sim$$

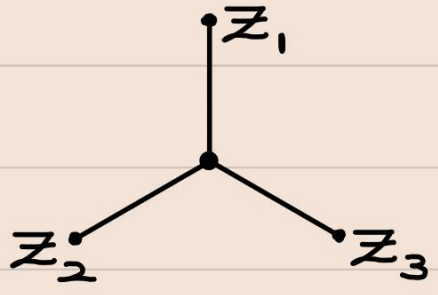


$$\Delta_F(x,y)\left[\int d^4w\, J(w)\,\Delta_F(w,x)\right]^2 \times \left[\int d^4z\, J(z)\,\Delta_F(z,y)\right]^2 \sim$$



$$\left[\int d^4w\, J(w)\,\Delta_F(w,x)\right]^3 \times \left[\int d^4z\, J(z)\,\Delta_F(z,y)\right]^3 \sim$$





Note that here we are just showing what types of diagrams upto 2nd order arises in the above figures. We have not yet established feynman rules here.

The above diagrams are called as the Source-to-Source diagrams where z's & w's are the Source points.

$$\Rightarrow Z[J] = \sum \left[\begin{array}{l}\text{All possible Connected \&} \\ \text{disconnected Vacuum and} \\ \text{Source-to-Source diagrams}\end{array}\right]$$

Perturbative expansion of n-Point Correlation for $\phi^3(x)$ interaction

We have already seen that the n-Point Correlation function is given by,

$$\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle = \left.\frac{(-i)^n \delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\right|_{J\to 0}$$

(The above is free-field normalized.)

In ϕ^3 theory, ground state expectation of $\phi(x)$ given by $\langle\Omega|\phi(x)|\Omega\rangle$ is non-zero.

$$\langle\Omega|\phi(x)|\Omega\rangle = \langle\phi(x)\rangle = \left.(-i)\frac{\delta Z[J]}{\delta J(x)}\right|_{J\to 0}$$

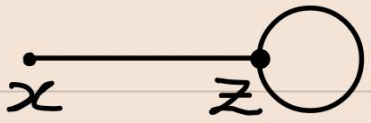
$$= \underbrace{\left.(-i)\frac{\delta Z_0[J]}{\delta J(x)}\right|_{J\to 0}}_{\text{No Contribution}} + \left.(-i)\frac{\delta Z^{(1)}[J]}{\delta J(x)}\right|_{J\to 0} + \left.(-i)\frac{\delta Z^{(2)}[J]}{\delta J(x)}\right|_{J\to 0}$$

$$\frac{\delta Z^{(1)}[J]}{\delta J(x)} = \frac{(-ig)}{2}\int \frac{d^4z}{i^3}\, \Delta_F(z,z)\Delta_F(x,z)$$

$$\Rightarrow (-i)\frac{\delta Z^{(1)}[J]}{\delta J(x)}\Big|_{J\to 0} = \frac{(-ig)}{2}\int d^4z\, \Delta_F(z,z)\Delta_F(x,z)$$

↳ Symmetry factor

$$\frac{(-ig)}{2}\int d^4z\, \Delta_F(z,z)\Delta_F(x,z) =$$



There is no 2nd order contribution to $\langle\phi\rangle$
None of the Even orders contribute to $\langle\phi\rangle$.

Note that the Symmetry factors for ϕ^3 theory is exactly Same as ϕ^4 theory.

$$\Rightarrow \langle\phi(x)\rangle =$$



$$+ \Theta(g^3)$$

$(S=2)$

Only odd order diagrams contribute to $\langle\phi(x)\rangle$

2-point correlation function

$$\langle \phi(x)\phi(y)\rangle = \left.\frac{(-i)^2 \delta^2 Z[J]}{\delta J(x)\,\delta J(y)}\right|_{J\to 0}$$

Upto 2nd order we get

$$Z[J] = Z_0[J] + Z^{(1)}[J] + Z^{(2)}[J]$$

$$\left.\frac{(-i)^2 \delta^n Z_0[J]}{\delta J(x)\,\delta J(y)}\right|_{J\to 0} = \Delta_F(x,y) = \quad x \bullet\!\!\longrightarrow\!\!\bullet y$$

$$\left.\frac{(-i)^2 \delta^n Z^{(1)}[J]}{\delta J(x)\,\delta J(y)}\right|_{J\to 0} = 0 \quad \text{(No contribution)}$$

Now let us find $\left.\frac{(-i)^2 \delta^2 Z^{(2)}[J]}{\delta J(x)\,\delta J(y)}\right|_{J\to 0}$

Observe that only those terms in '$Z^{(2)}[J]$' which are of order 0 & 2 in 'J' will Contribute.

$$Z^{(2)}[J] = \frac{1}{2!}\frac{(-ig)^2}{(3!)^2}\int \frac{d^4x d^4y}{i^6}\; Z_0[J]$$

$$\times \Big| 9\Delta_F(y,y)\Delta_F(x,x)\int d^4w\, J(w)\Delta_F(w,x)\int d^4z\, J(z)\Delta_F(z,y)$$
$$+18\left[\Delta_F(x,y)\right]^2\int d^4w\, J(w)\Delta_F(w,x)\int d^4z\, J(z)\Delta_F(z,y)$$
$$+18\,\Delta_F(x,y)\Delta_F(x,x)\left[\int d^4z\, J(z)\Delta_F(z,y)\right]^2$$
$$-6\Delta_F(x,y)^3 - 9\Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x) + \theta(J^4)\Big|$$

$$\frac{\delta Z^{(2)}[J]}{\delta J(x_1)} = \frac{1}{2!}\frac{(-ig)^2}{(3!)^2}\int \frac{d^4x d^4y}{i^6}\; Z_0[J]$$

$$\times \Big| 18\,\Delta_F(y,y)\Delta_F(x,x)\Delta_F(x_1,x)\int d^4z\, J(z)\Delta_F(z,y)$$
$$+36\left[\Delta_F(x,y)\right]^2\Delta_F(x_1,x)\int d^4z\, J(z)\Delta_F(z,y)$$
$$+36\,\Delta_F(x,y)\Delta_F(x,x)\Delta_F(x_1,y)\int d^4z\, J(z)\Delta_F(z,y)$$
$$+9\Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x)\int d^4z\, J(z)\Delta_F(z,x_1)$$
$$+6\Delta_F(x,y)^3\int d^4z\, J(z)\Delta_F(z,x_1) + \theta(J^3)\Big|$$

$$\left.\frac{\delta}{\delta J(x_2)}\frac{\delta Z^{(2)}[J]}{\delta J(x_1)}\right|_{J\to 0} =$$

$$= \frac{(-ig)^2}{72}\int\frac{d^4x\,d^4y}{i^6}\Big|\begin{array}{l} 18\,\Delta_F(y,y)\Delta_F(x,x)\Delta_F(x_1,x)\Delta_F(x_2,y) \\ +36\left[\Delta_F(x,y)\right]^2\Delta_F(x_1,x)\Delta_F(x_2,y) \\ +36\,\Delta_F(x,y)\Delta_F(x,x)\Delta_F(x_1,y)\Delta_F(x_2,y) \\ +9\Delta_F(y,y)\Delta_F(x,y)\Delta_F(x,x)\Delta_F(x_2,x_1) \\ +6\Delta_F(x,y)^3\Delta_F(x_2,x_1)\end{array}\Big|$$

$$\rightarrow \left.\frac{(-i)^2\delta^2 Z^{(2)}[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J\to 0} \quad \text{is given by,}$$

$$= \left[\frac{(-ig)}{2}\int d^4z\,\Delta_F(z,z)\Delta_F(z,x)\right]\left[\frac{(-ig)^2}{2}\int d^4w\,\Delta_F(w,w)\Delta_F(w,y)\right]$$

$$+\frac{(-ig)^2}{2}\int d^4z\,d^4w\,\Delta_F(x,z)\left[\Delta_F(z,w)\right]^2\Delta_F(w,y)$$

$$+\frac{(-ig)^2}{2}\int d^4z\,d^4w\,\Delta_F(x,w)\Delta_F(z,z)\Delta_F(z,w)\Delta_F(w,y)$$

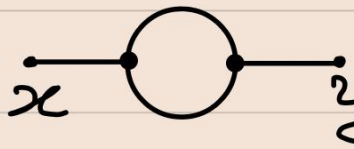
$$+\Delta_F(x,y)\frac{(-ig)^2}{8}\int d^4z\,d^4w\,\Delta_F(w,w)\Delta_F(z,w)\Delta_F(z,z)$$

$$+\Delta_F(x,y)\left[\frac{(-ig)^2}{12}\int d^4z\,d^4w\left[\Delta_F(z,w)\right]^3\right]$$

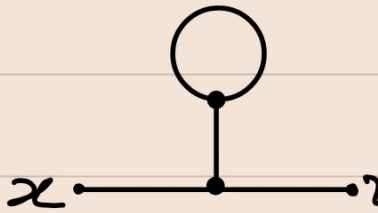
We define the following feynman diagrams



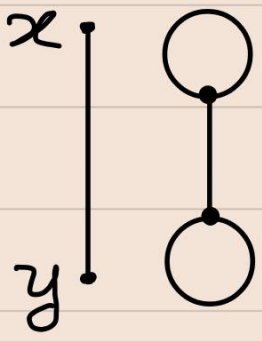
$$= \frac{(-ig)}{2}\int d^4z\, \Delta_F(z,z)\Delta_F(z,x)$$



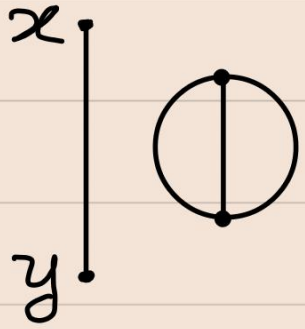
$$= \frac{(-ig)^2}{2}\int d^4z\, d^4w\, \Delta_F(x,z)\left[\Delta_F(z,w)\right]^2 \times \Delta_F(w,y)$$



$$= \frac{(-ig)^2}{2}\int d^4z\, d^4w\, \Delta_F(x,w)\Delta_F(z,z) \times \Delta_F(z,w)\Delta_F(w,y)$$



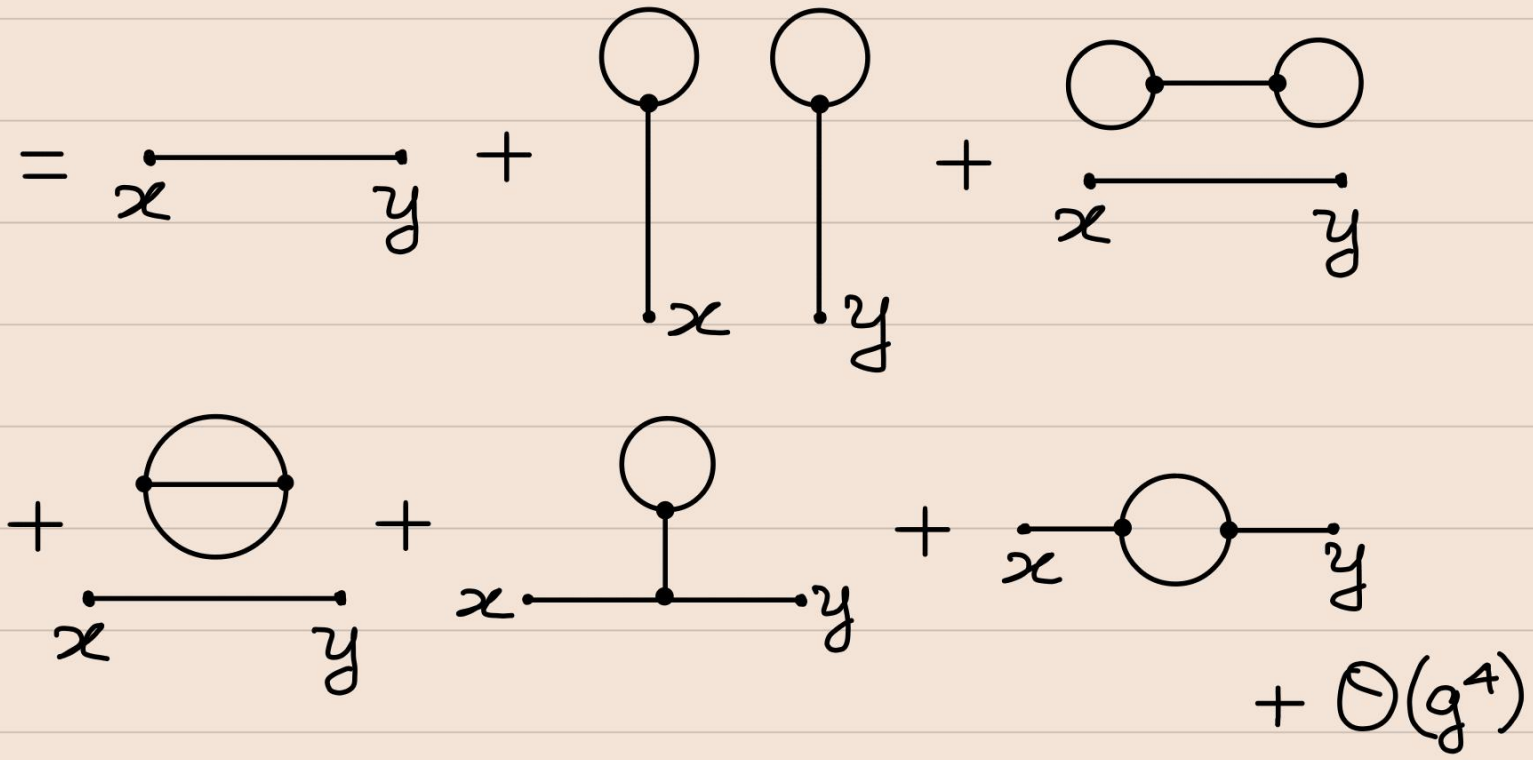
$$= \Delta_F(x,y)\frac{(-ig)^2}{8}\int d^4z\, d^4w\, \Delta_F(w,w) \Delta_F(z,w)\Delta_F(z,z)$$



$$= \Delta_F(x,y)\left[\frac{(-ig)^2}{12}\int d^4z\, d^4w\left[\Delta_F(z,w)\right]^3\right]$$

Thus we can finally write,

$$\langle \phi(x)\phi(y) \rangle = \left. \frac{(-i)^2 \delta^2 Z[J]}{\delta J(x)\, \delta J(y)} \right|_{J \to 0}$$



Note that we get vacuum diagrams here since the normalization for $Z[J]$ is such that $Z[0] \neq 1$. Thus, we should factor out the vacuum diagrams to get diagrams with external legs.

We saw previously that,

$$Z[0] = e^{\sum \left[\begin{array}{c} \text{all connected vacuum} \\ \text{diagrams without} \\ \text{any source} \end{array} \right]}$$

$$\Rightarrow Z[J] = \frac{Z[J]}{Z[0]} = \sum \left(\begin{array}{c} \text{All Connected or} \\ \text{disconnected diagrams} \\ \text{attached to atleast} \\ \text{One Source} \end{array} \right)$$

Such that $Z[0] = 1$

Now, $\ln(Z[J]) = \ln(Z[J]) - \ln(Z[0])$

Thus $Z[0]$'s will not contribute to the functional differentiation w.r.t J's. Hence, the correlations we get from $\ln(Z[J])$ will be free of vacuum bubbles.

Lets define $W[J] = \ln(Z[J])$. And the extract correlations from $W[J]$.

Moreover, as seen in the linked cluster theorem, we will generate only fully connected diagrams by using $W[J]$.

Connected Correlation functions (G)

The correlation function is said to be connected if all the external legs are connected to each other via some path.

$$G(x_1, x_2, \cdots x_n) = \frac{(-i)^n \delta^n W[J]}{\delta J(x_1)\delta J(x_2)\cdots\delta J(x_n)}\Bigg|_{J\to 0}$$

where $W[J] = \ln(Z[J])$

Upto 2nd order, we got,

$$Z[J] = Z_0[J] + Z^{(1)}[J] + Z^{(2)}[J]$$

$$\rightarrow \ln(Z[J]) = \ln(Z_0[J]) + \ln\left[1 + \frac{Z^{(1)}[J]}{Z_0[J]} + \frac{Z^{(2)}[J]}{Z_0[J]}\right]$$

Upto 2nd order,

$$\ln\left[1 + \frac{Z^{(1)}[J]}{Z_0[J]} + \frac{Z^{(2)}[J]}{Z_0[J]}\right] = \frac{Z^{(1)}[J]}{Z_0[J]} + \frac{Z^{(2)}[J]}{Z_0[J]} - \frac{1}{2}\left(\frac{Z^{(1)}[J]}{Z_0[J]}\right)^2$$

$$\rightarrow W[J] = \ln(Z_0[J]) + \frac{Z^{(1)}[J]}{Z_0[J]} + \frac{Z^{(2)}[J]}{Z_0[J]} - \frac{1}{2}\left(\frac{Z^{(1)}[J]}{Z_0[J]}\right)^2$$

$$G(x) = (-i)\frac{\delta W[J]}{\delta J(x)}\bigg|_{J\to 0} = \langle \phi(x) \rangle$$

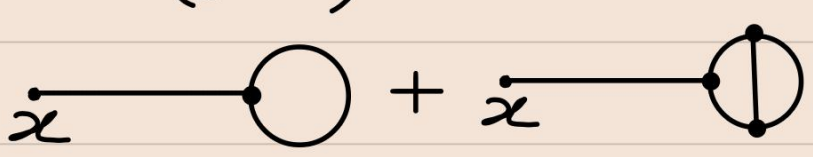
$$\Longrightarrow G(x) =$$



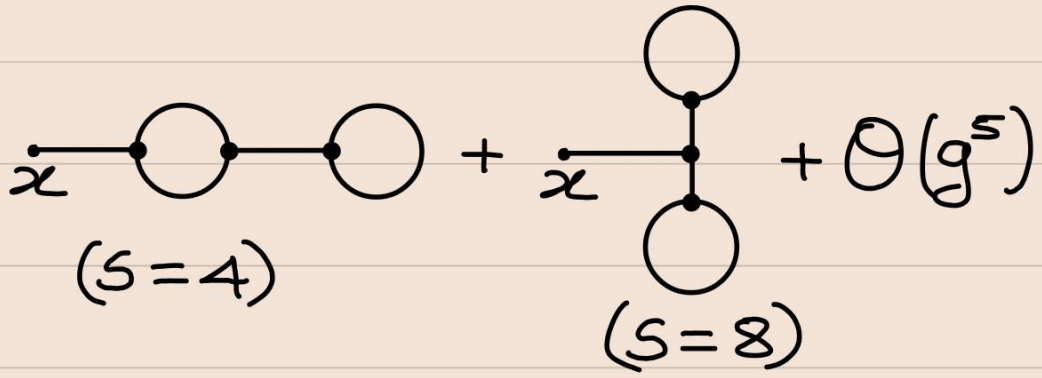
$$+ \mathcal{O}(g^3)$$

Only odd order diagrams contribute to this function. If we include 3rd order, then,

$$G(x) =$$



$$+$$



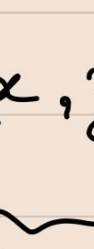
$$+ \mathcal{O}(g^5)$$

$$\rightarrow \boxed{G(x_1) = \sum \left(\begin{array}{c}\text{Sum of all connected} \\ \text{diagrams with one} \\ \text{external leg}\end{array}\right)}$$

Connected 2-point correlation function

It is given by $G(x,y) = \left.\frac{(-i)^2 \delta^2 W[J]}{\delta J(x)\delta J(y)}\right|_{J\to 0}$

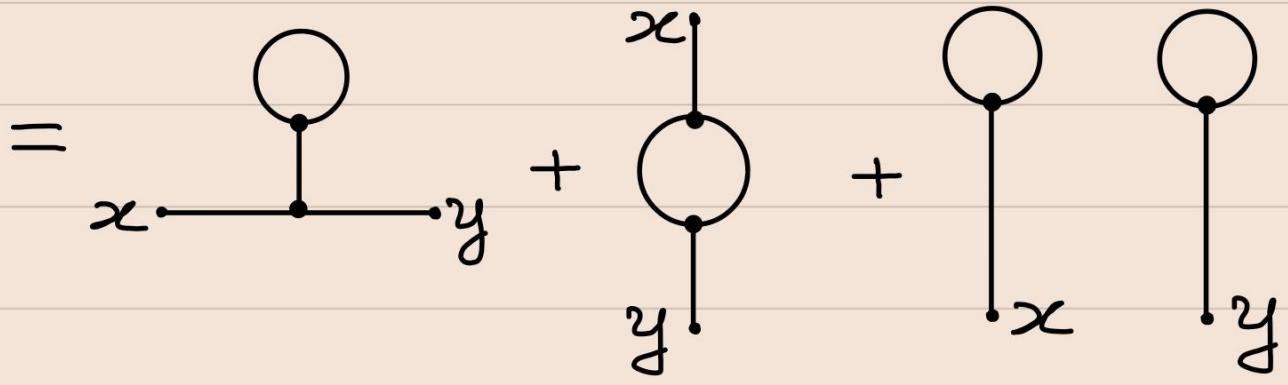
$$\longrightarrow \left.(-i)^2 \frac{\delta^2 \ln(Z_0[J])}{\delta J(x)\delta J(y)}\right|_{J\to 0} = \Delta_F(x,y) = \underbrace{\quad}_{(0^{th}\text{ order})}$$



$$\left.\frac{(-i)^2 \delta^2}{\delta J(x)\delta J(y)}\left(\frac{Z^{(1)}[J]}{Z_0[J]}\right)\right|_{J\to 0} = 0$$

(only even orders will contribute)

$$\left.\frac{(-i)^2 \delta^2}{\delta J(x)\delta J(y)}\left(\frac{Z^{(2)}[J]}{Z_0[J]}\right)\right|_{J\to 0}$$ gives the following

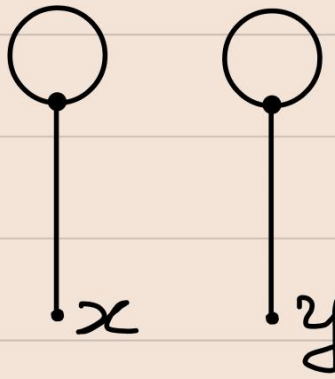


Observe that two vacuum bubble Contributions are removed.

Then finally, $\frac{(-i)^2 \delta^2}{\delta J(x) \delta J(y)} \left(-\frac{1}{2} \left[\frac{Z^{(1)}[J]}{Z_0[J]} \right]^2 \right) \Big|_{J \to 0}$

gives -ve of product of two 1st order one-point diagrams.

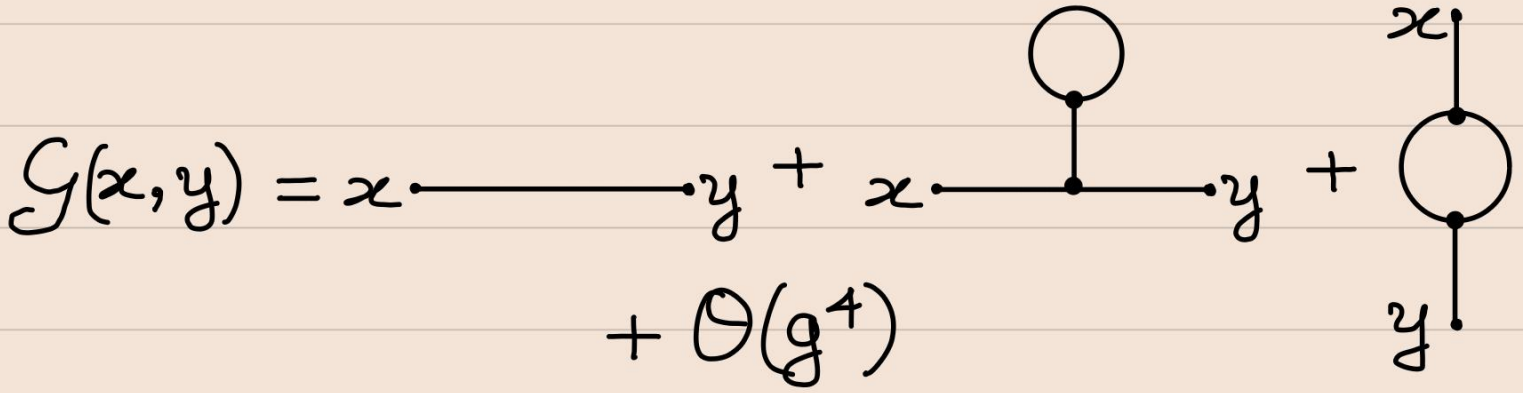
$$\rightarrow \frac{(-i)^2 \delta^2}{\delta J(x) \delta J(y)} \left(-\frac{1}{2} \left[\frac{Z^{(1)}[J]}{Z_0[J]} \right]^2 \right) \Big|_{J \to 0} = -$$



This term exactly cancells with the disconnected part seen previously.

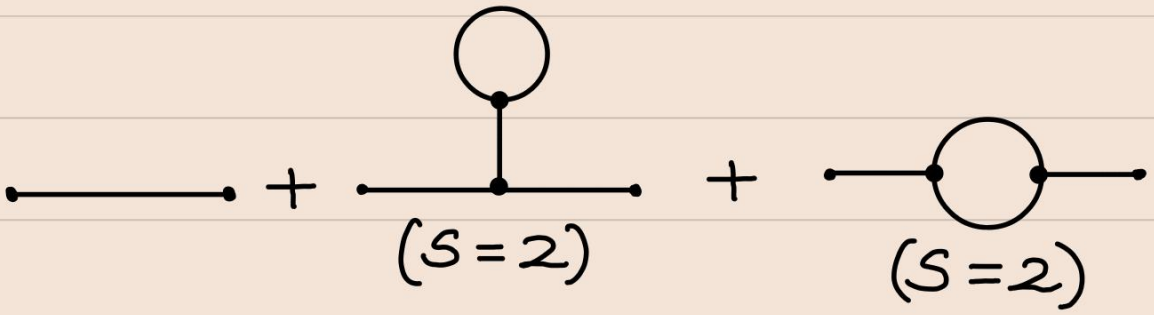
Thus upto 2nd order, we get,

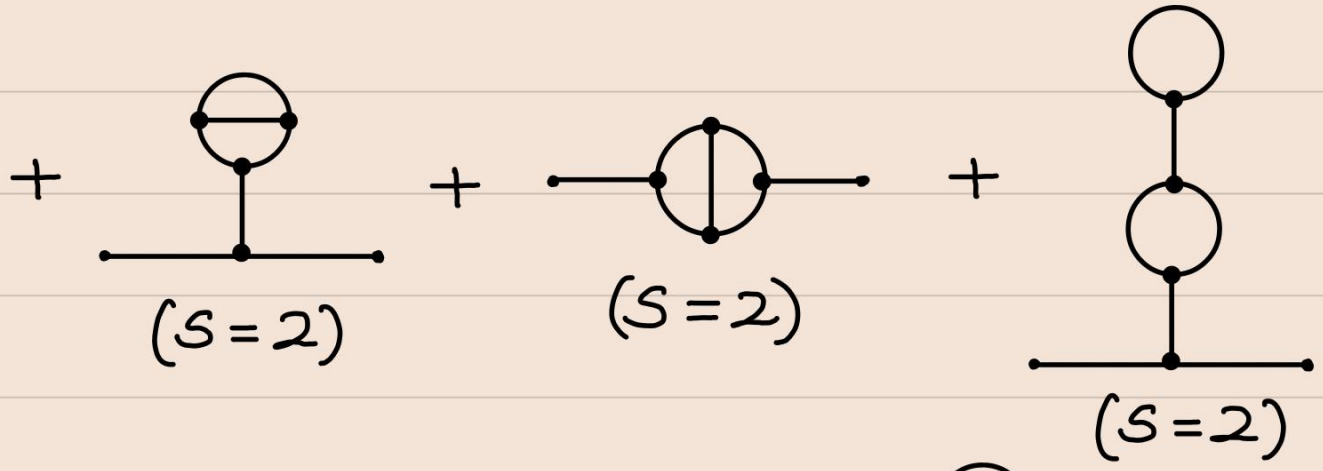
$G(x,y) =$

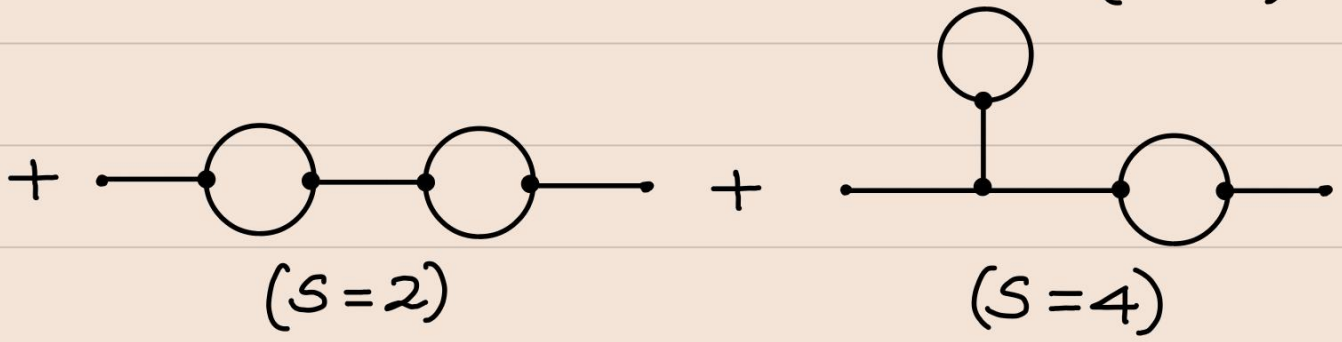


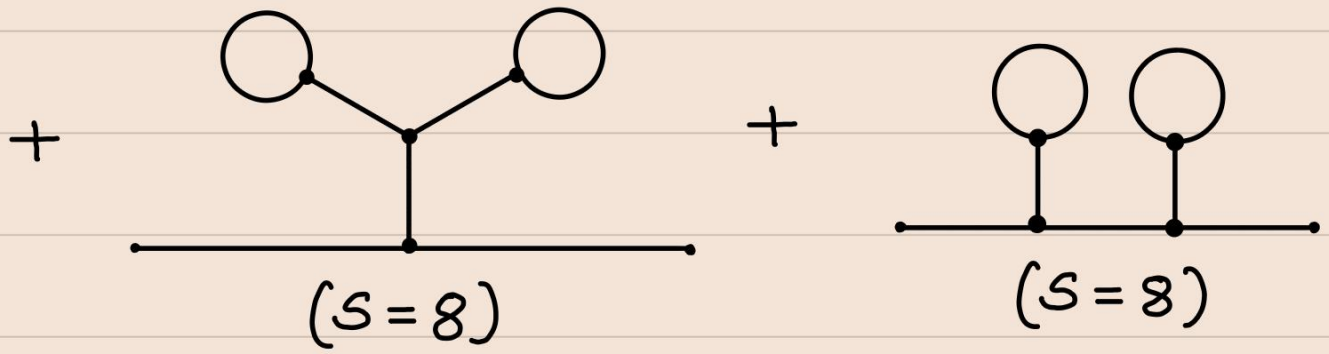
$+ O(g^4)$

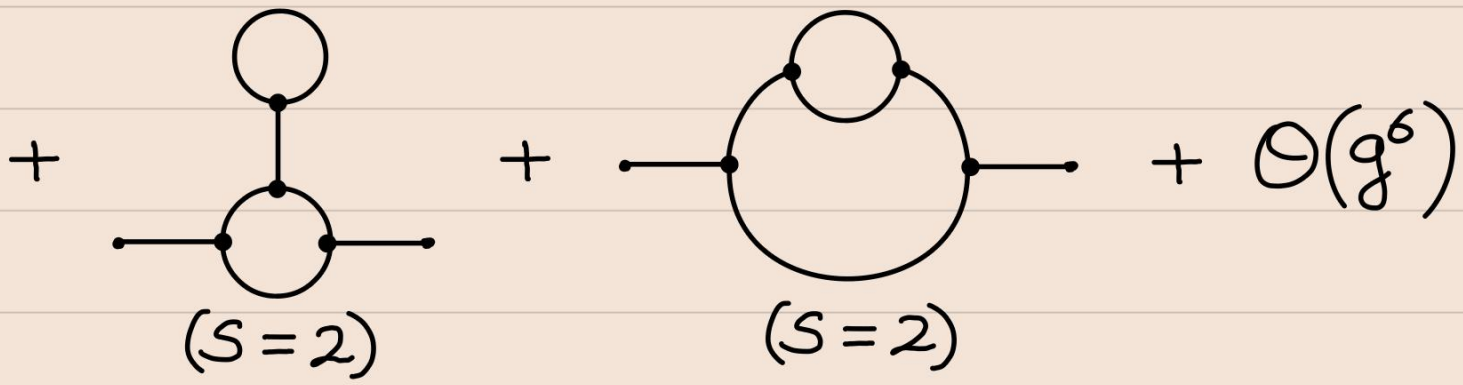
$$\rightarrow \boxed{G(x,y) = \sum \left(\begin{array}{l} \text{Sum of all fully} \\ \text{connected diagrams with} \\ \text{two external legs} \end{array} \right)}$$

$G(x,y) =$ 







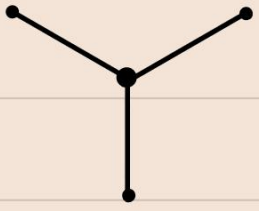
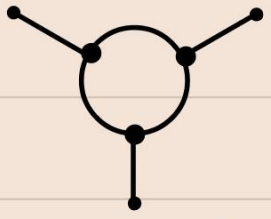
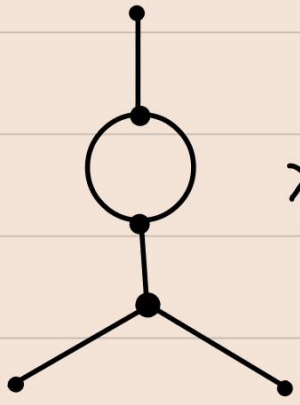


$+ \; \Theta(g^6)$

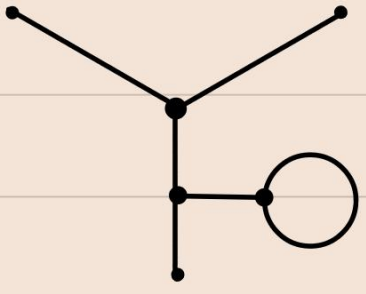
Connected 3-point correlation function

$$G(x,y,z) = \frac{(-i)^3 \delta^3 W[J]}{\delta J(x)\delta J(y)\delta J(z)}\Bigg|_{J\to 0}$$

$$\Rightarrow G(x,y,z) = \sum \left(\begin{array}{c} \text{Sum of all fully} \\ \text{connected diagrams with} \\ \text{three external legs} \end{array} \right)$$

$=$  $+$  $+$  $\times$ 3 perms.

$(S=1)$ $(S=1)$ $(S=2)$

$+$  $\times$ 3 perms. $+ \mathcal{O}(g^5)$

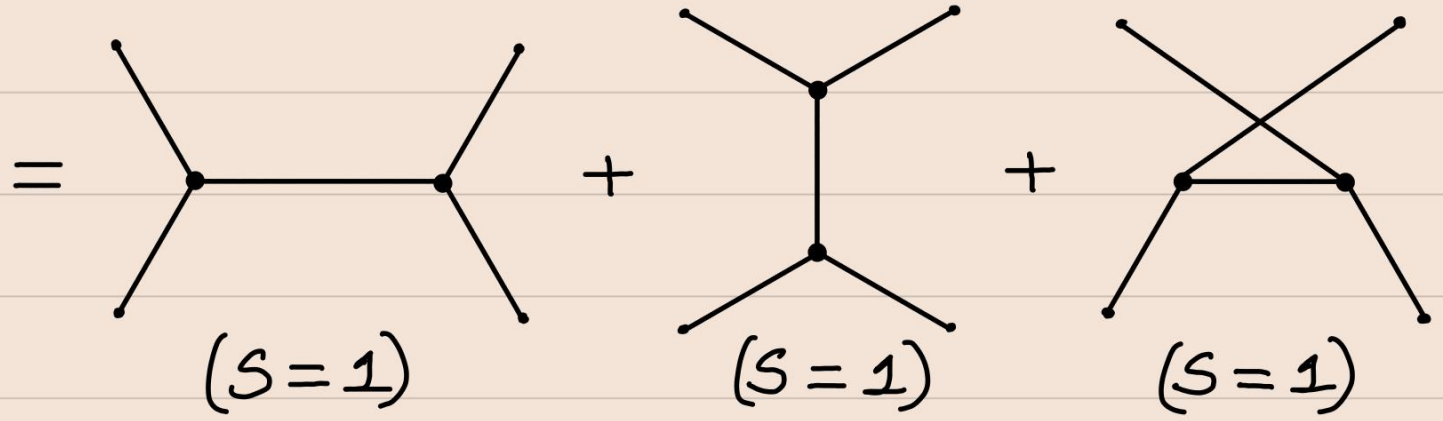
$(S=2)$

Note that only the odd orders will contribute to the 3 point Correlation.

Connected 4-point correlation function

$$G(x,y,z,w) = \left. \frac{(-i)^4 \, \delta^4 W[J]}{\delta J(x)\delta J(y)\delta J(z)\delta J(w)} \right|_{J \to 0}$$

(upto 2nd order)

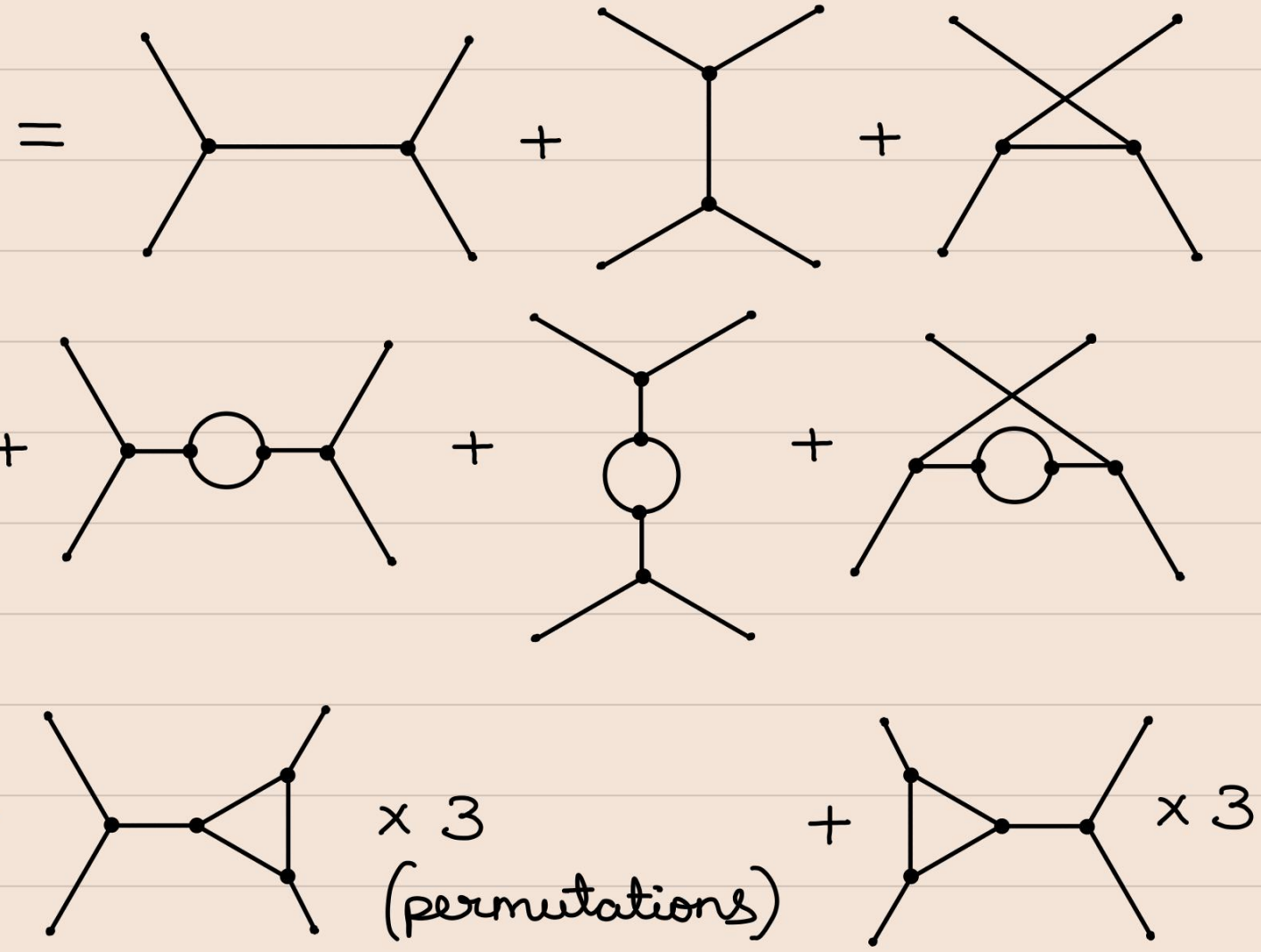


$$\Rightarrow G(x,y,z,w) = \sum \left(\begin{array}{l} \text{Sum of all connected} \\ \text{diagrams with four} \\ \text{external legs} \end{array} \right)$$

Only even orders of '2' & above will contribute.

For any n-point connected correlation of even 'n' only even orders of 'n-2' and above will contribute.

$G(x,y,z,w) =$



$+ \Theta(g^6)$

Now since $W[J]$ gives rise to only the connected diagrams, we can think of $Z[J]$ as a product of exponential of sum of vacuum diagram given by $Z[0]$ and exponential of sum of all fully connected source to source diagrams.

$$\Longrightarrow Z[J] = Z[0]\, e^{\sum \left[\begin{array}{c}\text{fully connected source} \\ \text{to source diagrams}\end{array}\right]}$$

where $Z[0] = e^{\sum \left[\begin{array}{c}\text{all connected vacuum} \\ \text{diagrams without} \\ \text{any source}\end{array}\right]}$

Thus we can define interacting field normalized generating functional $Z[J]$ such that $Z[0] = 1$ as,

$$Z[J] = \frac{Z[J]}{Z[0]} = e^{\sum \left[\begin{array}{c}\text{fully connected source} \\ \text{to source diagrams}\end{array}\right]}$$

[Note: These results are valid for any interacting theory]

Then if $W[J] = \ln(Z[J])$

$$\Rightarrow Z[J] = e^{W[J]}$$

$$\Rightarrow \boxed{W[J] = \sum \left[\begin{array}{l}\text{Sum of all fully Connected} \\ \text{Source to Source and} \\ \text{Connected Vacuum diagrams}\end{array}\right]}$$

Therefore we have demonstrated the <u>linked-Cluster theorem</u> in ϕ^4 theory.

$$Z[J] = \sum \left[\begin{array}{l}\text{Sum of all possible disconnected} \\ + \text{Connected Vacuum \& Source} \\ \text{to Source diagrams}\end{array}\right]$$

Also we showed,

$$Z[J] = \exp\left(\sum \left[\begin{array}{l}\text{Sum of all fully Connected} \\ \text{Source to Source and} \\ \text{Connected Vacuum diagrams}\end{array}\right]\right)$$

$$\boxed{\sum (\text{all diagrams}) = e^{\sum (\text{Connected diagrams})}}$$

Feynman Rules for ϕ^3 theory

The position Space feynman rules

- Every internal vertex is

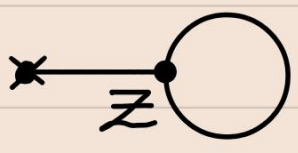


$$= (-ig)\int d^4z$$

- Each propagator b/w two vertices or external legs & vertex is a line

$$\Delta_F(z,\omega) =$$ 

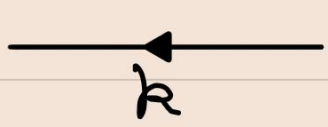
- Every Self loop at a vertex is always connected to either an internal vertex or an external leg. These factors are called as Tadpoles.

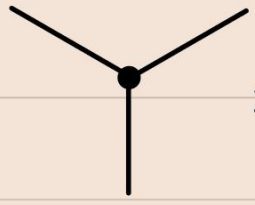


$$\rightarrow (-ig)\int d^4z\, \Delta_F(z,z)\, \Delta_F(z,x)$$

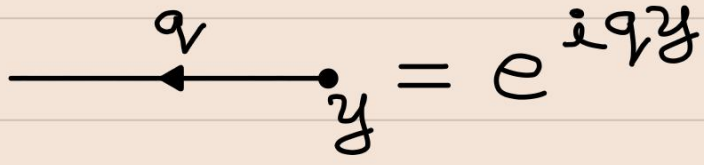
- Divide by the symmetry factor (S) (Same as that of ϕ^4 theory)

The momentum Space feynman rules

- Propagator:  $= \frac{i}{k^2 - m^2 + i\varepsilon}$

- Vertex:  $= -ig$ [Enforce momentum Conservation]

- External points carry a plane wave factor  $= e^{-ipx}$

 $= e^{iqy}$

- Integrate over all undetermined momenta 'k', $\int \frac{d^4k}{(2\pi)^4}$

- Divide by the symmetry factor (S).

- Every vacuum bubble, which is a disconnected piece of a diagram, that is not connected to esternal point will carry a factor of $(2\pi)^4 \delta(0)$