

Klein Gordon Equation

We know that the Hamiltonian operator is the generator of time translation & Momentum operator is the generator of space translation.

Let $|\psi(t, \vec{x})\rangle$ denote the state of a single particle system. Then

$$\hat{H}|\psi\rangle = i\hbar \frac{\partial |\psi\rangle}{\partial t}$$

$$\& \quad \hat{\vec{p}}|\psi\rangle = -i\hbar \vec{\nabla}|\psi\rangle$$

Focusing on a relativistic free particle, we are interested in the dispersion relation given by, $E_p = \sqrt{m^2 c^4 + p^2 c^2}$
So we need a Hamiltonian which yields this energy Eigen value for a particle in momentum Eigen state $|\vec{p}\rangle$ with momentum eigen value $\vec{p}$.

The approach of expanding the square root in series will violate locality & causality. Thus we choose to work with the square of Hamiltonian & try to get the dispersion relation

$$E^2 = m^2 c^4 + p^2 c^2$$

$$\Rightarrow \hat{H}^2 = m^2 c^4 + \hat{\vec{p}}^2 c^2$$

$$\Rightarrow \hat{H}^2 |\psi\rangle = m^2 c^4 |\psi\rangle + \hat{\vec{p}}^2 c^2 |\psi\rangle$$

In position basis we get

$$\left(i\hbar \frac{\partial}{\partial t} \right)^2 \psi(t, \vec{x}) = m^2 c^4 \psi(t, \vec{x}) + c^2 (-i\hbar \vec{\nabla})^2 \psi(t, \vec{x})$$

dividing throughout by $c^2 \hbar^2$ we get,

$$-\frac{\partial^2 \psi(t, \vec{x})}{\partial (ct)^2} = \left(\frac{mc}{\hbar} \right)^2 \psi(t, \vec{x}) - \nabla^2 \psi(t, \vec{x})$$

$$\Rightarrow \left[\frac{\partial^2}{\partial (ct)^2} - \nabla^2 + \left(\frac{mc}{\hbar} \right)^2 \right] \psi(t, \vec{x}) = 0$$

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial (ct)^2} - \nabla^2 \rightarrow \text{D'Alembertian}$$

$$\& \quad \mu^2 = \left(\frac{mc}{\hbar} \right)^2 \quad \text{where } \frac{\hbar}{mc} \text{ is}$$

the Compton wavelength which is the length scale of the particle.

Thus the resulting eqⁿ for the wave function $\psi(t, \vec{x})$ is,

$$\underline{(\partial_\mu \partial^\mu + \mu^2) \psi(t, \vec{x}) = 0}$$

This is the familiar Klein-Gordon eqⁿ.

Note: This eqⁿ is the governing eqⁿ for spinless scalar particles.

The solution to this eqⁿ are the plane waves

$$\psi_{(E, \vec{p})}(t, \vec{x}) = N e^{\frac{-iEt + i\vec{p} \cdot \vec{x}}{\hbar}}$$

In covariant notation we can write it as

$$\psi_{p^\mu}(x^\mu) = N e^{\frac{-i}{\hbar}(p_\mu x^\mu)}$$

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right)$$

$$\Rightarrow \partial_\mu \psi_{p^\mu}(x^\mu) = \frac{-i}{\hbar} p_\mu \psi_{p^\mu}(x^\mu)$$

$$\Rightarrow i\hbar \partial_\mu [\psi_{p^\mu}(x^\mu)] = p_\mu [\psi_{p^\mu}(x^\mu)]$$

Thus $\hat{p}_\mu = i\hbar \partial_\mu$ acts as the position representation of 4-momentum operator

$$\hat{p}_\mu = i\hbar \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right) = i\hbar \partial_\mu$$

$$\& \hat{p}^\mu = i\hbar \left(\frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right) = i\hbar \partial^\mu$$

$$\Rightarrow \partial_\mu \partial^\mu \psi_{p^\mu}(x^\mu) = -\frac{p_\mu p^\mu}{\hbar^2} \psi_{p^\mu}(x^\mu)$$

$$\text{But } p_\mu p^\mu = \frac{E^2}{c^2} - |\vec{p}|^2$$

The K.G. eqⁿ is

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2} \right) \psi_{p^\mu}(x^\mu) = 0$$

$$\Rightarrow -\frac{p_\mu p^\mu}{\hbar^2} + \frac{m^2 c^2}{\hbar^2} = 0$$

$$\Rightarrow -\frac{1}{\hbar^2} \left(\frac{E^2}{c^2} - |\vec{p}|^2 \right) + \frac{m^2 c^2}{\hbar^2} = 0$$

$$\Rightarrow \underline{E^2 = |\vec{p}|^2 c^2 + m^2 c^4}$$

$\hookrightarrow$ Necessary dispersion eqⁿ.

Thus for $(\partial_\mu \partial^\mu + \mu^2) \psi_{p^\mu}(x^\mu) = 0$

Solⁿ is $\psi_{(E, \vec{p})}(t, \vec{x}) = N e^{\frac{-iEt + i\vec{p} \cdot \vec{x}}{\hbar}}$

equivalently, $\psi_{p^\mu}(x^\mu) = N e^{\frac{-i(p_\mu x^\mu)}{\hbar}}$ which solves the K.G. eqⁿ with the dispersion relation $E^2 = |\vec{p}|^2 c^2 + m^2 c^4$.

For a plane wave function

$$\psi_{p^\mu}(x^\mu) = e^{\frac{-i p_\mu x^\mu}{\hbar}}$$

the energy dispersion relation is given by

$$E_p^2 = p^2 c^2 + m^2 c^4$$

$$\Rightarrow E_p = \pm \sqrt{p^2 c^2 + m^2 c^4}$$

Thus $\psi_{p^\mu}(x^\mu)$ also solves for negative energies for a free particle. If we neglect -ve energy solⁿ & keep only +ve ones, still we have another big issue of probability density being non positive definite.

What we are concerned is we need a proper way to represent the probability density of particles & a conservation law for probability through a Continuity eqⁿ.

$$(\partial_\mu \partial^\mu + m^2) \psi = 0$$

$$\& (\partial_\mu \partial^\mu + m^2) \psi^* = 0$$

$$\Rightarrow \psi^* (\partial_\mu \partial^\mu + m^2) \psi - \psi (\partial_\mu \partial^\mu + m^2) \psi^* = 0$$

$$\Rightarrow \psi^* (\partial_\mu \partial^\mu \psi) - \psi (\partial_\mu \partial^\mu \psi^*) = 0$$

$$\Rightarrow \partial_\mu (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) = 0$$

Multiply throughout by $\frac{-i\hbar}{2m}$, we get,

$$\partial_\mu \left(\frac{i\hbar}{2m} [\psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi] \right) = 0$$

Defining $J^\mu = \frac{i\hbar}{2m} [\psi \partial^\mu \psi^* - \psi^* \partial^\mu \psi]$

we get $\partial_\mu J^\mu = 0$

$J^\mu \Rightarrow$ Probability 4-current density.

$$J^\mu = (c\rho, \vec{J})$$

$$\rho = \frac{i\hbar}{2mc^2} \left[\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right]$$

$$\& \quad \vec{J} = \frac{i\hbar}{2m} (\psi \vec{\nabla} \psi^* - \psi^* \vec{\nabla} \psi)$$

& the Continuity eqⁿ we got can be written as,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

So in K.G theory the conserved quantity is

$$\rho = \frac{i\hbar}{2mc^2} \left[\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right]$$

instead of $\psi^* \psi$ as in the Schrodinger wave theory. The probability density ρ is non-positive definite.

For $\psi = N e^{-\frac{i}{\hbar} p_x x}$

$$\rightarrow \rho = \frac{i\hbar}{2mc^2} |N|^2 \left(-\frac{2iE}{\hbar} \right)$$

$$\Rightarrow \rho = \frac{E |N|^2}{mc^2}, \quad |N|^2 = \psi^* \psi$$

$$E = \pm \gamma mc^2 \quad \gamma \Rightarrow \text{Lorentz factor}$$

$$\Rightarrow \rho = \pm \gamma (\psi^* \psi)$$

Thus the K.G. probability density changes with the speed of the particle which reflects that the 3D volume element contracts by a factor of " γ " & hence ρ increases by the same factor.

But still as E can be +ve or -ve, interpreting ' ρ ' as probability density doesn't make sense.

Thus there is less scope in considering K.G. eqⁿ for relativistic quantum particles.

Weyl Equation

Weyl equation describes massless spin - $\frac{1}{2}$ particles called as the Weyl fermions.

For massless particles the expected dispersion relation is $E = pc$.

$$\text{Let } \hat{H} = c \vec{\alpha} \cdot \vec{\hat{p}}$$

$$\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$$

$$\text{we need } E^2 = p^2 c^2$$

$$\Rightarrow \hat{H}^2 = c^2 (\vec{\alpha} \cdot \vec{\hat{p}})^2$$

$$= c^2 (\alpha_i \hat{p}_i) (\alpha_j \hat{p}_j) \rightarrow \text{use einstein summation convention}$$

$$= c^2 \alpha_i \alpha_j \hat{p}_i \hat{p}_j \quad \text{assuming } [\alpha_i, \hat{p}_j] = 0$$

$$= \frac{c^2}{2} (\alpha_i \alpha_j \hat{p}_i \hat{p}_j + \alpha_j \alpha_i \hat{p}_j \hat{p}_i)$$

Since $[\hat{p}_i, \hat{p}_j] = 0$, we get,

$$\hat{H}^2 = c^2 \frac{[\alpha_i \alpha_j + \alpha_j \alpha_i]}{2} \hat{p}_i \hat{p}_j$$

But we need $\hat{H}^2 = c^2 \hat{p}^2$

$$\Rightarrow \frac{\{\alpha_i, \alpha_j\}}{2} \hat{p}_i \hat{p}_j = p_x^2 + p_y^2 + p_z^2$$

$$\Rightarrow \text{we need } \{\alpha_i, \alpha_j\} = 2\delta_{ij} \\ \& \alpha_i^2 = 1 //$$

So one such objects that follow above eqⁿ are the pauli matrices.

Thus $\vec{\alpha}$ has to be pauli matrices given by $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

$$\Rightarrow \underline{\hat{H} = c \vec{\sigma} \cdot \hat{\vec{p}}} \Rightarrow \text{Weyl Hamiltonian}$$

But if we had chosen the -ve energy solutions, then we should begin with the dispersion relation $E = -pc$

$$\Rightarrow \hat{H} = -c(\vec{\alpha} \cdot \hat{\vec{p}})$$

& $\vec{\alpha}$ turns out to be $\vec{\sigma}$.

$$\text{Thus } \underline{\hat{H} = \pm c \vec{\sigma} \cdot \hat{\vec{p}}}$$

Since $\hat{H}$ is now a 2×2 matrix inherently, it has to act on a 2-component Spinor $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$

called as Weyl spinors.

$$\Rightarrow \hat{H} \psi = \pm c \begin{bmatrix} \hat{p}_z & \hat{p}_x - i\hat{p}_y \\ \hat{p}_x + i\hat{p}_y & -\hat{p}_z \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

Since $\hat{H} = i\hbar \frac{\partial}{\partial t}$ we get

$$\frac{i\hbar}{c} \begin{bmatrix} \frac{\partial \psi_1}{\partial t} \\ \frac{\partial \psi_2}{\partial t} \end{bmatrix} = \pm \begin{bmatrix} \hat{p}_z & \hat{p}_x - i\hat{p}_y \\ \hat{p}_x + i\hat{p}_y & -\hat{p}_z \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$$

$$\Rightarrow \frac{i\hbar}{c} \frac{\partial \psi_1}{\partial t} = \pm (\hat{p}_z \psi_1 + \hat{p}_- \psi_2)$$

$$\& \frac{i\hbar}{c} \frac{\partial \psi_2}{\partial t} = \pm (\hat{p}_+ \psi_1 - \hat{p}_z \psi_2)$$

$$\text{where } \hat{p}_{\pm} = \hat{p}_x \pm i\hat{p}_y$$

In position representation $\vec{p} = -i\hbar \vec{\nabla}$

$$\Rightarrow \frac{1}{c} \frac{\partial \psi_1}{\partial t} \pm [\partial_z \psi_1 + (\partial_x - i\partial_y) \psi_2] = 0$$

$$\& \frac{1}{c} \frac{\partial \psi_2}{\partial t} \pm [(\partial_x + i\partial_y) \psi_1 - \partial_z \psi_2] = 0$$

These eqⁿ look very clumsy.
But we can simplify them using
the index notations.

Since $\hat{H}\psi = \pm c \vec{\sigma} \cdot \vec{p} \psi$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial (ct)} \pm i\hbar (\vec{\sigma} \cdot \vec{\nabla}) \psi = 0$$

$$\Rightarrow \left[\frac{\partial}{\partial x^0} \pm \vec{\sigma} \cdot \vec{\nabla} \right] \psi = 0$$

Let $\sigma^\mu = (\hat{I}, \sigma_x, \sigma_y, \sigma_z)$
 & $\bar{\sigma}^\mu = (\hat{I}, -\sigma_x, -\sigma_y, -\sigma_z)$

$\Rightarrow$ For +ve energy : $(\sigma^\mu \partial_\mu) \psi_R = 0$

& for -ve energy : $(\bar{\sigma}^\mu \partial_\mu) \psi_L = 0$

where ψ_R & ψ_L are right & left handed Weyl spinors respectively.

The handedness / helicity is because the operator $\vec{\sigma} \cdot \vec{p}$ is actually the projection of spin of a particle along the direction of motion.

If $\vec{\sigma} \cdot \hat{\vec{p}}$ is +ve i.e. if energy is +ve then Spin angular momentum of Spin-half particle is along the direction of propagation.



$\Rightarrow$ Right handed Particles (ψ_R)

Spin angular momenta is $\parallel$ to linear momenta

And if $\vec{\sigma} \cdot \hat{\vec{p}}$ is -ve i.e. if energy is -ve then Spin angular momentum of Spin-half particle is opposite to the direction of propagation.



$\Rightarrow$ left handed Particles (ψ_L)

$\vec{S}$ is anti-parallel to linear momenta.

Thus Weyl eqⁿ represents massless & Spin- $\frac{1}{2}$ particles with left & right handed helicity.

ψ_L & ψ_R are eigen states of Weyl eqⁿ.
 So left handed particles remain left
 handed & right handed particles remain
 right handed. They don't interchange
 with Space/Time. This is because
 the Helicity operator defined by

$$\hat{h} = \frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|}$$
 , it commutes

with the Hamiltonian. So $\hat{h}$ is
 a conserved quantity.

where
$$\left(\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \right) \psi_R = \psi_R$$

&
$$\left(\frac{\vec{\sigma} \cdot \vec{p}}{|\vec{p}|} \right) \psi_L = -\psi_L$$

Dirac Equation

To overcome the -ve probability density in Klein Gordon theory, Dirac tried to linearize the below operator relation

$$\hat{H}^2 = m^2 c^4 + \hat{p}^2 c^2$$

& tried to write an hamiltonian of the form

$$\underline{\hat{H}_D = c \vec{\alpha} \cdot \vec{\hat{p}} + \beta m c^2}$$

where $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z) = \alpha_i$
 $\vec{\hat{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z) = \hat{p}_i$

$\vec{\alpha}$ & β are unknowns that commute with $\vec{\hat{p}}$. They are found by putting a condition that

$$\hat{H}_D^2 = c^2 \hat{p}^2 + m^2 c^4$$

which gives Einstein's relation as the dispersion relation.

$$\Rightarrow c^2 \hat{p}^2 + m^2 c^4 = (c \vec{\alpha} \cdot \hat{\vec{p}} + \beta m c^2)^2$$

$$= c^2 (\vec{\alpha} \cdot \hat{\vec{p}})^2 + \beta^2 m^2 c^4 + \beta m c^3 (\vec{\alpha} \cdot \hat{\vec{p}}) + m c^3 (\vec{\alpha} \cdot \hat{\vec{p}}) \beta$$

$$= c^2 \hat{p}_i \hat{p}_j \alpha_i \alpha_j + \beta^2 m c^4 + m c^3 \hat{p}_i (\beta \alpha_i + \alpha_i \beta)$$

$$= c^2 \hat{p}_i \hat{p}_j \frac{(\alpha_i \alpha_j + \alpha_j \alpha_i)}{2} + \beta^2 m c^4 + m c^3 \hat{p}_i (\beta \alpha_i + \alpha_i \beta)$$

Thus we get conditions that $\vec{\alpha} \neq \beta$ should satisfy. They are,

$$(i) \alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \quad \& \quad \alpha_i^2 = 1$$

$$\beta^2 = 1$$

$$\beta \alpha_i + \alpha_i \beta = 0 \quad \checkmark \quad \text{Thus they are matrices}$$

(ii) $\vec{\alpha} \neq \beta$ should be Hermitian.

(iii) Since $B^2 = 1$ & $\alpha_i^2 = 1$
eigen values of α_i & B are ± 1

(iv) $\text{Trace}(\alpha_i) = \text{Trace}(B) = 0$

(Because of anti-commutativity)

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \quad (i \neq j)$$

$$\Rightarrow \alpha_i^2 \alpha_j = -\alpha_i \alpha_j \alpha_i$$

$$\Rightarrow \alpha_j = -\alpha_i \alpha_j \alpha_i$$

$$\begin{aligned} \Rightarrow \text{Trace}(\alpha_j) &= -\text{Trace}(\alpha_i \alpha_j \alpha_i) \\ &= -\text{Trace}(\alpha_i^2 \alpha_j) \end{aligned}$$

$$\Rightarrow \text{Trace}(\alpha_j) = -\text{Trace}(\alpha_j)$$

$$\Rightarrow \text{Trace}(\alpha_j) = 0 //$$

Similarly, $\alpha_i B + B \alpha_i = 0$

$$\Rightarrow \alpha_i^2 B + \alpha_i B \alpha_i = 0$$

$$\Rightarrow \text{trace}(B) = -\text{trace}(\alpha_i B \alpha_i)$$

$$\Rightarrow \text{trace}(B) = -\text{trace}(B)$$

$$\Rightarrow \text{trace}(B) = 0 //$$

(V) α & B matrices have even dimensions as their eigen values are ± 1 & trace is Zero.

(vi) $\vec{\alpha}$ & B cannot be 2×2 they have to be 4×4 or higher.

Because we can map $\vec{\alpha}$ to be pauli matrices $\vec{\sigma}$ which have all desired properties. But then we cannot find any suitable ' B ' to satisfy $B\vec{\sigma} + \vec{\sigma}B = 0$.

Since $\vec{\sigma}$ form a basis for 2×2 traceless hermitian matrices, if $\vec{\sigma}$ itself can't satisfy all the required property, then no other traceless hermitian matrix can.

$\vec{\alpha}$ & B matrices in Dirac basis

Dirac found the 4×4 $\vec{\alpha}$ & B matrices which satisfy all the above listed conditions. They are

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

In this basis 'B' is diagonal & $\vec{\alpha}$ are not diagonal.

So now the Dirac Hamiltonian is composed of 4×4 matrices of $\hat{\vec{p}}$. So then in the eqⁿ

$$\hat{H}_0 \psi = i\hbar \frac{\partial \psi}{\partial t}$$

we need ψ to be a 4-component object of the form,

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}$$

where ψ_i are functions of space & time (x^4).

ψ is called a spinor & forms a spinor field. In Dirac basis,

$$\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \quad \text{where } \psi_+ \Rightarrow \text{+ve energy sol}^n$$

& $\psi_- \Rightarrow \text{-ve energy sol}^n$.

Then we can write Dirac eqⁿ compactly starting from,

$$\hat{H}_0 \psi = c \vec{\alpha} \cdot \vec{\hat{p}} \psi + \beta mc^2 \psi$$

In position basis $\vec{\hat{p}} = -i\hbar \vec{\nabla}$
& we can also write $\hat{H}_0 = i\hbar \frac{\partial}{\partial t}$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c (\vec{\alpha} \cdot \vec{\nabla}) \psi + \beta mc^2 \psi$$

$$\rightarrow i\hbar \left[\frac{\partial}{\partial t} + c (\vec{\alpha} \cdot \vec{\nabla}) \right] \psi = \beta mc^2 \psi$$

multiply throughout by $\frac{\beta}{\hbar c}$

$$\Rightarrow i \left[\frac{\beta}{c} \frac{\partial}{\partial t} + (\beta \vec{\alpha} \cdot \vec{\nabla}) \right] \psi = \frac{mc}{\hbar} \psi$$

$$\Rightarrow i \left[\beta \partial_0 + \beta \vec{\alpha} \cdot \vec{\nabla} \right] \psi = \frac{mc}{\hbar} \psi$$

Defining $\gamma^\mu = (\beta, \beta \vec{\alpha})$

Dirac eqⁿ becomes,

$$\left(i \left[\gamma^0 \partial_0 + (\vec{\gamma} \cdot \vec{\nabla}) \right] - \frac{mc}{\hbar} \right) \psi = 0$$

$$\Rightarrow \left(i \gamma^\mu \partial_\mu - \frac{mc}{\hbar} \right) \psi = 0$$

If we set $c = \hbar = 1$ (natural units)
& denote $\gamma^\mu \partial_\mu = \not{\partial}$

$$\boxed{(i \not{\partial} - m) \psi = 0} \rightarrow \text{Dirac eq}^n.$$

$\hat{P}_\mu = i\hbar \partial_\mu$ is the energy-momentum operator.
 $\hat{P}_\mu = i\hbar \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right)$

$$\& \hat{P}^\mu = i\hbar \left(\frac{1}{c} \frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

$$\text{Dirac eq}^n \Rightarrow (\gamma^\mu \hat{P}_\mu - mc) \psi = 0$$

γ^μ are called Gamma matrices which satisfy

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \mathbb{I}_4$$

$$\Rightarrow B \alpha_i \alpha_j B + B \alpha_j \alpha_i B = 2\delta_{ij} \mathbb{I}_4$$

$$\Rightarrow \gamma^i (-\gamma^j) + \gamma^j (-\gamma^i) = 2\delta_{ij} \mathbb{I}_4$$

$$\Rightarrow \underline{\{\gamma^i, \gamma^j\} = -2\delta_{ij} \mathbb{I}_4}$$

$$B \alpha_i + \alpha_i B = 0$$

$$\Rightarrow B^2 \alpha_i + B \alpha_i B = 0$$

$$\Rightarrow \gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$$

$$\Rightarrow \underline{\{\gamma^0, \gamma^i\} = 0}$$

∴ Since $\gamma^0 = B$ & $B^2 = \mathbb{I}_4$
we get $\underline{\{\gamma^0, \gamma^0\} = 2\mathbb{I}_4}$

Since $\eta^{\mu\nu}$ is such that
 $\eta^{00} = 1$, $\eta^{ij} = -\delta_{ij}$ & $\eta^{0i} = 0$
 we can write

$$\underline{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}_4}$$

This is the defining property of the gamma matrices (γ^μ) which generates a Clifford-Algebra. This property is more fundamental than the numerical entries of γ^μ .

$$\gamma_\mu = \eta_{\mu\nu} \gamma^\nu$$

$$\Rightarrow (\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3)$$

- $\gamma^0 \Rightarrow$ Hermitian, eigen vals = ± 1 .
- $\gamma^i \Rightarrow$ Anti-Hermitian, $(\gamma^i)^2 = -1$
 $\rightarrow$ eigenvalues of $\gamma^i = \pm i$
- $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$
- $\text{trace}(\gamma^\mu) = 0$
- $\gamma^\mu \gamma_\mu = 4 \mathbb{I}_4$
- $\text{trace}(\gamma^\mu \gamma^\nu) = 4 \eta^{\mu\nu}$

In Dirac basis

$$\gamma^0 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}$$

$$\vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$$

where $\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix}$

$\psi_+ \Rightarrow$ +ve energy
Solⁿ
& $\psi_- \Rightarrow$ -ve energy
Solⁿ.

In Chiral / Weyl basis

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$$

where $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$ $\psi_L \Rightarrow$ Describes left handed particles

& $\psi_R \Rightarrow$ Describes right handed Particles

According to clifford Algebra, if two distinct set of matrices γ^μ & γ'^μ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, $\{\gamma'^\mu, \gamma'^\nu\} = 2\eta^{\mu\nu}$, then the two sets γ^μ & γ'^μ are connected by a similarity transform.

$$\gamma'^\mu = S^{-1} \gamma^\mu S$$

Solution of dirac eqⁿ also solves KG eqⁿ

Dirac eqⁿ is given by,

$$(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

$$\Rightarrow (i\hbar \gamma^\nu \partial_\nu + mc)(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

Expanding LHS we get,

$$[(i\hbar)^2 \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m^2 c^2]\psi = 0$$

$$\Rightarrow \left[\hbar^2 \frac{(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu)}{2} \partial_\nu \partial_\mu + m^2 c^2 \right] \psi = 0$$

$$\text{Using } \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu}$$

$$\Rightarrow (\hbar^2 \eta^{\mu\nu} \partial_\nu \partial_\mu + m^2 c^2) \psi = 0$$

Inverse of
Compton
wavelength
↑

$$\Rightarrow (\partial^\mu \partial_\mu + \mu^2) \psi = 0$$

$\mu = \frac{mc}{\hbar}$

But $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \rightarrow$ Each Component of Spinor ψ satisfies K.G. eqⁿ.

Hermitian adjoint of Dirac eqⁿ

Dirac's eqⁿ is $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$

The hermitian conjugate is given by

$$\psi^\dagger(-i\hbar(\gamma^\mu)^\dagger\overleftarrow{\partial}_\mu - mc) = 0$$

$$\text{where } \psi^\dagger\overleftarrow{\partial}_\mu = (\partial_\mu\psi)^\dagger$$

& also using $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ we get

$$i\hbar\psi^\dagger(\gamma^0\gamma^\mu\gamma^0)\overleftarrow{\partial}_\mu + mc\psi^\dagger = 0$$

multiply by γ^0 by right

$$\Rightarrow i\hbar\psi^\dagger[\gamma^0\gamma^\mu(\gamma^0)^2]\overleftarrow{\partial}_\mu + mc\psi^\dagger\gamma^0 = 0$$

defining $\bar{\psi} = \psi^\dagger\gamma^0$ we get

$$\Rightarrow i\hbar\bar{\psi}\gamma^\mu\overleftarrow{\partial}_\mu + mc\bar{\psi} = 0$$

$$\Rightarrow \bar{\psi}(i\hbar\gamma^\mu\overleftarrow{\partial}_\mu + mc) = 0$$

$$\Rightarrow \boxed{\bar{\psi}(i\hbar\overleftarrow{\not{D}} + mc) = 0}$$

Probability density & Continuity Eqⁿ

The dirac eqⁿ & its adjoint are given by,

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$$

$$\& \bar{\psi} (i\hbar\gamma^\mu\overleftarrow{\partial}_\mu + mc) = 0$$

Performing $\bar{\psi}(\text{dirac eq}^n) + (\text{dirac eq}^n)\psi$,

$$i\hbar [\bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi] = 0$$

$$\Rightarrow \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

Thus defining $J^\mu = c(\bar{\psi}\gamma^\mu\psi)$

Dirac eqⁿ predicts a Conserved four-current given by

$$\underline{\partial_\mu J^\mu = 0}, \text{ where } J^\mu = c\bar{\psi}\gamma^\mu\psi$$

writing $J^\mu = (c\rho, \vec{J})$

we get $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

where $\rho = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi$

Thus $\rho = \psi^\dagger \psi$ is the conserved probability density which is positive definite.

& $\vec{J} = c \bar{\psi} \vec{\gamma} \psi$

$$= c (\psi^\dagger \gamma^0) (\gamma^0 \vec{\alpha}) \psi$$

$$\Rightarrow \underline{\vec{J} = c \psi^\dagger \vec{\alpha} \psi}$$

$$\Rightarrow \underline{\frac{\partial}{\partial t} (\psi^\dagger \psi) + c \vec{\nabla} \cdot (\psi^\dagger \vec{\alpha} \psi) = 0}$$

Useful interpretation of J^μ

Consider the Dirac eqⁿ of the form,

$$(\gamma^\mu \hat{P}_\mu - mc)\psi = 0$$

If ψ is an eigen state of $\hat{P}_\mu$ with eigen value p_μ , then

$$(\gamma^\mu p_\mu - mc)\psi = 0$$

$$\Rightarrow \underline{\gamma^\mu p_\mu = mc}$$

$$\Rightarrow \underline{c = \frac{\gamma^\nu p_\nu}{m}}$$

$$\begin{aligned} \text{Now } J^\mu &= c(\bar{\psi} \gamma^\mu \psi) \\ &= \frac{c}{2} [\bar{\psi} \gamma^\mu \psi + \bar{\psi} \gamma^\mu \psi] \\ &= \frac{1}{2m} [\bar{\psi} \gamma^\mu \gamma^\nu p_\nu \psi + \bar{\psi} \gamma^\nu p_\nu \gamma^\mu \psi] \end{aligned}$$

$$\rightarrow J^\mu = \frac{p_\nu}{2m} \bar{\psi} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \psi$$

$$= \frac{p_\nu}{2m} \bar{\psi} (2\eta^{\mu\nu}) \psi$$

$$= \frac{p^\mu}{m} \bar{\psi} \psi$$

$$\rightarrow J^\mu = \frac{p^\mu}{m} (\bar{\psi} \psi)$$

$$J^0 = c\rho = \frac{p^0}{m} \bar{\psi} \psi$$

$$\Rightarrow \rho = \frac{E}{mc^2} (\bar{\psi} \psi) \quad \& \quad \vec{J} = \frac{\vec{p}}{m} (\bar{\psi} \psi)$$

$$E = \gamma mc^2 = \mathbf{k} \cdot \mathbf{E} + mc^2 = \sqrt{p^2 c^2 + m^2 c^4}$$

$$\Rightarrow \rho = \gamma (\bar{\psi} \psi)$$

$\rightarrow$ Since d^3x element shrink.

$\rightarrow$ Lorentz factor

Time-rate of Various Observables

a) $\hat{\vec{x}} \rightarrow$ Position operator

$$\begin{aligned}\frac{d\hat{x}_i}{dt} &= \frac{1}{i\hbar} [\hat{x}_i, \hat{H}_0] \\&= \frac{1}{i\hbar} [\hat{x}_i, c \vec{\alpha} \cdot \hat{\vec{p}} + \beta mc^2] \\&= \frac{c}{i\hbar} [\hat{x}_i, \vec{\alpha} \cdot \hat{\vec{p}}] \quad [\hat{x}_i, \beta] = 0 \\&= \frac{c}{i\hbar} \alpha_j \underbrace{[\hat{x}_i, \hat{p}_j]}_{i\hbar \delta_{ij}} \quad [\hat{x}_i, \vec{\alpha}] = 0 \\&= c \alpha_j \delta_{ij} = c \alpha_i\end{aligned}$$

$$\Rightarrow \underline{\frac{d\hat{\vec{x}}}{dt} = c \vec{\alpha}} \quad \rightarrow \vec{x} \text{ is not constant of motion}$$

$$\Rightarrow \left\langle \left(\frac{d\hat{\vec{x}}}{dt} \right)^2 \right\rangle = c^2 \Rightarrow \langle v \rangle_{\text{rms}} = c$$

Thus v_{rms} of relativistic free particle is 'c' (read about Zitterbewegung)

b) $\hat{\vec{p}} \rightarrow$ Momentum operator

$$\frac{d\hat{\vec{p}}}{dt} = \frac{1}{i\hbar} [\hat{\vec{p}}, \hat{H}_0]$$

$$\hat{H}_0 = c \vec{\alpha} \cdot \hat{\vec{p}} + \beta mc^2$$

Since $[\vec{p}, \vec{\alpha}] = 0$ & $[\vec{p}, \beta] = 0$
we get $[\vec{p}, \hat{H}_0] = 0$

$$\Rightarrow \frac{d\vec{p}}{dt} = 0$$

Thus $\vec{p}$ is a constant of motion & a conserved quantity.

c) $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} \rightarrow$ Orbital angular momentum

$$\frac{d\hat{\vec{L}}}{dt} = \frac{1}{i\hbar} [\hat{\vec{r}} \times \hat{\vec{p}}, \hat{H}_0]$$

$$= \frac{1}{i\hbar} [\hat{\vec{r}} \times \hat{\vec{p}}, c \vec{\alpha} \cdot \hat{\vec{p}}]$$

$$= \frac{c}{i\hbar} [\hat{\vec{r}}, (\vec{\alpha} \cdot \hat{\vec{p}})] \times \hat{\vec{p}}$$

$$\Rightarrow \frac{d\hat{\vec{L}}}{dt} = \frac{c}{i\hbar} \alpha_j [\hat{x}_k, \hat{p}_j] \hat{p}_l \epsilon_{kln}$$

$$= \frac{c}{(i\hbar)} \alpha_j (i\hbar) \delta_{kj} \hat{p}_l \epsilon_{kln} \vec{e}_n$$

$$= c \alpha_j p_l \epsilon_{jln} \vec{e}_n$$

$$= c (\vec{\alpha} \times \hat{\vec{p}})$$

$$\Rightarrow \underline{\frac{d\hat{\vec{L}}}{dt} = c \vec{\alpha} \times \hat{\vec{p}}}$$

Thus $\hat{\vec{L}}$ is not a conserved quantity as it doesn't commute with the Dirac Hamiltonian for a free particle. So we need to find a correction ($\hat{\vec{S}}$) such that we get a total angular momentum ($\hat{\vec{J}}$) which is conserved, where

$$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$$

We need $\hat{\vec{S}}$ such that $\frac{d\hat{\vec{S}}}{dt} = -c \vec{\alpha} \times \hat{\vec{p}}$

$$\text{or } [\hat{\vec{S}}, \hat{H}_0] = -i\hbar c (\vec{\alpha} \times \hat{\vec{p}})$$

d) Spin Angular Momentum ($\hat{\vec{S}}$)

$$\text{Let, } \vec{S} = \frac{\hbar}{2} \vec{\Sigma} = \frac{\hbar}{2} \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

we should check if it satisfies the required condition.

$$\hat{H}_0 = \begin{bmatrix} mc^2 & c(\vec{\sigma} \cdot \hat{\vec{p}}) \\ c(\vec{\sigma} \cdot \hat{\vec{p}}) & -mc^2 \end{bmatrix}$$

$$[\hat{\vec{S}}, \hat{H}_0] = \frac{\hbar}{2} \left[\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \begin{pmatrix} mc^2 & c(\vec{\sigma} \cdot \hat{\vec{p}}) \\ c(\vec{\sigma} \cdot \hat{\vec{p}}) & -mc^2 \end{pmatrix} \right]$$

$$= \frac{\hbar}{2} \left\{ \begin{bmatrix} \vec{\sigma} mc^2 & c\vec{\sigma}(\vec{\sigma} \cdot \hat{\vec{p}}) \\ c\vec{\sigma}(\vec{\sigma} \cdot \hat{\vec{p}}) & -\vec{\sigma} mc^2 \end{bmatrix} - \begin{bmatrix} \vec{\sigma} mc^2 & c(\vec{\sigma} \cdot \hat{\vec{p}})\vec{\sigma} \\ c(\vec{\sigma} \cdot \hat{\vec{p}})\vec{\sigma} & -\vec{\sigma} mc^2 \end{bmatrix} \right\}$$

$$\Rightarrow [\hat{S}, \hat{H}_0] = \frac{\hbar c}{2} \begin{bmatrix} 0 & [\vec{\sigma}, (\vec{\sigma} \cdot \hat{p})] \\ [\vec{\sigma}, (\vec{\sigma} \cdot \hat{p})] & 0 \end{bmatrix}$$

$$[\vec{\sigma}, \vec{\sigma} \cdot \hat{p}] = [\vec{\sigma}, \sigma_j \hat{p}_j]$$

$$= [\vec{\sigma}, \sigma_j] \hat{p}_j = \vec{e}_k [\sigma_k, \sigma_j] \hat{p}_j$$

Using $[\sigma_k, \sigma_j] = 2i \epsilon_{kjl} \sigma_l$

$$\begin{aligned} \Rightarrow [\vec{\sigma}, \vec{\sigma} \cdot \hat{p}] &= 2i \epsilon_{kjl} \sigma_l \hat{p}_j \vec{e}_k \\ &= 2i \epsilon_{jlk} \hat{p}_j \sigma_l \vec{e}_k \end{aligned}$$

$$\Rightarrow [\vec{\sigma}, \vec{\sigma} \cdot \hat{p}] = 2i (\hat{p} \times \vec{\sigma})$$

$$\Rightarrow [\hat{S}, \hat{H}_0] = \frac{\hbar c}{2} \begin{bmatrix} 0 & 2i (\hat{p} \times \vec{\sigma}) \\ 2i (\hat{p} \times \vec{\sigma}) & 0 \end{bmatrix}$$

$$\Rightarrow [\hat{S}, \hat{H}_0] = i\hbar c (\hat{p} \times \vec{\sigma})$$

Thus
$$\frac{d\hat{\vec{S}}}{dt} = \frac{1}{i\hbar} [\hat{\vec{S}}, \hat{H}_0]$$

$$\rightarrow \frac{d\vec{S}}{dt} = -c(\vec{\alpha} \times \vec{p})$$

Previously we got,
$$\frac{d\hat{\vec{L}}}{dt} = c \vec{\alpha} \times \vec{p}$$

$$\rightarrow \frac{d}{dt} (\hat{\vec{L}} + \hat{\vec{S}}) = 0$$

Thus $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ is such that

$$\frac{d\hat{\vec{J}}}{dt} = 0 \Rightarrow [\hat{\vec{J}}, \hat{H}_0] = 0$$

where $\hat{\vec{L}} = \vec{r} \times \vec{p} \rightarrow$ orbital angular momentum

$$\& \hat{\vec{S}} = \frac{\hbar}{2} \hat{\vec{\Sigma}} = \frac{\hbar}{2} \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix} \rightarrow \text{Spin angular momentum}$$

& $\hat{\vec{J}}$ is a conserved quantity.

Helicity of Particles

Helicity is defined as the projection of total angular momentum in the direction of propagation of particle.

Direction of Propagation is given by the direction of momentum operator which is $\frac{\hat{\vec{p}}}{|\vec{p}|}$

$$\Rightarrow \text{Helicity operator} = \frac{\hat{\vec{J}} \cdot \hat{\vec{p}}}{|\vec{p}|}$$

$$\text{But } \vec{J} = \vec{L} + \vec{S}$$

$$\& \vec{L} \cdot \vec{p} = (\vec{r} \times \vec{p}) \cdot \vec{p} = 0$$

$$\Rightarrow \text{Helicity operator, } \hat{h} = \frac{\hat{\vec{S}} \cdot \hat{\vec{p}}}{|\vec{p}|}$$

$$\hat{h} = \frac{\hbar}{2} \frac{\hat{\vec{\Sigma}} \cdot \hat{\vec{p}}}{|\vec{p}|}$$

$$[\hat{h}, \hat{H}_0] = \frac{\hbar}{2} \left[\frac{\vec{\Sigma} \cdot \hat{\vec{p}}}{|\vec{p}|}, \underbrace{Bmc^2 + c\vec{\alpha} \cdot \vec{p}}_{\text{vanishing contribution}} \right]$$

$$= \frac{\hbar c}{2} \left[\frac{\vec{\Sigma} \cdot \hat{\vec{p}}}{|\vec{p}|}, \vec{\alpha} \cdot \vec{p} \right]$$

$$= \frac{\hbar c}{2|\vec{p}|} \left[\begin{pmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \cdot \hat{\vec{p}} \\ \vec{\sigma} \cdot \hat{\vec{p}} & 0 \end{pmatrix} \right]$$

$$= \frac{\hbar c}{2|\vec{p}|} \left[\begin{pmatrix} 0 & (\vec{\sigma} \cdot \hat{\vec{p}})^2 \\ (\vec{\sigma} \cdot \hat{\vec{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\vec{\sigma} \cdot \hat{\vec{p}})^2 \\ (\vec{\sigma} \cdot \hat{\vec{p}})^2 & 0 \end{pmatrix} \right]$$

$$\Rightarrow \underline{[\hat{h}, \hat{H}_0] = 0}$$

Eigen values of Helicity operator

$$\hat{h} = \frac{\hbar}{2} \frac{\vec{\Sigma} \cdot \hat{\vec{p}}}{|\vec{p}|}$$

$$\hat{h}^2 = \frac{\hbar^2}{4|\vec{p}|^2} \begin{bmatrix} \vec{\sigma} \cdot \hat{\vec{p}} & 0 \\ 0 & \vec{\sigma} \cdot \hat{\vec{p}} \end{bmatrix}^2$$

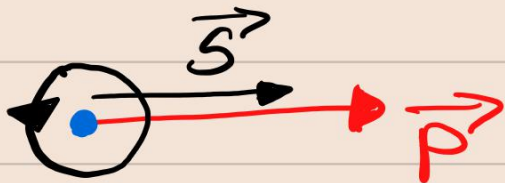
$$\hat{h}^2 = \frac{\hbar^2}{4|\vec{p}|^2} \begin{bmatrix} (\vec{\sigma} \cdot \vec{p})^2 & 0 \\ 0 & (\vec{\sigma} \cdot \vec{p})^2 \end{bmatrix}$$

$$(\vec{\sigma} \cdot \vec{p})^2 = |\vec{p}|^2$$

$$\Rightarrow \hat{h}^2 = \frac{\hbar^2}{4} \hat{I}_4$$

$\rightarrow$ eigen values of $\hat{h}$ are $\pm \frac{\hbar}{2}$ //

Thus a free particle can have helicity of only $\pm \frac{\hbar}{2}$ (quantized) & is a conserved quantity as it commutes with $\hat{H}_0$.



$\Rightarrow$ Right handed
Particles (ψ_R)

Spin angular momenta
is $\parallel$ to linear momenta



$\Rightarrow$ left handed
Particles (ψ_L)

$\vec{S}$ is anti-parallel
to linear momenta.

Lagrangian formulation of Dirac eqⁿ

The appropriate lagrangian that gives rise to dirac eqⁿ is,

$$\mathcal{L} = \bar{\Psi}(i\hbar c \not{\partial} - mc^2)\Psi$$

treating Ψ & $\bar{\Psi}$ as dynamical fields we can write the Euler lagrange eqⁿ of motion as,

$$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \right] = 0$$

$$\& \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi})} \right] = 0$$

$$\mathcal{L} = i\hbar c \bar{\Psi} \gamma^\mu \partial_\mu \Psi - mc^2 \bar{\Psi} \Psi$$

$$\frac{\partial \mathcal{L}}{\partial \Psi} = -mc^2 \bar{\Psi}, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} = i\hbar c \bar{\Psi} \gamma^\mu$$

$$\Rightarrow -mc^2 \bar{\psi} = \partial_\mu (i\hbar c \bar{\psi} \gamma^\mu)$$

$$\Rightarrow i\hbar (\partial_\mu \bar{\psi}) \gamma^\mu + mc \bar{\psi} = 0$$

$$\Rightarrow \underline{\bar{\psi} (i\hbar \overleftarrow{\partial} + mc) = 0}$$

↳ Hermitian adjoint of Dirac eqⁿ.

Now the $\bar{\psi}$ eqⁿ is fairly simple.

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\hbar c \overleftarrow{\partial} - mc^2) \psi$$

$$\& \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} = 0$$

$$\rightarrow \text{Dirac eq}^n : \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$$

$$\Rightarrow \underline{(i\hbar \overleftarrow{\partial} - mc) \psi = 0}$$

Global $U(1)$ Symmetry

The Dirac Lagrangian is invariant under global phase rotation.

$$\mathcal{L} = \bar{\Psi}(i\hbar c \not{\partial} - mc^2)\Psi$$

$$\Rightarrow \mathcal{L} = \Psi^\dagger \gamma^0 (i\hbar c \gamma^\mu \partial_\mu \Psi) - mc^2 \Psi^\dagger \gamma^0 \Psi$$

$$\text{If } \Psi \rightarrow e^{i\alpha} \Psi \Rightarrow \Psi^\dagger \rightarrow e^{-i\alpha} \Psi^\dagger \\ \Rightarrow \underline{\mathcal{L} \rightarrow \mathcal{L}} \text{ as } e^{i\alpha} = \text{constant}$$

Hence there exists $U(1)$ symmetry. The corresponding conservation law is given by the following when we do an infinitesimal $U(1)$ rot.

$$e^{i\alpha} \sim 1 + i\alpha$$

$$\Rightarrow \Psi \rightarrow \Psi' = \Psi + i\alpha \Psi \Rightarrow \delta \Psi = i\alpha \Psi \\ \Psi^\dagger \rightarrow \Psi'^\dagger = \Psi^\dagger - i\alpha \Psi^\dagger \Rightarrow \delta \Psi^\dagger = -i\alpha \Psi^\dagger$$

$$\& \delta \mathcal{L} = 0.$$

$$\Rightarrow J_N^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (\partial \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} (\partial \bar{\psi})$$

$$= (i\hbar c \bar{\psi} \gamma^\mu) i\alpha \psi$$

$$= (-\hbar\alpha) (c \bar{\psi} \gamma^\mu \psi)$$

& $\partial_\mu J_N^\mu = 0 \Rightarrow$ by Noether's theorem

$$\Rightarrow -\hbar\alpha \partial_\mu (c \bar{\psi} \gamma^\mu \psi) = 0$$

Probability Current density

Thus $J_N^\mu \propto J_{\text{Prob.}}^\mu$

Hence, global $U(1)$ symmetry of Dirac Lagrangian results in a conserved probability current given by

$$\underline{J_{\text{Prob.}}^\mu = c \bar{\psi} \gamma^\mu \psi}$$

This also leads to probability (conserved Noether charge) to conserve.

Local $U(1)$ Symmetry in the presence of a gauge Field

In the presence of a gauge field A^μ , the normal derivative ∂_μ is replaced by a covariant derivative D_μ where,

$$D_\mu \psi = \left(\partial_\mu + \frac{iq}{\hbar} A_\mu \right) \psi$$

$$\& (D_\mu \psi)^\dagger = \left(\partial_\mu - \frac{iq}{\hbar} A_\mu \right) \psi^\dagger$$

The gauge invariant dirac lagrangian is given by

$$\mathcal{L} = \bar{\psi} (i\hbar c \not{D} - mc^2) \psi$$

$$\begin{aligned} \Rightarrow \mathcal{L} &= \bar{\psi} (i\hbar c \gamma^\mu D_\mu - mc^2) \psi \\ &= \bar{\psi} (i\hbar c \gamma^\mu \partial_\mu - mc^2) \psi - qc \bar{\psi} \gamma^\mu A_\mu \psi \end{aligned}$$

$$\rightarrow \mathcal{L} = \bar{\psi} (i\hbar c \not{D} - mc^2) \psi - qc \bar{\psi} \not{A} \psi$$

$\hookrightarrow$ Gauge Invariant lagrangian

Now if $\psi \rightarrow e^{i\alpha(x)}\psi$ & if $A_\mu \rightarrow A_\mu - \frac{1}{q} \partial_\mu \alpha(x)$, then we get $\mathcal{L} \rightarrow \mathcal{L} \Rightarrow \delta \mathcal{L} = 0$. Thus this lagrangian has local gauge symmetry.

If the gauge field is EM field then the e.m. lagrangian in the absence of external source, we have $\mathcal{L}_{\text{e.m.}} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Also the interaction term is given by the minimal coupling prescription as

$$\begin{aligned} \mathcal{L}_{\text{int}} &= -q c \bar{\psi} \cancel{A} \psi, \text{ where } \cancel{A} = \gamma^\mu A_\mu \\ &= -q c A_\mu (\bar{\psi} \gamma^\mu \psi) \end{aligned}$$

Notice that $c \bar{\psi} \gamma^\mu \psi = J^\mu$ is the conserved probability density current.

$$\Rightarrow \mathcal{L}_{\text{int}} = -q J^\mu A_\mu$$

Thus qJ^μ acts as the charge current density that acts as a source for e.m. $\Rightarrow J_{e.m.}^\mu = qJ^\mu$

$$\Rightarrow \mathcal{L}_{int} = J_{e.m.}^\mu A_\mu \Rightarrow \text{determined by strength of coupling constant 'q'}$$

Then the total lagrangian given by the sum of dirac particle, gauge field & minimal coupling interaction term is the QED lagrangian.

$$\mathcal{L}_{QED} = \mathcal{L}_{dirac} + \mathcal{L}_{e.m.} + \mathcal{L}_{int.}$$

$$\Rightarrow \boxed{\mathcal{L}_{QED} = \bar{\psi}(i\hbar c \not{\partial} - mc^2)\psi - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - qJ^\mu A_\mu}$$

Energy Momentum tensor for Dirac field

The dirac lagrangian is given by

$$\mathcal{L} = \bar{\Psi} (i\hbar c \gamma^\mu \partial_\mu - mc^2) \Psi$$

The energy-momentum tensor is given by

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} (\partial^\nu \Psi) + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\Psi})}}_0 (\partial^\nu \bar{\Psi}) - \eta^{\mu\nu} \mathcal{L}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi)} = \bar{\Psi} (i\hbar c \gamma^\mu)$$

$$\Rightarrow \Theta^{\mu\nu} = i\hbar c \bar{\Psi} \gamma^\mu (\partial^\nu \Psi) - \eta^{\mu\nu} \mathcal{L}$$

This is not symmetric & also is imaginary. Hence we need to add a four divergence term to $\mathcal{L}$ or $\Theta^{\mu\nu}$ & symmetrize the energy-momentum

Notice that adding a four divergence to $\mathcal{L}$ doesn't change the physics as the eqⁿ of motion remain same. So adding a term $-\frac{i\hbar c}{2} \partial_\mu (\bar{\Psi} \gamma^\mu \Psi)$ will generate a hermitian $\mathcal{L}$.

$$\text{Let, } \mathcal{L} \rightarrow \mathcal{L} - \frac{i\hbar c}{2} \partial_\mu (\bar{\Psi} \gamma^\mu \Psi)$$

$$\Rightarrow \mathcal{L} = \frac{i\hbar c}{2} [\bar{\Psi} \vec{\partial} \Psi - \bar{\Psi} \overleftarrow{\partial} \Psi] - mc^2 \bar{\Psi} \Psi$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \Psi} = -\frac{i\hbar c}{2} (\partial_\mu \bar{\Psi}) \gamma^\mu - mc^2 \bar{\Psi}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\Psi}} = \frac{i\hbar c}{2} \gamma^\mu (\partial_\mu \Psi) - mc^2 \Psi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} = \frac{i\hbar c}{2} \bar{\Psi} \gamma^\mu, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi})} = -\frac{i\hbar c}{2} \gamma^\mu \Psi$$

EOM are given by

$$\frac{\partial \mathcal{L}}{\partial \Psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \right) = 0$$

$$\Rightarrow -\frac{i\hbar c}{2} (\partial_\mu \bar{\Psi}) \gamma^\mu - mc^2 \bar{\Psi} - \frac{i\hbar c}{2} \partial_\mu \bar{\Psi} \gamma^\mu = 0$$

$$\Rightarrow \underline{\bar{\Psi} (i\hbar \vec{\partial} - mc) = 0}$$

Similarly, $\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} \right) = 0$

$$\Rightarrow \frac{i\hbar c}{2} \gamma^\mu (\partial_\mu \psi) - mc^2 \psi - \partial_\mu \left[-\frac{i\hbar c}{2} \gamma^\mu \psi \right] = 0$$

$$\rightarrow \underline{(i\hbar \not{\partial} - mc) \psi = 0}$$

Thus the modified lagrangian is a equally good lagrangian which reproduces dirac eqⁿ & its conjugate.

The momentum density four vectors are,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} = \frac{i\hbar c}{2} \bar{\psi} \gamma^\mu$$

$$\& \bar{\pi}^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} = -\frac{i\hbar c}{2} \gamma^\mu \psi$$

Then we can finally calculate the generalized Symmetric energy-momentum tensor,

$$\underline{T^{\mu\nu} = \pi^\mu (\partial^\nu \psi) + (\partial^\nu \bar{\psi}) \bar{\pi}^\mu - \eta^{\mu\nu} \mathcal{L}}$$

$$\begin{aligned}
 \rightarrow T^{\mu\nu} &= \left(\frac{i\hbar c}{2} \bar{\psi} \gamma^\mu \right) (\partial^\nu \psi) \\
 &\quad - (\partial^\nu \bar{\psi}) \left(\frac{i\hbar c}{2} \gamma^\mu \psi \right) - \eta^{\mu\nu} \mathcal{L} \\
 &= \frac{i\hbar c}{2} \left[\bar{\psi} \gamma^\mu (\partial^\nu \psi) - (\partial^\nu \bar{\psi}) \gamma^\mu \psi \right] \\
 &\quad - \eta^{\mu\nu} \left(\frac{i\hbar c}{2} \left[\bar{\psi} \gamma^\sigma \partial_\sigma \psi - (\partial_\sigma \bar{\psi}) \gamma^\sigma \psi \right] - mc^2 \bar{\psi} \psi \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{i\hbar c}{2} \left[\bar{\psi} (\gamma^\mu \partial^\nu - \eta^{\mu\nu} \not{\partial}) \psi \right] \\
 &\quad - \frac{i\hbar c}{2} \left[\bar{\psi} (\partial^\nu \gamma^\mu - \eta^{\mu\nu} \not{\partial}) \psi \right] + mc^2 \eta^{\mu\nu} \bar{\psi} \psi
 \end{aligned}$$

Thus the fully symmetric & real Energy momentum tensor is

$$\begin{aligned}
 T^{\mu\nu} &= \frac{i\hbar c}{2} \left[\bar{\psi} (\gamma^\mu \partial^\nu - \eta^{\mu\nu} \not{\partial}) \psi \right] + h.c. \\
 &\quad + mc^2 \eta^{\mu\nu} \bar{\psi} \psi
 \end{aligned}$$

where h.c. is hermitian conjugate of the preceding term.

$$T^{00} = \frac{i\hbar c}{2} (\bar{\psi} \gamma^0 \dot{\psi} - \dot{\bar{\psi}} \psi) - \frac{i\hbar c}{2} [\dot{\bar{\psi}} \gamma^0 \psi - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] + mc^2 \bar{\psi} \psi$$

$$= \frac{i\hbar c}{2} \left[\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \gamma^0 \gamma^\mu (\partial_\mu \psi) \right] - \dot{\psi}^\dagger \psi + (\partial_\mu \psi^\dagger) \gamma^0 \gamma^\mu \psi + mc^2 \psi^\dagger \gamma^0 \psi$$

$$= \frac{i\hbar c}{2} \left[\cancel{\psi^\dagger \dot{\psi}} - \cancel{\dot{\psi}^\dagger} \psi + \cancel{\dot{\psi}^\dagger} \psi - \cancel{\psi^\dagger \dot{\psi}} \right] + (\partial_i \psi^\dagger) \gamma^0 \gamma^i \psi - \psi^\dagger \gamma^0 \gamma^i \partial_i \psi + mc^2 \psi^\dagger \gamma^0 \psi$$

$$= \frac{i\hbar c}{2} [(\vec{\nabla} \psi^\dagger) \cdot \vec{\alpha} \psi - \psi^\dagger \vec{\alpha} \cdot (\vec{\nabla} \psi)] + mc^2 \psi^\dagger \gamma^0 \psi$$

$$= \frac{i\hbar c}{2} [\vec{\nabla} \cdot (\psi^\dagger \vec{\alpha} \psi) - 2 \psi^\dagger \vec{\alpha} \cdot \vec{\nabla} \psi] + mc^2 \psi^\dagger \gamma^0 \psi$$

$$T^{00} = \mathcal{H}, \quad \vec{j} \Rightarrow \text{Probability current}$$

$$\Rightarrow \mathcal{H} = \frac{i\hbar}{2} (\vec{\nabla} \cdot \vec{j}) - i\hbar c (\psi^\dagger \vec{\alpha} \cdot \vec{\nabla} \psi) + mc^2 \psi^\dagger \gamma^0 \psi$$

$$\mathcal{H} = -\frac{1}{2} \hat{\vec{p}} \cdot \vec{j} + \psi^\dagger (c \vec{\alpha} \cdot \hat{\vec{p}}) \psi + \psi^\dagger \beta mc^2 \psi$$

$$= -\frac{1}{2} (\hat{\vec{p}} \cdot \vec{j}) + \psi^\dagger (\hat{H}_D \psi)$$

$$\Rightarrow \int \mathcal{H} d^3x = \int \psi^\dagger (\hat{H}_D \psi) d^3x = \underline{\underline{\langle \hat{H}_D \rangle}}$$

since $\int \vec{p} \cdot \vec{j} d^3x \propto \int \underbrace{\vec{\nabla} \cdot \vec{j}}_{\text{Zero}} d^3x$

as $\vec{j}$ at surface = 0 //

Free particle Solution of Dirac Equation

Consider a free particle dirac eqⁿ given by,

$$(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

$$\text{Let, } \psi(x^\mu) = u(p) e^{\frac{-ip_\mu x^\mu}{\hbar}}$$

where $u(p)$ is a 4-spinor independent of x^μ . Thus $\psi(x^\mu)$ is a plane wave solution for dirac eqⁿ.

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right) \Rightarrow 4\text{-momentum}$$

$$\Rightarrow \psi(x^\mu) = e^{\frac{-ip_\mu x^\mu}{\hbar}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

In dirac repⁿ:

$$\gamma^0 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$$

Let $\mathcal{U}$ be a bispinor, $\Rightarrow \mathcal{U} = \begin{bmatrix} \mathcal{U}_+ \\ \mathcal{U}_- \end{bmatrix}$

where $\mathcal{U}_+ = \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix}$ & $\mathcal{U}_- = \begin{bmatrix} \mathcal{U}_3 \\ \mathcal{U}_4 \end{bmatrix}$

$\Rightarrow$ Dirac eqⁿ is

$$(i\hbar \gamma^\mu \partial_\mu - mc)\psi = 0$$

$$\Rightarrow (i\hbar \gamma^0 \partial_0 + i\hbar \vec{\gamma} \cdot \vec{\nabla})\psi = mc\psi$$

$$\Rightarrow i\hbar \partial_0 \psi + i\hbar \vec{\alpha} \cdot \vec{\nabla} \psi = mc\beta \psi$$

$$\frac{\hat{H}\psi}{c} - \vec{\alpha} \cdot \hat{\vec{p}} \psi = mc\beta \psi$$

$$\Rightarrow \underline{\hat{H}\psi = c(\vec{\alpha} \cdot \hat{\vec{p}}) + mc^2\beta \psi}$$

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \quad \& \quad \beta = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}$$

Since $\psi = \mathcal{U} e^{\frac{-i}{\hbar} \vec{p}_u \cdot \vec{x}^u}$ is a plane wave, it is eigenfunc. of both $\hat{H}$ & $\hat{\vec{p}}$.

$$\Rightarrow \begin{bmatrix} E\mathbb{I}_2 & 0 \\ 0 & E\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \mathcal{U}_+ \\ \mathcal{U}_- \end{bmatrix} = \begin{bmatrix} 0 & c\vec{\sigma} \cdot \vec{p} \\ c\vec{\sigma} \cdot \vec{p} & 0 \end{bmatrix} + \begin{bmatrix} mc^2\mathbb{I}_2 & 0 \\ 0 & -mc^2\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \mathcal{U}_+ \\ \mathcal{U}_- \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (E-mc^2)\mathbb{I}_2 & -c(\vec{\sigma} \cdot \vec{p}) \\ -c(\vec{\sigma} \cdot \vec{p}) & (E+mc^2)\mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \mathcal{U}_+ \\ \mathcal{U}_- \end{bmatrix} = 0$$

$$\Rightarrow (E-mc^2)\mathcal{U}_+ - c(\vec{\sigma} \cdot \vec{p})\mathcal{U}_- = 0$$

$$\& -c(\vec{\sigma} \cdot \vec{p})\mathcal{U}_+ + (E+mc^2)\mathcal{U}_- = 0$$

$$\rightarrow \psi_+ = \frac{c(\vec{p} \cdot \vec{\sigma}) \psi_-}{E - mc^2}$$

$$\& \psi_- = \frac{c(\vec{p} \cdot \vec{\sigma}) \psi_+}{E + mc^2}$$

$$\Rightarrow \psi_+ = \frac{[c(\vec{p} \cdot \vec{\sigma})]^2 \psi_+}{E^2 - m^2 c^4}$$

$$\Rightarrow E^2 - m^2 c^4 = c^2 |\vec{p}|^2$$

$$\Rightarrow \underline{E^2 = m^2 c^4 + p^2 c^2}$$

$$\Rightarrow \underline{E = \pm \sqrt{m^2 c^4 + p^2 c^2}}$$

Thus even if we choose a first order linear hamiltonian, the -ve energy eigen values are not avoidable & are a part of the complete energy spectra.

To find U -Spinor components

We have the relation given by,

$$u_+ = \frac{c(\vec{p} \cdot \vec{\sigma}) u_-}{E - mc^2}$$

$$\& u_- = \frac{c(\vec{p} \cdot \vec{\sigma}) u_+}{E + mc^2}$$

$$\text{where } u_+ = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \& u_- = \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{c}{E - mc^2} \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

$$\& \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \frac{c}{E + mc^2} \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\text{If } u_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow u_3 = \frac{cp_z}{E+mc^2}$$

$$\& u_4 = \frac{c(p_x + ip_y)}{E+mc^2}$$

$$\Rightarrow u_R^{(+)} = N_R^{(+)} \begin{bmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x + ip_y)}{E+mc^2} \end{bmatrix} \quad \left(N_R^{(+)} \text{ is some normalizing constant} \right)$$

$\Rightarrow$ This is a +ve energy solⁿ with right handed helicity.
(E is +ve)

Similarly, if $u_+ = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then $u_3 = \frac{c(p_x - ip_y)}{E + mc^2}$

& $u_4 = \frac{-cp_z}{E + mc^2}$

$$\Rightarrow u_L^{(+)} = N_L^{(+)} \begin{bmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E + mc^2} \\ \frac{-cp_z}{E + mc^2} \end{bmatrix}$$

This is a +ve energy particle with left handed helicity.

(E is +ve)

Now if we set $U_- \neq$
then find U_+ , we get,

for $U_- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

for $U_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$U_R^{(-)} = N_R^{(-)} \begin{bmatrix} \frac{c p_z}{E - mc^2} \\ \frac{c(p_x + i p_y)}{E - mc^2} \\ 1 \\ 0 \end{bmatrix}$$

$$U_L^{(-)} = N_L^{(-)} \begin{bmatrix} \frac{c(p_x - i p_y)}{E - mc^2} \\ \frac{-c p_z}{E - mc^2} \\ 0 \\ 1 \end{bmatrix}$$

This is a -ve
energy particle
with right handed
helicity.
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