Functional

Functional is a mathematical object that maps a function to a number.

Example:

a) Average Kinetic energy along a trajectory $\langle T \rangle$.

$$\langle T \rangle = \frac{1}{\tau} \int_0^{\tau} \frac{1}{2} m [\dot{x}(t)]^2 \, dt$$

For $x(t) = A e^{-\omega t}$ (Function)

$$\Rightarrow \dot{x}(t) = -A\omega e^{-\omega t}$$

$$\langle T \rangle = \frac{A^2 \omega m}{4\tau} \left(1 - e^{-2\omega\tau}\right)$$

↳ Number

b) Action is a functional

$$\mathcal{A} = \int_{t_i}^{t_f} \mathcal{L}(x(t), \dot{x}(t), t) \, dt$$

↳ Lagrangian

Classical field

It is a vector or a scalar field which is defined at all points in Space-time. They are deterministic & are continuous in all points of Space - time. They don't incorporate principles of quantum mechanics.

Eg: Electromagnetic field, vector potential, Scalar potential, velocity field, temperature field, etc

We shall denote a classical field by $\phi(x^\mu)$ which could be a Single Scalar field or a component of a vector field.

A functional in classical field theory depends on classical fields & gives out a number.

A classical field is a trivial functional defined as

$$\phi(x^\mu) = \int d^4y \; \phi(y^\mu)\, \delta^4(x^\mu - y^\mu)$$

Action for a classical field is also a functional.

$$\text{Action} = \int d^4x \; \underbrace{\mathcal{L}(\phi, \partial_\mu \phi, x^\mu)}_{\text{Lagrangian density}}$$

Note: In field theory the notion of a single particle will not make any sense. The field dynamics is more concerned about the value field in space-time. So quantities like trajectories given by $\vec{x}(t)$ will not have any meaning. $\{ct, x, y, z\}$ are all in an equal footing now as just the coordinates of a frame.

Functional derivative

Let $F = F[f(x^\mu)]$ be a functional that depends on $f(x^\mu)$ then the functional derivative of F w.r.t 'f' is given by,

$$\frac{\partial F}{\partial f} = \lim_{\varepsilon \to 0} \frac{F[f(x') + \varepsilon \delta(x - x')] - F[f(x)]}{\varepsilon}$$

The functional derivative tells how the output number returned by the functional $F[f(x)]$ changes as the input function $f(x)$ slightly changes.

(a) Consider $F[f(x)] = \int g[f(x)]\, dx$

$$\Rightarrow F[f(x') + \varepsilon \delta(x - x')] = \int g[f(x') + \varepsilon \delta(x - x')]\, dx'$$

$$= \int [g[f(x')] + \varepsilon g'[f(x')]\, \delta(x - x')]\, dx'$$

$$\Rightarrow F[f(x') + \varepsilon \delta(x - x')]$$

$$= F[f(x')] + \varepsilon \int g' f(x') \delta(x - x') dx'$$

$$\Rightarrow F[f(x') + \varepsilon \delta(x - x')] - F[f(x)]$$
$$= \varepsilon g'[f(x)]$$

$$\Rightarrow \frac{\delta F[f(x)]}{\delta f(x)} = g'[f(x)]$$

Note $F[f(x')] = F[f(x)]$

Thus for $F[f(x)] = \int g[f(x)] dx$

$$\Rightarrow \frac{\delta F[f(x)]}{\delta f(x)} = g'[f(x)]$$

Eg: $F[f(x)] = \int [f(x)]^p \, dx$

$$\Rightarrow \frac{\delta F}{\delta f} = p[f(x)]^{p-1}$$

(b) Consider $H[f(x)] = \int g[f'(x)]\,dx$

where $f'(x) = \frac{df}{dx}$

Now, $g\left(\frac{d}{dx}\left[f(x') + \varepsilon\,\delta(x-x')\right]\right)$

$$= g\left[f'(x') + \varepsilon \frac{d}{dx}\delta(x-x')\right]$$

$$= g[f'(x')] + \varepsilon \frac{\partial g}{\partial f'(x)} \frac{d}{dx}\delta(x-x')$$

$$\Rightarrow H[f(x') + \varepsilon\,\delta(x-x')] - H[f(x)]$$

$$= \varepsilon \int dx' \frac{\partial g}{\partial f'(x)} \frac{d}{dx}\delta(x-x')$$

$$\Rightarrow \frac{\delta H}{\delta f} = \int dx' \frac{\partial g}{\partial f'(x)} \frac{d}{dx'}\delta(x-x')$$

(Use integration by parts)

$$= \left[\frac{\partial g}{\partial f'(x)} \delta(x - x') \right]_{x' = \text{lower limit}}^{x' = \text{upper limit}}$$

$$- \int dx' \, \delta(x - x') \frac{d}{dx'} \left[\frac{\partial g}{\partial f'(x)} \right]$$

Usually the functions are taken such that they vanish at boundries. So first term = 0. & also if we assume that lower lim. $< x <$ upper lim.

Thus we get finally

For $H[f(x)] = \int g[f'(x)] dx$

where $f'(x) = \frac{df}{dx}$,

$$\frac{\partial H}{\partial f} = - \frac{d}{dx} \left[\frac{\partial g[f'(x)]}{\partial f'(x)} \right] //$$

Eg: For $H[f(x)] = \int [f'(x)]^p dx$

$$\frac{\partial H}{\partial f} = -\frac{d}{dx}\left[p f'(x)^{p-1}\right]$$

$$= -p(p-1)\left[f'(x)\right]^{p-2} \frac{d^2 f(x)}{dx^2}$$

For $\langle T \rangle = \frac{1}{\tau} \int_0^{\tau} \frac{1}{2} m[\dot{x}(t)]^2 dt$

$$\frac{\partial \langle T \rangle}{\partial x(t)} = \frac{1}{\tau}\left[-m\ddot{x}(t)\right]$$

Note: All these Identities can be generalized to any dimensions & this generalization to 4D Space-time will lead to Euler-Lagrange eqn of motion for fields.

A field $\phi^A(x^\mu)$ can also be treated as a trivial functional

$$\phi^A(x^\mu) = \int d^4\bar{x}\ \phi^A(\bar{x}^\mu)\,\delta^{(4)}(\bar{x}^\mu - x^\mu)$$

then $\dfrac{\delta\phi^{(A)}(x^\mu)}{\delta\phi^{(B)}(y^\mu)}$ is given by

$$\frac{\delta^A_B}{\varepsilon}\left[\int d^4\bar{x}\left[\phi^{(A)}(\bar{x}^\mu) + \varepsilon\,\delta^{(4)}(\bar{x}^\mu - y^\mu)\right]\delta^{(4)}(\bar{x}^\mu - x^\mu) - \int d^4\bar{x}\,\phi^A(\bar{x}^\mu)\,\delta^{(4)}(\bar{x}^\mu - x^\mu)\right]$$

$$\Longrightarrow$$

$$\frac{\delta\phi^{(A)}(x^\mu)}{\delta\phi^{(B)}(y^\mu)} = \delta^A_B \int d^4\bar{x}\ \delta^{(4)}(\bar{x}^\mu - y^\mu)\,\delta^{(4)}(\bar{x}^\mu - x^\mu)$$

$$\Longrightarrow \boxed{\frac{\delta\phi^{(A)}(x^\mu)}{\delta\phi^{(B)}(y^\mu)} = \delta^A_B\ \delta^{(4)}(x^\mu - y^\mu)}$$

We have also introduced δ^A_B as different fields governing the system are independent of each other.

This analysis can also be extended to the functional derivative of $\partial_\nu \phi^A$

$$\partial_\nu \phi^A(x^\mu) = \int d^4\bar{x}\, [\partial_\nu \phi^A(\bar{x}^\mu)]\, \delta^{(4)}(\bar{x}^\mu - x^\mu)$$

$\Rightarrow$

$$\frac{\delta[\partial_\nu \phi^A(x^\mu)]}{\delta[\partial_\sigma \phi^B(y^\mu)]} = \delta^A_B \delta^\sigma_\nu \int d^4\bar{x}\, [\partial_\mu \delta^{(4)}(\bar{x}^\mu - y^\mu)] \times \delta^{(4)}(\bar{x}^\mu - x^\mu)$$

$$\Rightarrow \boxed{\frac{\delta[\partial_\nu \phi^A(x^\mu)]}{\delta[\partial_\sigma \phi^B(y^\mu)]} = \delta^A_B \delta^\sigma_\nu \delta^{(4)}(x^\mu - y^\mu)}$$

Lagrangian formulation

Consider a system described by a set of dynamical fields $\{\phi^A(x^\mu)\}$.

Let $\mathcal{L}[\phi^A(x^\mu), \partial_\mu \phi^A(x^\mu)]$ be the lagrangian density of the system. and lagrangian $L = \int d^3x\, \mathcal{L}$.

The action is given by $\int dt\, L$

$$\Rightarrow S[\phi^A] = \int d^4x\, \mathcal{L}\left[\phi^A(x^\mu), \partial_\mu \phi^A(x^\mu)\right]$$

$$S\left[\phi^A + \varepsilon\, \delta(\bar{x}^\mu - x)\right] =$$

$$= \int d^4\bar{x}\, \mathcal{L}\left[\begin{array}{l}\left[\phi^A(\bar{x}^\mu) + \varepsilon\, \delta^{(A)}(\bar{x}^\mu - x^\mu)\right], \\ \bar{\partial}_\mu\left[\phi^A(\bar{x}^\mu) + \varepsilon\, \delta^{(A)}(\bar{x}^\mu - x^\mu)\right]\end{array}\right]$$

$$= \int d^4\bar{x}\, \mathcal{L}\left[\phi^A(\bar{x}^\mu), \bar{\partial}_\mu \phi^{(A)}(\bar{x}^\mu)\right] \Big\} S[\phi^{(A)}]$$

$$+ \varepsilon \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi^A}\right) \delta^{(4)}(\bar{x}^\mu - x^\mu)$$

$$+ \varepsilon \int d^4x \left[\frac{\partial \mathcal{L}}{\partial(\bar{\partial}_\mu \phi^A)}\right] \varepsilon\, \bar{\partial}_\mu \underbrace{\left[\delta^{(4)}(\bar{x}^\mu - x^\mu)\right]}_{\text{Perform by parts}}$$

The fields at boundry goes to zero.

$$\Longrightarrow S[\phi^A + \varepsilon\,\delta(\bar{x}^\mu - x)] - S[\phi^A]$$

$$= \varepsilon\left(\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\right]\right)$$

$$\Longrightarrow \frac{\delta S}{\delta \phi^A} = \lim_{\varepsilon \to 0} \frac{S[\phi^A + \varepsilon\,\delta(\bar{x}^\mu - x)] - S[\phi^A]}{\varepsilon}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\right]$$

By variational principle, the fields in the System are such that $\frac{\delta S}{\delta \phi^A} = 0$. Thus the fields obey the eq^n $\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\right] = 0$

Note that the above derivation works even if $\mathcal{L}$ explicitly depends on x^μ.

Eg: Consider $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$

$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$

$m \rightarrow$ reciprocal of compton length of K.G field mass term $\left(m \rightarrow \frac{mc}{\hbar}\right)$

& $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = (\partial^\mu \phi)$

EoM is given by $\partial_\mu(\partial^\mu \phi) + m^2 \phi = 0$

$\Rightarrow \underline{(\Box + m^2)\phi = 0}$

$\hookrightarrow$ Klein-Gordan eq^n.

This eq^n in the non-relativistic limit ($c \rightarrow \infty$) reduces to Schrodinger eq^n for a free particle where

$\phi(x^\mu) = e^{-i\omega t}\psi(\vec{x})$, $E = \hbar\omega = mc^2$

$\psi(\vec{x})$ is the wave function for the particle.

Hamiltonian formulation

Consider a lagrangian density $\mathcal{L}$ where

$$\mathcal{L} \to \mathcal{L}\left(\phi^A(x^\mu), \partial_\mu \phi^A(x^\nu), x^\mu\right)$$

where $\{\phi^A\}$ are set of classical fields governing the system.

The conjugate momentum field is defined by $\Pi^0_A = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A}$ where $\dot{\phi}^A = \frac{\partial \phi^{(A)}}{\partial x^0}$

time derivative

Thus Π^0_A is a functional derivative of the lagrangian (not lagrangian density) w.r.t the time derivative of ϕ^A.

Then the hamiltonian density is given by $\mathcal{H}(\phi^A, \nabla\phi^A, \Pi^0_A)$ as a legendre transformation of $\mathcal{L}$.

$$\mathcal{H}(\phi^A, \nabla\phi^A, \Pi_A^0) = \Pi_A^0 \dot{\phi}^A - \mathcal{L}(\phi^A, \partial_\mu \phi^A)$$

Then the total hamiltonian is defined as

$$H = \int d^3x \, \mathcal{H}(\phi^A, \nabla\phi^A, \Pi_A^0)$$

$$\delta H = \int d^3x \left[\left(\frac{\partial \mathcal{H}}{\partial \phi^A}\right) \delta\phi^A + \left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right) \cdot \delta(\vec{\nabla}\phi^A) + \left(\frac{\partial \mathcal{H}}{\partial \Pi_A^0}\right) \delta\Pi_A^0 \right]$$

$$= \int d^3x \left[\left(\frac{\partial \mathcal{H}}{\partial \phi^A}\right) \delta\phi^A + \left(\frac{\partial \mathcal{H}}{\partial \Pi_A^0}\right) \delta\Pi_A^0 \right] + \int d^3x \left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right) \cdot \delta(\vec{\nabla}\phi^A)$$

Consider the last term, we can use $\delta(\vec{\nabla}\phi^A) = \vec{\nabla}(\delta\phi^A)$

$$\Rightarrow \int d^3x \left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right) \cdot \delta(\vec{\nabla}\phi^A) = \int d^3x \left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right) \cdot \vec{\nabla}(\delta\phi^A)$$

we can use integration by parts & also if the fields are such that they vanish at $\pm\infty$, then,

$$\int d^3x \left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right) \cdot \delta(\vec{\nabla}\phi^A) = \int d^3x \left[-\vec{\nabla}\cdot\left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right)\right] \delta\phi^A$$

$$\Rightarrow \delta H = \int d^3x \left[\left(\frac{\partial \mathcal{H}}{\partial \phi^A}\right) - \vec{\nabla}\cdot\left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right)\right] \delta\phi^A + \int d^3x \left(\frac{\partial \mathcal{H}}{\partial \pi^\circ_A}\right) \delta\pi^\circ_A$$

Since ϕ^A & π_A are independent of each other, we get,

$$\Rightarrow \frac{\delta H}{\delta \phi^A} = \left(\frac{\partial \mathcal{H}}{\partial \phi^A}\right) - \vec{\nabla}\cdot\left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right)$$

$$\& \quad \frac{\delta H}{\delta \pi^\circ_A} = \left(\frac{\partial \mathcal{H}}{\partial \pi^\circ_A}\right)$$

The usual Hamiltonian eqⁿ of motions are given now as the functional derivatives of H as

$$\dot{\phi}^A = \frac{\partial H}{\partial \Pi_A^0} \quad \& \quad \dot{\Pi}_A^0 = -\frac{\partial H}{\partial \phi^A}$$

Thus we have,

$$\boxed{\dot{\phi}^A = \left(\frac{\partial \mathcal{H}}{\partial \Pi_A^0}\right)}$$

$$\& \quad \boxed{\dot{\Pi}_A^0 = \vec{\nabla}\cdot\left(\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi^A)}\right) - \frac{\partial \mathcal{H}}{\partial \phi^A}}$$

The above eqⁿ are Hamiltonian field equations.

Poission Brackets

Consider two functionals F & G_1 which are given by

$$F = \int d^3x \, \mathcal{F}(\phi^A, \vec{\nabla}\phi^A, \pi^0_A)$$

& $$G_1 = \int d^3x \, \mathcal{G}(\phi^A, \vec{\nabla}\phi^A, \pi^0_A)$$

where $\mathcal{F}$ & $\mathcal{G}$ are the respective functional densities.

The poission bracket b/w F & G_1 is then given by,

$$\{F, G_1\}_P = \int d^3x \left[\underbrace{\left(\frac{\partial F}{\partial \phi^A}\right)\left(\frac{\partial G_1}{\partial \pi^0_A}\right) - \left(\frac{\partial F}{\partial \pi^0_A}\right)\left(\frac{\partial G_1}{\partial \phi^A}\right)} \right]$$

all evaluated at same time 't'

Then we can find the fundamental poission bracket relations.

$$\{\phi^A(t,\vec{x}), \phi^B(t,\vec{y})\}_p = 0$$

Since $\frac{\partial\phi}{\partial\pi}$ terms are zero.

Also $\{\pi^\circ_A(t,\vec{x}), \pi^\circ_B(t,\vec{y})\}_p = 0$

as $\frac{\partial\pi}{\partial\phi}$ terms are zero.

Then finally we can find,

$\{\phi^A(t,\vec{x}), \pi^\circ_B(t,\vec{y})\}_p$ as follows

$$\int d^3z \left(\frac{\partial\phi^A(t,\vec{x})}{\partial\phi^C(t,\vec{z})}\right)\left(\frac{\partial\pi^\circ_B(t,\vec{y})}{\partial\pi^\circ_C(t,\vec{z})}\right)$$

(other term vanishes)

Using $\left(\frac{\partial\phi^A(t,\vec{x})}{\partial\phi^C(t,\vec{z})}\right) = \delta^A_C\,\delta^{(3)}(\vec{x}-\vec{z})$

& $\left(\frac{\partial\pi^\circ_B(t,\vec{y})}{\partial\pi^\circ_C(t,\vec{z})}\right) = \delta^C_B\,\delta^{(3)}(\vec{y}-\vec{z})$

$$\Rightarrow \{\phi^A(t,\vec{x}), \Pi^0_B(t,\vec{y})\}_P$$

$$= \int d^3z \; \delta^A_C \; \delta^{(3)}(\vec{x}-\vec{z}) \; \delta^C_B \; \delta^{(3)}(\vec{y}-\vec{z})$$

$$\Rightarrow \boxed{\{\phi^A(t,\vec{x}), \Pi^0_B(t,\vec{y})\}_P = \delta^A_B \; \delta^{(3)}(\vec{x}-\vec{y})}$$

Thus the fields ϕ^A & Π^0_A (conjugate momentum field) mimics the classical generalized coordinates & their conjugate momentum in both Hamilton eqⁿ of motion & the fundamental poission brackets.

Note: Poission brackets are used to describe the dynamics of fields & their interactions and the evolution of these fields over time. They are evaluated at same time because it captures the instantaneous relationships b/w different spatial points of field.

Hamiltonian density for Scalar Klein - Gordon field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

$$\rightarrow \mathcal{L} = \frac{1}{2}\left[\dot{\phi}^2 - (\vec{\nabla}\phi)^2\right] - \frac{1}{2}m^2\phi^2$$

$$\Pi^0 = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$$

$$\mathcal{H} = \Pi^0\dot{\phi} - \mathcal{L}$$

$$\Rightarrow \mathcal{H} = \frac{1}{2}(\Pi^0)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2$$

The Hamiltonian eqⁿ of motion are:

$$\dot{\phi} = \Pi^0 \quad \& \quad \dot{\Pi}^0 = \vec{\nabla}\cdot(\vec{\nabla}\phi) - m^2\phi$$

$$\Rightarrow \underline{\dot{\Pi}^0 = \nabla^2\phi - m^2\phi}$$

or $\partial_0^2\phi - \nabla^2\phi + m^2\phi = 0$

$$\Rightarrow (\partial_\mu\partial^\mu + m^2)\phi = 0 \Rightarrow \text{K.G. eqⁿ.}$$

Noether's Theorem

It states that for every continuous symmetry, there exists an associated conserved quantity & a conservation law.

Consider a system of fields ϕ^A & their derivatives $\partial_\mu \phi^A$. They are governed by a lagrangian $\mathcal{L}$ according to the equation,

$$\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \right) = 0$$

Let there be a small change in the field given by,

$$\phi \rightarrow \overline{\phi} = \phi + \underbrace{\delta\phi}_{\text{arbitrarily small}}$$

$\delta\phi$ is such that it vanishes at the boundries.

Then the change in lagrangian density is given by,

$$\delta\mathcal{L} = \mathcal{L}\left[\phi^A + \delta\phi^A, \partial_\mu(\phi^A + \delta\phi^A)\right] - \mathcal{L}\left[\phi^A, \partial_\mu\phi^A\right]$$

$$\Rightarrow \delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\phi^A}\right)\delta\phi^A + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\right)\delta(\partial_\mu\phi^A)$$

Introducing Π^μ as the four momentum density by,

$$\Pi^\mu_A = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}$$

where Π^0_A = Conjugate momentum field

& using $\delta(\partial_\mu\phi^A) = \partial_\mu(\delta\phi^A)$, we get,

$$\delta\mathcal{L} = \left(\frac{\partial\mathcal{L}}{\partial\phi^A}\right)\delta\phi^A + \Pi^\mu_A\,\partial_\mu(\delta\phi^A)$$

Also $\Pi^\mu_A\,\partial_\mu(\delta\phi^A) = \partial_\mu\left(\Pi^\mu_A\,\delta\phi^A\right) - \left(\partial_\mu\Pi^\mu_A\right)\delta\phi^A$

$$\Rightarrow \delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \Pi^\mu_A\right)\delta\phi^A + \partial_\mu\left(\Pi^\mu_A \delta\phi^A\right)$$

But since ϕ^A satisfy Lagrangian eqⁿ of motion, $\frac{\partial \mathcal{L}}{\partial \phi^A} - \partial_\mu \Pi^\mu_A = 0$

$$\Rightarrow \delta\mathcal{L} = \partial_\mu\left(\Pi^\mu_A \delta\phi^A\right)$$

$\Rightarrow$ The change in action because of this transformation is given by,

$$\delta S = \int_\Omega d^4x\, \delta\mathcal{L}$$

$\Omega \rightarrow$ volume

If we allow the change in lagrangian by a four-divergence of some vector K^μ,

i.e. if $\delta\mathcal{L} = \partial_\mu K^\mu$, then,

$$\Rightarrow \delta S = \int_\Omega d^4x \left(\partial_\mu K^\mu\right)$$

Using 4-D Gauss divergence theorem we get, $\delta S = \int_{\partial\Omega} (dA)\, n_\mu K^\mu$

$\partial\Omega \rightarrow$ boundry

$n_\mu \rightarrow$ surface normal vector

Since ϕ^A is constant at boundry surface $\Rightarrow (\delta\phi^A)_{surface} = 0$.

$\Rightarrow S'_{new} = S_{old} + \text{Some Constant}$

Thus if S_{old} is extremized with some conditions, then S'_{new} is also extremized with same conditions.

Thus we keep the physics invariant under the transformation Provided $\delta\mathcal{L} = \partial_\mu K^\mu$. This is the underlying symmetry of the system. This should lead to a conserved quantity.

But we already got $\delta\mathcal{L} = \partial_\mu(\Pi^\mu_A \delta\phi^A)$

$$\Rightarrow \quad \partial_\mu \left(\Pi^\mu_A \delta\phi^A \right) = \partial_\mu K^\mu$$

$$\Rightarrow \quad \partial_\mu \left(\Pi^\mu_A \delta\phi^A - K^\mu \right) = 0$$

If $J^\mu_N = \Pi^\mu_A \delta\phi^A - K^\mu \Rightarrow \partial_\mu J^\mu_N = 0$

Thus there exists a conserved 4-current J^μ_N called as the Noether's current, which is locally conserved.

Statement of Noether's theorem

"If a continuous symmetry transform $\phi^A \rightarrow \phi^A + \delta\phi^A$ only changes $\mathcal{L}$ by the addition of a 4-divergence $(\partial_\mu K^\mu)$, there by doesn't change the physics, then this implies the existence of a 4-current $J^\mu_N = \Pi^\mu_A \delta\phi^A - K^\mu$ & if ϕ^A obey's lagrange E.o.M, then the 4-current J^μ_N is conserved, i.e. $\partial_\mu J^\mu_N = 0 \rightarrow$ Conservation law"

Conserved Noether's charge

Consider $Q_N(t) = \int_{\Omega} d^3x \; J_N^0(t, \vec{x})$

↳ time component at a time 't'

$$\Rightarrow \frac{dQ_N(t)}{dt} = \int_{\Omega} d^3x \left(\partial_0 J_N^0\right)$$

But $\partial_0 J_N^0 = -\vec{\nabla}\cdot\vec{J}$

$$\Rightarrow \frac{dQ_N}{dt} = -\int_{\Omega} d^3x \left(\vec{\nabla}\cdot\vec{J}\right)$$

$$= -\int_{d\Omega} d\vec{A}\cdot\vec{J}$$

If $\vec{J} = 0$ at Surface, i.e. there is no flux of charges moving in or out of Surface, then, $\frac{dQ_N}{dt} = 0 \Rightarrow Q_N$ is a conserved quantity.

Space-time translations & Energy-Momentum Tensor ($\Theta^{\mu\nu}$)

Consider infinitesimal global space-time translation given by,

$$x^\mu \longrightarrow \bar{x}^\mu = x^\mu + \varepsilon^\mu$$

$$\rightarrow \quad \delta x^\mu = \bar{x}^\mu - x^\mu = \varepsilon^\mu$$

$$\Rightarrow \quad \phi^A(x^\mu) \longrightarrow \phi'^A(\bar{x}^\mu) = \phi^A(x^\mu + \varepsilon^\mu)$$

$$\phi'^A(\bar{x}^\mu) = \phi^A(x^\mu) + \varepsilon^\nu \left[\partial_\nu \phi^A(x^\mu)\right]$$

$$\rightarrow \delta\phi^A = \varepsilon^\nu (\partial_\nu \phi^A)$$

$$\mathcal{L}\left[\phi^A(x^\mu), \partial_\sigma \phi^A(x^\mu)\right] \longrightarrow \mathcal{L} + \delta\mathcal{L}$$

where $$\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi^A}\right)(\delta\phi^A) + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\right]\delta(\partial_\mu \phi^A)$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \phi^A}\right)\varepsilon^\nu(\partial_\nu \phi^A) + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\right]\partial_\mu\left[\varepsilon^\nu(\partial_\nu \phi^A)\right]$$

$$\Rightarrow \delta\mathcal{L} = \varepsilon^\nu \left[\left(\frac{\partial \mathcal{L}}{\partial \phi^A} \right) (\partial_\nu \phi^A) + \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \right] (\partial_\mu \partial_\nu \phi^A) \right]$$

Using $\Pi_A^\mu = \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^A)} \right]$

& also requiring $\delta\mathcal{L} = \partial_\mu K^\mu$ we get

$$\partial_\nu K^\nu = \varepsilon^\nu \left[\frac{\partial \mathcal{L}}{\partial \phi^A} (\partial_\nu \phi^A) + \Pi_A^\mu (\partial_\mu \partial_\nu \phi^A) \right]$$

Now $\partial_\nu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} (\partial_\nu \phi^A) + \Pi_A^\mu \partial_\nu (\partial_\mu \phi^A)$

$$\Rightarrow \partial_\nu K^\nu = \varepsilon^\nu \partial_\nu \mathcal{L} = \partial_\nu (\varepsilon^\nu \mathcal{L})$$

$$\Rightarrow \underline{K^\nu = \varepsilon^\nu \mathcal{L}}$$

Thus $J_N^\mu = \Pi_A^\mu (\delta\phi^A) - K^\mu$

$$= \Pi_A^\mu \varepsilon^\nu (\partial_\nu \phi^A) - \varepsilon^\mu \mathcal{L}$$

$$= \varepsilon_\nu [\Pi^\mu (\partial^\nu \phi^A) - \eta^{\mu\nu} \mathcal{L}]$$

$$\Rightarrow J_N^\mu = \varepsilon_\nu \left[\Pi_A^\mu \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L} \right]$$

Defining $\Theta^{\mu\nu} = \Pi_A^{\mu} \partial^{\nu} \phi^A - \eta^{\mu\nu} \mathcal{L}$ as the energy Momentum Tensor, we get

$$J_N^{\mu} = \varepsilon_{\nu} \Theta^{\mu\nu}$$

Since $\partial_{\mu} J_N^{\mu} = 0 \Rightarrow \underline{\partial_{\mu} (\varepsilon_{\nu} \Theta^{\mu\nu}) = 0}$

Energy-Momentum tensor is a quantity that describes the density & the flux of energy & momentum in Space time.

$$\underline{\Theta^{\mu\nu} = \Pi_A^{\mu} \partial^{\nu} \phi^A - \eta^{\mu\nu} \mathcal{L}}$$

For $\mu = \nu = 0$

$$\Theta^{00} = \Pi_A^{0} (\partial^0 \phi^A) - \eta^{00} \mathcal{L}$$

$$\partial^0 \phi^A = \partial_t \phi^A = \dot{\phi}^A \quad , \eta^{00} = 1$$

$$\Rightarrow \Theta^{00} = \Pi_A^{0} \dot{\phi}^A - \mathcal{L} = \mathcal{H} \Rightarrow \text{Hamiltonian density}$$

Thus Θ^{00} is the Hamiltonian density of the System.

For $\underline{\mu = 0}$, $\nu = i$ $(i = 1,2,3)$

$$\Theta^{0i} = \Pi_A^0 \partial^i \phi^A - \eta^{0i} \mathcal{L} \qquad (\eta^{0i} = 0)$$

$$\Rightarrow \Theta^{0i} = \Pi_A^0 \partial^i \phi^A = \text{Spatial component of relativistic momentum density}$$

$$\Rightarrow \vec{p} = -\int d^3x \, \Pi_A^0 (\vec{\nabla} \phi^A) \rightarrow \text{Canonical Momenta}$$

This implies that $\Theta^{0\nu}$ is the four-momentum density.

$$\Rightarrow P^\nu = \int d^3x \, \Theta^{0\nu}$$

P^ν ↓ four momenta

Consider the poission bracket of P^ν with $\phi^A(x)$

$$\{P^\nu, \phi^A(x)\}_P =$$

$$= \int d^3z \left[\left(\frac{\partial P^\nu}{\partial \phi^B(z)} \right) \left(\frac{\partial \phi^A(x)}{\partial \Pi^0_B(z)} \right) - \left(\frac{\partial \phi^A(x)}{\partial \phi^B(z)} \right) \left(\frac{\partial P^\nu}{\partial \Pi^0_B(z)} \right) \right]_{\text{at } z}$$

$$\frac{\partial P^\nu}{\partial \phi^B(z)} = \int d^3x \, \frac{\partial \left[\Pi^0_c \, \partial^\nu \phi^c - \eta^{0\nu} \mathcal{L} \right]}{\partial \phi^B(z)}$$

$$= \int d^3x \left\{ \underbrace{\frac{\partial \left[\Pi^0_c \partial^\nu \phi^c \right]}{\partial \phi^B(z)}}_{0} - \eta^{0\nu} \frac{\partial \mathcal{L}}{\partial \phi^B(z)} \right\}$$

$$\Rightarrow \frac{\partial P^\nu}{\partial \phi^B(z)} = -\eta^{0\nu} \int \partial^3 x \, \frac{\partial \mathcal{L}}{\partial \phi^B(z)}$$

$$\Rightarrow \frac{\partial P^\nu}{\partial \phi^B(z)} = \frac{\partial \mathcal{L}}{\partial \phi^B(z)}$$

$$\frac{\delta \phi^A(x)}{\delta \Pi_B^0(z)} = 0$$

$$\frac{\delta \phi^A(x)}{\delta \phi^B(z)} = \delta_B^A \delta^{(3)}(\vec{x} - \vec{z})$$

$$\frac{\delta P^\nu}{\delta \Pi_B^0(z)} = \partial^\nu \phi^B(z)$$

(note: $\frac{\partial \mathcal{L}}{\partial \Pi_B^0} = 0$; $\mathcal{L}$ doesn't depend on Π_B^μ)

$$\Rightarrow \{P^\nu, \phi^A(x)\}_P = -\int d^3z \, \delta_B^A \delta^3(\vec{x} - \vec{z}) \partial^\nu \phi^B(z)$$

$$\Rightarrow \underline{\{P^\nu, \phi^A(x)\}_P = -\partial^\nu \phi^A(x)}$$

Thus P^ν acts as the generator for infinitesimal translation of space-time.

Conservation of Energy-momentum Tensor

$$\Theta^{\mu\nu} = \Pi_A^{\ \mu}\,\partial^\nu\phi^A - \eta^{\mu\nu}\mathcal{L}$$

$$\partial_\mu \Theta^{\mu\nu} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\,\partial^\nu\phi^A\right] - \eta^{\mu\nu}\partial_\mu\mathcal{L}$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\right]\partial^\nu\phi^A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\left(\partial_\mu\partial^\nu\phi^A\right)$$

$$- \eta^{\mu\nu}\left[\left(\frac{\partial\mathcal{L}}{\partial\phi^A}\right)\left(\partial_\mu\phi^A\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi^A)}\left(\partial_\mu\partial_\sigma\phi^A\right)\right]$$

$$= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\right]\partial^\nu\phi^A + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\left(\partial_\mu\partial^\nu\phi^A\right)$$

$$- \frac{\partial\mathcal{L}}{\partial\phi^A}\left(\partial^\nu\phi^A\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\sigma\phi^A)}\left(\partial^\nu\partial_\sigma\phi^A\right)$$

$$= \left[\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^A)}\right] - \left(\frac{\partial\mathcal{L}}{\partial\phi^A}\right)\right]\partial^\nu\phi^A = 0$$

$$\underbrace{\qquad\qquad}_{0 \text{ (as } \phi^A \text{ satisfies EoM)}}$$

Thus we proved $\partial_\mu \Theta^{\mu\nu} = 0$

So, if 'μ' is taken as the component of noether current, then, $\mu = 0$ will correspond to conserved charge.

$\Rightarrow$ The corresponding Noether charge is given by

$$P^\nu(t) = \int_\Omega d^3x \; \Theta^{0\nu}$$

(at an instant of time 't')

$$\frac{1}{c}\frac{dP^\nu(t)}{dt} = \int_\Omega d^3x \; (\partial_0 \Theta^{0\nu})$$

$$= \int_\Omega d^3x \; (-\partial_i \Theta^{i\nu})$$

$$\Rightarrow \frac{dP^\nu}{c\,dt} = -\int_{\partial\Omega} d\mathcal{A} \; \Theta^{i\nu}$$

at surface

$\hookrightarrow$ Surface boundry in 3D

Thus if Θ^{iv} at boundry surface is zero, i.e. no flux of any component of four momentum "ν", then,

$$\Rightarrow \frac{dP^{\nu}}{dt} = 0 \quad (\text{for all } \nu)$$

Thus total energy & the momentum are conserved.

$$P^{0} = E/c \Rightarrow \frac{dE}{dt} = 0$$

$P^{i} = \vec{p}$ (canonical momenta) is also a conserved quantity

$$\Rightarrow \frac{d\vec{p}}{dt} = 0.$$

Θ^{ij} $(i,j = 1,2,3)$ give the stress tensor (pressure & sheer stress)

$\Theta^{\mu 0}$ gives the Energy flux densities

$\Theta^{0\nu}$ gives the four momentum density

Symmetrization of Energy-Momentum Tensor

Generally the canonical Energy-momentum tensor is not Symmetric.

i.e., $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$

But if $\Theta^{\mu\nu}$ & $\Theta^{\nu\mu}$ describe the same physical observable then we expect $\int d^3x \left(\Theta^{\mu\nu} - \Theta^{\nu\mu}\right)$ to vanish

i.e. If $\Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\sigma H^{\sigma\mu\nu}$

(four-divergence of some tensor field)

Then $\int d^3x \left(\Theta^{\mu\nu} - \Theta^{\nu\mu}\right)$ is given by the volume integral of four divergence of some tensor field components which by stokes theorem is proportional to surface integral of the components which can be taken to be zero.

To construct a generalized, and a symmetric energy momentum tensor $T^{\mu\nu}$ such that $T^{\mu\nu} = T^{\nu\mu}$ we need $\Theta^{\mu\nu}$ such that, there exists a tensor $H^{\sigma\mu\nu}$, where,

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = -\partial_\sigma H^{\sigma\mu\nu}$$

& if $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma B^{\sigma\mu\nu}$

where $B^{\sigma\mu\nu}$ is such that it is anti-symmetric with respect to its first two indices, $B^{\sigma\mu\nu} = -B^{\mu\sigma\nu}$

then $T^{\mu\nu}$ is symmetric, provided

$$B^{\sigma\mu\nu} = \frac{1}{2}\left(H^{\sigma\mu\nu} + H^{\mu\nu\sigma} - H^{\nu\sigma\mu}\right)$$

(By definition, $B^{\sigma\mu\nu} = -B^{\mu\sigma\nu}$)

& $B^{\sigma\mu\nu} - B^{\sigma\nu\mu} = H^{\sigma\mu\nu}$

Thus we can show that.

$$T^{\mu\nu} - T^{\nu\mu} = \Theta^{\mu\nu} + \partial_\sigma B^{\sigma\mu\nu} - \Theta^{\nu\mu} - \partial_\sigma B^{\sigma\nu\mu}$$

$$= \Theta^{\mu\nu} - \Theta^{\nu\mu} + \partial_\sigma \left[B^{\sigma\mu\nu} - B^{\sigma\nu\mu} \right]$$

$$= -\partial_\sigma H^{\sigma\mu\nu} + \partial_\sigma H^{\sigma\mu\nu} = 0 \,//$$

The physical observable represented by tensors T & Θ are the same using the similar argument as before where $\int_{\Omega} d^3x \, \partial_\sigma B^{\sigma\mu\nu} = 0$.

The tensor field $B^{\sigma\mu\nu}$ is called as Bilinfante tensor. And $T^{\mu\nu}$ is the generalized Symmetric Energy-momentum tensor, where as $\Theta^{\mu\nu}$ is the canonical momentum tensor.

For a given system, we can calculate,

$$\Theta^{\mu\nu} = \Pi_A^{\ \mu} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}$$

$$\Theta^{\nu\mu} = \Pi_A^{\ \nu} \partial^\mu \phi^A - \eta^{\nu\mu} \mathcal{L}$$

Since $\eta^{\mu\nu} = \eta^{\nu\mu}$ we have

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = \Pi_A^{\ \mu} \partial^\nu \phi^A - \Pi_A^{\ \nu} \partial^\mu \phi^A$$

$$\Rightarrow -\partial_\sigma H^{\sigma\mu\nu} = \Pi_A^{\ \mu} \partial^\nu \phi^A - \Pi_A^{\ \nu} \partial^\mu \phi^A$$

Given $H^{\sigma\mu\nu}$, we construct a Belinfante tensor

$$B^{\sigma\mu\nu} = \frac{1}{2}\left(H^{\sigma\mu\nu} + H^{\mu\nu\sigma} - H^{\nu\sigma\mu}\right)$$

where $B^{\sigma\mu\nu} - B^{\sigma\nu\mu} = H^{\sigma\mu\nu}$

$\Rightarrow T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma B^{\sigma\mu\nu}$ is a symmetric tensor.

Eg 1 : Klein Gordon field (Scalar field)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$$

Here $\Pi^\mu = \partial^\mu\phi$

& Eom $\rightarrow$ $(\partial^\mu\partial_\mu + m^2)\phi = 0$

$$\Theta^{\mu\nu} = \Pi^\mu_A \partial^\nu\phi^A - \eta^{\mu\nu}\mathcal{L}$$

$$\Theta^{\mu\nu} = (\partial^\mu\phi)(\partial^\nu\phi) - \frac{\eta^{\mu\nu}}{2}\left[(\partial_\circ\phi)(\partial^\circ\phi) - m^2\phi^2\right]$$

$$\Theta^{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 = \mathcal{H}$$

$$\Theta^{0\nu} = \dot{\phi}(\partial^\nu\phi) - \eta^{0\nu}\mathcal{L}$$

$$\Rightarrow \Theta^{0i} = \dot{\phi}(\partial^i\phi)$$

$$\vec{P} = \int d^3x \qquad {}^{0i} = -\int d^3x \; (\partial_\circ\phi)(\vec{\nabla}\phi)$$

$$\underline{\Theta^{0i} = \Theta^{i0}}$$

$$\Theta^{ij} = (\partial^i\phi)(\partial^j\phi) - \eta^{ij}\mathcal{L}$$

$$\Theta^{ij} = (\partial_i\phi)(\partial_j\phi) + \delta_{ij}\mathcal{L}$$

$$\Theta^{ii} = (\partial_i\phi)^2 + \frac{1}{2}\left[\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\right]$$

$$\Theta^{11} = \frac{1}{2}\left[\dot{\phi}^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 - m^2\phi^2\right]$$

$$\Theta^{22} = \frac{1}{2}\left[\dot{\phi}^2 - (\partial_1\phi)^2 - (\partial_3\phi)^2 - m^2\phi^2\right]$$

$$\Theta^{33} = \frac{1}{2}\left[\dot{\phi}^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 - m^2\phi^2\right]$$

$$\Theta^{ij}_{i\neq j} = (\partial_i\phi)(\partial_j\phi)$$

Z_2 Symmetry of K.G. field

$\mathcal{L}$ is invariant under $\phi \rightarrow -\phi$
This is called as Z_2 Symmetry.

Eg 2: Complex Klein-Gordon field

The lagrangian density is given by, $\mathcal{L} = (\partial_\mu \psi)(\partial^\mu \psi^+) - m^2 \psi \psi^+$

The eqⁿ of motion are given by,

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right] = 0 \quad \& \quad \frac{\partial \mathcal{L}}{\partial \psi^+} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^+)}\right] = 0$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$-m^2 \psi^+ - \partial_\mu \partial^\mu \psi^+ = 0 \;, \quad -m^2 \psi - \partial_\mu \partial^\mu \psi = 0$$

Thus we see that both ψ & ψ^+ behave individually as K.G fields which satisfy

$$(\partial_\mu \partial^\mu + m^2)\psi = 0 \quad \& \quad (\partial_\mu \partial^\mu + m^2)\psi^+ = 0$$

$$\Pi^\mu_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \partial^\mu \psi^+$$

$$\& \quad \Pi^\mu_{\psi^+} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^+)} = \partial^\mu \psi$$

Thus the corresponding momentum density field is given by $\Pi^\mu = \partial^\mu \psi^+$ & its complex conjugate $\Pi^{\mu+} = \partial^\mu \psi$

The hamiltonian density is given by

$$\mathcal{H} = \Pi_A^\circ \dot{\phi}^A - \mathcal{L}$$

$$= \Pi^\circ \dot{\psi} + (\Pi^\circ)^+ \dot{\psi}^+ - \mathcal{L}$$

$$= \dot{\psi}^+ \dot{\psi} + \dot{\psi}(\dot{\psi}^+) - \dot{\psi}\psi^+ + (\vec{\nabla}\psi).(\vec{\nabla}\psi^+) + m^2 \psi \psi^+$$

$$\rightarrow \underline{\mathcal{H} = \dot{\psi}^+ \dot{\psi} + (\vec{\nabla}\psi^+).(\vec{\nabla}\psi) + m^2 \psi^+ \psi}$$

Energy-Momentum tensor is given by,

$$\Theta^{\mu\nu} = \Pi_A^\mu (\partial^\nu \phi^A) - \eta^{\mu\nu} \mathcal{L}$$

$$= \Pi^\mu (\partial^\nu \psi) + (\Pi^\mu)^+ (\partial^\nu \psi^+) - \eta^{\mu\nu} \left[(\partial_\sigma \psi)(\partial^\sigma \psi^+) - m^2 \psi \psi^+ \right]$$

$$\Theta^{\mu\nu} = (\partial^\mu \psi^+)(\partial^\nu \psi) + (\partial^\mu \psi)(\partial^\nu \psi^+) - \eta^{\mu\nu} \left[(\partial_\sigma \psi)(\partial^\sigma \psi^+) - m^2 \psi \psi^+ \right]$$

Symmetries of Complex Scalar field

a) $\mathbb{Z}_2$ - Symmetry

Consider the lagrangian

$$\mathcal{L} = (\partial_\mu \psi)(\partial^\mu \psi^+) - m^2 \psi \psi^+$$

$\mathcal{L}$ is invariant under $\psi \to -\psi$

This is called as $\mathbb{Z}_2$ Symmetry.

b) U(1) global Symmetry

But it also has a Superior Symmetry of U(1) Symmetry.

i.e., $\mathcal{L}$ is invariant under the U(1) transformation given by

$$\psi \longrightarrow e^{i\alpha} \psi \Rightarrow \psi^+ \longrightarrow e^{-i\alpha} \psi^+$$

provided α is a constant phase.

So $\partial_\mu (e^{i\alpha} \psi) = e^{i\alpha} \partial_\mu \psi$ which keeps $\mathcal{L}$ invariant.

This is a global transformation.

To get the conserved current & charge for U(1) Symmetry, we consider an infinitesimal 'α' where $e^{i\alpha} \sim 1 + i\alpha$

$$\Rightarrow \quad \psi \to \bar{\psi} = \psi + i\psi\alpha$$

$$\Rightarrow \quad \delta\psi = i\alpha\psi$$

$$\& \quad \bar{\psi}^+ = \psi^+ - i\psi^+\alpha$$

$$\delta\bar{\psi}^+ = -i\alpha\bar{\psi}^+$$

Since $\delta\mathcal{L} = 0$, $(\partial_\mu K^\mu = 0)$

$$\Rightarrow J_N^\mu = \Pi_A^\mu (\delta\phi^A) \qquad (\phi^A = \{\psi, \psi^+\})$$

$$= (\partial^\mu\psi^+)(i\alpha\psi) + \partial^\mu\psi(-i\alpha\psi^+)$$

$$\Rightarrow \underline{J_N^\mu = i\alpha\left[\psi(\partial^\mu\psi^+) - \psi^+(\partial^\mu\psi)\right]}$$

$$\& \quad Q_N = i\alpha\int d^3x\,(\psi\dot{\psi}^+ - \psi^+\dot{\psi})$$

↙ Nother charge

c) Gauge transformation & Local U(1) Symmetry

If $\psi \rightarrow e^{i\alpha(x^\mu)}\psi$ then the usual lagrangian given by where $\alpha(x)$ is a real field.

$(\partial^\mu \psi^+)(\partial_\mu \psi) - m^2 \psi^+ \psi$ is not invariant Since,

$$\partial_\mu\left[e^{i\alpha(x^\mu)}\psi\right] = e^{i\alpha(x^\mu)}\left[(\partial_\mu\psi) + i\psi[\partial_\mu\alpha(x)]\right]$$

$$\partial^\mu\left[e^{-i\alpha(x^\mu)}\psi^+\right] = e^{-i\alpha(x^\mu)}\left[(\partial^\mu\psi^+) - i\psi^+(\partial^\mu\alpha(x))\right]$$

$$\Rightarrow \partial^\mu\left[e^{-i\alpha(x^\mu)}\psi^+\right]\partial_\mu\left[e^{i\alpha(x^\mu)}\psi\right]$$

$$= (\partial^\mu\psi^+)(\partial_\mu\psi) + (\partial_\mu\alpha(x))(\partial^\mu\alpha(x))\psi^+\psi$$

$$+ i\,\partial_\mu\alpha(x)\left[\psi(\partial^\mu\psi^+) - \psi^+(\partial^\mu\psi)\right]$$

If we neglect the 2nd order term $(\partial_\mu\alpha(x))(\partial^\mu\alpha(x))\psi^+\psi$, Still $\mathcal{L}$ varies.

$$\Rightarrow \delta\mathcal{L} = i\partial_\mu\alpha(x)\left[\psi(\partial^\mu\psi^+) - \psi^+(\partial^\mu\psi)\right]$$

This change is mainly because,

$$\partial_\mu\psi \longrightarrow e^{i\alpha(x^\mu)}\left[\partial_\mu + i\partial_\mu\alpha(x)\right]\psi$$

Thus to preserve the form of the lagrangian if we had initially started with the lagrangian given by

$$\mathcal{L} = (D^\mu\psi)^\dagger(D_\mu\psi) - m^2\psi^+\psi$$

where $D_\mu = \partial_\mu + \frac{iqA_\mu(x)}{\hbar}$

then under local $U(1)$ transform

$\psi \to e^{i\alpha(x)}\psi$ the new lagrangian will remain invariant provided the field $A_\mu(x)$ transforms by

$A_\mu \longrightarrow A_\mu - \frac{\hbar}{q}\partial_\mu\alpha(x) \to$ <u>Gauge transform</u>

along with $\psi \to e^{i\alpha(x)}\psi \to U(1)$ transform

So $D_\mu\psi = \left[\partial_\mu + \frac{iqV}{\hbar}A_\mu(x)\right]\psi$

$\downarrow U(1)$ + gauge transform

$$\left(\left[\partial_\mu + i\partial_\mu\alpha(x)\right] + \frac{iq}{\hbar}\left[A_\mu(x) - \frac{\hbar}{q}\partial_\mu\alpha(x)\right]\right)\psi$$

$$= \left[\partial_\mu + i\partial_\mu\alpha(x) + \frac{iqV}{\hbar}A_\mu(x) - i\partial_\mu\alpha(x)\right]\psi$$

$$= \left[\partial_\mu + \frac{iqV}{\hbar}A_\mu(x)\right]\psi = (D_\mu\psi)$$

$$\Rightarrow D_\mu\psi \xrightarrow[\text{gauge.}]{U(1)} (D_\mu\psi)$$

Thus the theory described by

$$\mathcal{L} = (D^\mu\psi)^\dagger(D_\mu\psi) - m^2\psi^\dagger\psi$$

is invariant under local $U(1)$ & gauge transformation.

Note: A theory which has a field $A^\mu(x)$, introduced to produce an invariance with respect to local transformations is called gauge Theory. & the field A^μ is called as the gauge field.

Covariant Formulation of ED

Electromagnetic theory or Electrodyna-mics is is one of the simplest gauge theory which is built on the 4-vector potential (A^μ) formulation.

The observable fields are the Electric field ($\vec{E}$) & magnetic field($\vec{B}$) which have three spatial components

$$\vec{E}(x^\mu) = [E_x(x^\mu), E_y(x^\mu), E_z(x^\mu)] = E^i$$

& $$\vec{B}(x^\mu) = [B_x(x^\mu), B_y(x^\mu), B_z(x^\mu)] = B^i$$

In the usual electro-magnetic theory these fields are derived by the scalar potential (ϕ) & the vector potential ($\vec{A}$) which are functions of (x^μ), $\phi \to \phi(x^\mu)$

$$\vec{A} \longrightarrow \vec{A}(x^\mu) = [A_x(x^\mu), A_y(x^\mu), A_z(x^\mu)] = A^i$$

where $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ & $\vec{B} = \vec{\nabla} \times \vec{A}$

$$E^i = -\partial_i \phi - \frac{\partial A^i}{\partial t}, \quad B^i = \varepsilon^{ij}{}_k \partial_j A^k$$

or $$\frac{E^i}{c} = -\partial_i (\phi/c) - \frac{\partial A^i}{\partial (ct)}$$

& $$B^1 = \partial_2 A^3 - \partial_3 A^2$$
$$B^2 = \partial_3 A^1 - \partial_1 A^3$$
$$B^3 = \partial_1 A^2 - \partial_2 A^1$$

If we define a 4-vector potential as $A^\mu = (A^0, A^1, A^2, A^3) = (\phi/c, A_x, A_y, A_z)$

$$\Rightarrow A_\mu = (A_0, A_1, A_2, A_3) = (\phi/c, -A_x, -A_y, -A_z)$$

where $A_\mu = \eta_{\mu\nu} A^\nu \Rightarrow A^0 = A_0$
& $A^i = -A_i$

Thus $$\frac{E^i}{c} = -\partial_i A^0 - \partial_0 A^i = -\partial_i A_0 + \partial_0 A_i$$

$$\Rightarrow \left(\frac{E^i}{c}\right) = \partial_0 A_i - \partial_i A_0$$

$$E_x/c = \partial_0 A_1 - \partial_1 A_0$$
$$E_y/c = \partial_0 A_2 - \partial_2 A_0$$
$$E_z/c = \partial_0 A_3 - \partial_3 A_0$$

$$B_x = \partial_2 A^3 - \partial_3 A^2 = \partial_3 A_2 - \partial_2 A_3$$
$$B_y = \partial_3 A^1 - \partial_1 A^3 = \partial_1 A_3 - \partial_3 A_1$$
$$B_z = \partial_1 A^2 - \partial_2 A^1 = \partial_2 A_1 - \partial_1 A_2$$

Thus if a structure of the form "$\partial_\mu A_\nu - \partial_\nu A_\mu$" is defined, then $\vec{E}$ & $\vec{B}$ are the elements of this structure.

Electromagnetic Stress tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$$

$$\Rightarrow F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

& $F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$= \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

Dual Electromagnetic stress tensor

By definition of $\vec{E}$ & $\vec{B}$ we can derive two eqⁿ,

Since $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

& Since $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

These eqⁿ can be derived from the electromagnetic stress tensor or its dual.

By defⁿ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Since $\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha = 0$ (commutative)

$$\Rightarrow \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0$$

This is called as the Jacobi identity of EM tensor.

For $\sigma = 0$ & $\mu, \nu = \{1, 2, 3\}$

$$\partial_0 F_{ij} + \partial_j F_{0i} + \partial_i F_{j0} = 0$$

$$F_{0i} = \frac{E^i}{c} \qquad F_{j0} = -\frac{E^j}{c}$$

$$\Rightarrow \quad \partial_0 F_{ij} = \frac{(\partial_i E^j - \partial_j E^i)}{c}$$

$$\Rightarrow \frac{1}{c} \frac{\partial}{\partial t} (\partial_i A_j - \partial_j A_i) = \frac{(\partial_i E^j - \partial_j E^i)}{c}$$

$$\Rightarrow -\frac{\partial}{\partial t} (\partial_i A^j - \partial_j A^i) = (\partial_i E^j - \partial_j E^i)$$

$$B^k = \varepsilon_{ijk} \partial_i A^j$$

$$(\vec{\nabla} \times \vec{E})^k = \varepsilon_{ijk} \partial_i E^j$$

$$\Rightarrow -\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \vec{E} \quad //$$

For $(\sigma, \mu, \nu) = (1,2,3)$

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0$$

$$\rightarrow \partial_1 F_{23} + \partial_3 F_{12} + \partial_2 F_{31} = 0$$

$$\Rightarrow \quad \partial_x(-B_x) + \partial_z(-B_z) + \partial_2(-B_y) = 0$$

$$\Rightarrow \quad \underline{\vec{\nabla} \cdot \vec{B} = 0}$$

Thus the Jacobi identity given by

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0$$

incorporates the Homogenous Maxwell equations in it. We can derive Electromagnetic dual tensor from this identity by multiplying it by $\frac{\varepsilon^{\alpha\sigma\mu\nu}}{3!}$ throughout where

$\varepsilon^{0123} = 1$ $\rightarrow$ 4D - completely anti-Symmetric tensor

$$\Rightarrow \frac{\varepsilon^{\alpha\sigma\mu\nu}}{3!}\left(\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma}\right) = 0$$

$$\Rightarrow \frac{1}{3!}\Big[\varepsilon^{\alpha\sigma\mu\nu}\partial_\sigma F_{\mu\nu} + \overbrace{\varepsilon^{\alpha\sigma\mu\nu}\partial_\nu F_{\sigma\mu}}^{\substack{\nu\to\sigma\\ \sigma\to\mu\\ \mu\to\nu}} + \underbrace{\varepsilon^{\alpha\sigma\mu\nu}\partial_\mu F_{\nu\sigma}}_{\substack{\mu\to\sigma\\ \nu\to\mu\\ \sigma\to\nu}}\Big] = 0$$

$$\Rightarrow \frac{\partial_\sigma F_{\mu\nu}}{3!}\left(\varepsilon^{\alpha\sigma\mu\nu} + \underbrace{\varepsilon^{\alpha\mu\nu\sigma}}_{\varepsilon^{\alpha\sigma\mu\nu}} + \underbrace{\varepsilon^{\alpha\nu\sigma\mu}}_{\varepsilon^{\alpha\sigma\mu\nu}}\right) = 0$$

$$\Rightarrow \partial_\sigma\left[\frac{\varepsilon^{\alpha\sigma\mu\nu}}{2} F_{\mu\nu}\right] = 0$$

Defining $G^{\alpha\sigma} = \frac{1}{2}\varepsilon^{\alpha\sigma\mu\nu}F_{\mu\nu}$ as the dual EM tensor, we get

$$\partial_\sigma G^{\alpha\sigma} = 0$$

& $G^{\alpha\sigma} = -G^{\sigma\alpha}$

$$G^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ B_y & -\frac{E_z}{c} & 0 & \frac{E_x}{c} \\ B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{bmatrix}$$

The eqn $\partial_\mu G^{\mu\nu} = 0$ give the Homogenous maxwell equation

for $\nu = 0 \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$

for $\nu = \{1,2,3\} \Rightarrow \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$$G_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta} G^{\alpha\beta}$$

$$G_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ -B_y & -\frac{E_z}{c} & 0 & E_x/c \\ -B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{bmatrix}$$

Properties of EM tensors (F & G)

(i) $F_{\mu\nu}F^{\mu\nu} = 2\left(|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2}\right)$

(ii) $F_{\mu\nu}G^{\mu\nu} = F^{\mu\nu}G_{\mu\nu} = -\frac{4}{c}\left(\vec{E}\cdot\vec{B}\right)$

(iii) $G_{\mu\nu}G^{\mu\nu} = 2\left(\frac{|\vec{E}|^2}{c^2} - |\vec{B}|^2\right)$

(iv) $\eta_{\mu\nu}F^{\mu\nu} = F^{\mu}{}_{\mu} = \text{Trace}(F) = 0$

(v) $\text{Det}(F) = -\varepsilon_{\mu\nu\alpha\beta}F^{\mu 0}F^{\nu 1}F^{\alpha 2}F^{\beta 3}$

$\Rightarrow \text{Det}(F) = \frac{1}{c^2}\left(\vec{B}\cdot\vec{E}\right)^2$

$\varepsilon_{\mu\nu\alpha\beta}$ = 4D levi-cevita tensor with $\varepsilon_{0123} = 1$

Lorentz transformation of EM tensor

The electric & magnetic fields, though they are 3-vectors, they are not part of any 4-vectors, rather they are part of the six independent components of the Electromagnetic Stress tensor (F). So the lorentz transformation of rank-2 tensor (F) has to be performed as follows.

Then the lorentz transformation along the boost B_x along the x-direction is given by the lorentz-transformation matrise,

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\gamma B_x & 0 & 0 \\ -\gamma B_x & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

lorentz transformed $\rightarrow \bar{F}^{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta} F^{\alpha\beta}$

This gives the lorentz transform of $\vec{E}$ & $\vec{B}$ fields as

$\bar{E}_x = E_x$	$\bar{B}_x = B_x$
$\bar{E}_y = \gamma(E_y - v_x B_z)$	$\bar{B}_y = \gamma(B_y - v_x E_z)$
$\bar{E}_z = \gamma(E_z + v_x B_y)$	$\bar{B}_z = \gamma(B_z - v_x E_y)$

Thus lorentz transformation mixes $\vec{E}$ & $\vec{B}$ fields. So the electric & magnetic fields are not lorentz invariants. The only two lorentz invariant quantities are

$\left(|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2}\right) \rightarrow$ Proportional to $F^{\mu\nu}F_{\mu\nu}$ & $G^{\mu\nu}G_{\mu\nu}$

& $(\vec{E} \cdot \vec{B}) \Rightarrow$ Proportional to $F^{\mu\nu}G_{\mu\nu}$ & $\sqrt{\det(F)}$

Source for $\vec{E}$ & $\vec{B}$ fields

The charge & current densities act as the source for $\vec{E}$ & $\vec{B}$ respectively as shown by the non-homogenous maxwell's eq^n,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(x^\mu)}{\varepsilon_0}, \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

where $\mu_0\varepsilon_0 = \frac{1}{c^2}$ $\qquad \frac{1}{\varepsilon_0} = c^2\mu_0$

$$\Rightarrow \vec{\nabla}\cdot\frac{\vec{E}}{c} = \mu_0(c\rho)$$

$$\& \; -\frac{\partial}{\partial(ct)}\left(\frac{\vec{E}}{c}\right) + \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

These equations can be incorporated into a tensor eq^n of $F^{\mu\nu}$ as,

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

provided $J^\nu = (c\rho, J_x, J_y, J_z)$

where $\partial_\mu F^{\mu 0} = \mu_0 J^0$ gives the eqn $\vec{\nabla}\cdot\vec{E} = 0$ & $\partial_\mu F^{\mu i} = \mu_0 J^i$ will result in $\vec{\nabla}\times\vec{B} - \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$.

J^ν acts as the Source for the electric & magnetic fields, where the four-divergence of EM field tensor is proportional to the Source field called as the four-current density.

$$J^\nu = (c\rho, \vec{J}) = (c\rho, J_x, J_y, J_z)$$

The charge-continuity equation which signifies the conservation of charge is given by,

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \frac{\partial(c\rho)}{\partial(ct)} + \partial_i J^i = 0$$

$$\Rightarrow \boxed{\partial_\mu J^\mu = 0}$$

This also follows from the Maxwell's eqn directly.

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\partial_\nu \partial_\mu F^{\mu\nu} = \mu_0 (\partial_\nu J^\nu)$$

$$\partial_\nu \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 (\partial_\nu J^\nu)$$

$$(\partial_\mu \partial^\mu) \partial_\nu A^\nu - (\partial_\nu \partial^\nu) \partial_\mu A^\mu = \mu_0 \partial_\nu J^\nu$$

Since μ & ν are dummy indices

LHS = 0

$$\Rightarrow \quad \underline{\partial_\nu J^\nu = 0}$$

Thus Continuity eqn is not an additional constraint. It is just a resultant of maxwell's eqn & the definition of $\vec{E}$ & $\vec{B}$ (or F-tensor).

Gauge invariance of EM theory

A^μ, i.e., the four vector potential is a gauge field and the theory described by A^μ is gauge independent. i.e., the EM theory is the same if we choose the four vector potential to be

$$A^\mu \text{ or } \tilde{A}^\mu = A^\mu - \partial^\mu \chi$$

where 'χ' is any twice differentiable function of x^μ.

The transformation, $A^\mu \longrightarrow \tilde{A}^\mu = A^\mu - \partial^\mu \chi$ is called as gauge transformation. And EM theory is invariant if it is written in terms of either A^μ or $\tilde{A}^\mu = A^\mu - \partial^\mu \chi$. This is because the EM field tensor $F_{\mu\nu}$ & hence $\vec{E}$ & $\vec{B}$ are invariant under gauge transformation, since,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \tilde{F}_{\mu\nu}.$$

Thus $F^{\mu\nu}$ & $G^{\mu\nu}$ are gauge invariants & also J^{μ} (sources).

Lorenz Gauge

Consider a four potential A^{μ} that the EM theory is built on.

We can perform a gauge transformation $A^{\mu} \longrightarrow A^{\mu}_{lorenz}$ such that $\partial_{\mu} A^{\mu}_{lorenz} = 0$, then it is called as the lorenz gauge condition. This is achieved by performing $A^{\mu}_{lorenz} = A^{\mu} - \partial^{\mu}\chi$ where we set

$$\partial_{\mu}\partial^{\mu}\chi = \partial_{\mu}A^{\mu}$$

Lorenz gauge is a special gauge since the maxwell's eqn reduce to wave eqn for potential components.

The Maxwell's eqn are given by

$$\partial_{\mu} F^{\mu\nu} = \mu_0 J^{\nu}$$

$$\Rightarrow \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 J^\nu$$

$$\Rightarrow (\partial_\mu \partial^\mu) A^\nu - \partial^\nu(\partial_\mu A^\mu) = \mu_0 J^\nu$$

If we choose χ such that

$$\partial_\mu \partial^\mu \chi = \partial_\mu A^\mu$$

& then use $A^\mu_{lorenz} = A^\mu - \partial_\mu \chi$ & if we work with lorenz gauge, then maxwell's eq^n reduce to,

$$(\partial_\mu \partial^\mu) A^\nu_{lorenz} = \mu_0 J^\nu$$

or $\underline{\Box A^\nu_{lorenz} = \mu_0 J^\nu}$ → Maxwell's eq^n in Potential form.

In the absence of source, we get,

$$\Box A^\mu_{lorenz} = 0$$

Thus every component of A^μ_{lorenz} will satisfy massless K-G eq^n,

$$\frac{1}{c^2}\frac{\partial^2 A^\mu_{lorenz}}{\partial t^2} = \nabla^2 A^\mu_{lorenz}$$

$\Rightarrow A^\mu_{lorenz}$ are waves in 4D.

The condition $\partial_\mu A^\mu_{lorenz} = 0$ makes the four components of A^μ_{lorenz} to relate by one equation & hence the number of independent components is just three.

But still A^μ_{lorenz} is not unique as we can perform another transform on it such that the resultant four potential is again another lorenz gauge $\tilde{A}^\mu_{lorenz}$.

i.e. $\tilde{A}^\mu_{lorenz} = A^\mu_{lorenz} - \partial^\mu \varepsilon_s$

if $\partial_\mu \partial^\mu \varepsilon_s = 0$, then we have both $\tilde{A}^\mu_{lorenz}$ & A^μ_{lorenz} to satisfy lorenz gauge condition. Thus A^μ_{lorenz} is not unique.

But if we set "ε_s" such that $\partial^0 \varepsilon_s = A^0_{lorenz}$, then $\tilde{A}^0_{lorenz} = 0$

$\Rightarrow \tilde{A}^\mu_{lorenz}$ is such that $\vec{\nabla} \cdot \vec{A} = 0$

This is Coulomb gauge, which has only two degree's of freedom.

Lagrangian formulation of EM theory

The lagrangian for a system has to follow some principles like

(i) Locality : $\mathcal{L} \longrightarrow \mathcal{L}(A^\mu, \partial_\nu A^\mu)$

(ii) lorentz invariance : $\mathcal{L}$ should be function of quantities like $A_\mu A^\mu$, $\partial_\mu A^\mu$, etc.

(iii) Gauge invariance : $\mathcal{L}$ should be such that the action is gauge invariant. i.e. $\mathcal{L} \to \mathcal{L}(F_{\mu\nu}, G_{\mu\nu})$

(iv) Incorporate charge / Sources: So we need a term of the form $\sim J^\mu A_\mu$ in $\mathcal{L}$.

(v) $\mathcal{L}$ has to respect the inbuilt Symmetries of the System.

The terms like $F^{\mu\nu} G_{\mu\nu}$ are not allowed as it is a pseudo scalar (Proportional to $\vec{E} \cdot \vec{B}$).

Thus an appropriate lagrangian for EM field, whose action

$S_{EM} = \int d^4x \, \mathcal{L}_{EM}$ is gauge invariant provided $J^\mu \to 0$ at $\pm\infty$, is given by,

$$\boxed{\mathcal{L}_{EM} = \frac{-1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$

$J^\mu = (c\rho, \vec{J})$

μ_0 = Permeability of free space

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

ε_0 = Permittivity of free space

$$\mathcal{L}_{EM} = \frac{1}{2\mu_0}\left(\frac{|\vec{E}|^2}{c^2} - |\vec{B}|^2\right) - J^\mu A_\mu$$

$$\mathcal{L}_{EM} = \frac{1}{2}\varepsilon_0 |\vec{E}|^2 - \frac{1}{2}\frac{|\vec{B}|^2}{\mu_0} - J^\mu A_\mu$$

Consider $F^{\mu\nu}F_{\mu\nu} = F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)$

$= F^{\mu\nu}\partial_\mu A_\nu - F^{\mu\nu}\partial_\nu A_\mu$

$= F^{\mu\nu}\partial_\mu A_\nu + F^{\nu\mu}\partial_\nu A_\mu$

(Since μ & ν are dummy indices)

$$\Longrightarrow F^{\mu\nu}F_{\mu\nu} = 2F^{\mu\nu}\partial_\mu A_\nu$$

Thus the EM lagrangian can be written as,

$$\mathcal{L} = \frac{-1}{2\mu_0}F^{\mu\nu}(\partial_\mu A_\nu) - J^\mu A_\mu$$

$$= \frac{-1}{2\mu_0}\left[(\partial^\alpha A^\beta)(\partial_\alpha A_\beta) - (\partial^\beta A^\alpha)(\partial_\alpha A_\beta)\right] - J^\alpha A_\alpha$$

$$= \frac{-1}{2\mu_0}\left[(\partial_\alpha A^\beta)(\partial^\alpha A_\beta) - (\partial_\beta A^\alpha)(\partial_\alpha A^\beta)\right] - J_\alpha A^\alpha$$

The eqn of motion is given by

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -J_\nu$$

& $$\Pi^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}$$

$$= \frac{-1}{2\mu_0}\left[2\,\delta^\mu_\alpha \delta^\beta_\nu\, \partial^\alpha A_\beta - 2\,\partial_\beta A^\alpha \delta^\mu_\alpha \delta^\beta_\nu\right]$$

$$\Pi^\mu{}_\nu = \frac{-1}{\mu_0}\left(\partial^\mu A_\nu - \partial_\nu A^\mu\right)$$

$$\Rightarrow \Pi^{\mu\nu} = \frac{-1}{\mu_0}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

$$\Rightarrow \boxed{\Pi^{\mu\nu} = \frac{-1}{\mu_0} F^{\mu\nu}}$$

EoM is given by

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \Pi^{\mu\nu} = 0$$

$$\Rightarrow \quad -J^{\nu} = \partial_{\mu}\Pi^{\mu\nu} = -\frac{1}{\mu_0}\partial_{\mu}F^{\mu\nu}$$

$$\Rightarrow \quad \boxed{\partial_{\mu}F^{\mu\nu} = \mu_0 J^{\nu}}$$

Thus we retrived non-homogenous maxwell's eqn that govern the EM fields from $\mathcal{L}_{EM}$. Thus the $\mathcal{L}_{EM}$ we wrote was right.

Electromagnetic Energy momentum Tensor

$$\textcircled{H}^{\mu\nu} = \Pi^{\mu}_{\sigma}(\partial^{\nu}A^{\sigma}) - \eta^{\mu\nu}\mathcal{L}$$

$$= \Pi^{\mu\sigma}(\partial^{\nu}A_{\sigma}) - \eta^{\mu\nu}\mathcal{L}$$

$$\rightarrow \quad \textcircled{H}^{\mu\nu} = -\frac{1}{\mu_0}\left(F^{\mu\sigma}\partial^{\nu}A_{\sigma}\right) - \eta^{\mu\nu}\mathcal{L}$$

The canonical energy-momentum tensor is not symmetric and also it is unphysical as it is also gauge dependent ($\partial^{\nu}A_{\sigma}$ term).

But we can construct a Symmetric general energy momentum tensor using a Bifilante tensor $B^{\sigma\mu\nu}$ such that $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma B^{\sigma\mu\nu}$ is Symmetric & Gauge independent.

where $B^{\sigma\mu\nu} = \frac{1}{\mu_0} F^{\mu\sigma} A^\nu$

$$\Rightarrow \partial_\sigma B^{\sigma\mu\nu} = \frac{1}{\mu_0}\left[F^{\mu\sigma}(\partial_\sigma A^\nu) + A^\nu(\partial_\sigma F^{\mu\sigma})\right]$$

$$\Rightarrow \partial_\sigma B^{\sigma\mu\nu} = \frac{1}{\mu_0} F^{\mu\sigma}(\partial_\sigma A^\nu) + A^\nu J^\mu$$

But in the absence of Source field J^μ we get

$$\partial_\sigma B^{\sigma\mu\nu} = \frac{1}{\mu_0} F^{\mu\sigma}(\partial_\sigma A^\nu)$$

$$\Rightarrow T^{\mu\nu} = -\frac{1}{\mu_0} F^{\mu\sigma}(\partial^\nu A_\sigma) - \eta^{\mu\nu}\mathcal{L} + \frac{1}{\mu_0} F^{\mu\sigma}(\partial_\sigma A^\nu)$$

$$\Rightarrow T^{\mu\nu} = \frac{F^{\mu\sigma}(\partial_\sigma A^\nu - \partial^\nu A_\sigma)}{\mu_0} - \eta^{\mu\nu}\mathcal{L}$$

$$= \frac{F^{\mu}{}_{\sigma}(\partial^\sigma A^\nu - \partial^\nu A^\sigma)}{\mu_0} - \eta^{\mu\nu}\mathcal{L}$$

$$\Rightarrow T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}{}_{\sigma} F^{\sigma\nu}\right) - \eta^{\mu\nu}\mathcal{L}$$

To show $\partial_\mu T^{\mu\nu} = 0$

We have already proved that $\partial_\mu \mathcal{H}^{\mu\nu} = 0$. Since $T^{\mu\nu}$ has an extra term given by

$$\partial_\sigma \mathcal{B}^{\sigma\mu\nu} = \frac{1}{\mu_0} F^{\mu\sigma}(\partial_\sigma A^\nu),$$

we should check if $\partial_\mu(\partial_\sigma \mathcal{B}^{\sigma\mu\nu}) = 0$

$$\partial_\mu(\partial_\sigma \mathcal{B}^{\sigma\mu\nu}) = \partial_\mu\left[\frac{1}{\mu_0} F^{\mu\sigma}(\partial_\sigma A^\nu)\right]$$

In the absence of sources,

$$\partial_\mu F^{\mu\sigma} = 0$$

$$\Rightarrow \partial_\mu\left(\partial_\sigma B^{\sigma\mu\nu}\right) = \frac{1}{\mu_0} \underbrace{F^{\mu\sigma}}_{\text{anti symmetric w.r.t } \mu \& \sigma} \underbrace{\left(\partial_\mu \partial_\sigma A^\nu\right)}_{\text{Symmetric w.r.t } \mu \& \sigma}$$

$$\Rightarrow F^{\mu\sigma}\left(\partial_\mu \partial_\sigma A^\nu\right) = 0 \;//$$

$$\Rightarrow \boxed{\partial_\mu T^{\mu\nu} = 0} \text{ provided } \underline{J^\mu = 0}$$

(Source-less EM field).

where,

$$\boxed{T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\sigma} F_\sigma{}^\nu\right) - \frac{\eta^{\mu\nu}}{4\mu_0}\left(F^{\alpha\beta} F_{\alpha\beta}\right)}$$

↳ Symmetric, Gauge invariant & conserved $(\partial_\mu T^{\mu\nu} = 0)$

But in the presence of J^μ the energy-momentum tensor is not Conserved as there is exchange of energy-Momentum b/w fields & Sources in the form of lorentz-force & work done on charges by EM field.

Hamiltonian density given by T^{00}

$$T^{00} = \frac{1}{\mu_0} F^{0\sigma} F_{\sigma}{}^{0} - \frac{1}{4\mu_0} \left(F^{\alpha\beta} F_{\alpha\beta} \right)$$

$$\mathcal{H} = \frac{1}{\mu_0} \left(F^{01}F_{10} + F^{02}F_{20} + F^{03}F_{30} \right) - \left[\frac{1}{2} \varepsilon_0 |\vec{E}|^2 - \frac{1}{2} \frac{|\vec{B}|^2}{\mu_0} \right]$$

$$= \frac{1}{\mu_0 c^2} \left(E_x^2 + E_y^2 + E_z^2 \right) - \frac{1}{2} \varepsilon_0 |\vec{E}|^2 + \frac{1}{2} \frac{|\vec{B}|^2}{\mu_0}$$

$$= \varepsilon_0 |\vec{E}|^2 - \frac{1}{2} \varepsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2$$

$$\Rightarrow \mathcal{H} = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2$$

Thus hamiltonian density is nothing but the total energy density of EM fields.

$$T^{i0} = \frac{1}{\mu_0}\left(F^{i\sigma}F_{\sigma 0}\right)$$

$$= \frac{1}{\mu_0}\left(F^{i1}F_{10} + F^{i2}F_{20} + F^{i3}F_{30}\right)$$

$$\Rightarrow T^{10} = \frac{1}{\mu_0}\left(F^{12}F_{20} + F^{13}F_{30}\right)$$

$$= \frac{1}{\mu_0}\left(B_z E_y - B_y E_z\right)$$

$$T^{10} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)^1$$

Similarly $T^{i0} = \frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)^i$

$\downarrow$

i^{th}-component of $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$

$\vec{S} \Rightarrow$ Poynting vector that gives the momentum density which has the information of energy flux of EM field, it describes the rate at which energy flowing per area per unit time $\perp^{r}$ to the direction of propagation.

Relativistic charged particle in Electromagnetic fields

The lagrangian of a free relativistic particle with mass 'm' is given by

$$L_{free} = -mc^2\sqrt{1-\frac{v^2}{c^2}} = \frac{-mc^2}{\gamma}$$

$$\Rightarrow S_{free} = \int L_{free}\, dt$$

But in presence of A^μ, i.e., the EM potentials, it the action is given by

$$S_{charged} = -q\int A_\mu dx^\mu$$

$$\Rightarrow S = S_{free} + S_{charged}$$

$$\Rightarrow S_{\substack{(charged\ particle\\ in\ EM\ field)}} = -mc^2\int\frac{dt}{\gamma} - q\int A_\mu dx^\mu$$

$$\Rightarrow S = \int \left(-\frac{mc^2}{\gamma} - q A_\mu \frac{dx^\mu}{dt} \right) dt$$

$$= \int \left[-\frac{mc^2}{\gamma} - q A_\mu \underbrace{\left(\frac{d\tau}{dt}\right)}_{\gamma^{-1}} \left(\frac{dx^\mu}{d\tau}\right) \right] dt$$

$$\Rightarrow S = \int \frac{1}{\gamma} \left(-mc^2 - q A_\mu u^\mu \right) dt$$

$$u^\mu = \gamma (c, v_x, v_y, v_z)$$

$$\Rightarrow \frac{A_\mu u^\mu}{\gamma} = (\phi - \vec{A} \cdot \vec{v})$$

$$\Rightarrow L_{charge} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q(\phi - \vec{A} \cdot \vec{v})$$

---

eqⁿ of motion for charged particle is given by,

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0$$

[3-vector notation, no contra/co-variant notation]

$$L_{charge} = -mc^2\sqrt{1-\beta_{jk}v_j v_k} - q\left(\phi - A_j v_j\right)$$

$$\frac{\partial L}{\partial x_i} = -q\left[\partial_i\phi - v_j\left(\partial_i A_j\right)\right]$$

$$\frac{\partial L}{\partial v_i} = -mc^2\gamma\left(-\frac{v_i}{c^2}\right) + qA_i$$

$$\Rightarrow \frac{\partial L}{\partial v_i} = m\gamma v_i + qA_i$$

π_i (canonical momenta) $= p_i + qA_i$

p_i → relativistic momentum

$$\Rightarrow \pi_i = p_i + qA_i$$

π_i → canonical momenta

p_i → relativistic momenta

EoM is given by,

$$\frac{d\pi_i}{dt} = \frac{\partial L}{\partial x_i} = -q\left[\partial_i\phi - v_j\left(\partial_i A_j\right)\right]$$

$$\Rightarrow \frac{d}{dt}\left(p_i + qA_i\right) = -q\left(\partial_i \phi\right) + q v_j \left(\partial_i A_j\right)$$

$$\Rightarrow \frac{dp_i}{dt} = q\left[-\frac{dA_i}{dt} - \partial_i \phi + v_j \partial_i A_j\right]$$

$$= q\left[-\frac{\partial A_i}{\partial t} - v_j \partial_j A_i - \partial_i \phi + v_j \partial_i A_j\right]$$

$$= q\left[-\frac{\partial A_i}{\partial t} - \partial_i \phi + v_j \left(\partial_i A_j - \partial_j A_i\right)\right]$$

$$\frac{dp_i}{dt} = F_i \Rightarrow \text{force (lorentz force)}$$

$$E_i = -\partial_i \phi - \frac{\partial A_i}{\partial t}$$

$$\vec{v} \times \vec{B} = \vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)$$

$$= v_j \hat{e}_j \times \partial_k A_l \varepsilon_{klm} \hat{e}_m$$

$$= v_j \partial_k A_l \varepsilon_{mkl} \varepsilon_{mnj} \hat{e}_n$$

$$= v_j \partial_k A_l (\delta_{kn}\delta_{lj} - \delta_{kj}\delta_{ln}) \hat{e}_n$$

$$= v_j \partial_i \hat{e}_i A_j - v_j \partial_j A_i e_i$$

$$\Rightarrow \underline{(\vec{v} \times \vec{B})_i = v_j(\partial_i A_j - \partial_j A_i)}$$

$$\Rightarrow \underline{\vec{F} = q\left[\vec{E} + \vec{v} \times \vec{B}\right]}$$

Thus we arrive at lorentz force expression for eqn of motion of a charged particle in EM field. But the entire procedure could have been done covariantly which results in

$$\text{Four force} \Rightarrow \underline{\frac{dP_\mu}{d\tau} = q F_{\mu\nu} u^\nu}$$

$u^\nu \Rightarrow$ four velocity

$F_{\mu\nu} \Rightarrow$ EM tensor

Manifestly Covariant lagrangian for a charged particle in EM field

$$L_{charge} = -mc^2\sqrt{1-\frac{v^2}{c^2}} - q\left(\phi - \vec{A}\cdot\vec{v}\right)$$

$$= -\frac{mc^2}{\gamma} - \frac{q}{\gamma}\left[\frac{\phi}{c}(\gamma c) - \vec{A}\cdot(\gamma\vec{v})\right]$$

using $u^\mu = \gamma(c, \vec{v})$
with $u^\mu u_\mu = c^2$

$$\Rightarrow L_{charge} = -\frac{m u^\mu u_\mu}{\gamma} - \frac{q A^\mu u_\mu}{\gamma}$$

$$S = \int L_{charge}\, dt$$

$$S = \int\left[\frac{m u^\mu u_\mu}{2} + q A^\mu u_\mu\right] d\tau$$

↳ ½ factor to get correct physics

$$L(\tau) = \frac{m u^\mu u_\mu}{2} + q A^\mu u_\mu$$

(neglect overall -ve)

For $\mathcal{L}(\tau)$ we can write eqⁿ of motion as

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial u^\mu}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial u^\mu} = \Pi_\mu \Rightarrow \text{canonical momenta}$$

$$\Pi_\mu = m u_\mu + q A_\mu$$

↳ relativistic 4-momenta

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = q u^\nu (\partial_\mu A_\nu)$$

$$\Rightarrow q u^\nu (\partial_\mu A_\nu) = \frac{d}{d\tau}\left[m u_\mu + q A_\mu\right]$$

$$= \frac{d(m u_\mu)}{d\tau} + q\left[\frac{dt}{d\tau}\frac{\partial A_\mu}{\partial t} + \frac{dx^i}{d\tau}(\partial_i A_\mu)\right]$$

$$= \frac{d(p_\mu)}{d\tau} + q\,\frac{dx^\nu}{d\tau}(\partial_\nu A_\mu)$$

$$\Rightarrow q\,u^\nu(\partial_\mu A_\nu) = \frac{dp_\mu}{d\tau} + q\,u^\nu(\partial_\nu A_\mu)$$

$$\Rightarrow \frac{dp_\mu}{d\tau} = q\,u^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= q\,u^\nu F_{\mu\nu}$$

$$\Rightarrow \boxed{\underbrace{\frac{dp_\mu}{d\tau}}_{\text{Four-force}} = q\,F_{\mu\nu}\,u^\nu}$$

The Zero-th component is

$p_0 = \frac{E}{c}$ → total energy

$$\frac{dE}{c\,d\tau} = q\,F_{oi}\,u^i = q\,\frac{\vec{E}\cdot(\gamma\vec{v})}{c}$$

$$\Rightarrow \underbrace{\frac{dt}{d\tau}}_{\gamma}\frac{dE}{dt} = q\gamma(\vec{E}\cdot\vec{v})$$

$$\vec{F} = q\vec{E}$$

→ coulomb force.

$$\Rightarrow \frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

→ work done per unit time

Thus the zero-th component of eqⁿ of motion is just an energy balance eqⁿ which relates the total work per unit time done by the field on the charged particle is given by $\frac{dE}{dt} = q\,\underbrace{\vec{E}\cdot\vec{v}}_{\text{work done per time}}$

Thus among the fields $\vec{E}$ & $\vec{B}$ only $\vec{E}$ does work on charge & can increase its total energy.

The other Spatial component of four-force eqⁿ gives the expression of lorentz force acting on the charged particle by the $\vec{E}$ & $\vec{B}$ fields.

Minimal Coupling

Minimal coupling is a prescription to write the lagrangian of two systems along with the interaction terms between fields & particles.

It is a systematic way of incorporating gauge fields into the lagrangian densities.

The term minimal refers to the fact that only the simplest of the interaction terms are kept & higher multipole moments of distribution are neglected.

In minimal coupling of a gauge field to a matter field (scalar/spinor) we usually replace normal derivatives with covariant derivative with gauge field in it.

Complex Scalar field in EM fields

$$\mathcal{L}_{\text{Complex}} = (\partial^\mu \psi^+)(\partial_\mu \psi) - m^2 \psi^+ \psi$$

$$\mathcal{L}_{EM} = \frac{-1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

But to incorporate a gauge field in complex scalar field theory we make $\mathcal{L}_{\text{complex}}$ gauge invariant by replacing normal partial derivatives (∂_μ) to covariant derivatives, $D_\mu = \partial_\mu + \frac{iq}{\hbar} A_\mu$

$$\Rightarrow \underset{\text{(Complex + EM)}}{\mathcal{L}} = (D^\mu \psi)^+ (D_\mu \psi) - m^2 \psi^+ \psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L} = (\partial^\mu \psi^+)(\partial_\mu \psi) - m^2 \psi^+ \psi - \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu}$$
$$+ \frac{iqV}{\hbar} A^\mu \left[\psi (\partial_\mu \psi^+) - \psi^+ (\partial_\mu \psi)\right]$$
$$+ q^2 \frac{A^\mu A_\mu}{\hbar^2} \psi^+ \psi$$

First term $\Rightarrow$ Complex Scalar field
Second term $\Rightarrow$ EM field
last two terms $\Rightarrow$ Minimal Coupling b/w the two fields

If we observe the interaction terms have added a current density J_N^μ which is a Conserved noether Current corresponding to the global U(1) Symmetry of $\mathcal{L}_{complex}$.

$$J_N^\mu = \frac{iq}{\hbar}\left[\psi(\partial^\mu\psi^+) - \psi^+(\partial^\mu\psi)\right]$$

$$\Rightarrow \text{Interaction} \Rightarrow J_N^\mu A_\mu$$

& also a mass term $\frac{q^2}{\hbar^2}A^\mu A_\mu \psi^+\psi$ proportional to Square of the Coupling strength (usually neglected)

$$\Rightarrow \mathcal{L} = \mathcal{L}_{complex} + \mathcal{L}_{EM} + J_N^\mu A_\mu + \frac{q^2}{\hbar^2}A^\mu A_\mu \psi^+\psi$$