

Frame of Reference

A frame of reference is an observer with a coordinate system which extends over all space and time & can define the position of an event in space & time.

Inertial frame of reference

It is a frame whose observer feels no acceleration. It is in these frames that Newton's laws of motion are valid. An inertial observer experiences zero proper acceleration.

- Eg:-
- Observer in deep space at rest or at uniform motion.
 - An observer in free fall who is experiencing weightlessness.

Einstein's principle of Relativity

Equivalence Principle

- The laws of physics are same in all inertial reference frames.
- All inertial frames are equivalent. There is no absolute reference frame.

Constancy of Speed of light

Speed of light ' c ' as observed by any inertial observer is the same. And it is the ultimate cut-off for velocity of a physical object or information as observed by an inertial observer.

Note: The Speed of light can vary for non-inertial observer if the measurement is non-local.

Lorentz transformations

Observer O' is moving at a constant velocity of ' v ' in +ve x -direction as measured by the observer O . Then the space & time coordinates of two observers are related by the Lorentz transformations as below

$$\underbrace{L(ct, x, y, z)}_{\text{measured by } O} = \underbrace{(ct', y', z', x')}_{\text{measured by } O'}$$

$$ct' = \gamma (ct - \beta x)$$

$$x' = \gamma [x - \beta (ct)]$$

$$y' = y$$

$$z' = z$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\beta = \frac{v}{c}$$

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Lorentz matrix}} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

The above is not a unique way of defining the Lorentz transform.

Lorentz transformation is defined as those transformation that will keep the space-time displacement defined by $(ct)^2 - x^2 - y^2 - z^2$ a constant independent of inertial frames. i.e.,

$$(ct')^2 - (x')^2 - (y')^2 - (z')^2 = (ct)^2 - (x)^2 - (y)^2 - (z)^2$$

This quantity is called $(s)^2$.

$(s)^2 = \text{preserved} \Rightarrow \text{Lorentz transform}$

Four vector

A vector ' x ' is said to be four vector if it has four components (x^0, x^1, x^2, x^3) & has its norm defined by $\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}$ preserved under Lorentz transform.

$$\Rightarrow x' = L(x) \text{ \& norm}(x') = \text{norm}(x)$$

$$(x)^2 = (x^0)^2 - |\vec{x}|^2 = (x^0)^2 - \vec{x} \cdot \vec{x}$$

$$\text{where } |\vec{x}|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

So we get,

$$(x)^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

as the square of the norm of the four vector ' x '.

If $(x)^2 > 0 \Rightarrow$ time-like vector
 $(x)^2 = 0 \Rightarrow$ light-like vector
 $(x)^2 < 0 \Rightarrow$ space like vector

Tensor Notations

A four vector in its component form be expressed as

$$x = x^\mu e_\mu \quad \text{where}$$

$e_\mu \rightarrow$ basis vectors $\mu = (0, 1, 2, 3)$

$e_0 \equiv e_t \Rightarrow$ time direction

$e_i \equiv e_{\vec{x}} \Rightarrow$ space direction ($i = 1, 2, 3$)

$$\Rightarrow \underbrace{x^\mu = (x^0, x^1, x^2, x^3)}$$

Contravariant components

$$\underbrace{x_\mu = (x_0, x_1, x_2, x_3)}$$

Covariant components

$$(x)^2 = x^\mu x_\mu$$

$$= x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3$$

Defining

$$x_0 = x^0, \quad x^1 = -x_1$$

$$x^2 = -x_2, \quad x^3 = -x_3$$

$$\text{we get, } (x)^2 = (x^0)^2 - |\vec{x}|^2.$$

$$\text{Let } x = (x^0, x^1, x^2, x^3) \\ y = (y^0, y^1, y^2, y^3)$$

$$\begin{aligned} \text{then } xy &= x^\mu y_\mu \\ &= x^0 y_0 + x^1 y_1 + x^2 y_2 + x^3 y_3 \\ &= x^0 y^0 - \vec{x} \cdot \vec{y} \\ \vec{x} \cdot \vec{y} &= x^1 y^1 + x^2 y^2 + x^3 y^3 \end{aligned}$$

$\Rightarrow xy$ is the 4-dot product of four vectors x & y .

Minkowski Metric $[\eta_{\mu\nu}]$

$$[\eta_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \eta_{\mu\nu} = \begin{cases} 1 & ; \mu = \nu = 0 \\ -1 & ; \mu = \nu = (1, 2, 3) \\ 0 & ; \mu \neq \nu \end{cases}$$

$$x_\mu = \eta_{\mu\nu} x^\nu$$

$$\Rightarrow x^\mu y_\mu = x^\mu \eta_{\mu\nu} y^\nu$$

Since $x_i = -x^i \Rightarrow x^i = -x_i$
 & $x_0 = x^0 \Rightarrow x^0 = x_0$

we can define $x^\mu = \eta^{\mu\nu} x_\nu$
 where $\eta^{\mu\nu} \equiv \eta_{\mu\nu}$

$$\Rightarrow x^\mu y_\mu = x^\mu \eta_{\mu\nu} y^\nu$$

$$& x^\mu y_\mu = x_\nu \eta^{\nu\mu} y_\mu$$

$$\Rightarrow x^\mu y_\mu = x^\mu \eta_{\mu\nu} \eta^{\nu\sigma} y_\sigma$$

$$\Rightarrow y_\mu = \delta_\mu^\sigma y_\sigma = \eta_{\mu\nu} \eta^{\nu\sigma} y_\sigma$$

$$\Rightarrow \underline{\eta_{\mu\nu} \eta^{\nu\sigma} = \delta_\mu^\sigma}$$

Four displacement

$$\underbrace{dx^\mu}_{\text{It is a 4-vector}} = (dx^0, dx^1, dx^2, dx^3) = (cdt, dx, dy, dz)$$

$$\Rightarrow dx_\mu = (cdt, -dx, -dy, -dz)$$

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$(ds)^2 = dx^\mu \eta_{\mu\nu} dx^\nu = d\bar{x}^\mu \eta_{\mu\nu} d\bar{x}^\nu$$

where $d\bar{x}^\mu \Rightarrow$ Lorentz transformed four displacement

$$d\bar{x}^\mu = \left(\frac{\partial \bar{x}^\mu}{\partial x^\sigma} \right) dx^\sigma$$

$$\Lambda^\mu_\sigma = \frac{\partial \bar{x}^\mu}{\partial x^\sigma} \Rightarrow \text{Lorentz matrix} \left(\begin{array}{l} \text{Jacobian} \\ \text{matrix of} \\ \text{transform} \end{array} \right)$$

$$\Rightarrow \underline{d\bar{x}^\mu = \Lambda^\mu_\nu dx^\nu}$$

Since $d\bar{x}^\mu d\bar{x}_\mu = dx^\alpha dx_\alpha$

$$\Rightarrow d\bar{x}^\mu \eta_{\mu\nu} d\bar{x}^\nu = dx^\alpha \eta_{\alpha\beta} dx^\beta$$

$$\Rightarrow dx^\alpha \Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta dx^\beta = dx^\alpha \eta_{\alpha\beta} dx^\beta$$

$$\Rightarrow \underbrace{\Lambda^\mu_\alpha \eta_{\mu\nu} \Lambda^\nu_\beta}_{= \eta_{\alpha\beta}}$$

Thus if any transformation has a jacobian matrix which satisfies the above condition, then this transformation is said to be lorentz transformation.

Proper time

Consider an observer O with coordinates $(ct, x, y, z) \Rightarrow$ lab frame & an observer O' moving at a speed $\vec{v}$ w.r.t lab frame. The coordinates of $O' : (ct', x', y', z')$

The Space time displacement of the observer O' as seen by the lab frame is dx^μ .
 $dx^\mu = (cdt, dx, dy, dz)$.

Let O' have a clock which is at rest with respect to him.

The Space-time interval of the clock/observer as measured by the observer himself is given by $dt' = d\tau$ &
 $dx' = dy' = dz' = 0$.

$d\tau \Rightarrow$ Proper time interval. It is the time interval as measured by the observer by a clock at rest w.r.t himself.

$$\Rightarrow (ds')^2 = (ds)^2$$

$$\Rightarrow c^2(d\tau)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

$$= c^2(dt)^2 \left(1 - \frac{1}{c^2} \underbrace{\left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right]}_{v^2} \right)$$

$$= c^2(dt)^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow d\tau = dt \sqrt{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \frac{dt}{d\tau} = \gamma \rightarrow \text{length contraction}$$

Four gradient

Let $\varphi \rightarrow$ be a scalar field on a 4D space-time.

i.e. $\varphi \rightarrow \varphi(ct, x, y, z)$

then the four gradient of ' φ ' is given as follows,

$$\partial_\mu \varphi = \left(\frac{\partial \varphi}{\partial x^0}, \frac{\partial \varphi}{\partial x^1}, \frac{\partial \varphi}{\partial x^2}, \frac{\partial \varphi}{\partial x^3} \right)$$

$$\partial_\mu \varphi = \left(\frac{1}{c} \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$

These components act as Covariant components.

Since $\partial_\mu \varphi = \frac{\partial \varphi}{\partial x^\mu}$

under a coordinate change from $x^\mu \longrightarrow \bar{x}^\mu$

$$\bar{x}^\mu = \bar{x}^\mu(x^0, x^1, x^2, \dots)$$

$$\frac{\partial \varphi}{\partial \bar{x}^\mu} = \underbrace{\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial \varphi}{\partial x^\alpha}}_{\text{chain rule.}}$$

$$\Rightarrow \bar{\partial}_\mu \varphi = \left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \right) \partial_\alpha \varphi$$

Inverse of Jacobian matrix

Thus ' $\partial_\mu \varphi$ ' $\Rightarrow$ acts as a covariant vector.

Four Laplacian ($\square$) / Box operator

$$\square = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \underbrace{\nabla^2}_{3\text{-laplacian}}$$

Four velocity (u^μ)

$$\text{Def}^n: - u^\mu = \frac{dx^\mu}{d\tau}$$

$$= \frac{dt}{d\tau} \left(\frac{dx^\mu}{dt} \right)$$

$$\Rightarrow \underline{u^\mu = \gamma(c, v_x, v_y, v_z)}$$

$$u^\mu u_\mu = \gamma^2 (c^2 - v^2) = c^2 //$$

$$\Rightarrow \underline{\underline{\text{norm}(u^\mu) = c}}$$

$$\Rightarrow \text{For an object at rest, } \vec{v} = 0. \Rightarrow u^\mu = (c, 0, 0, 0)$$

Still its four velocity $\neq 0$. It moves along the time direction with speed 'c' in space-time.

Four Momentum (P^μ)

It is defined as $P^\mu = mU^\mu$

$$\Rightarrow P^\mu = m\gamma(c, v_x, v_y, v_z)$$

$$P^\mu P_\mu = m^2 c^2$$

Consider $P^0 = m\gamma c$

$$= mc \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$P^0 = mc \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= mc \left[1 + \frac{1}{2} \left(\frac{v^2}{c^2}\right) + \frac{3}{8} \left(\frac{v^2}{c^2}\right)^2 + \dots \right]$$

$$\Rightarrow cP^0 = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots$$

$$\Rightarrow cP^0 = mc^2 + K.E$$

Higher order
K.E term

$$\Rightarrow cP^0 = E \Rightarrow \text{total energy.}$$

$P^i = \gamma m v^i \Rightarrow$ Relativistic momentum

$$\Rightarrow P^\mu = \left(\frac{E}{c}, \vec{P} \right) \xrightarrow{3 \text{ momentum}}$$

$$\text{So } P^\mu P_\mu = \frac{E^2}{c^2} - |\vec{P}|^2 = m^2 c^2$$

$$\Rightarrow \underline{E^2 = m^2 c^4 + p^2 c^2} \quad \left. \vphantom{E^2 = m^2 c^4 + p^2 c^2} \right\} \text{gapped band because of rest mass.}$$

Einstein's Mass energy Equivalence

$$E = mc^2 + K \cdot E$$

$$E = \gamma mc^2$$

$$E = \sqrt{m^2 c^4 + p^2 c^2}$$

Four frequency (K^μ)

$$K^\mu = \frac{P^\mu}{\hbar} = \left(\frac{\omega}{c}, \vec{k} \right)$$

$$\text{Since, } E = \hbar \omega, \quad \vec{p} = \hbar \vec{k}$$

Fourier transform for a Relativistic field

Let $\phi(x)$ be a relativistic field that depends on x^μ in position space-time $(ct, \vec{x})$.

$$\Rightarrow \mathcal{F}[\phi(x^\mu)] = \tilde{\phi}(k) = \int d^4x e^{-ikx} \phi(x)$$

$$\Rightarrow \mathcal{F}'[\tilde{\phi}(k)] = \phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \tilde{\phi}(k)$$

where $kx = k^0 x^0 - \vec{k} \cdot \vec{x} = \omega t - \vec{k} \cdot \vec{x}$

$$\delta^{(4)}(x - x') = \frac{1}{(2\pi)^4} \int d^4k e^{ik(x - x')}$$
