

Group Theory

A group is an algebraic structure consisting of a set ' G ' along with a binary operation '*' which follows

- (i) $\forall g, h \in G, g * h \in G$
- (ii) $\forall g, h, k \in G, g * (h * k) = (g * h) * k$
- (iii) $\forall g \in G, \exists e \in G : g * e = e * g = g$
- (iv) $\forall g \in G, \exists g^{-1} \in G : g * g^{-1} = g^{-1} * g = e$

Note:

- a) Identity element ' e ' is unique.
- b) Inverse of a group element is unique.
- c) A group is usually denoted as $(G, *)$
- d) $(\{e\}, *)$ is called as trivial group.

Abelian group

- (i) If a group $(G, *)$ is such that for all $g, h \in G$,
$$g * h = h * g \quad (\text{commutative})$$

Sub group

A non-empty subset S of G is said to form a subgroup of $(G, *)$, if $(S, *)$ is itself a group under the same binary operation '*'.

Equivalently, $(S, *)$ is a subgroup of $(G, *)$, if & only if $\forall g, h \in S$, $g * h^{-1} \in S$.

Normal / Invariant Sub groups

Let $(H, *)$ be a subgroup of $(G, *)$. $\forall g \in G$ & $h \in H$, if $ghg^{-1} \in H$, then $(H, *)$ is called as a normal subgroup of $(G, *)$.

$(\{e\}, *)$ is a trivial normal subgroup. As $G \subset G$, a group is also its own subgroup which is normal.

Direct product of groups

Consider two groups $(G, *)$ & $(H, \cdot)$. The direct product between these two groups, denoted by $(G, *) \otimes (H, \cdot)$ is another group $(F, \odot)$, whose elements are given by $f = (g, h)$, where $g \in G$ & $h \in H$ and the group operation ' $\odot$ ' between $f_1 = (g_1, h_1)$ & $f_2 = (g_2, h_2)$ is given by,

$$f_1 \odot f_2 = (g_1 * g_2, h_1 \cdot h_2)$$

The identity element of F : $e_F = (e_G, e_H)$.
The inverse of $f = (g, h)$ is $f^{-1} = (g^{-1}, h^{-1})$.

Note: If G has ' m ' elements & H has ' n ' elements, then $F = G \otimes H$ ' mn ' elements.

Elements of G can be represented as (g, e_H) is a sub group of F . Like wise $(e_G, h) \in H$ is a subgroup of F .

Simple group

A simple group is a non trivial group whose normal subgroups are only $\{e\}$ and the group itself.

Center of a group

The center of a group $(G, *)$ which is denoted by Z is the set of those elements of G that commute with every element of G .

$$Z = \{z \in G : zg = gz \ \forall g \in G\}$$

- $(Z, *)$ forms an abelian & normal subgroup of $(G, *)$.
- Identity element, $e \in Z$
- G is abelian $\iff Z = G$
- The center of non-abelian simple group is just $\{e\}$.

Cyclic Subgroup

Cyclic Subgroup of $(G, *)$ is a Subgroup that can be generated by a single element of the group.

Let $a \in G$, then set of all possible integer powers of 'a' is denoted by $\langle a \rangle$.

$$\langle a \rangle = \{ a^k : k \in \mathbb{Z} \}$$

where, $a^0 = e$.

$$a^k = (a * a * a \dots * a)_{k \text{ times}}, k \in \mathbb{Z}^+.$$

$$a^{-k} = (a^{-1} * a^{-1} * a^{-1} \dots * a^{-1})_{k \text{ times}}, k \in \mathbb{Z}^+.$$

$\langle a \rangle$ forms a cyclic subgroup of G which is also abelian.

Cyclic group

A group is said to be cyclic if all the elements of the group is generated by taking integer power of a generator element 'a'.

$(G, *)$ is cyclic if $G = \langle a \rangle$.

where $\langle a \rangle = \{a^k : k \in \mathbb{Z}\}$

Order of a group is the number of elements in the group.

A finite cyclic group can be represented by C_n , $n \in \mathbb{Z}^+$, where

$$C_n = \{a^k : k \in \mathbb{Z}, a^n = e\}$$

$$\Rightarrow C_n = \{e, a, a^2, \dots, a^{n-1}\}$$

Cayley table for C_n

*	e	a	a ²		a ^m		a ⁿ⁻¹
e	e	a	a ²		a ^m		a ⁿ⁻¹
a	a	a ²	a ³		a ^{m+1}		e
a ²	a ²	a ³	a ⁴		a ^{m+2}		a
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
a ^m	a ^m	a ^{m+1}	a ^{m+2}		a ^{2m}		a ^{m-1}
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
a ⁿ⁻¹	a ⁿ⁻¹	e	a		a ^{m-1}		a ⁿ⁻²

Cayley table for C_7

*	e	a	a ²	a ³	a ⁴	a ⁵	a ⁶
e	e	a	a ²	a ³	a ⁴	a ⁵	a ⁶
a	a	a ²	a ³	a ⁴	a ⁵	a ⁶	e
a ²	a ²	a ³	a ⁴	a ⁵	a ⁶	e	a
a ³	a ³	a ⁴	a ⁵	a ⁶	e	a	a ²
a ⁴	a ⁴	a ⁵	a ⁶	e	a	a ²	a ³
a ⁵	a ⁵	a ⁶	e	a	a ²	a ³	a ⁴
a ⁶	a ⁶	e	a	a ²	a ³	a ⁴	a ⁵

Cosets of a group

Let $(H, *)$ be a subgroup of $(G, *)$. Given an element $g \in G$, the cosets of H in G are the sets obtained by multiplying each element of H by g .

There are two types of cosets:

- Left coset: $gH = \{g * h : \forall h \in H\}$
- Right coset: $Hg = \{h * g : \forall h \in H\}$

Property of cosets (same for right cosets)

- $g \in gH$ as $e \in H$
- $gH = H$ iff $g \in H$
- $(g_1 g_2)H = g_1(g_2 H)$ ($g_1 g_2 \equiv g_1 * g_2$)
- $g_1 H = g_2 H$, iff $g_1 \in g_2 H$
- Either $g_1 H = g_2 H$ or $g_1 H \cap g_2 H = \{ \}$
- $g_1 H = g_2 H \iff g_1 g_2^{-1} \in H$
- $g_1 H \neq g_2 H$ have same no: of elements

Conjugacy between two elements

Two elements 'a' & 'b' of a group are said to be conjugate if there exists a third element 'g' where,
 $a = g b g^{-1}$ ('a' is conjugate to 'b')

- Conjugacy is symmetric:

If 'a' is conjugate to 'b', then 'b' is conjugate to 'a'.

- Conjugacy is reflexive:

'g' is conjugate to 'g' itself.

- Conjugacy is transitive:

If 'a' is conjugate to 'b' & if 'b' is conjugate to 'c', then 'a' is conjugate to 'c'.

Hence conjugacy forms an equivalence relation (reflexive, symmetric & transitive)

Conjugacy Class

Since conjugacy is an equivalence relation, it partitions a group into different classes (equivalence classes), such that every element of the class are conjugate to each other.

The conjugacy class of ' a ' $\in G$ is,
$$Cl(a) = \{gag^{-1} \mid \forall g \in G\}$$

Two conjugacy classes $Cl(a) = Cl(b)$ iff ' a ' is conjugate to ' b '. Either two conjugacy classes are equal or disjoint. So no different conjugacy classes overlap. Every group element belongs to only one conjugacy class.

The class number of ' G ' is the number of different conjugacy classes that G can be partitioned into.

Properties of Conjugacy classes:

- $Cl(e) = \{e\}$ where e is Identity.
- G is abelian $\Leftrightarrow Cl(g) = \{g\}, \forall g \in G$
- If ' a ' & ' b ' are conjugate, then even a^n & b^n ($n \in \mathbb{Z}$) are conjugate.
- $Cl(a) = \{a\} \Leftrightarrow 'a' \in \text{centre}, Z(G)$.
- Set of inverses of elements of a class also form a class.
- $\forall g_1, g_2 \in G, g_1 g_2 \in Cl(g_2 g_1)$.
- Two subsets S & T of G need not be subgroups are conjugate if $\exists g \in G : T = g S g^{-1}$. And $Cl(S)$ is the set of all subsets which are conjugate to S .

Quotient groups

Let $(N, *)$ be a normal subgroup of $(G, *)$. Then the quotient group is the set denoted by G/N is the set of all cosets of N in G . i.e.,

$$G/N = \{gN, \forall g \in G\}$$

Note: $gN = Ng \quad \forall g \in G$.

We can define a binary operation called Coset multiplication as follows. For each $aN \neq bN \in G/N$, $(aN)(bN) = (aNb)N = (ab)N$. This is because N is normal subgroup.

Then the set of cosets G/N , with binary operation as Coset multiplication will form a group structure called as Quotient group / factor group.

Group Homomorphism

Let $(G, *)$ & $(H, \cdot)$ be two groups. A group homomorphism from $(G, *)$ to $(H, \cdot)$ is a function $f: G \rightarrow H$ such that for all $g_1, g_2 \in G$ & $f(g_1), f(g_2) \in H$,

$$f(g_1 * g_2) = f(g_1) \cdot f(g_2)$$

$$\Rightarrow f(e_G) = e_H \text{ \& } f(g^{-1}) = [f(g)]^{-1}$$

Group Isomorphism

It is a bijective (one-one & onto) group homomorphism. If there exists group isomorphism between two groups G & H , then the groups G & H are said to be isomorphic & have exactly same properties and need not be distinguished in terms of group theory.

Kernel of group homomorphism

Consider two groups $(G, *)$ & $(H, \cdot)$ which have a group homomorphism ' f '. Then the kernel of group homomorphism ' f ' is the set of all those elements of G which are mapped to the identity element of H .

$$K = \text{Ker}(f: G \rightarrow H) = \{g \in G : f(g) = e_H\}$$

$(K, *)$ is a normal subgroup of $(G, *)$.

$$\begin{aligned} \text{Let } k \in K &\Leftrightarrow f(k) = e_H \\ \Leftrightarrow f(gkg^{-1}) &= f(g) \cdot f(k) \cdot f(g)^{-1} \\ &= f(g) \cdot f(g)^{-1} \\ &= e_H \end{aligned}$$

$\Leftrightarrow gkg^{-1} \in K$. Thus K is normal.

Note: Kernel of group isomorphism $f: G \rightarrow H$ is just $\{e_G\}$.

Additive group of Integer Modulo n

It is a group based on modular arithmetic. This type of groups are denoted by $\mathbb{Z}_n$ $n \in \mathbb{Z}$ (Integers).

$$\mathbb{Z}_n = \{m : m \in \mathbb{Z}, 0 \leq m < n\}$$

The elements of $\mathbb{Z}_n$ are given by all possible remainders when dividing any integer by ' n '. $\mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$

The group operation is given by addition modulo n , denoted by $\oplus_n$.

$$\forall a, b \in \mathbb{Z}_n, a \oplus_n b = (a+b)_{\text{mod } n}$$

- Additive identity is 0.
- Additive inverse of $m \in \mathbb{Z}_n$ is $(n-m)$

$\mathbb{Z}_n$ is a cyclic group of order n .

Symmetric group (S_n)

Consider a set $X_n = \{1, 2, 3, \dots, n\}$. Then the permutation of a finite set X_n is a bijective function (map) from the set X_n to itself.

Symmetric group S_n is a group on a finite set $X_n = \{1, 2, 3, \dots, n\}$ whose elements are all bijective functions from $X_n \rightarrow X_n$ & whose group operation is the composition of functions, i.e., S_n contains all the possible permutations of the set X_n , first 'n' positive integers.

Thus the order of S_n is $n!$ which is the total number of permutations of 'n' elements in 'n' slots.

S_n is non abelian group.

Eg: Consider the Symmetric group on the set $\{1, 2, 3, 4\}$

Let f & g be two bijective functions from $X_4 \rightarrow X_4$ of the form

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

Then,

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$f \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad g \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$f \circ f \circ f = \mathbb{I} \\ \Rightarrow f^{-1} = f \circ f$$

$$g \circ g \circ g = \mathbb{I} \\ \Rightarrow g^{-1} = g \circ g$$

$$g^{-1} \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \quad f^{-1} \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$f \circ g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad g \circ f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

Permutation group

A subgroup of a Symmetric group is called as Permutation group. A permutation group is essentially a selection of a set of permutations that are closed under the operation of composition of permutations.

By Cayley's theorem, every group is isomorphic to some permutation group.

Consider a cyclic subgroup of order 4.
 $\langle a \rangle = \{e, a, a^2, a^3\}$ where $a^4 = 1$.

This is isomorphic to the following permutations of $\{1, 2, 3, 4\}$.

If, $(1234) \xrightarrow{a} (2341)$, then

$(1234) \xrightarrow{a^2} (3412), (1234) \xrightarrow{a^3} (4132)$

& $(1234) \xrightarrow{a^4} (1234)$.

$\rightarrow \{(1234), (2341), (3412), (4132)\}$ is isomorphic to the group $\{e, a, a^2, a^3\}$.

A permutation is even if it decomposes into even number of exchanges.

A permutation is odd if it decomposes into odd number of exchanges.

$$\text{Eg: } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \Rightarrow \text{even : 0 exchanges}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \Rightarrow \text{odd : 1 exchange}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \Rightarrow \text{even : 2 exchanges}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \Rightarrow \text{even : 2 exchanges}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \Rightarrow \text{odd : 3 exchanges}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \Rightarrow \text{even : 4 exchanges}$$

Alternating group (A_n)

Alternating group (A_n) is a group of all the possible even permutations of a set $X_n = \{1, 2, 3, \dots, n\}$.

It is a subgroup of S_n .

S_n has $n!$ number of elements. And they can be of two types:
 $\frac{n!}{2}$ of them are odd permutations.

$\frac{n!}{2}$ of them are even permutations.

All these even permutations form a group called Alternating group ' A_n ' or order $\frac{n!}{2}$.

A_n is abelian iff $n \leq 3$ & Simple only if $n = 3$ or $n \geq 5$.

Matrix Addition group

Consider a set of all $m \times n$ Matrices whose entries are taken from another group $(G, +)$. This set is denoted as $M_{n \times m}(G)$. An element of this set is typically of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} = [a_{ij}]_{m \times n}$$

where an entry at i^{th} row & j^{th} column is given by $a_{ij} \in (G, +)$

Then, along with matrix addition as the binary operation $(M_{n \times m}(G), +)$ forms a Matrix addition group.

$$\forall A, B \in (M_{n \times m}(G), +), A = [a_{ij}] \text{ and } B = [b_{ij}] \Rightarrow A + B = [a_{ij} + b_{ij}]$$

Field (F)

It is a non empty set with two binary operations called addition & multiplication satisfy the following.

Let $(F, +, \cdot)$ be a field. Then
 $\forall a, b, c \in F$, addition: $a+b$
& multiplication: $a \cdot b$ or ab .

(i) $(F, +)$ forms an abelian group with '0' as the additive identity.

(ii) $(F - \{0\}, \cdot)$ forms an abelian group with '1' as the multiplicative identity.

(iii) Multiplication distributes over addition.
i.e. $(a+b)c = ac + bc$

Eg: Rational numbers, Real numbers
& Complex numbers are some fields.

General Linear Groups 'GL'

They are groups formed by a set of all invertible matrices with matrix multiplication as the binary operation.

$GL(n, \mathbb{F})$ is a set of all $n \times n$ matrices with entries from field $\mathbb{F}$ which are all invertible, i.e. whose determinant is not zero.

For $n > 2$, $GL(n, \mathbb{F})$ is not abelian.

Identity element is $I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n \times n}$
where 1 is identity of $\mathbb{F}$.

The center of $GL(n, \mathbb{F})$ is $\alpha I_n = \begin{bmatrix} \alpha & 0 & 0 & \dots & 0 \\ 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha \end{bmatrix}$
where $\alpha \in \mathbb{F}$.

The subgroups of $GL(n, \mathbb{F})$ are usually called matrix groups.

Special Linear Group 'SL'

Special linear group is a subgroup of General linear group such that the determinant is $+1$. It is given by $SL(n, F)$ which is the set of all $n \times n$ matrices with entries from field F such that the determinant is 1 .

$SL(n, F)$ is normal subgroup of $GL(n, F)$.

for $A \in SL(n, F)$ & any $B \in GL(n, F)$

$$\begin{aligned} \det(BAB^{-1}) &= \det(B) \det(A) \det(B^{-1}) \\ &= \det(B) \frac{1}{\det(B)} \det(A) = \det(A) = 1 \end{aligned}$$

$$\Rightarrow BAB^{-1} \in SL(n, F) \quad \forall A \in SL(n, F) \\ \& \quad \forall B \in GL(n, F)$$

The center of $SL(n, F)$ is just $\{I_n\}$

The proper subgroups of $SL(n, F)$ which are normal is just $\{I_n, -I_n\}$

Cyclic group of order n ($n \rightarrow \text{finite}$)

A cyclic group is a symmetry group of discrete rotations which corresponds to symmetric rotations of a n -sided polygon.

$$C_n = \{e, a, a^2, a^3, \dots, a^{n-1} : a^n = e\}$$

A representation of cyclic group of order n is given by complex numbers of magnitude 1 or the rotation matrices, etc.

$$a \approx R\left(\frac{2\pi}{n}\right) \approx e^{i\left(\frac{2\pi}{n}\right)}$$

where $R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

$$\& e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Dihedral group of order $2n$ (D_n)

This is a symmetry group of regular polygon of n -sides which consists of n -discrete rotational symmetries & n -reflection symmetries.

The set of all n -rotation symmetry forms C_n which is subgroup of D_n .

$D_n = C_n \cup bC_n$ where it has two generators a & b , which satisfy $a^n = e$, $b^2 = e$, $(ba)^2 = e \Rightarrow aba = b$
 $\Rightarrow ab = ba^{-1}$, $a^2b = b(a^{-1})^2$, $a^mb = b(a^{-1})^m$

$a \Rightarrow$ represents rotation

$b \Rightarrow$ represents reflection about an axis.

$$D_n = \{e, a, a^2, \dots, a^{n-1}, b, ba, ba^2, \dots, ba^{n-1}\}$$

$(n > 3) \quad \text{order}(D_n) = 2n$

Cayley table for D_3

*	e	a	a ²	b	ba	ba ²
e	e	a	a ²	b	ba	ba ²
a	a	a ²	e	ba ²	b	ba
a ²	a ²	e	a	ba	ba ²	b
b	b	ba	ba ²	e	a	a ²
ba	ba	ba ²	b	a ²	e	a
ba ²	ba ²	b	ba	a	a ²	e

Cayley table for D_4

*	e	a	a ²	a ³	b	ba	ba ²	ba ³
e	e	a	a ²	a ³	b	ba	ba ²	ba ³
a	a	a ²	a ³	e	ba ³	b	ba	ba ²
a ²	a ²	a ³	e	a	ba ²	ba ³	b	ba
a ³	a ³	e	a	a ²	ba	ba ²	ba ³	b
b	b	ba	ba ²	ba ³	e	a	a ²	a ³
ba	ba	ba ²	ba ³	b	a ³	e	a	a ²
ba ²	ba ²	ba ³	b	ba	a ²	a ³	e	a
ba ³	ba ³	b	ba	ba ²	a	a ²	a ³	e

Conjugate relations in D_n

- $b^{-1}ab = a^{-1} \Rightarrow a \sim a^{-1}$
 $\Rightarrow b^{-1}a^m b = (a^{-1})^m \Rightarrow \underline{\underline{a^m \sim (a^{-1})^m}}$
- $aba = b \Rightarrow a(ba)a^{-1} = ba^{-1} \Rightarrow ba \sim ba^{-1}$
- $aba = b \Rightarrow aba^2 = ba$
 $\Rightarrow a(ba^2)a^{-1} = b \Rightarrow \underline{\underline{ba^2 \sim b}}$
- $aba = b \Rightarrow aba^3 = ba^2$
 $\Rightarrow a(ba^3)a^{-1} = ba \Rightarrow (ba^3) \sim ba \sim ba^{-1}$
- $a(ba^4)a^{-1} = ba^2 \Rightarrow (ba^4) \sim (ba^2) \sim b$
 $\Rightarrow b \sim ba^2 \sim ba^4 \sim ba^6 \dots$
 $ba \sim ba^3 \sim ba^5 \sim ba^7 \dots$

If n is even then reflections divide in two conjugate classes. But if n is odd, then all reflections are equivalent.

Conjugacy classes in dihedral group for even n are,

$$\{e\}, \{a, a^{n-1}\}, \{a^2, a^{n-2}\}, \dots, \{a^{n/2}\} \\ \{b, ba^2, \dots, ba^{n-2}\}, \{ba, ba^3, \dots, ba^{n-1}\}$$

$$\left. \begin{array}{l} 1: \text{identity class} \\ \frac{n}{2}: \text{rotation class} \\ 2: \text{reflection class} \end{array} \right\} \frac{n}{2} + 3 \text{ no: of conjugacy classes}$$

Conjugacy classes in dihedral group for odd n are,

$$\{e\}, \{a, a^{n-1}\}, \{a^2, a^{n-2}\}, \dots, \{a^{\frac{n-1}{2}}, a^{\frac{n+1}{2}}\} \\ \{b, ba, ba^2, ba^3, \dots, ba^{n-1}\}$$

$$\left. \begin{array}{l} 1: \text{identity class} \\ \frac{n-1}{2}: \text{rotation classes} \\ 1: \text{reflection class} \end{array} \right\} \frac{n+3}{2} \text{ no: of conjugacy classes.}$$

Vector Space

A non-empty set V consisting of vectors along with a field F consisting of scalars with two binary operations called Vector addition and Scalar multiplication are said to form a Vector Space if they satisfy the following.

Let $\vec{u}, \vec{v} \in V$ and $\alpha, \beta \in F$,
vector addition is denoted by $\vec{v} + \vec{u}$
& scalar multiplication is done b/w
a scalar & a vector as $\alpha \vec{v}$.

- (i) $(V, +)$ form an additive abelian group.
with $\vec{0}$ as additive identity.
- (ii) $\alpha \vec{v} = \vec{v} \alpha \in V$
- (iii) $\alpha(\beta \vec{v}) = (\alpha\beta) \vec{v}$
- (iv) $1(\vec{v}) = \vec{v}$
- (v) $\alpha(\vec{v} + \vec{u}) = \alpha \vec{v} + \alpha \vec{u}$
- (vi) $(\alpha + \beta) \vec{v} = \alpha \vec{v} + \beta \vec{v}$

Operators of a vector space

Consider a n -dim Vector Space V with an associated field F .

A linear operation is defined as a linear map $M: V \rightarrow V$ which maps every vector $\vec{v}$ to some other vector in the same vector space.

Any linear operation can be recognized as a $n \times n$ matrix acting on a vector that can be recognized as a $n \times 1$ column matrix in some basis, whose matrix entries are taken from the associated field ' F '.

Thus every square matrix represents some operation on the vectors of some vector space.

General Linear Group of a Vector Space

Consider a n -dim Vector Space V with an associated field F .

An invertible linear operation is a linear map $M: V \rightarrow V$ which is also a bijection. So for every map M there also exists an inverse map M^{-1} such that the composition of maps $M^{-1} \circ M = M \circ M^{-1} = I$ (identity map).

All invertible linear operators on an n -dim vector space can be recognised as a set of all $n \times n$ invertible matrices whose entries are from the field F , & their determinant is non-zero, with matrix multiplication as group operation.

This constitutes the general linear group on a vector space $GL(V)$

Representation Theory of Groups

A representation of a group $(G, *)$ on a vector space V over a field F is a group homomorphism $\varphi: G \rightarrow GL(V)$.

Thus representation of a group is just mapping of $g \in G$ to $\varphi(g) \in GL(V)$ such that $\forall g_1, g_2 \in G, \exists \varphi(g_1), \varphi(g_2)$ in $GL(V)$ such that,

$$\varphi(g_1 * g_2) = \varphi(g_1) \overset{\text{matrix multiplication}}{\varphi(g_2)}$$

Here V is called as representation space & dimension of V is also the dimension of representation.

Kernel of representation is the set of all those $g \in G$ whose $\varphi(g) = I_n$.

If $\text{Ker}(\varphi) = \{e\}$ then φ is faithful.

Equivalent Representations of a Group in Same Dimensions

The matrix representation of a group on a vector space V depends on the choice of basis. Since two bases are related by a similarity transformation, then we can have many possible representation for the same group.

i.e. If $g \rightarrow \rho(g)$ is a particular choice of representation that depends on the chosen basis for ' V ', then under the change of basis by a similarity transformation ' S ', every matrix operator $M \rightarrow S^{-1}MS$. So $\rho(g) \rightarrow \rho'(g)$ where $\rho'(g) = S^{-1}\rho(g)S$ is also an equivalent representation of $(G, *)$ on V .

$\{\rho(g)\}$ & $\{S^{-1}\rho(g)S\}$, $S \in GL(V)$ are equivalent representations of $(G, *)$ on V .

Isomorphic Representations of a Group

(applicable for different dimensions)

Let $(G, *)$ have two representations on the vector spaces V & W over some field F .
Then the representations $\rho(g) \in GL(V)$ & $\rho(g) \in GL(W)$ are said to be isomorphic, if there exists a vector space isomorphism $V \rightarrow W$ such that there exists a bijective map b/w two representations $\{\rho(g)\}$ & $\{\rho(g)\}$.

Eg: Consider $U(1)$ group which is
$$U(1) = \{e^{i\theta} : i^2 = -1, \theta \in \mathbb{R}\}$$

This group is isomorphic to the set of all matrices of the form $R(\theta)$:

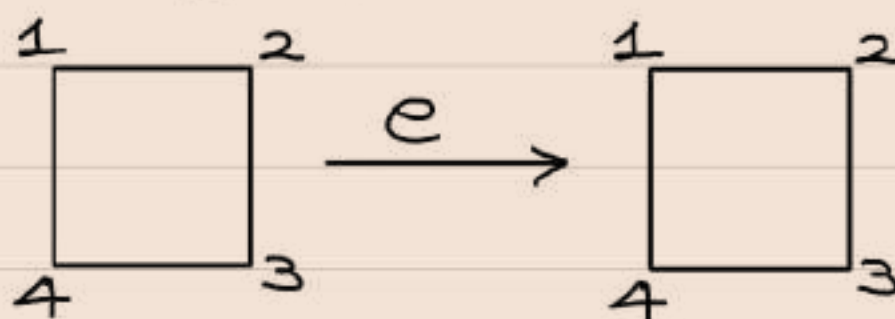
$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(2).$$

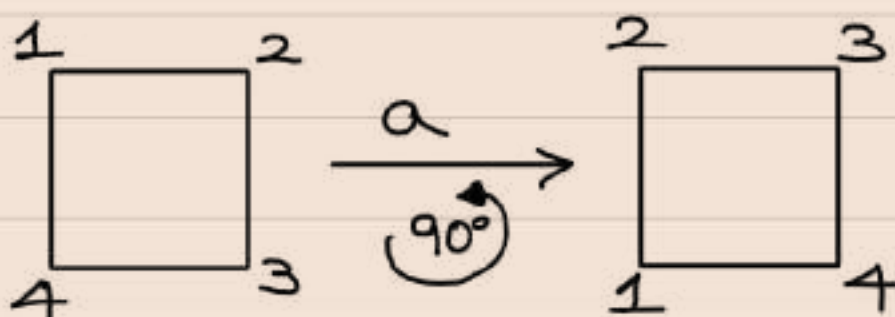
Thus for the group $C_n = \langle a \rangle : a^n = e$, the representations $\{R(\frac{2\pi}{n})\}$ & $\{e^{i(\frac{2\pi}{n})}\}$ are isomorphic.

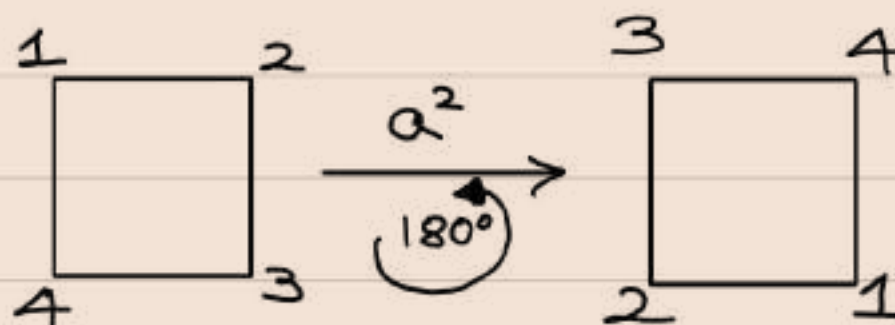
2D representation for D_4 group

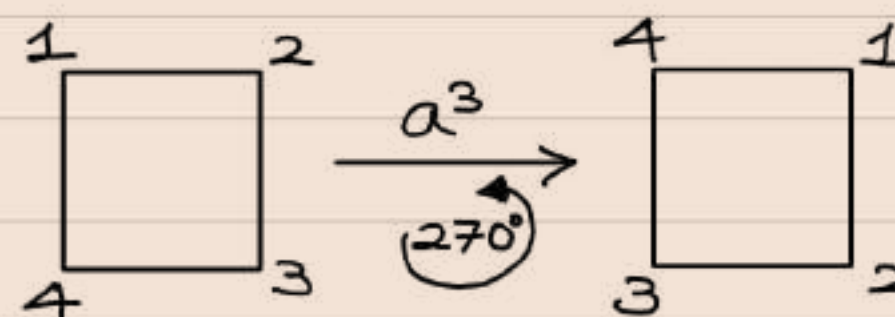
D_4 consists of 8 Symmetry transformation of a Square. Thus it has 8 elements with 4 rotations & 4 reflections.

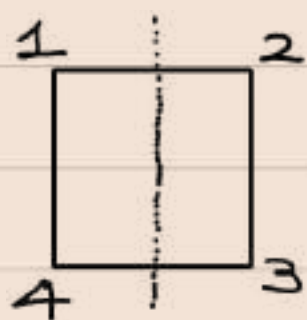
The group elements & representations.


$$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

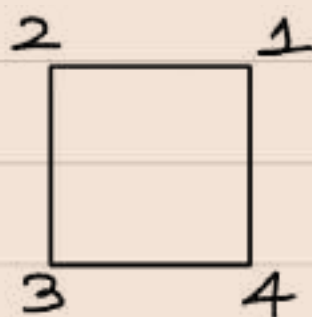

$$a = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$


$$a^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

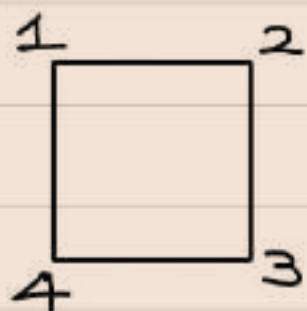

$$a^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



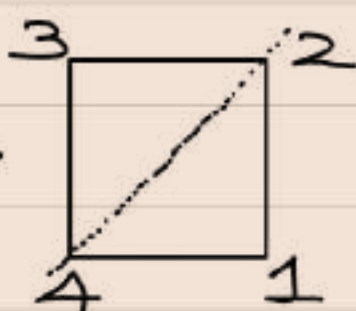
$b \rightarrow$



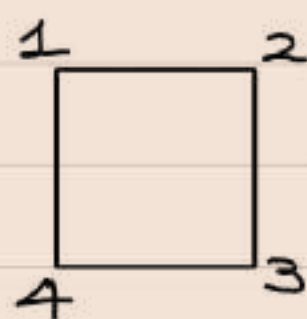
$$b = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$ba \rightarrow$



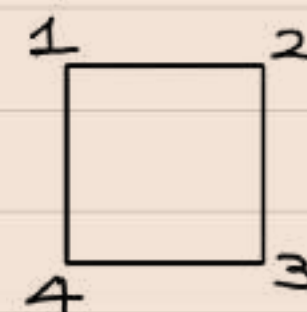
$$ba = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



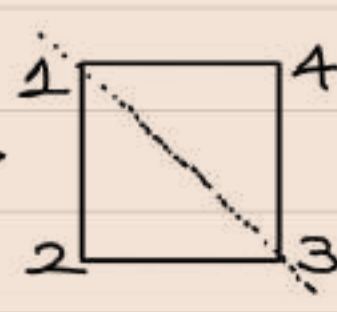
$ba^2 \rightarrow$



$$ba^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$ba^3 \rightarrow$



$$ba^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{2D \text{ representation}}$

4D representations

$$\mathbb{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$a^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$a^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$ba = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ba^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$ba^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The AD representation of D_4 above is done by permutation matrices which are orthogonal i.e. Inverse = Transpose.

Permutation Matrix Group

Permutation matrices are defined as a set of $n \times n$ matrices whose entries are all 0's or 1's such that in each row and in each column, '1' appears only once & all the other entries are 0.

The set of all $n \times n$ permutation matrices forms a faithful representation of the symmetric group S_n . Hence the total no: of $n \times n$ permutation matrices is given by $n!$. Thus $n \times n$ permutation matrices form a group $P(n)$ of order ' $n!$ '.

Every odd (or even) permutation has a corresponding permutation matrix whose determinant is -1 (or $+1$). Thus the alternating group A_n is the set of all permutations whose corresponding permutation matrix has determinant $+1$.

Direct Sum of Representations

A matrix direct sum is given by the following. Consider two matrices $A_{m \times n}$ & $B_{k \times l}$. Then their direct sum is,

$$A_{m \times n} \oplus B_{k \times l} = \begin{bmatrix} A_{m \times n} & \overset{\text{Zero matrix}}{O_{m \times l}} \\ O_{k \times n} & B_{k \times l} \end{bmatrix}_{(m+k) \times (n+l)}$$

Thus the direct sum of two square matrices of dim $(m \times m)$ & $(n \times n)$ results in a bigger matrix of dimensions of $(m+n) \times (m+n)$ written in the block diagonal form. Similarly, the result of many matrices can be written as a block diagonal matrix.

$$A_1 \oplus A_2 \oplus A_3 \dots \oplus A_p = \begin{bmatrix} A_1 & O & O & \dots & O \\ O & A_2 & O & \dots & O \\ O & O & A_3 & \dots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & A_p \end{bmatrix}$$

Reducibility of a Representation

A representation of a group is said to be reducible if there exists a similarity matrix 'S' such that, $\forall g \in G$, the matrix $\rho(g)$ can be made into another equivalent representation $S^{-1}\rho(g)S$ such that $S^{-1}\rho(g)S$ can be written as block diagonal matrix, where each diagonal block is also a representation of $g \in G$, but of a lower lower dimension.

$\rho: G \rightarrow GL(V)$ is reducible, if $\exists S \in GL(V)$ such that $\forall g \in G$,

$$S^{-1}\rho(g)S = \begin{bmatrix} \rho^{(1)}(g) & 0 & 0 & \dots & 0 \\ 0 & \rho^{(2)}(g) & 0 & \dots & 0 \\ 0 & 0 & \rho^{(3)}(g) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \rho^{(k)}(g) \end{bmatrix}$$

block diagonal $\leftarrow$

Where $\{\rho^{(i)}(g)\}$ is some lower dimensional representation of G over a vector space $W_i \Rightarrow \rho^{(i)}: G \rightarrow GL(W_i)$ such that $\sum_i \dim(W_i) = \dim(V)$.

$\rho^{(i)}: G \rightarrow GL(W_i)$ is called as Subrepresentation of $\rho: G \rightarrow GL(V)$.

A representation is reducible if it can be decomposed into atleast two subrepresentations which can be same (as in multiple copy of some smaller subrepresentation) or different ones.

Irreducible representation

A representation is said to be irreducible if it is not reducible. i.e. $\{\rho(g)\}$ can't be transformed into block diagonal form by any similarity transformation. (It has no proper sub repsⁿ).

Eg: $U(1)$ group can be represented by $\{e^{i\theta}\}$ as a 1×1 matrix. So it is a 1D representation. But it also has a 2D representation given by $R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. But using

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \text{ \& } S^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

we get $S^{-1}R(\theta)S$ as $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$

Thus $S^{-1}R(\theta)S$ is block diagonal & $\{e^{i\theta}\}, \{e^{-i\theta}\}$ are 1D repⁿ of $U(1)$. Thus $R(\theta)$ is a reducible repⁿ for $U(1)$ in field of $\mathbb{C}$ numbers.

But $R(\theta)$ is irreducible over a field of Real numbers as no real matrix 'S' exists to diagonalize $R(\theta)$.

Group Action

Consider a group $(G, *)$ & some set X .

The left action of the group on X is a map $A: G \times X \longrightarrow X$ such that

$$a) A(e, x) = x \quad \forall x \in X, \text{ with } e = \text{Identity of } G.$$

$$b) A(g_1 * g_2, x) = A(g_1, A(g_2, x)) \\ \forall g_1, g_2 \in G \quad \& \quad x \in X.$$

The right action of the group on X is a map $A: X \times G \longrightarrow X$ such that

$$a) A(x, e) = x \quad \forall x \in X, \text{ with } e = \text{Identity of } G.$$

$$b) A(x, g_1 * g_2) = A(A(x, g_1), g_2) \\ \forall g_1, g_2 \in G \quad \& \quad x \in X.$$

Thus the group is said to act on X .

The action of a group on a set X is just a set of bijective functions that also has a group structure with function composition as operation. Thus we can also define group action as a group homomorphism from G to symmetric group of X of all bijections $X \rightarrow X$.

Orbit of $x \in X$; $A(G, x)$

$$A(G, x) = \{A(g, x), \forall g \in G\}$$

$A(G, x)$ is the set of elements in X to which x can be mapped to by $g \in G$.

Stabilizers of $x \in X$; G_x

$G_x = \{g \in G : A(g, x) = x\}$. Thus G_x are those group elements which keep x invariant. G_x forms a subgroup.

Free Action: A is free iff $G_x = \{e\} \forall x \in X$.

Group Representation as a Group Action

Any representation $\varphi: G \rightarrow GL(V)$ defines an action Φ of the group $(G, *)$ on the vector space V , where $\Phi: G \times V \rightarrow V$ such that

$$\underline{\Phi(g, \vec{v}) = \varphi(g) \vec{v}}$$

The transformation of an operator $\hat{T}$ of the vector space V under a group G is given by $\varphi(g) \hat{T} \varphi(g)^{-1}$.

So consider a system with a governing hamiltonian $\hat{H}$. We can say that the system or the Hamiltonian has a symmetry of a group G , if the operator $\hat{H}$ is invariant under the action of any group element on it.

$$\left. \begin{aligned} \forall g \in G \rightarrow \varphi(g) \hat{H} \varphi(g)^{-1} &= \hat{H} \\ \rightarrow \underline{\varphi(g) \hat{H} &= \hat{H} \varphi(g)} \end{aligned} \right\} \hat{H} \text{ commutes with } \{ \varphi(g) \}.$$

Schur's Lemma 1

Let $\varphi: G \rightarrow GL(V)$ & $\rho: G \rightarrow GL(W)$ be two irreducible representations of a group $(G, *)$. Let M be a linear map $M: V \rightarrow W$ such that

$$M\varphi(g) = \rho(g)M \quad \forall g \in G$$

then either M is an isomorphism, i.e., φ & ρ are equivalent or $M = 0$ map.

Schur's Lemma 2

Let $\varphi: G \rightarrow GL(V)$ be an irreducible representation of a group $(G, *)$ and let $T \in$ Some operator on V . If $T\varphi(g) = \varphi(g)T \quad \forall g \in G$ then T is either identity or it is a diagonal matrix with all entries same (Scalar matrix) of the form λI_n [$n \rightarrow \dim(V)$].

Character of a group element

Let $\varphi: G \rightarrow GL(V)$ be a representation of a group $(G, *)$ on V over a field F .

Then the character of an element is a map $\text{Char}: G \rightarrow F$ given by,

$$\underline{\text{Char}(g) = \text{Trace}[\varphi(g)]}$$

Thus $\text{Char}(g)$ is independent of the choice of basis for V .

The character of identity element is always $\text{Trace}(\mathbb{I}_n) = \dim(V)$.

If two elements of ' G ' belong to same conjugacy class, then the two elements have equal character.

Irreducible repⁿ $\Rightarrow$ Simple character.

Reducible repⁿ $\Rightarrow$ Compound character.

Orthogonality of Irreducible Repⁿ

Let $\varphi: G \rightarrow GL(V)$ be a reducible representation of a group $(G, *)$ on V over field F . Thus without loss of generality we can write

$$\varphi(g) = \begin{bmatrix} P^{(1)}(g) & 0 & 0 & \cdots & 0 \\ 0 & P^{(2)}(g) & 0 & \cdots & 0 \\ 0 & 0 & P^{(3)}(g) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P^{(k)}(g) \end{bmatrix}$$

Where $\{P^{(\alpha)}(g)\}$ is some lower dimensional irreducible representation of G over a vector space W_α where $P^{(\alpha)}: G \rightarrow GL(W_\alpha)$ is such that

$$\sum_{\alpha} \dim(W_\alpha) = \dim(V).$$

$P^{(\alpha)}: G \rightarrow GL(W_\alpha)$ is called as the subrepresentation of $\varphi: G \rightarrow GL(V)$.

Consider two irreducible repⁿ of G given by $\{\rho^{(\alpha)}(g)\} \neq \{\rho^{(\beta)}(g)\}$. Let $M^{(\alpha\beta)}$ be a linear map $M^{(\alpha\beta)}: W_\alpha \rightarrow W_\beta$.

Define

$$A^{(\alpha\beta)} = \sum_{g \in G} \rho^{(\alpha)}(g) M^{(\alpha\beta)} \rho^{(\beta)}(g^{-1})$$

$$\text{Then } \rho^{(\alpha)}(\tilde{g}) A^{(\alpha\beta)} = \sum_{g \in G} \rho^{(\alpha)}(\tilde{g}) \rho^{(\alpha)}(g) M^{(\alpha\beta)} \rho^{(\beta)}(g^{-1})$$

$$= \sum_{g \in G} \rho^{(\alpha)}(\tilde{g}g) M^{(\alpha\beta)} \rho^{(\beta)}(g^{-1})$$

$$= \sum_{g \in G} \rho^{(\alpha)}(\tilde{g}g) M^{(\alpha\beta)} \rho^{(\beta)}((\tilde{g}g)^{-1}) \rho^{(\beta)}(\tilde{g}g) \rho^{(\beta)}(g^{-1})$$

$$= \left[\sum_{g \in G} \rho^{(\alpha)}(\tilde{g}g) M^{(\alpha\beta)} \rho^{(\beta)}((\tilde{g}g)^{-1}) \right] \rho^{(\beta)}(\tilde{g})$$

Sum over all $g \in G$

$$= A^{(\alpha\beta)} \rho^{(\beta)}(\tilde{g})$$

$$\Rightarrow \text{If } A^{(\alpha\beta)} = \sum_{g \in G} \rho^{(\alpha)}(g) M^{(\alpha\beta)} \rho^{(\beta)}(g^{-1})$$

$$\underline{\rho^{(\alpha)}(g) A^{(\alpha\beta)} = A^{(\alpha\beta)} \rho^{(\beta)}(g)} \quad (\text{for any } M)$$

But $\{\rho^{(\alpha)}(g)\} \neq \{\rho^{(\beta)}(g)\}$ is irreducible repⁿ of G . By Schur's lemma,

If $\rho^{(\alpha)}(g) \cong \rho^{(\beta)}(g)$ i.e. they are equivalent then for $\alpha = \beta$ we get,

$$\Rightarrow A^{\alpha\beta} = \lambda_{M^{(\alpha\beta)}} I_{n_\alpha}$$

If they are not equivalent, then, for $\alpha \neq \beta$ $A^{\alpha\beta} = 0$.

Thus we can combine two relations

$$\sum_{g \in G} \rho^{(\alpha)}(g) M^{(\alpha\beta)} \rho^{(\beta)}(g^{-1}) = \lambda_{M^{(\alpha\beta)}} I_{n_\alpha} \delta_{\alpha\beta}$$

Choosing $M^{(\alpha\beta)}$ such that it is all zero except at l^{th} row & m^{th} column

$$\Rightarrow \sum_{g \in G} \rho_{il}^{(\alpha)}(g) \rho_{mj}^{(\beta)}(g^{-1}) = \lambda_{ml} \delta_{ij}^{(\alpha)} \delta_{\alpha\beta}$$

If $\alpha = \beta$ & summing over 'i'
 λ_{ml} can be obtained as follows.

$$\lambda_{ml} \delta_{ij}^{(\alpha)} \delta_{\alpha\beta} = \sum_{g \in G} \rho_{il}^{(\alpha)}(g) \rho_{mj}^{(\beta)}(g^{-1})$$

$$\sum_i \lambda_{ml} \delta_{ii}^{(\alpha)} = \sum_{g \in G} \sum_i \underbrace{\rho_{mi}^{(\alpha)}(g^{-1}) \rho_{il}^{(\alpha)}(g)}_{\text{matrix multiplication}}$$

$$\lambda_{ml} \underbrace{\sum_i \delta_{ii}^{(\alpha)}}_{n_\alpha} = \sum_{g \in G} \left[\rho^{(\alpha)}(g^{-1}) \rho^{(\alpha)}(g) \right]_{ml}$$

$$\lambda_{ml} n_\alpha = \sum_{g \in G} \rho_{ml}^{(\alpha)}(e)$$

$$\lambda_{ml} = \frac{\delta_{ml}^{(\alpha)} |G|}{n_\alpha}$$

$$\rightarrow \sum_{g \in G} \rho_{il}^{(\alpha)}(g) \rho_{mj}^{(\beta)}(g)^{-1} = \frac{\delta_{ml}^{(\alpha)} |G| \delta_{ij}^{(\alpha)} \delta_{\alpha\beta}}{n_\alpha}$$

Now if we set $i=l$ & $m=j$

$$\sum_{g \in G} \rho_{ii}^{(\alpha)}(g) \rho_{jj}^{(\beta)}(g)^{-1} = \frac{[\delta_{ij}^{(\alpha)}]^2 |G| \delta_{\alpha\beta}}{n_\alpha}$$

& Sum over i & j we get,

$$\sum_{g \in G} \text{Char}^{(\alpha)}(g) \text{Char}^{(\beta)}(g^{-1}) = |G| \delta_{\alpha\beta}$$

Thus characters of different irreps of a group are orthogonal.

If a representation $\rho^{(\alpha)}$ is irreducible, then we get

$$\sum_{g \in G} \text{Char}^{(\alpha)}(g) \text{Char}^{(\alpha)}(g^{-1}) = |G|$$

In unitary repⁿ $\rho(g^{-1}) = \rho(g)^{\dagger}$
 $\Rightarrow \text{char}^{\alpha}(g^{-1}) = \text{char}^{(\alpha)}(g)^{*}$ $\downarrow$
complex conjugate

All representations are isomorphic to some unitary representation.

$$\text{So } \sum_{g \in G} \text{Char}^{(\alpha)}(g) \text{Char}^{(\alpha)}(g)^{*} = |G|$$

$$\text{or } \left\{ \sum_{g \in G} |\text{Char}^{(\alpha)}(g)|^2 = |G| \right\}$$

$\hookrightarrow$ Criteria for Irreducibility

Generating a Vector Space from a Group

Let $(G, *)$ be a group and let F be a field. Then the group vector space $F[G]$ is defined as the set of all linear combinations of elements of G with coefficients from F , where

$$F[G] = \left\{ \sum_{g \in G} a_g g : a_g \in F \text{ \& } g \in G \right\}$$

An element of $F[G]$ looks like

$$\sum_{g \in G} a_g g = a_{g_1} g_1 + a_{g_2} g_2 + a_{g_3} g_3 + \dots$$

(Note: $g \in G$, but $1g \in F[G]$, $1 = \text{unity of } F$.)
($0 \cdot g = \vec{0}$, $0 \in F$, $\vec{0} \in F[G]$, $g \in G$)

The set $F[G]$ forms a vector space under the vector addition '+' between two elements of $F[G]$ and scalar multiplication between an element of $F[G]$ & F . The group elements act as the basis of vector space.

Let $\vec{x} = \sum_{g \in G} a_g g$ & $\vec{y} = \sum_{g \in G} b_g g$

$\vec{x}, \vec{y} \in F[G]$ & Let $c \in F$, then

Vector addition: $\vec{x} + \vec{y} = \sum_{g, h \in G} (a_g + b_g) g$

& gh represents $g * h$
& $a_g b_h$ is the field multiplication

Scalar multiplication: $c\vec{x} = \sum_{g \in G} (ca_g) g$

We can also extend the idea to a vector multiplication ($\cdot$).

Let $\vec{x} = \sum_{g \in G} a_g g$ & $\vec{y} = \sum_{h \in G} b_h h$

$\Rightarrow \vec{x} \cdot \vec{y} = \sum_{g, h \in G} a_g b_h (gh)$ } Vector multiplication need not commute

Thus $F[G]$ also forms a Ring structure.
So $F[G]$ is also called group ring.

Representation of Group Ring $F[G]$

Representation of $F[G]$ can be done on an n -dimensional vector space V over the field F as a map φ from $F[G]$ to the set of all possible operations from $V \rightarrow V$ (Endomorphisms)

Let $x, y \in F[G]$ & $c \in F$, then

$$\varphi(\vec{x} + \vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$$

$$\varphi(c\vec{x}) = c\varphi(\vec{x})$$

$$\varphi(\vec{x} \cdot \vec{y}) = \varphi(\vec{x}) \circ \varphi(\vec{y})$$

↪ composition

In matrix repⁿ of V , the set of all operations will be $n \times n$ matrices & the vectors will be $n \times 1$ column vectors.

Thus representation of $F[G]$ can also be recognized as a map from $F[G]$ to all $n \times n$ matrices with entries from field F . So every element of $F[G]$ represent a linear matrix operator on V .

Algebra over a Vector Space

A vector space over a field $\mathbb{F}$ is said to form an algebra 'A' if it is equipped with an additional operation called as Algebra multiplication which is a map from $A \times A \rightarrow A$ which satisfies bilinearity.

Let $\vec{x}, \vec{y} \neq \vec{z} \in A, \alpha \in \mathbb{F}$

- $\vec{x} + \vec{y}$ denotes vector addition
- $\alpha \vec{x}$ denotes scalar multiplication
- $\vec{x} \times \vec{y}$ denotes Algebra multiplication

Bilinearity:

$$(i) \vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$$

$$(ii) (\vec{x} + \vec{y}) \times \vec{z} = \vec{x} \times \vec{z} + \vec{y} \times \vec{z}$$

$$(iii) (\alpha \vec{x}) \times \vec{y} = \vec{x} \times (\alpha \vec{y}) = \alpha (\vec{x} \times \vec{y})$$

(Associativity & commutativity not needed)

Associative algebra

Algebra multiplication is associative
i.e. $(\vec{x} \times \vec{y}) \times \vec{z} = \vec{x} \times (\vec{y} \times \vec{z})$

Unital Algebra

There exists an identity element
with respect to algebra multiplication
 $\exists 1_A \in A : \vec{x} \times 1_A = 1_A \times \vec{x} = \vec{x} \in A$

Commutative Algebra

$$\forall \vec{x}, \vec{y} \in A, \vec{x} \times \vec{y} = \vec{y} \times \vec{x}$$

Alternative Algebra

Algebra multiplication need not be
associative, but is alternative. i.e.

- $\vec{x} \times (\vec{x} \times \vec{y}) = (\vec{x} \times \vec{x}) \times \vec{y}$
- $(\vec{y} \times \vec{x}) \times \vec{x} = \vec{y} \times (\vec{x} \times \vec{x})$

Algebra of Matrix Multiplication

Set of all $n \times n$ square matrices with entries from a field F forms a vector space with usual Matrix addition & matrix scalar multiplication.

But the matrix multiplication is a bilinear operation between matrices that can be adopted as Algebra multiplication.

Since matrix multiplication is associative, and also as there exists Identity matrix for matrix multiplication, this forms an associative + unital algebra.

This is the broadest & and the most general algebraic structure of matrices which encompasses all other algebraic structures.

Group Algebra

As we saw earlier, a group 'G' along with a field $\mathbb{F}$ forms a vector space denoted by $\mathbb{F}[G]$ of the form.

$$\mathbb{F}[G] = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{F} \text{ \& } g \in G \right\}$$

So here the group serves as the basis of the vector space. The algebra multiplication is given as follows.

$$\text{Let } \vec{x} = \sum_{g \in G} a_g g \text{ \& } \vec{y} = \sum_{h \in G} b_h h$$

$$\vec{x}, \vec{y} \in \mathbb{F}[G] \text{ \& } a_g, b_h \in \mathbb{F} \text{ \& } g, h \in G$$

then, algebra multiplication can be,

$$\vec{x} \times \vec{y} = \sum_{g, h \in G} \underbrace{a_g b_h}_{\substack{\text{multiplication} \\ \text{of the field}}} (gh) \quad \xrightarrow{\text{group operation}}$$

- Group algebras are always associative since group & field elements are associative.
- Group algebras are always unital since $1_A = 1 \cdot e \in \mathbb{F}[G]$ such that $\vec{x} \times 1_A = 1_A \times \vec{x} = \vec{x}$
- If group is abelian, then the group algebra is commutative

Eg:

- Polynomials: form an associative, commutative & unital algebra.
- Consider a general linear group over a field $\mathbb{F}$, $GL(n, \mathbb{F})$. Then the group algebra $\mathbb{F}[GL(n, \mathbb{F})]$ forms a associative + unital algebra.

Structure Coefficients

For algebras A over a field F , the algebra multiplication $A \times A \rightarrow A$ is completely determined by how the algebra multiplication between the basis elements of A is defined.

Consider an algebra A with the basis set $\{\vec{e}_i\}$. Then the elements of A are given by, $\vec{x} = \sum_i x_i \vec{e}_i$

$$(x_i, y_j \in F) \quad \& \quad \vec{y} = \sum_j y_j \vec{e}_j$$

$$\Rightarrow \vec{x} \times \vec{y} = \sum_{i,j} x_i y_j (\vec{e}_i \times \vec{e}_j)$$

Then $\vec{x} \times \vec{y}$ also belongs to F

$$\Rightarrow \underline{(\vec{e}_i \times \vec{e}_j) = \sum_k C_{ijk} \vec{e}_k}, \quad C_{ijk} \in F$$

C_{ijk} are called Structure coefficients.

Lie Algebra

A lie algebra is a vector space $\mathfrak{g}$ over a field F together with a bilinear algebra multiplication specified by a lie bracket $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying,
($\vec{x}, \vec{y}, \vec{z} \in \mathfrak{g}, \alpha \in F$)

- Bilinearity:

$$[\vec{x}, \vec{y} + \vec{z}] = [\vec{x}, \vec{y}] + [\vec{x}, \vec{z}]$$

$$[\vec{x} + \vec{y}, \vec{z}] = [\vec{x}, \vec{z}] + [\vec{y}, \vec{z}]$$

$$[\alpha \vec{x}, \vec{y}] = [\vec{x}, \alpha \vec{y}] = \alpha [\vec{x}, \vec{y}]$$

- Anti-commutativity: $[\vec{x}, \vec{y}] = -[\vec{y}, \vec{x}]$
 $\Rightarrow [\vec{x}, \vec{x}] = 0$

- Jacobi-Identity:

$$[\vec{x}, [\vec{y}, \vec{z}]] + [\vec{z}, [\vec{x}, \vec{y}]] + [\vec{y}, [\vec{z}, \vec{x}]] = 0$$

A lie algebra need not be associative or commutative nor unital.

If there exists an associative algebra multiplication, we can generate a lie algebra where the lie bracket is given by the commutator.

$$\text{i.e. } \underbrace{[\vec{x}, \vec{y}]}_{\text{lie algebra multiplication}} = \vec{x} \overset{\text{original algebra operation}}{\vec{y}} - \vec{y} \vec{x}$$

Matrix lie algebra

- Matrix multiplication is associative, so $[X, Y] = XY - YX$
- The space of $n \times n$ matrices over a field F denoted by $GL(n, F)$ is a lie algebra with commutator as lie bracket.
- $gl(n, F)$ is the lie algebra of general linear group $GL(n, F)$ whose group operation is matrix multiplication.

Lie Groups

Lie groups are groups whose elements depend continuously & differentiably on one or more real parameters, such that group operations - multiplication & inversion are smooth (Infinitely differentiable) functions of these parameters.

Thus Lie groups can be thought of smooth manifolds with a compatible group structure. And the mathematical theory of continuous groups is usually called theory of lie groups.

Physically, lie groups are mathematical structures that describe continuous symmetries, i.e. a system is invariant after applying transformations, which are the group elements, that depend on smooth parameterization of elements.

Lie Group Theory of $GL(n, F)$

$$GL(n, F) = \{ A \in F^{n \times n} : \det(A) \neq 0 \}$$

It consists all $n \times n$ matrices with entries from a field F ($\mathbb{R}$ or $\mathbb{C}$) which are invertible.

Every matrix in $GL(n, \mathbb{R})$ has n^2 real continuous parameters that form a smooth real manifold of n^2 dimensions. But for $GL(n, \mathbb{C})$, it has n^2 complex parameters which is $2n^2$ real parameters.

The Lie algebra of $GL(n, F)$ is given by $gl(n, F)$ which is a Lie algebra of $n \times n$ matrices over field ' F '.

The exponential map of $gl(n, F)$ always lies in the group $GL(n, F)$ but the converse need not be true.

Consider the exponent of $A \in \text{gl}(2, \mathbb{F})$

$$\text{Let } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{gl}(2, \mathbb{F})$$

where $a, b, c, d \in \mathbb{F}$.

$$\text{Let, } T = \text{tr}(A) = a + d$$

$$\Delta = \det(A) = ad - bc$$

$$\theta = \sqrt{\Delta - \left(\frac{T}{2}\right)^2}$$

$$\text{Consider } B = A - \frac{T}{2} I_2$$

$$\Rightarrow \text{trace}(B) = T - \frac{T}{2}(2) = 0$$

$$\Rightarrow e^B = e^A e^{-\frac{T}{2}} \Rightarrow e^A = e^{\frac{T}{2}} e^B$$

then we should find 'B'.

$$B = \begin{bmatrix} \left(\frac{a-d}{2}\right) & b \\ c & -\left(\frac{a-d}{2}\right) \end{bmatrix}$$

$$\begin{aligned}
\det(B) &= \frac{-(a-d)^2}{4} - bc \\
&= \frac{-a^2 - d^2 + 2ad - bc}{4} \\
&= \frac{-a^2 - d^2 - 2ad + 4ad - bc}{4} \\
&= \frac{-(a+d)^2}{4} + ad - bc \\
&= \frac{-I^2}{4} + \Delta
\end{aligned}$$

$$\Rightarrow \det(B) = \Delta - \frac{I^2}{4} = \theta^2$$

$$\begin{aligned}
B^2 &= \begin{bmatrix} \left(\frac{a-d}{2}\right) & b \\ c & -\left(\frac{a-d}{2}\right) \end{bmatrix} \begin{bmatrix} \left(\frac{a-d}{2}\right) & b \\ c & -\left(\frac{a-d}{2}\right) \end{bmatrix} \\
&= \left[\left(\frac{a-d}{2}\right)^2 + bc \right] \mathbb{I} = -\theta^2 \mathbb{I}
\end{aligned}$$

$$\Rightarrow B^2 = -\theta^2 \mathbb{I}$$

$$\begin{aligned}
\Rightarrow B^0 &= \mathbb{I} & B^1 &= B & B^2 &= -\theta^2 \mathbb{I}_2 \\
B^3 &= -\theta^2 B & B^4 &= \theta^4 \mathbb{I}_2 & B^5 &= \theta^4 B \\
B^6 &= -\theta^6 \mathbb{I}_2 & B^7 &= -\theta^6 B & B^8 &= \theta^8 \mathbb{I}_2
\end{aligned}$$

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} + \frac{B^5}{5!} + \dots$$

$$= I + B - \frac{\theta^2 I}{2!} - \frac{\theta^2 B}{3!} + \frac{\theta^4 I}{4!} + \frac{\theta^4 B}{5!} - \frac{\theta^6 I}{6!}$$

$$= \left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \right] I_2 + \left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right] \frac{B}{\theta}$$

$$\Rightarrow e^B = \cos(\theta) I_2 + \frac{\sin(\theta)}{\theta} B$$

So, given $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $T = \frac{a+d}{2}$, $\Delta = ad - bc$

$$\neq \theta = \sqrt{\Delta - \left(\frac{T}{2}\right)^2} = \sqrt{\det\left(A - \frac{T}{2} I_2\right)}$$

$$e^A = e^{\frac{T}{2}} \left[\cos(\theta) I_2 + \frac{\sin(\theta)}{\theta} \left(A - \frac{T}{2} I_2 \right) \right]$$

$$\det(e^A) = e^{\text{Trace}(A)} \neq 0$$

Thus $e^A \in GL(n, F)$ where $A \in gl(n, F)$.

Lie Group Theory of $SL(n, F)$

$SL(n, F)$ is the set of all $n \times n$ invertible matrices whose entries are from field F ($\mathbb{R}$ or $\mathbb{C}$) such that they have determinant '1'.

The constraint that determinant = 1 makes the no: of independent parameters in F to $n^2 - 1$.

The lie algebra of $SL(n, F)$ is given by $sl(n, F)$ which composes of all $n \times n$ traceless matrices over field F . & these matrices form a vector space with commutator as algebra multiplication (trace of commutator = 0)

Exponentiating a matrix from $sl(n, F)$ results in a matrix from $SL(n, F)$, but the converse is not true.

Consider the exponential map of all traceless 2×2 matrices.

$$\text{Let } A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, A^2 = (a^2 + bc)I_2$$

$$\text{Let } \theta^2 = a^2 + bc$$

$$\Rightarrow e^A = \begin{cases} \cosh(\theta)I_2 + \frac{\sinh(\theta)}{\theta}A, & \theta^2 > 0 \\ \cos(\theta)I_2 + \frac{\sin(\theta)}{\theta}A, & \theta^2 < 0 \\ 1 + A, & \theta^2 = 0 \end{cases}$$

$$\text{Since } \det(e^A) = e^{\text{trace}(A)}$$

$$\text{as } \text{trace}(A) = 0 \Rightarrow \det(e^A) = 1$$

$$\Rightarrow \underline{e^A \in SL(2, F)}.$$

Orthogonal Group $O(n, F)$

The set of all $n \times n$ matrices with entries from a field F such that its inverse equals its transpose, will form a matrix group under multiplication called as Orthogonal Group $O(n, F)$ of dim ' n '. It is a subgroup of $GL(n, F)$.

$$\rightarrow O(n, F) = \{ A \in GL(n, F) : A^T A = A A^T = I_n \}$$

A real $n \times n$ orthogonal matrix can be parameterized by $\frac{n(n-1)}{2}$ independent real parameters.

$$\text{Let } A = [a_{ij}]_{n \times n} \quad i, j = \{1, 2, 3, \dots, n\}$$

$$\Rightarrow A^T = [a_{ji}]_{n \times n}$$

$$A^T A = [\delta_{ij}] = \left[\sum_k (a_{ik})^T a_{kj} \right]$$

$$\Rightarrow \sum_k a_{ki} a_{kj} = \delta_{ij}$$

For $i=j = \{1, 2, 3, \dots, n\}$ $\Rightarrow \sum_k a_{ki}^2 = 1$

i.e. Every column is such that sum of square of all elements of a column is 1. Thus columns are normalized to 1.

Since $A^T A = A A^T \Rightarrow \sum_k a_{ik}^2 = 1$

$\Rightarrow$ Every row is such that sum of square of all elements of a row is 1. Thus rows are normalized to 1.

For $i \neq j$: There are $n^2 - n$ non diagonal conditions for $\sum_k a_{ki} a_{kj} = \delta_{ij}$ ($i \neq j$)

But out of $n^2 - n$ such conditions, half of them are repeated since $A^T A$ is equal to $A A^T$. So useful constraint are $\frac{n^2 - n}{2}$ of them.

Thus there are ' n ' number of diagonal constraints & $(n^2 - n)/2$ number of non-diagonal constraints. Hence the total no: of constraints: $\frac{n^2 - n}{2} + n$

So, out of n^2 entries, there are $\frac{n^2 + n}{2}$ constraints. Then the no: of independent parameters are $n^2 - \left(\frac{n^2 + n}{2}\right) = \frac{n(n-1)}{2}$.

Thus $O(1) = \{1, -1\}$ no parameter.

We can also see that an orthogonal matrix is such that: All columns of $O(n)$ are orthonormal to each other where the metric for norm is I_n . Similarly all rows of $O(n)$ are also orthonormal to each other.

Thus columns (or rows) of an orthogonal real matrix forms an orthonormal basis for a real vector space.

The determinant of an orthogonal matrix is given as follows.

$$\begin{aligned} \text{Since } & \det(A^T) = \det(A) \\ & \& A^T A = I_n \\ & \Rightarrow [\det(A)]^2 = 1 \end{aligned}$$

$$\text{If } A \in O(n) \Rightarrow \underline{\det(A) = \pm 1}$$

Thus every orthogonal matrix is of two types whose determinant is 1 that preserve orientation of space which is called $SO(n)$ ($\det = 1$) & others whose determinant is -1 that includes reflections.

$$\Rightarrow O(n) = SO(n) \overset{\rightarrow \text{union}}{\cup} \gamma SO(n)$$

$SO(n) \Rightarrow$ orthogonal matrix of $\det 1$

$\gamma \Rightarrow$ reflection matrix about a fixed axis $\gamma = \begin{bmatrix} -1 & 0 \\ 0 & I_{n-1} \end{bmatrix}$ is allowed.

then $\xrightarrow{\text{coset}} \gamma SO(n) \Rightarrow$ orthogonal matrices whose determinant is -1 .

Thus union of $SO(n)$ & $\gamma SO(n)$ gives $O(n)$. Multiplication of any two elements of $\gamma SO(n)$ results in an element of $SO(n)$.

$O(n)$ is a symmetry group of $(n-1)$ -Hyper Sphere, since every orthogonal transformation $A \in O(n)$ preserves the length of the vector. So $O(n)$ carries the points on the $(n-1)$ Sphere (i.e. Surface of n -Sphere) to the points which again lie on $(n-1)$ Sphere.

The Lie algebra of $O(n)$ is $o(n)$ which is given by the set of all anti-symmetric matrices $X^T = -X$, & the Lie bracket is the commutator which is also anti-symmetric (closed).

Special Orthogonal Group $SO(n)$

The set of all $n \times n$ orthogonal matrices such that their determinant is 1 forms a matrix group under multiplication called $SO(n)$.

$$SO(n) = \{A \in O(n) : \det(A) = 1\}$$

$SO(n)$ is a normal subgroup of $O(n)$ as $\forall M \in O(n) \ \& \ A \in SO(n)$, we have $\det(M^{-1}AM) = \det(A) = 1 \Rightarrow M^{-1}AM \in SO(n)$.

$SO(n)$ is the kernel of Homomorphism map $\det: O(n) \rightarrow \{1, -1\}$. $SO(2)$ is abelian where as $SO(n)$, $n > 2$ is not.

A real $SO(n)$ matrix has $\frac{n(n-1)}{2}$ real independent parameters.

$$SO(1) = \{1\} \quad (\text{no free parameter})$$

The lie algebra of $SO(n)$ is $so(n)$ which is the set of all anti-symmetric $n \times n$ matrices.

$$so(n) = \{ X \in \mathbb{R}^{n \times n} : X^T = -X \}$$

This also ensures that all matrices in $so(n)$ are traceless. This also makes sure that lie algebra given by matrix commutation is closed, as commutator of any two matrices is traceless, as $\text{trace}(XY) = \text{trace}(YX)$, so $\text{trace}([X, Y]) = \text{trace}(XY - YX) = \underline{\underline{0}}$

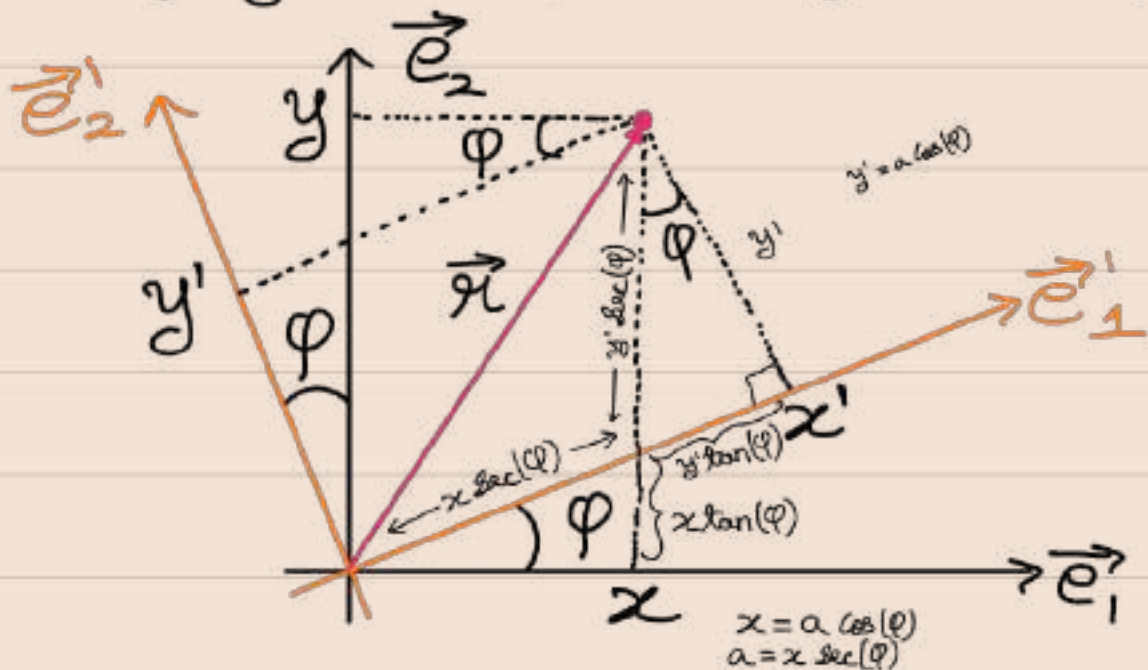
The no: of free parameters for all anti-symmetric $n \times n$ matrices = $\frac{n(n-1)}{2}$

$SO(n)$ is given by exponentiating the matrices of $so(n)$. i.e.,

$$\forall R \in SO(n), \exists X \in so(n), X^T = -X : e^X = R$$

The 2D rotation group $SO(2)$

The rotation of a 2D-euclidian real space is done using a 2×2 rotation matrix acting on a 2D cartesian basis vectors lying on a plane by an angle ' φ '.



$$\vec{r} = x\vec{e}_1 + y\vec{e}_2 = x'\vec{e}'_1 + y'\vec{e}'_2$$

$\{\vec{e}_1, \vec{e}_2\} \Rightarrow$ original coordinates basis

$\{\vec{e}'_1, \vec{e}'_2\} \Rightarrow$ rotated coordinate basis

By diagram

$$\begin{aligned} x' &= x \sec(\varphi) + y' \tan(\varphi) \\ y &= y' \sec(\varphi) + x \tan(\varphi) \end{aligned}$$

$$\begin{aligned}
 1^{\text{st}} \text{ relation: } x &= x' \cos(\varphi) - y' \sin(\varphi) \\
 \Rightarrow x' &= x \cos(-\varphi) - y \sin(-\varphi) \\
 \Rightarrow x' &= \underline{x \cos(\varphi) + y \sin(\varphi)}
 \end{aligned}$$

$$2^{\text{nd}} \text{ relation: } y' = y \cos(\varphi) - x \sin(\varphi)$$

$$\begin{aligned}
 \Rightarrow x' &= x \cos(\varphi) + y \sin(\varphi) \\
 y' &= y \cos(\varphi) - x \sin(\varphi)
 \end{aligned}$$

$$\rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix}}_{\text{Jacobian matrix}} \begin{bmatrix} x \\ y \end{bmatrix}$$

This is the rotation of coordinates by an angle ' θ ' anti-clockwise. But the rotation of basis is given by inverse of Jacobian matrix.

$$\underline{\rightarrow \begin{bmatrix} \vec{e}_1' \\ \vec{e}_2' \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \end{bmatrix}}$$

Thus the rotation matrix is given by,

$$R(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}, \quad R(2\pi + \varphi) = R(\varphi)$$

$R(\varphi)$ rotates the coordinates by an angle ' θ ', & $R(\varphi)\vec{r}$ gives the new coordinates with respect to rotated basis vectors.

$R(\varphi)$ is an Orthogonal matrix, it has only one independent parameter (φ), its determinant is 1. Thus $R(\theta)$ belongs to $SO(2)$.

$$R(\varphi_1)R(\varphi_2) = R(\varphi_1 + \varphi_2) = R(\varphi_2)R(\varphi_1)$$

Thus 2D rotation matrices commute.

$$R(0) = I_2 \Rightarrow \text{Identity exist}$$

$$R(\varphi)^{-1} = (R(\varphi))^T = R(-\varphi) \Rightarrow \text{Inverse exist}$$

Consider a general matrix from $SO(2)$,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}$$

constraints are $ad - bc = 1$

$$a^2 + b^2 = 1$$

$$a^2 + c^2 = 1$$

$$b^2 + d^2 = 1$$

$$c^2 + d^2 = 1$$

$$ab + cd = 0$$

$$ac + bd = 0$$

$$\text{let } b = \pm \sqrt{1 - a^2}, \quad c = \pm \sqrt{1 - a^2}$$

$$d = \pm \sqrt{1 - b^2} = \pm a$$

$$ad = 1 + bc$$

$$\Rightarrow a(\pm a) = 1 \pm (1 - a^2)$$

$$\pm a^2 = 1 \pm 1 \mp a^2$$

$$\Rightarrow \pm 2a^2 = 1 \pm 1 \quad \left. \vphantom{\pm 2a^2 = 1 \pm 1} \right\} \begin{array}{l} \text{Some values} \\ \text{can't be -ve} \end{array}$$

$$\text{So } \underline{\underline{d = +a}}$$

$$\Rightarrow bc = a^2 - 1$$

$$= -(1 - a^2)$$

$$\Rightarrow bc = -b^2$$

$$\Rightarrow \underline{c = -b}$$

Thus we get $A = \begin{bmatrix} a & -\sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{bmatrix}$

if $A \in SO(2)$. Where 'a' is a free parameter $-1 < a < +1$.

$$\Rightarrow \text{Let } a = d = \cos(\varphi) \Rightarrow c = -b = \sin(\varphi)$$

$$\Rightarrow A \text{ is of the form } \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$$

Thus any matrix in $SO(2)$ can be written using a rotation matrix $R(\varphi)$. where $\varphi + 2\pi = \varphi \pmod{2\pi}$. This corresponds to all points of a unit circle which is its topological manifold.

Consider an infinitesimal rotation $R(2\varphi)$ such that $(2\varphi)^2 \sim 0$. Then

$$R(2\varphi) = \begin{bmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ \sin(2\varphi) & \cos(2\varphi) \end{bmatrix}$$

$$\cos(2\varphi) = 1 - \underbrace{\frac{(2\varphi)^2}{2!}}_{\text{neglect}} + \dots$$

$$\sin(2\varphi) = 2\varphi - \underbrace{\frac{(2\varphi)^3}{3!}}_{\text{neglect}} + \dots$$

$$\Rightarrow R(2\varphi) = \begin{bmatrix} 1 & -2\varphi \\ 2\varphi & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbb{I}_2} + 2\varphi \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\mathcal{J}}$$

Thus Lie algebra belongs to these $\left\{ \begin{array}{l} \text{Anti-Symmetric} \\ \& \text{traceless} \end{array} \right.$

$$\Rightarrow R(\delta\varphi) = \mathbb{I}_2 + \delta\varphi \mathcal{J}$$

($\mathcal{J}$ is independent on rotation angle.)

Consider $R(\varphi + \delta\varphi) = R(\varphi)R(\delta\varphi)$

$$\Rightarrow R(\varphi + \delta\varphi) = R(\varphi) + \delta\varphi R(\varphi)\mathcal{J}$$

But by Taylor expansion

$$R(\varphi + \delta\varphi) = R(\varphi) + \delta\varphi \frac{dR(\varphi)}{d\varphi}$$

$$\Rightarrow \frac{dR(\varphi)}{d\varphi} = R(\varphi)\mathcal{J}$$

Using the boundary condition $R(0) = \mathbb{I}_2$
we get, $R(\varphi) = e^{\varphi\mathcal{J}}$

$\varphi\mathcal{J} \Rightarrow$ general form of anti-symmetric matrix.

$\varphi\mathcal{J} = \begin{bmatrix} 0 & -\varphi \\ \varphi & 0 \end{bmatrix} \in \mathfrak{so}(2)$ lie algebra.
(2×2) $\mathcal{J}$ is called as the generator of group

$$\Rightarrow SO(2) = \{e^A : A \in \mathfrak{so}(2)\}$$

Irreducible repⁿ of $SO(2)$

$R(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}$ is already an irreducible repⁿ over a real space.

But since it is abelian, it should have some 1 dim representation, so we shall try finding a repⁿ in the complex vector space (or Hilbert space)

Let $\hat{R}(\varphi)$ correspond to an operator which maps to $R(\varphi)$ in real space which is Euclidian. Since $R(\varphi) \neq \hat{R}(\varphi)$ should be related by a homomorphism,

$$\hat{R}(\varphi_1) \hat{R}(\varphi_2) = \hat{R}(\varphi_1 + \varphi_2) = \hat{R}(\varphi_2) \hat{R}(\varphi_1)$$

$$\hat{R}(\varphi + 2\pi) = \hat{R}(\varphi)$$

$$\hat{R}(0) = \hat{I}$$

$\rightarrow$ Identity operator

So for very small rotation, even the rotation operator should be of the below form

$$\hat{R}(\theta) = \hat{I} + \theta \hat{g}$$

Since we are in a complex space, we choose $\hat{g}$ to be anti hermitian.

This is analogous to finding

$$R(\theta) = I_2 + \theta g \quad \text{where } g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

But also note that

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i^2 \\ -i^2 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\Rightarrow g = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \text{anti-Hermitian even in } R(\theta) \text{ expansion.}$$

$$\Rightarrow g = -i J, \quad J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \text{traceless Hermitian}$$

So we can write

$$\hat{R}(\varphi) = \hat{I} - i\varphi \hat{J}$$

where $\hat{J}$ is some Hermitian operator

$$\Rightarrow \underline{\hat{R}(\varphi) = e^{-i\varphi \hat{J}}} \Rightarrow \text{operator eqn.}$$

Since $\hat{J}$ is hermitian, the vector space can be described by the span of eigen kets of $\hat{J}$.

$$\Rightarrow \hat{J}|m\rangle = m|m\rangle$$

eigen value $\swarrow$ $\searrow$ eigen kets

$m \rightarrow$ some label for indication.

$$\Rightarrow \hat{R}(\varphi)|m\rangle = e^{-i\varphi m}|m\rangle$$

as $\hat{R}(\varphi)$ is a function of $\hat{J}$.

$$\text{But } R(\varphi + 2\pi) = R(\varphi)$$

$$\Rightarrow \hat{R}(\varphi + 2\pi)|m\rangle = e^{i(\varphi + 2\pi)m}|m\rangle$$

$$= e^{i\varphi m}|m\rangle$$

$$\Rightarrow e^{i(2\pi)m} = 1 \Rightarrow \underline{m \in \mathbb{Z}}$$

$\Rightarrow$ Eigen values of $\hat{R}(\varphi)$ are $\{e^{im\varphi}\}$

They are the 1 dimensional representations of $SO(2)$ group.

- $m=0 \Rightarrow \{1\}$ is the trivial repⁿ
- $m=1 \Rightarrow \{e^{-i\varphi}\}$. This is the isomorphism map to $R(\varphi)$ in the clock-wise direction
- $m=-1 \Rightarrow \{e^{i\varphi}\}$. This is also another isomorphism map to $R(\varphi)$ but in the counter-clockwise sense.
- $m=\pm 2 \Rightarrow \{e^{\pm 2i\varphi}\}$. This is a homomorphism to $R(\varphi)$, covering twice in 1 real rot.

The $U(1)$ group

It is the set of all complex numbers whose magnitude is 1. And any member of $U(1)$ is of the form $e^{i\varphi}$ where $\varphi = \text{real no.}$

$$U(1) = \{e^{i\varphi} : \varphi \in \mathbb{R}\}$$

Thus $U(1)$ is a 1 parameter family of transformations that corresponds to phase rotation in complex plane.

The topology of $U(1)$ group is that of a unit circle. So, it is also called as circle group.

$U(1)$ is an abelian group. And its 1d repⁿ is $\{e^{i\theta} : \theta \in \mathbb{R}\}$ itself.

It is isomorphic to $SO(2)$ group.

$\forall e^{i\theta} \in U(1) \exists R(\theta)$ (2D rot.).

$$\text{where } R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So $R(\theta)$ matrices is 2D repⁿ of $U(1)$ but it is reducible in complex space.

$$\exists V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \text{ \& } V^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$\text{Such that } V^{-1}R(\theta)V = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

This acts on the complex plane spanned by $(x+iy, x-iy)$

$$\begin{pmatrix} x'+iy' \\ x'-iy' \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} x+iy \\ x-iy \end{pmatrix}$$

$$\Rightarrow x'+iy' = e^{i\theta}(x+iy)$$

$$\& x'-iy' = e^{-i\theta}(x-iy)$$

$$\Rightarrow 2x' = (e^{i\theta} + e^{-i\theta})x + iy(e^{i\theta} - e^{-i\theta})$$

$$\Rightarrow x' = x \cos(\theta) - y \sin(\theta)$$

$$\& 2iy' = (e^{i\theta} - e^{-i\theta})x + iy(e^{i\theta} + e^{-i\theta})$$

$$\Rightarrow y' = x \sin(\theta) + y \cos(\theta)$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus we recovered rotation of (x, y) from a representation of $U(1)$. The above example tells that in complex plane phase rot. in cartesian basis is similar to 2D rot. in x - y plane.

Thus $U(1) \& SO(2)$ are isomorphic to each other.

The 3D rotation group $SO(3)$

$SO(3)$ is defined as the set of all 3×3 matrices with determinant 1.

A more physical definition is,

" $SO(3)$ is the set of all the possible continuous linear transformations in 3D real Euclidean space that preserve the Euclidean norm (length) of the vector."

Consider the cartesian basis vectors for a Euclidean space ($\mathbb{R}^3$) $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Under rotation they transform to $\{\vec{e}'_1, \vec{e}'_2, \vec{e}'_3\}$ according to the inverse Jacobian matrix

$$\vec{e}'_i = \sum_j \underbrace{\left(\frac{\partial x_j}{\partial x'_i} \right)}_{\text{Inverse of Jacobian}} \vec{e}_j$$

Then, the coordinates also undergo transformation according to the Jacobian matrix as follows,

$$x'_i = \underbrace{\left(\frac{\partial x'_i}{\partial x_j} \right)}_{\text{Jacobian Matrix}} x_j =$$

The transformation preserves $\|\vec{x}\|$

$$\text{i.e. } (x'_i)^T x'_i = (x_i)^T x_i$$

$$\Rightarrow \left(\frac{\partial x'_i}{\partial x_j} \right)^T (x_j)^T \left(\frac{\partial x'_i}{\partial x_k} \right) x_k = (x_j)^T x_j \\ = (x_j)^T x_k \delta_{jk}$$

$$\Rightarrow \underbrace{\left(\frac{\partial x'_i}{\partial x_j} \right)^T \left(\frac{\partial x'_i}{\partial x_k} \right)} = \delta_{jk}$$

Thus the Jacobian matrix is an orthogonal matrix if it preserves the Euclidean norm (length).

But to preserve orientation of the coordinate space (no reflections), we need determinant of Jacobian matrix to be '1'.

Such matrices are called as the rotation matrices in 3D space. And constitutes the $SO(3)$ group.

The Jacobian matrix and its inverse are nothing but the 3D rotation matrices which belongs to $SO(3)$.

$$\Rightarrow \underbrace{R_{ij} = \frac{\partial x'_i}{\partial x_j}}_{\text{Elements of 3D rotation matrix}}$$

$$x'_k = R_{kj} x_j \Rightarrow \text{Coordinates rotation}$$

$$\vec{e}'_k = (R_{kj})^T \vec{e}_j \Rightarrow \text{Rotation of basis vectors}$$

$$\text{They obey: } \underline{R R^T = R^T R = \mathbb{I}_3}$$

Axis angle parameterization of R

Any rotation in 3D can be described uniquely by the axis of rotation which is along some $\hat{n}$ direction & the rotation angle (ψ). where $0 < \psi < 2\pi$

But any $\hat{n}$ is determined by two angles $(\theta, \varphi) \Rightarrow$ Polar & Azimuthal angles. In cartesian basis

$$\hat{n} = [\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta)]$$
$$(0 < \theta < \pi, 0 < \varphi < 2\pi)$$

Thus a 3D rotation matrix can be labelled by $\hat{n}$ & ψ as $R_{\hat{n}}(\psi)$, where $\hat{n}$ can be labelled by (θ, φ) . This is also known as the "Euler's rotational theorem". Every element of $SO(3)$ has an axis-angle pair.

But the axis angle pair is not unique. Since, $R_{\hat{n}}(\psi) \neq R_{-\hat{n}}(-\psi)$ will represent exact same rotation.

$SO(3)$ has three independent real parameters, according to the formula $\frac{n(n-1)}{2}$ with $n=3$.

Consider a rotation matrix which is labelled by $R_{\hat{n}}(\psi)$

Let $A \in SO(3)$, then A corresponds to some rotation in 3D. It can be seen geometrically, that,

$$\underline{R_{\hat{n}'}(\psi) = A R_{\hat{n}}(\psi) A^T} \quad \text{where } \hat{n}' = A \hat{n}$$

Thus all rotation matrices with same angle of rotation belong to same conjugacy class.

Consider the rotation in x - y plane.
So axis of rotation is $\hat{z}$ ($\theta=0$).

$$\Rightarrow R_{\hat{z}}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then rotation about any other axis $\hat{n}$ given by rotating $\hat{z}$ along that axis is

$$R_{\hat{n}}(\psi) = R[R_{\hat{z}}(\psi)]R^{-1}$$

where

$$\hat{n} = R\hat{z}, \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \hat{n} = \begin{bmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{bmatrix}$$

To get $\hat{n}$ from $\hat{z}$, first apply polar rotation of ' θ ' about y -axis & then a azimuthal rotation ' φ ' about z -axis.

$$\Rightarrow \hat{n} = R_{\hat{z}}(\varphi)R_{\hat{y}}(\theta)\hat{z}$$

$$\hat{R}_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\hat{R}_z(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow R_{\hat{n}}(\psi) = R_z(\varphi) R_y(\theta) R_z(\psi) R_y^T(\theta) R_z^T(\varphi)$$

Thus any element of $SO(3)$ can be parametrized by $(\theta, \varphi) = \hat{n} \neq \psi$ using the expression $R_{\hat{n}}(\psi)$ given above. It results in Rodrigues rotation formula

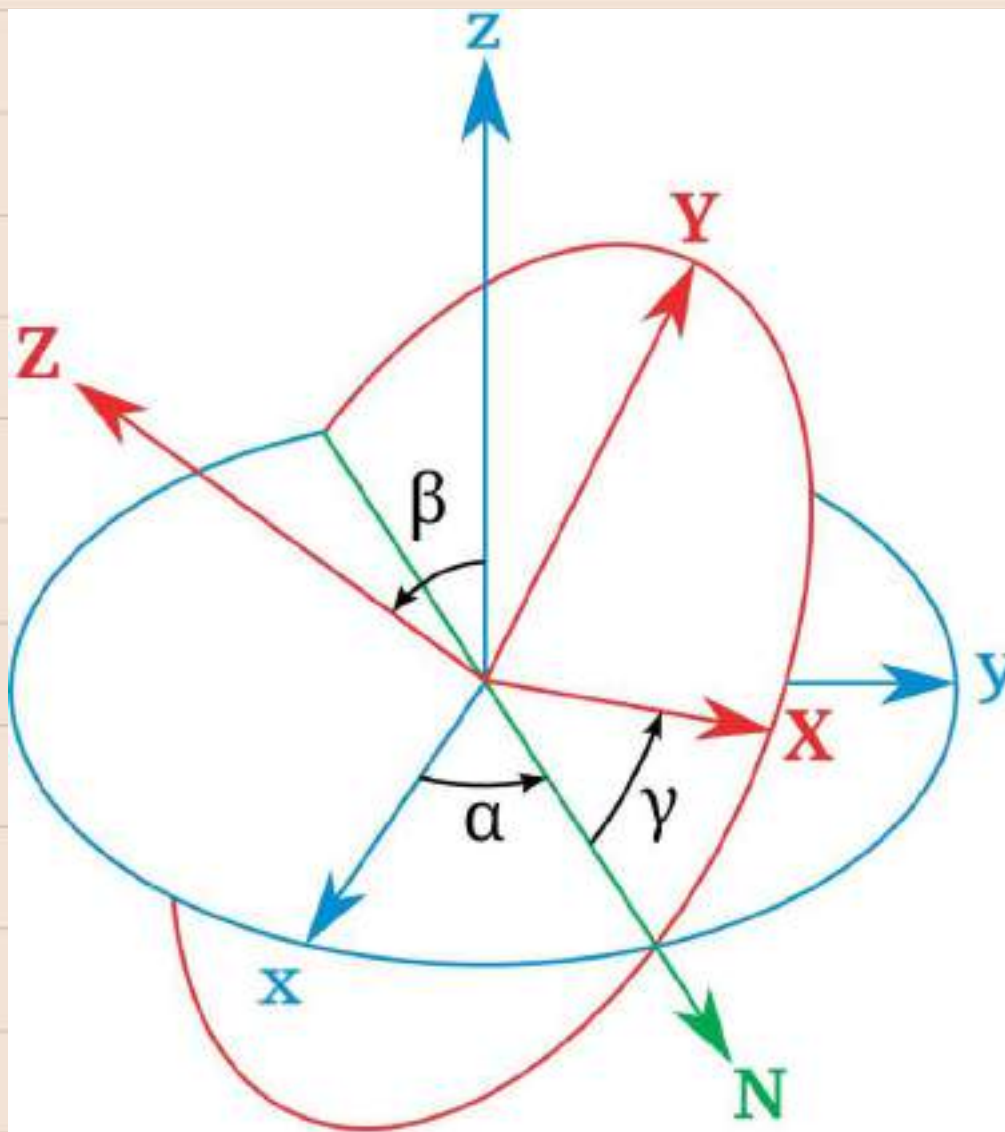
$$R_{\hat{n}}(\psi) = \mathbb{I}_3 + \sin(\psi) K + [1 - \cos(\psi)] K^2$$

$$K = \begin{bmatrix} 0 & -\cos(\theta) & \sin(\theta)\sin(\varphi) \\ \cos(\theta) & 0 & -\sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) & -\sin(\theta)\cos(\varphi) & 0 \end{bmatrix}$$

Euler angle Parameterization of 'R'

A general rotation can also be described relative to two different coordinate systems with same origin.

Consider two cartesian coordinate system with same origin. Thus the systems are rotated with respect to each other.



(x, y, z)
↳ old / fixed

(X, Y, Z)
↳ new / rotated

$N \rightarrow$ Intersection of xy & XY planes called Nodal line.

$$(0 \leq \alpha < 2\pi, 0 \leq \beta < \pi, 0 \leq \gamma < 2\pi)$$

The general 3D rotation matrix can be written in euler angle parameterization (α, β, γ) as,

$$R(\alpha, \beta, \gamma) = R_{\hat{z}}(\alpha) R_{\hat{y}}(\beta) R_{\hat{z}}(\gamma)$$

$$\begin{bmatrix} -\sin(\alpha)\sin(\gamma) + \cos(\alpha)\cos(\beta)\cos(\gamma) & -\sin(\alpha)\cos(\gamma) - \sin(\gamma)\cos(\alpha)\cos(\beta) & \sin(\beta)\cos(\alpha) \\ \sin(\alpha)\cos(\beta)\cos(\gamma) + \sin(\gamma)\cos(\alpha) & -\sin(\alpha)\sin(\gamma)\cos(\beta) + \cos(\alpha)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ -\sin(\beta)\cos(\gamma) & \sin(\beta)\sin(\gamma) & \cos(\beta) \end{bmatrix}$$

The relation b/w euler angles (α, β, γ) & angle-axis parameters (ψ, θ, φ) is,

$$R_{\hat{n}}(\psi) = R_z(\alpha) R_y(\beta) R_z(\gamma)$$

$$\hat{n} = (\theta, \varphi)$$

$$\varphi = \frac{(\pi + \alpha - \gamma)}{2}$$

$$\tan(\theta) = \frac{\tan(\beta/2)}{\sin(\frac{\alpha + \gamma}{2})}$$

$$\cos(\psi) = 2 \cos^2\left(\frac{\beta}{2}\right) \cos^2\left(\frac{\alpha + \gamma}{2}\right) - 1$$

SO(3) Lie Algebra

Given a fixed axis along the direction $\hat{n}$, the rotations about $\hat{n}$ forms a subgroup of $SO(3)$ which is isomorphic to $SO(2)$, which represents rotations in a plane $\perp$ to $\hat{n}$.

Similar to how the elements of $SO(2)$ were generated by exponentiating a anti-symmetric, we can say that

Elements of $\underline{R_n(\psi)}$ are $\{e^{\psi \mathcal{J}_{\hat{n}}} : \mathcal{J}_{\hat{n}}^T = -\mathcal{J}_{\hat{n}}\}$

For a fixed $\hat{n}$, the $\{R_{\hat{n}}\}$ forms a 1 parameter subgroup isomorphic to $SO(2)$. For SO_2 $\mathcal{J}_{\hat{n}} \rightarrow \mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ could be written as $-i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -i \mathcal{J}$

Similarly we need $\mathcal{J}_{\hat{n}} = -i \mathcal{J}_{\hat{n}}$ where $\mathcal{J}_{\hat{n}}$ is Hermitian matrix.

$\Rightarrow J_{\hat{n}}$ is the generator of rotation about an axis $\hat{n}$.

$$\text{As } R_{\hat{n}'}(\psi) = R[R_{\hat{n}}(\psi)]R^T$$

$$\Rightarrow R J_{\hat{n}} R^T = J_{\hat{n}'}, \quad \hat{n}' = R \hat{n}$$

Thus $J_{\hat{n}}$ transforms as a vector in the direction of $\hat{n}$.

The rotation about the coordinate basis is,

$$\hat{R}_{\hat{z}}(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) \\ 0 & \sin(\psi) & \cos(\psi) \end{bmatrix}$$

$$\hat{R}_{\hat{y}}(\psi) = \begin{bmatrix} \cos(\psi) & 0 & \sin(\psi) \\ 0 & 1 & 0 \\ -\sin(\psi) & 0 & \cos(\psi) \end{bmatrix}$$

$$\hat{R}_{\hat{x}}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

But by just performing a small rotation of $\delta\psi$ about these axes & keeping upto 1st order of $\delta\psi$,

$$R_{\hat{x}}(\delta\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta\psi \\ 0 & \delta\psi & 1 \end{bmatrix} = I_3 + (\delta\psi) \mathcal{J}_{\hat{x}}$$

$$R_{\hat{y}}(\delta\psi) = \begin{bmatrix} 1 & 0 & \delta\psi \\ 0 & 1 & 0 \\ -\delta\psi & 0 & 1 \end{bmatrix} = I_3 + (\delta\psi) \mathcal{J}_{\hat{y}}$$

$$R_{\hat{z}}(\delta\psi) = \begin{bmatrix} 1 & -\delta\psi & 0 \\ \delta\psi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 + (\delta\psi) \mathcal{J}_{\hat{z}}$$

$$\text{where } \mathcal{J}_{\hat{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathcal{J}_{\hat{y}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\& \mathcal{J}_{\hat{z}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow (g_k)_{lm}$: Element of g_k matrix along l^{th} row & m^{th} column
($k=1 \Rightarrow \hat{x}$, $k=2 \Rightarrow \hat{y}$, $k=3 \Rightarrow \hat{z}$)

$(g_k)_{lm} = -\epsilon_{klm}$: Levi-civita Symbol.

g_k matrices are anti-symmetric.
We can define $g_k = -i J_k$

$$\text{or } \underline{(J_k)_{lm} = -i \epsilon_{klm}}$$

where J_k is Hermitian matrix.

$$\Rightarrow \underline{R_k(\psi) = e^{-i\psi J_k}}$$

$\{J_k\}$ behave as coordinate vectors $\vec{e}_k$ under rotation i.e.

$$\underline{R J_k R^{-1} = J_l R_{lk}}$$

Thus (J_x, J_y, J_z) behave as vector.

Then a any rotation about $\hat{n}$ direction should be given by

$$R_{\hat{n}}(\psi) = e^{-i\psi J_{\hat{n}}}$$

where $J_{\hat{n}} = \hat{n} \cdot \vec{J} \Rightarrow \underline{R_{\hat{n}}(\psi) = e^{-i\psi(\hat{n} \cdot \vec{J})}}$

Consider any Euclidean vector $\vec{n} = \|\vec{n}\| \hat{n}$ which is along $\hat{n}$ direction. Let, $\psi = \|\vec{n}\| \Rightarrow \vec{n} = \psi \hat{n}$. Then, $\vec{n} \cdot \vec{J} = \psi \hat{n} \cdot \vec{J}$.
If $\vec{n} = (n_x, n_y, n_z)$, we get,

$$-i\vec{n} \cdot \vec{J} = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix} \rightarrow \text{general form of a } 3 \times 3 \text{ anti-symmetric matrix.}$$

$$\Rightarrow \underline{R_{\hat{n}}(\psi) = e^{-i\vec{n} \cdot \vec{J}}}$$

Thus any element of $SO(3)$ is given by the exponential of a 3×3 anti-symmetric-traceless matrix.

Hence the $so(3)$ lie algebra is the set of all 3×3 traceless anti-symmetric matrices.

$\vec{J} = (J_x, J_y, J_z)$ forms the basis for the space of generators of rot^n .

And the generator of rot^n along an $\hat{n}$ direction is $\vec{J} \cdot \hat{n}$.

To find algebra multiplication relations for $\{J_{\hat{n}}\}$.

Use $R J_k R^{-1} = J_l R_{lk}$,
 $k=1, R=R_y, \psi \rightarrow \partial\psi$.

$$\Rightarrow R_{\hat{y}}(\partial\psi) J_{\hat{x}} R_{\hat{y}}(-\partial\psi) = J_l [R_{\hat{y}}(\partial\psi)]_{l1}$$

$$R_{\hat{y}}(\partial\psi) = I_3 - i\partial\psi J_{\hat{y}}$$

$$\text{LHS, } (I_3 - i\partial\psi J_{\hat{y}}) J_{\hat{x}} (I_3 + i\partial\psi J_{\hat{y}})$$

$$\text{LHS} = J_{\hat{x}} - i\delta\psi J_{\hat{y}} J_{\hat{x}} + i\delta\psi J_{\hat{x}} J_{\hat{y}}$$

$$\text{RHS} = J_{\ell} (\delta_{\ell 1} - i\delta\psi (J_{\hat{y}})_{\ell 1})$$

$$\Rightarrow J_{\hat{x}} + i\delta\psi [J_{\hat{x}}, J_{\hat{y}}] = J_{\hat{x}} - i\delta\psi J_{\ell} (J_{\hat{y}})_{\ell 1}$$

using $(J_k)_{\ell m} = -i\epsilon_{k\ell m}$

$$\Rightarrow \underline{[J_{\hat{x}}, J_{\hat{y}}] = iJ_3}$$

By cyclic permutation of indices

we get, $[J_j, J_k] = i\epsilon_{jkl} J_{\ell}$

Thus the lie algebra is closed.

$$[-iJ_j, -iJ_k] = -i\epsilon_{jkl} J_{\ell}$$

$$\Rightarrow \underline{[g_j, g_k] = \epsilon_{jkl} g_{\ell}}$$

↪ In terms of anti-symmetric matrices

Now let $\vec{u} = (u_x, u_y, u_z)$ be any vector. Under rotation it transforms as

$$R u_k R^{-1} = u_l R_{lk}$$

If we follow similar steps, then

$$\Rightarrow R_{\hat{y}}(\partial\psi) u_x R_{\hat{y}}(-\partial\psi) = u_l [R_{\hat{y}}(\partial\psi)]_{l1}$$

$$\text{LHS} = u_x - i\partial\psi J_y u_x + i\partial\psi u_x J_y$$

$$\text{RHS} = u_l (\delta_{l1} - i\partial\psi (J_{\hat{y}})_{l1})$$

$$\Rightarrow [u_x, J_y] = i u_z$$

$$\Rightarrow \underline{[u_j, J_k] = i \epsilon_{jkl} u_l}$$

Thus given any vector $\vec{u}$,

$$[u_j, J_k] = i \epsilon_{jkl} u_l$$

where $\{J_k\} \Rightarrow$ generator of rotations.

Irreducible representations of $SO(3)$

The $so(3)$ algebra is given by the fundamental commutation relations

$$[J_j, J_k] = i \epsilon_{jkl} J_l$$

which are the generators for $R_{\hat{n}}(\psi)$.

Now we wish to get the irreducible matrix repⁿ of both $SO(3)$ & $so(3)$.

Consider the group & algebra elements acting on a complex Hilbert space. So we have a set of three operators that have the relation,

$$[\hat{J}_j, \hat{J}_k] = i \epsilon_{jkl} \hat{J}_l$$

where $\hat{\vec{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$

Defining $\hat{J}^2 = \hat{\vec{J}} \cdot \hat{\vec{J}}$ we get

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

& $[\hat{J}_k, \hat{J}^2] = 0$. Thus $\hat{J}^2$ will commute with all components of $\hat{\vec{J}}$, but no component of $\hat{\vec{J}}$ will commute with other component of $\hat{\vec{J}}$.

$\hat{J}^2$ commutes with any $\hat{\vec{J}} \cdot \vec{n}$, i.e. with all elements of lie algebra, thus with lie group elements.

Hence, $\hat{J}^2$ is called as Casimir operator.

Thus by Schur's lemma in all the irreducible repⁿ of $SO(3)$, $\hat{J}^2$ is a scalar matrix proportional to identity.

Now we can simultaneously diagonalize only $\hat{J}^2$ along with one other component.

By convention we choose to diagonalize $\hat{J}^2$ & $\hat{J}_z$ operators.

Defining $\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y$ we get some commutation relations,

$$[\hat{J}_z, \hat{J}_{\pm}] = \pm \hat{J}_{\pm}, \quad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_z$$

$$\hat{J}^2 = \frac{1}{2} [\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+] + \hat{J}_z^2$$

$$\Rightarrow \hat{J}^2 = \hat{J}_- \hat{J}_+ + \hat{J}_z + \hat{J}_z^2$$

Choosing $\hat{J}_z |m\rangle = m |m\rangle$

$$\hat{J}_{\pm} |m\rangle \propto |m \pm 1\rangle$$

Then using the condition that the expectation value of $\hat{J}^2 - \hat{J}_z^2 \geq 0$, we get the value of 'm' to be in between $-j$ & $+j$ where $j = \frac{n}{2}$ where $n = \{0, 1, 2, 3, \dots\}$

Then finally we can get, the basis of Hilbert space to be labelled by the kets $|j, m\rangle$ where

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

For a given j , there exists $2j+1$ values of m , with
 $m = j, j-1, j-2, \dots, -j, -j-1, -j-2, \dots$

$$\hat{J}^2 |j, m\rangle = j(j+1) |j, m\rangle$$

$$\hat{J}_z |j, m\rangle = m |j, m\rangle$$

$$\hat{J}_+ |j, m\rangle = \sqrt{(j-m+1)(j+m)} |j, m+1\rangle$$

$$\hat{J}_- |j, m\rangle = \sqrt{(j+m+1)(j-m)} |j, m-1\rangle$$

Since $|j, m\rangle$ are eigen kets of Hermitian matrices $\hat{J}^2$ & $\hat{J}_z$, we need to set $\langle j, m | j', m' \rangle = \delta_{jj'} \delta_{mm'}$

Then for a particular 'j' value we get a $(2j+1) \times (2j+1)$ matrix repⁿ for all operators, elements of $SO(3)$ group & $so(3)$ algebra, acting on the corresponding Hilbert sub space of $2j+1$ dimensions.

In the subspace of a particular 'j' & basis $\{|j, j\rangle, |j, j-1\rangle, \dots, |j, -j\rangle\}$

$$\hat{J}^2 : j(j+1) \delta_{m', m} \sim j(j+1) \mathbb{I}_{2j+1}$$

$$\hat{J}_z : m \delta_{m', m} \sim \begin{bmatrix} j & 0 & \dots & 0 & 0 \\ 0 & j-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-j & 0 \\ 0 & 0 & \dots & 0 & -j \end{bmatrix}$$

$$\hat{J}_{\pm} : \sqrt{(j \pm m + 1)(j \mp m)} \delta_{m', m \pm 1}$$

Thus for all possible j values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ we get a $(2j+1)$ -dim irreducible representation of $SO(3)$ group & $so(3)$ algebra.

For $j=0$: We get identity repⁿ.

For $j=\frac{1}{2}$: We get 2-dim repⁿ.

$$\hat{J}_x = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \hat{J}^2 = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

$$\hat{J}_y = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad J_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{J}_z = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad J_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{J} = \frac{1}{2} \vec{\sigma} \quad (\vec{\sigma} = \text{Pauli matrices})$$

$$\{\sigma_k, \sigma_l\} = 2\delta_{kl}, \quad \sigma_k \sigma_l = \delta_{kl} + i\epsilon_{klm} \sigma_m$$

The rotation matrix in axis angle parameterization is given by,

$$R_{\hat{n}}(\psi) = e^{-i\psi \vec{J} \cdot \hat{n}} = e^{-i\frac{\psi}{2} (\vec{\sigma} \cdot \hat{n})}$$

$$\vec{\sigma} \cdot \hat{n} = \begin{bmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{bmatrix}$$

$$\hat{n} = [\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta)]$$

$$\Rightarrow \vec{\sigma} \cdot \hat{n} = \begin{bmatrix} \cos(\theta) & e^{-i\varphi} \sin(\theta) \\ e^{i\varphi} \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

$$\text{Thus } J_{\hat{n}} = \frac{1}{2} \begin{bmatrix} \cos(\theta) & e^{-i\varphi} \sin(\theta) \\ e^{i\varphi} \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

is the 2×2 representation of $so(3)$

Then $e^{-i\frac{\psi}{2} (\vec{\sigma} \cdot \hat{n})}$ is then given by

$$R_{\hat{n}}(\psi) = \cos\left(\frac{\psi}{2}\right) \mathbb{I}_2 - i (\vec{\sigma} \cdot \hat{n}) \sin\left(\frac{\psi}{2}\right)$$

Notice that for $\psi = 2\pi$,
 $R_{\hat{n}}(2\pi) = -\mathbb{I}_2$ instead of $\mathbb{I}_2$.

But for $\psi = 4\pi$, $R_{\hat{n}}(4\pi) = \mathbb{I}_2$.

Thus $j = 1/2$ yields a double valued representation for $SO(3)$. So $SO(3)$ has no faithful 2-dim repⁿ.

Euler-angle parameterization

$$R(\alpha, \beta, \gamma) = R_{\hat{z}}(\alpha) R_{\hat{y}}(\beta) R_{\hat{z}}(\gamma)$$

$$(0 \leq \alpha < 2\pi, 0 \leq \beta < \pi, 0 \leq \gamma < 2\pi)$$

$\Rightarrow$ Denoting $\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma)$ as the matrix element of rotation in $|j, m\rangle$ basis, we get,

$$\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = \langle j, m' | e^{-i\hat{J}_z\alpha} e^{-i\hat{J}_y\beta} e^{-i\hat{J}_z\gamma} | j, m \rangle$$

$\hookrightarrow$ Wigner D-matrix

$$e^{-i\hat{J}_z\gamma} |j, m\rangle = e^{-im\gamma} |j, m\rangle$$

$$\langle j, m' | e^{-i\hat{J}_z\alpha} = e^{-im'\alpha} \langle j, m' |$$

$$\Rightarrow \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = e^{-i(m'\alpha + m\gamma)} d_{m'm}^{(j)}$$

$$\text{where } d_{m'm}^{(j)} = \langle j, m' | e^{-i\hat{J}_y\beta} | j, m \rangle$$

$$\text{For } j = 1/2, \quad \hat{J}_y = \frac{\beta}{2} \sigma_y$$

$$\Rightarrow e^{-i\frac{\beta}{2}\sigma_y} = \cos\left(\frac{\beta}{2}\right) \mathbb{I}_2 - i\sigma_y \sin\left(\frac{\beta}{2}\right)$$

$$\Rightarrow d^{(1/2)} = \begin{bmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{bmatrix}$$

Thus the 2×2 repⁿ of $SO(3)$ is

$$\mathcal{D}^{(1/2)}(\alpha, \beta, \gamma) = \begin{bmatrix} e^{-\frac{i}{2}(\alpha+\gamma)} \cos\left(\frac{\beta}{2}\right) & -e^{\frac{i}{2}(-\alpha+\gamma)} \sin\left(\frac{\beta}{2}\right) \\ e^{\frac{i}{2}(\alpha-\gamma)} \sin\left(\frac{\beta}{2}\right) & e^{\frac{i}{2}(\alpha+\gamma)} \cos\left(\frac{\beta}{2}\right) \end{bmatrix}$$

For $j=1 \Rightarrow 3D \text{ rep}^n \text{ of } SO(3)$

$$\hat{J}_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \hat{J}^2 = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\hat{J}_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \hat{J}_+ = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{J}_z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \hat{J}_- = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

Repⁿ of rotation elements

By Euler-angle parameterization,

$$R(\alpha, \beta, \gamma) = R_{\hat{z}}(\alpha) R_{\hat{y}}(\beta) R_{\hat{z}}(\gamma)$$

$$(0 \leq \alpha < 2\pi, 0 \leq \beta < \pi, 0 \leq \gamma < 2\pi)$$

Denoting $\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma)$ as the matrix element of rotation in $|j, m\rangle$ basis

$$\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = \langle j, m' | e^{-i\hat{J}_z\alpha} e^{-i\hat{J}_y\beta} e^{-i\hat{J}_z\gamma} | j, m \rangle$$

$$\Rightarrow \mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = e^{-i(m'\alpha + m\gamma)} d_{m'm}^{(j)}$$

$$\text{where } d_{m'm}^{(j)} = \langle j, m' | e^{-i\hat{J}_y\beta} | j, m \rangle$$

now $\hat{J}_y^{(j=1)}$ is such that $\hat{J}_y^3 = \hat{J}_y$

$$\Rightarrow e^{-i\hat{J}_y\beta} = I_3 - i\hat{J}_y \sin(\beta) + \hat{J}_y^2 (\cos(\beta) - 1)$$

This is equivalent to Rodrigues rotation formula for rotation matrices.

$$d^{(1)} = \begin{bmatrix} \frac{1}{2}[1 + \cos(\beta)] & -\frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}[1 - \cos(\beta)] \\ \frac{\sin(\beta)}{\sqrt{2}} & \cos(\beta) & -\frac{\sin(\beta)}{\sqrt{2}} \\ \frac{1}{2}[1 - \cos(\beta)] & \frac{\sin(\beta)}{\sqrt{2}} & \frac{1}{2}[1 + \cos(\beta)] \end{bmatrix}$$

Thus $j=1$ repⁿ is a faithful & a single valued repⁿ.

For $j = 3/2 \Rightarrow 4D \text{ rep}^n \text{ of } SO(3)$

$$\hat{J}^2 = \frac{15}{4} \hat{I}_4, \quad \hat{J}_+ = \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\hat{J}_- = \hat{J}_+^\dagger$$

$$\hat{J}_x = \frac{1}{2} \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\hat{J}_y = \frac{i}{2} \begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\hat{J}_z = \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$R(\alpha, \beta, \gamma) = R_{\hat{z}}(\alpha) R_{\hat{y}}(\beta) R_{\hat{z}}(\gamma)$$

Denoting $\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma)$ as the matrix element of rotation in $|j, m\rangle$ basis

$$\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = \langle j, m' | e^{-i\hat{J}_z\alpha} e^{-i\hat{J}_y\beta} e^{-i\hat{J}_z\gamma} | j, m \rangle$$

$$\mathcal{D}_{m'm}^{(j)}(\alpha, \beta, \gamma) = e^{-i(m'\alpha + m\gamma)} \langle j, m' | e^{-i\hat{J}_y\beta} | j, m \rangle$$

$$\text{Let } d_{m'm}^{(j)} = \langle j, m' | e^{-i\hat{J}_y\beta} | j, m \rangle$$

The eigen value of $\hat{J}_y$ is $\pm\frac{3}{2}, \pm\frac{1}{2}$
then the minimal polynomial is,

$$\left(\hat{J}_y^2 - \frac{9\mathbb{I}_4}{4}\right) \left(\hat{J}_y^2 - \frac{\mathbb{I}_4}{4}\right) = 0$$

Thus $e^{-i\hat{J}_y\beta}$ can be grouped into

$\mathbb{I}_4, i\hat{J}_y, \hat{J}_y^2, i\hat{J}_y^3$ terms.

Then the elements for $d^{(3/2)}$ matrix are given by,

$$d_{3/2, 3/2}^{(3/2)} = \frac{1}{2} [1 + \cos(\beta)] \cos\left(\frac{\beta}{2}\right)$$

$$d_{3/2, 1/2}^{(3/2)} = -\frac{\sqrt{3}}{2} [1 + \cos(\beta)] \sin\left(\frac{\beta}{2}\right)$$

$$d_{3/2, -1/2}^{(3/2)} = \frac{\sqrt{3}}{2} [1 - \cos(\beta)] \cos\left(\frac{\beta}{2}\right)$$

$$d_{3/2, -3/2}^{(3/2)} = -\frac{1}{2} [1 - \cos(\beta)] \sin\left(\frac{\beta}{2}\right)$$

$$d_{1/2, 1/2}^{(3/2)} = \frac{1}{2} [3\cos(\beta) - 1] \cos\left(\frac{\beta}{2}\right)$$

$$d_{1/2, -1/2}^{(3/2)} = -\frac{1}{2} [3\cos(\beta) + 1] \sin\left(\frac{\beta}{2}\right)$$

And other elements are given by,

$$\underline{d_{m', m}^{(j)} = (-1)^{m' - m} d_{m, m'}^{(j)} = d_{-m, -m'}^{(j)}}$$

We should note that for every given ' j ' = $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, there exists a $(2j+1)$ -dimensional irreducible repⁿ of $SO(3)$ group & $so(3)$ algebra.

For half-integer j -values, the repⁿ is double valued & for integer value of j , the repⁿ is single valued. And there are no multiple valued repⁿ for $SO(3)$. This is because of the double connectedness of $SO(3)$ manifold.

It is also possible to create reducible representations of $SO(3)$ by the procedure of tensor product of two irreducible representations, which correspond to addition of two angular momentum of j & j' quantum numbers.

Rotations in 4D Euclidean Space

The $SO(4)$ group

The $SO(4)$ group is the group of all rotations in a 4 dimensional Euclidean Space which preserve the length of the vector & its orientation along with fixing the origin.

$$SO(4) = \{R \in GL(4, \mathbb{R}) : R^T = R^{-1}, \det(R) = 1\}$$

The number of real independent parameters to describe any element of the group is '6' according to the formula $\frac{n(n-1)}{2}$ with $n=4$.

Each rotation has to be thought of happening in a plane like $x-y$, $y-z$, $x-z$, etc. We can label the 4-orthogonal axis in 4D space as $\{1, 2, 3, 4\}$.

Then there can be rotations in six different orthogonal planes:

$$\{(1,2), (2,3), (3,1), (1,4), (2,4), (3,4)\}$$

The generators can be labelled as J_{ab} where $(a,b) \in \text{orthogonal planes}$.

J_{ab} is the generator of rotations in the plane (a,b) .

The $so(4)$ lie algebra consists of the set of all 4×4 traceless anti-symmetric matrices such that exponentiating any matrix from $so(4)$ lies in the $SO(4)$ group.

$$\Rightarrow so(4) = \{A \in \mathbb{R}^{4 \times 4} : A^T = -A\}$$

Any general $so(4)$ matrix also has 6 independent real parameters.

It can be represented as,

$$A = \begin{bmatrix} 0 & -n_{12} & n_{13} & n_{14} \\ n_{12} & 0 & -n_{23} & n_{24} \\ -n_{13} & n_{23} & 0 & n_{34} \\ -n_{14} & -n_{24} & -n_{34} & 0 \end{bmatrix}$$

So any $A \in \text{so}(4)$ can be written as,

$$A = n_{23} \mathcal{J}_{23} + n_{13} \mathcal{J}_{13} + n_{12} \mathcal{J}_{12} + n_{14} \mathcal{J}_{14} + n_{24} \mathcal{J}_{24} + n_{34} \mathcal{J}_{34}$$

$$\mathcal{J}_{12} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{13} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{J}_{14} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$g_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$g_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Any $g_{ab} = -i(i g_{ab}) = -i J_{ab}$
 $\Rightarrow \underline{J_{ab} = i g_{ab}}$

$$\Rightarrow A = -i \sum_k n_k J_k$$

$k \in \{(1,2), (2,3), (3,1), (1,4), (2,4), (3,4)\}$

J_{ab} is called as the generator of rotations in (a,b) plane.

Then the commutators b/w J_{ab} are

$$[J_{23}, J_{13}] = i J_{12}$$

$$[J_{13}, J_{12}] = i J_{23}$$

$$[J_{12}, J_{23}] = i J_{13}$$

Resembles

$$[J_j, J_k] = i \epsilon_{jkl} J_l$$

$$[J_{14}, J_{24}] = iJ_{12}$$

$$[J_{14}, J_{13}] = iJ_{34}$$

$$[J_{24}, J_{34}] = iJ_{23}$$

$$[J_{24}, J_{12}] = iJ_{14}$$

$$[J_{34}, J_{14}] = iJ_{13}$$

$$[J_{34}, J_{23}] = iJ_{24}$$

Thus if we label the generators of rotations in x - y - z plane as

$$J_{23} = J_1 \equiv J_x$$

$$J_{13} = J_2 \equiv J_y$$

$$J_{12} = J_3 \equiv J_z$$

$$J_{14} = K_1 \equiv K_x$$

$$J_{24} = K_2 \equiv K_y$$

$$J_{34} = K_3 \equiv K_z$$

Then there are three sets of commutation relations,

$$[J_j, J_k] = i\epsilon_{jkl} J_l$$

$$[K_j, J_k] = i\epsilon_{jkl} K_l$$

$$[K_j, K_k] = i\epsilon_{jkl} J_l$$

The first set of commutations states that $SO(4)$ encompasses a $SO(3)$ or rotation in 3D with angular momentum as generators.

The second set of commutations shows that (K_x, K_y, K_z) act as a set of vector operators under the $SO(3)$ sub algebra.

The third set of commutations shows that the algebra is closed. i.e. nothing new can be got by commuting K_j 's, we just get J_i 's

Note that the algebra is invariant under the discrete transformation $K_j \rightarrow -K_j$ which follows from the reflection of 4th axis while keeping $x-y-z$ unchanged.

If we define $\vec{N} = \frac{\vec{J} + \vec{K}}{2}$

$$\& \vec{M} = \frac{\vec{J} - \vec{K}}{2}$$

then $[N_j, M_k] = 0$

$$[N_j, N_k] = i \epsilon_{jkl} N_l$$

$$[M_j, M_k] = i \epsilon_{jkl} M_l$$

Thus $so(4)$ can encompass two independent $so(3)$ algebras.

$$\Rightarrow so(4) = so(3) \oplus so(3)$$

Thus near the identity or locally there is an isomorphism between $SO(4)$ & $SO(3) \otimes SO(3)$

Unitary Group $U(n)$

The unitary group $U(n)$ is defined as the set of all $n \times n$ matrices which are unitary, & they form a group under matrix multiplication.

$$U(n) = \{ U \in GL(n, \mathbb{C}) : U^\dagger U = I_n \}$$

The $U(n)$ group is a Lie group of dimension n^2 over $\mathbb{R}$.

$$\text{Since } U^\dagger U = U U^\dagger = I_n$$

All columns of ' U ' forms an O.N. set
& All rows of ' U ' forms another orthonormal set.

$$\text{i.e. } \sum_n U_{jn}^* U_{kn} = \delta_{jk} \quad \left. \vphantom{\sum_n} \right\} \text{rows are O.N.}$$

$$\& \sum_m U_{mj}^* U_{mk} = \delta_{jk} \quad \left. \vphantom{\sum_m} \right\} \text{Columns are O.N.}$$

Due to Symmetry $U^\dagger U = U U^\dagger$, we can just focus on only either row or column orthonormality.

$$U = [U_1 \quad U_2 \quad U_3 \dots U_n]$$

$U_i \Rightarrow i^{\text{th}}$ column of U .

now $\langle u_j | u_k \rangle = \delta_{jk}$

For $j=k$ (which occurs n -times) we have $\langle u_j | u_j \rangle = 1$. Thus there are n -diagonal constraints.

For $j \neq k$ we can just focus on only upper diagonal which are $\frac{n(n-1)}{2}$ of them (due to Symmetry)

$$\text{Re}(\langle u_i | u_j \rangle) = 0 : \frac{n(n-1)}{2} \text{ constraints}$$

$$\text{Im}(\langle u_i | u_j \rangle) = 0 : \frac{n(n-1)}{2} \text{ constraints}$$

Thus total no: of constraints are $n + \frac{n(n-1)}{2} + \frac{n(n-1)}{2} = n + n^2 - n = n^2$

So out of $2n^2$ real numbers that make up a unitary matrix only n^2 of them are independent.

Thus a $n \times n$ unitary matrix has n^2 independent real parameters.

The eigenvalues of a unitary matrix are complex numbers of unit magnitude. And the eigen vectors corresponding of different eigen values are orthogonal.

$$\text{i.e. } U|q\rangle = q|q\rangle \Rightarrow q^*q = 1 \\ \Rightarrow q = e^{i\varphi}, \varphi \in \text{real} \text{ \& } \langle q|q'\rangle = \delta_{qq'}$$

Thus the determinant of unitary matrix is a complex number of unit modulus.
 $\Rightarrow \det(U)[\det(U)]^* = 1.$

Let $\{e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3} \dots e^{i\varphi_n}\}$ be the eigen values of U , with or without degeneracy.

$$\text{Then } \det(U) = e^{i \sum_j \varphi_j} = e^{i \varphi_{\text{net}}}$$

$$\rightarrow \det: U(n) \rightarrow U(1).$$

Any $n \times n$ unitary matrix can be written as an exponential of a anti-Hermitian matrix.

$$\Rightarrow U = e^{iH} \text{ where } H \text{ is hermitian.}$$

Since $[U, U^\dagger] = 0 \Rightarrow U$ is diagonalizable

$\Rightarrow$ There exists a matrix $V \in U(n)$ such that

$$U = V^\dagger D V$$

where D is diagonal matrix of all eigen values of ' U '.

$$\Rightarrow U = V^+ \begin{bmatrix} e^{i\varphi_1} & 0 & 0 & \dots & 0 \\ 0 & e^{i\varphi_2} & 0 & \dots & 0 \\ 0 & 0 & e^{i\varphi_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{i\varphi_n} \end{bmatrix} V$$

$$\Rightarrow U = V^+ e^{i\Phi} V$$

where $\Phi = \begin{bmatrix} \varphi_1 & 0 & 0 & \dots & 0 \\ 0 & \varphi_2 & 0 & \dots & 0 \\ 0 & 0 & \varphi_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \varphi_n \end{bmatrix}$

$$\Rightarrow U = V^+ \left[\sum_k \frac{(i\Phi)^k}{k!} \right] V = \sum_k \frac{(iV^+ \Phi V)^k}{k!}$$

$$\Rightarrow U = e^{iV^+ \Phi V}$$

Let $H = V^+ \Phi V$, $H^\dagger = H$

$$\Rightarrow \underline{U = e^{iH}} \quad (iH \Rightarrow \text{anti hermitian})$$

Thus U is exponential of A : $A^\dagger = -A$.

The commutator of two anti-hermitian matrices is also anti-hermitian.

$$\text{Let } A^\dagger = -A \text{ \& } B^\dagger = -B \\ \Rightarrow [A, B]^\dagger = [B^\dagger, A^\dagger] = [B, A] = -[A, B].$$

Thus the lie-algebra of $U(n)$ is $\mathfrak{u}(n)$ is the vector space of all the $n \times n$ anti-Hermitian matrices with commutator as the lie bracket.

The unitary group of dim-1 is $U(1)$ which is the circle group.

$$U(1) = \{e^{i\varphi} : \varphi \in \mathbb{R}\}$$

$U(1)$ represents the symmetry of a unit circle in the complex plane.

$U(1)$ is isomorphic to $SO(2)$.

Special Unitary Group $SU(n)$

Special unitary group is the set of all $n \times n$ unitary matrices whose determinant is 1. So $SU(n)$ is the kernel of the map $\det: U(n) \rightarrow U(1)$.

$$SU(n) = \{V \in GL(n, \mathbb{C}) : V^\dagger V = I_n, \det(V) = 1\}$$

$SU(n)$ is a normal subgroup of $U(n)$.

$SU(1) = \{1\}$ is just identity.

A $SU(n)$ matrix has $n^2 - 1$ real independent parameters. One less than that of $U(n)$, since there is an extra constraint that the eigen values of any $SU(n)$ matrix are such that they are given by $\{e^{i\varphi_1}, e^{i\varphi_2}, \dots, e^{i\varphi_n}\}$ with a constraint that $\sum_i \varphi_i = 0$.

Same like how any element of $U(n)$ can be written as e^{iH} where $H^\dagger = H$, we can extend this idea to $SU(n)$ matrices which can be written as e^{iH} but since $\det(e^{iH}) = 1$, it results in $e^{\text{trace}(H)} = 1 \Rightarrow H$ is traceless Hermitian matrix.

Thus any element of $SU(n)$ is given by exponential of some traceless, anti-Hermitian matrix.

$$\Rightarrow SU(n) = \{e^{iH} : H \in \mathbb{C}^{n \times n}, H^\dagger = H, \text{tr}(H) = 0\}$$

Thus the $SU(n)$ Lie algebra is the set of all traceless, anti-Hermitian matrices.

$$\Rightarrow su(n) = \{H \in \mathbb{C}^{n \times n} : H^\dagger = -H, \text{tr}(H) = 0\}$$

Any element of $SU(n)$ algebra also has $n^2 - 1$ real independent parameters.

The $SU(2)$ group

$SU(2)$ is the set of all 2×2 complex matrices that are unitary & with determinant 1.

Consider $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $U \in SU(2)$

$$\Rightarrow \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\& \quad ad - bc = 1$$

$$\Rightarrow |a|^2 + |c|^2 = 1$$

$$a^*b + c^*d = 0 \quad \& \quad ad - bc = 1$$

$$|b|^2 + |d|^2 = 0$$

$$\Rightarrow a(a^*b + c^*d) = 0$$

$$\Rightarrow |a|^2 b + c^*(ad) = 0$$

$$\Rightarrow |a|^2 b + c^*(1+bc) = 0$$

$$\Rightarrow |a|^2 b + c^* + b|c|^2 = 0$$

$$\Rightarrow \underbrace{(|a|^2 + |c|^2)}_1 b + c^* = 0$$

$$\Rightarrow b + c^* = 0$$

$$\Rightarrow \underline{\underline{c = -b^*}}$$

$$\nexists \quad a^* b + (-b^* d) = 0$$

$$\Rightarrow (a^* - d)b = 0$$

$$\Rightarrow \underline{\underline{a^* = d}}$$

Thus $U = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$ Such that $|a|^2 + |b|^2 = 1$.

Let $a = x + iy$, $b = z - iw$
 they should satisfy $x^2 + y^2 + z^2 + w^2 = 1$

This shows the topology of a 3D sphere of unit radius embedded in a 4D euclidian space.

Euler angle parametrization of $SU(2)$

$$\text{Let } U \in U(2) \quad , \Rightarrow U = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

$$\text{Such that } |a|^2 + |b|^2 = 1$$

$$\Rightarrow a = e^{i\zeta} \cos(\theta) \quad \& \quad b = e^{i\eta} \sin(\theta)$$

$$0 \leq \theta \leq \frac{\pi}{2} \quad \& \quad 0 \leq (\zeta, \eta) \leq 2\pi$$

$$\Rightarrow U = \begin{bmatrix} \cos(\theta) e^{i\zeta} & \sin(\theta) e^{i\eta} \\ \sin(\theta) e^{-i\eta} & \cos(\theta) e^{-i\zeta} \end{bmatrix}$$

It has three real independent parameters. This parameterization gives rise to the Wigner $'D^{(1/2)}'$ matrix of $SO(3)$.

$$\text{If } \theta = \frac{\beta}{2}, \quad \xi = -\frac{(\alpha + \gamma)}{2} \text{ \& } \eta = \frac{(-\alpha + \gamma)}{2}$$

then the matrix becomes $D^{(\frac{1}{2})}(\alpha, \beta, \gamma)$

$$D^{(\frac{1}{2})}_{(\alpha, \beta, \gamma)} = \begin{bmatrix} e^{\frac{-i(\alpha + \gamma)}{2}} \cos\left(\frac{\beta}{2}\right) & -e^{\frac{i(-\alpha + \gamma)}{2}} \sin\left(\frac{\beta}{2}\right) \\ e^{\frac{i(\alpha - \gamma)}{2}} \sin\left(\frac{\beta}{2}\right) & e^{\frac{i(\alpha + \gamma)}{2}} \cos\left(\frac{\beta}{2}\right) \end{bmatrix}$$

$$0 \leq \alpha \leq 2\pi, \quad 0 \leq \beta \leq \pi, \quad 0 \leq \gamma \leq 4\pi$$

Thus we got that any $SU(2)$ matrix is of the form $D^{(\frac{1}{2})}(\alpha, \beta, \gamma)$ where the range of $0 \leq \gamma \leq 4\pi$ which encompasses the double valued representation of $SO(3)$.

$$\Rightarrow SO(3) = \frac{SU(2)}{\mathbb{Z}_2}$$

So $SO(3)$ is quotient group of $SU(2)$ with $\mathbb{Z}_2 = \{1, -1\}$. $SU(2)$ is double cover of $SO(3)$. Every rotⁿ in $SO(3)$ has two corresponding U \& $-U$ in $SU(2)$.

Pauli-Parameterization of $SU(2)$

Any $SU(2)$ matrix can be written as an exponential of traceless and anti-hermitian 2×2 matrix.

A general traceless-hermitian matrix is

$$X = \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}$$

$$X = x_1 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\sigma_1} + x_2 \underbrace{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}_{\sigma_2} + x_3 \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\sigma_3}$$

$$\Rightarrow X = x^k \sigma_k \quad \begin{array}{l} \text{is Hermitian} \\ \rightarrow \text{Pauli matrices} \end{array}$$

then $-ix^k \sigma_k$ is anti-Hermitian.

$$\text{Let } \vec{x} = (x_1, x_2, x_3) = \theta \hat{n}$$

where $\hat{n} \Rightarrow$ direction along $\vec{x}$
& $\psi = |\vec{x}| \Rightarrow$ length of $\vec{x}$.

Then
$$e^{-i\chi} = e^{-i\psi \hat{n} \cdot \vec{\sigma}}$$

$$= \mathbb{I}_2 \cos(\psi) - i(\hat{n} \cdot \vec{\sigma}) \sin(\psi)$$

$$= \begin{bmatrix} \cos(\psi) - i\hat{n}_3 \sin(\psi) & -i(n_1 - in_2) \sin(\psi) \\ -i(n_1 + in_2) \sin(\psi) & \cos(\psi) + i\hat{n}_3 \sin(\psi) \end{bmatrix}$$

If $\hat{n} = (\theta, \varphi)$ in polar coordinates, then $\exp(i\psi \hat{n} \cdot \vec{\sigma})$ can be written as,

$$\begin{bmatrix} \cos(\psi) - i\cos(\theta) \sin(\psi) & -ie^{i\varphi} \sin(\theta) \sin(\psi) \\ -ie^{i\varphi} \sin(\theta) \sin(\psi) & \cos(\psi) + i\cos(\theta) \sin(\psi) \end{bmatrix}$$

Thus $e^{-i\psi \vec{\sigma} \cdot \hat{n}}$ represents some 2×2 unitary matrix 'SU(2)' whose determinant is 1.

This is exactly the same rotation matrix $R_{\hat{n}}(2\psi)$ which we got in 2×2 repⁿ of SO(3).

Here (x_1, x_2, x_3) can be thought of as a coordinate vector in a 3D euclidian space.

Then any 3D coordinate vector can be mapped to a 2×2 hermitian & a traceless matrix using,

$$X = x_k \sigma_k = \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}$$

$$\det |X| = -(x_1^2 + x_2^2 + x_3^2) = -|\vec{x}|^2$$

Conversely, any traceless hermitian matrix X can be written as $x_k \sigma_k$.

where
$$x_k = \frac{1}{2} \text{trace} [(X \sigma_k)]$$

Thus there is 1-to-1 mapping between 3D coordinate vectors and 2×2 traceless-hermitian matrices & length of the vector mapped to the square root of -ve of determinant.

Now consider an arbitrary $SU(2)$ matrix $U \in SU(2)$ which will transform $X \rightarrow X' = UXU^\dagger$.

Now $(X')^\dagger = (UXU^\dagger)^\dagger = UXU^\dagger = X'$

Thus X' is hermitian & can be mapped to any 3D coordinate vector

$$\underline{x'_k = \frac{1}{2} \text{trace}(UXU^\dagger \sigma_k)}$$

$$\begin{aligned} \text{Then } \det(X') &= \det(UXU^\dagger) \\ &= \det(UU^\dagger X) \end{aligned}$$

$$\Rightarrow \det(X') = \det(X)$$

$$\begin{aligned} \text{But } |\vec{x}'|^2 &= -\det(X') = -\det(X) \\ \Rightarrow \underline{\underline{|\vec{x}'|^2 &= |\vec{x}|^2}} \end{aligned}$$

Thus a matrix $U \in SU(2)$ induces a rotation on $\vec{x}$ through a 2×2 traceless hermitian matrix.

Now the matrices U & $-U$ will induce same rotation on $\vec{x}$. Thus there is a two $\rightarrow$ one mapping from $SU(2) \rightarrow SO(3)$. Thus $SU(2)$ is called as double cover of $SO(3)$.

The mapping $U \in SU(2) \rightarrow R \in SO(3)$ can be written as,

$$\text{Since } x'_k = \frac{1}{2} \text{trace}(U X U^\dagger \sigma_k)$$

$$\& x'_k = R_{kl} x_l$$

$$\Rightarrow R_{kl} x_l = \frac{1}{2} \text{trace}(U x_l \sigma_l U^\dagger \sigma_k)$$

$$\Rightarrow R_{kl} = \frac{1}{2} \text{trace}(U \sigma_l U^\dagger \sigma_k)$$

$$\Rightarrow \underline{R_{kl} = \frac{1}{2} \text{trace}(\sigma_k U \sigma_l U^\dagger)}$$

Both U & $-U$ gives same R .

So $SU(2)$ is not globally isomorphic to $SO(3)$, but only locally.

Two lie groups are locally isomorphic if their lie algebras are globally isomorphic. So that,

- Near the identity element, both groups behave similarly.
- Their infinitesimal transformations of generators & commutation relations, are same, which preserves the local structure of the group.

This can be shown explicitly by isomorphism of generators between the groups $SO(3)$ & $SU(2)$.

The elements of $SU(2)$ are given by exponentiating $SU(2)$.

$SU(2)$ is given by all elements of the type $-iX = -iX_k \sigma_k$

$$\Rightarrow SU(2) = \{ e^{-iX_k \sigma_k} : X_k \in \mathbb{R}^3 \}$$

Thus the generators of $SU(2)$ are the pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$.

Pauli matrices have the relation

$$[\sigma_j, \sigma_k] = 2i \epsilon_{jkl} \sigma_l$$

$$\Rightarrow \left[\frac{\sigma_j}{2}, \frac{\sigma_k}{2} \right] = i \epsilon_{jkl} \left(\frac{\sigma_l}{2} \right)$$

If we define $T_k = \frac{\sigma_k}{2}$, then

$$\underline{[T_j, T_k] = i \epsilon_{jkl} T_l}$$

This is the relation followed by the generators of $SU(2)$.

But the algebra generators of $SO(3)$ given by $\{J_k\}$, where

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad J_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}$$

$$\& J_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Any element of $SO(3)$ is got by exponentiating a matrix 'N' of the form $N = -i n_k J_k$

$$\Rightarrow N = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix} \left. \vphantom{\begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}} \right\} \begin{array}{l} 3 \times 3 \text{ anti} \\ \text{symmetric} \\ \text{matrix} \end{array}$$

$$\Rightarrow \underline{SO(3) = \{ e^{-i n_k J_k} : n_k \in \mathbb{R}^3 \}}$$

The generators of $SO(3)$ also satisfy,

$$[J_i, J_k] = i\epsilon_{jkl} J_l$$

which is exactly same as that of $SU(2)$ generators. $T_k = \frac{\sigma_k}{2}$.

Thus the lie algebras are identical & manifestly isomorphic. Thus near the identity any $A \in$ lie group is given by $I + X$, where $X \in$ lie algebra. Since lie algebra $so(3) \cong su(2)$ are isomorphic, the lie groups $SO(3) \cong SU(2)$ are isomorphic near the identity / locally isomorphic.

The $SU(3)$ group

$SU(3)$ is a group of all possible 3×3 matrices that are unitary & has determinant '1'.

$$SU(3) = \{U \in GL(3, \mathbb{C}) : U^\dagger U = \mathbb{I}_3, \det(U) = 1\}$$

It is a lie group of dimension '8', there are 8 independent real parameters to describe any $U \in SU(3)$.

Any $SU(3)$ element can be written as the exponential of a 3×3 traceless anti-Hermitian matrix or e^{iH} where H is 3×3 traceless Hermitian matrix.

The general form of a 3×3 hermitian matrix also requires 8 independent real parameters.

$$H = \begin{bmatrix} a & c & d \\ c^* & b & f \\ d^* & f^* & -a-b \end{bmatrix}$$

a, b are real (2-parameters)
 c, d, f are complex (6-parameters).

The set of all such matrices form a 8 dimensional vector space over the field of real numbers. The suitable basis to such vector space which extends the idea of Pauli matrices are Gell-Mann matrices. They are represented by $\{\lambda_i\}$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

$$\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Any traceless-hermitian matrix can be written as the linear combination over real field as,

$$\underline{H} = \theta_k \lambda_k \quad (\theta_k \in \mathbb{R})$$

$\Rightarrow iH$ will be anti-Hermitian & traceless.

Then $e^{iH} = e^{i\theta_k \lambda_k}$ will be an $SU(3)$ element. Thus λ_k are the generators of $SU(3)$ lie group.

The Gell-mann matrices have trace orthogonality i.e. $\text{trace}(\lambda_j \lambda_k) = 2\delta_{jk}$

$SU(2)$ algebras are embedded in $SU(3)$

- $\lambda_1, \lambda_2, \lambda_3$ together constitute a $SU(2)$ algebra in 1-2 sector as they are just pauli matrices with 3rd sector all zero.
- λ_4 & λ_5 are same as λ_1 & λ_2 but they live in 1-3 sector and $\lambda_{[4,5]} = \frac{\lambda_3 + \sqrt{3}\lambda_8}{2}$ is same as λ_3 living in 1-3 sector.
So λ_4, λ_5 & $\lambda_{[4,5]}$ forms another $SU(2)$ algebra in 1-3 sector.
- Similarly λ_6, λ_7 & $\lambda_{[6,7]} = \frac{-\lambda_3 - \sqrt{3}\lambda_8}{2}$ forms another $SU(2)$ algebra in the 2-3 sector.

Thus $SU(3)$ has three overlapping $SU(2)$ sub-algebras as there are only two diagonal matrices λ_3 & λ_8 which are shared by three subalgebras.

The structure constants for the Gell-mann matrices are

$$[\lambda_j, \lambda_k] = 2if_{jkl} \lambda_l$$

where f_{jkl} is anti-symmetric in all indices.

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$

Other combinations which don't appear even upon switching the indices are in general zero.

The Lorentz Group $O(3,1)$

Here we look into the symmetries in 4D Minkowski Space with one time-like coordinate ' x^0 ' & three space-like coordinates (x^1, x^2, x^3) . Together they form 4D Space-time coordinate denoted as a 4-vector $x \equiv x^\mu = (x^0, x^1, x^2, x^3)$

The Minkowski norm of a 4-vector $x^\mu = (x^0, x^1, x^2, x^3)$ is given by $(x)^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$

Normality of a vector:

Time-like normalized, if $(x)^2 = 1$

Space-like normalized, if $(x)^2 = -1$

Light / Null like: If $(x)^2 = 0$.

In matrix notation, we can denote the norm operation as follows.

$$\begin{bmatrix} x^0 & x^1 & x^2 & x^3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{\text{Metric}} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

If we denote the elements of the metric as $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, then,

$$\underline{(x)^2 = x^\mu \eta_{\mu\nu} x^\nu}$$

The inner product b/w two 4-vectors x^μ & y^ν can be written as

$$\underline{xy = x^\mu y^\nu \eta_{\mu\nu}}$$

Properties of Inner product in Minkowski Space

Let x, y, z denote 4-vectors

$$(i) \quad xy = yx$$

$$(ii) \quad (\alpha x + \beta y)z = \alpha(xz) + \beta(yz)$$

$$(iii) \quad (x \pm y)^2 = (x)^2 + (y)^2 \pm 2(xy)$$

$$(iv) \quad (x + y)(x - y) = (x)^2 - (y)^2$$

$$(v) \quad \frac{1}{2}[(x + y)^2 + (x - y)^2] = (x)^2 + (y)^2$$

Orthogonality of two 4-vectors

If $xy = 0$, then x^μ & y^μ are orthogonal to each other.

Lorentz transformations

These are the transformations on a 4D Minkowski space which keeps the Minkowski norm invariant.

i.e. for $x^\mu = (x^0, x^1, x^2, x^3)$, Under Lorentz transformation $x^\mu \rightarrow \bar{x}^\mu$, we get $\bar{x}^\mu = (\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$ such that,

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (\bar{x}^0)^2 - (\bar{x}^1)^2 - (\bar{x}^2)^2 - (\bar{x}^3)^2$$

Thus $\eta_{\mu\nu} = \bar{\eta}_{\mu\nu}$ is preserved.

Lorentz transformation in tensor notation is

$$\bar{x}^\mu = L^\mu{}_\nu x^\nu$$

$$\text{Since } (\bar{x})^2 = (x)^2$$

$$\Rightarrow \bar{x}^\mu \bar{x}^\nu \eta_{\mu\nu} = x^\mu x^\nu \eta_{\mu\nu}$$

$$\Rightarrow L^\mu{}_\rho x^\rho L^\nu{}_\sigma x^\sigma \eta_{\mu\nu} = x^\rho x^\sigma \eta_{\rho\sigma}$$

$$\Rightarrow \underline{L^\mu_\rho \eta_{\mu\nu} L^\nu_\sigma = \eta_{\rho\sigma}}$$

If $L = [L^\mu_\nu]_{4 \times 4}$ matrix, then, the above condition becomes

$\boxed{L^T \eta L = \eta} \Rightarrow$ Any Lorentz transformation matrix has to obey this eqⁿ. This becomes the defⁿ for a transformation to be called as Lorentz transformation.

- Lorentz transformation is Linear.
i.e., $\forall x, y \in 4\text{-vectors}$
 & $\forall \alpha, \beta \in \mathbb{R}$

$$L(\alpha x + \beta y) = \alpha(Lx) + \beta(Ly)$$

- Lorentz transformation preserves inner product relation.
i.e. $\underline{(Lx)(Ly) = xy}$

- Determinant of L matrix is ± 1 .

$$\text{Since } L^T \eta L = \eta$$

$$\Rightarrow \det(L^T \eta L) = \det(\eta)$$

$$\Rightarrow \det(L^T) \cancel{\det(\eta)} \det(L) = \cancel{\det(\eta)}$$

$$\Rightarrow \det(L^T) \det(L) = 1$$

For any matrix its determinant is equal to its transpose's determinant.

$$\Rightarrow (\det(L))^2 = 1$$

$$\Rightarrow \underline{\det(L) = \pm 1}$$

- Lorentz transformations are orthogonal matrices with respect to Minkowski metric $\eta = \text{diag}(1, -1, -1, -1)$.

Since $(L^T)\eta L = \eta$

$$\Rightarrow \eta^{-1}(L^T)\eta L = \mathbb{I}_4$$

$$\Rightarrow [\eta^{-1}L^T\eta]L = \mathbb{I}_4$$

But $\eta^{-1} = \eta$ $\Rightarrow (\eta L^T \eta)L = \mathbb{I}_4$

$$\Rightarrow (\eta L \eta)^T L = \mathbb{I}_4$$

Let $\tilde{L} = \eta L \eta$, then

$$\Rightarrow \underline{\tilde{L}^T L = \mathbb{I}_4}$$

Thus 'L' is loosely an orthogonal matrix where the orthogonality is determined by Minkowski metric instead of Identity.

- The columns of 'L' matrix are minkowski orthonormal to each other because of $\underline{L^\mu_\sigma \eta_{\mu\nu} L^\nu_\rho = \eta_{\sigma\rho}}$

For $\sigma = \rho = 0$

$$L^\mu_0 \eta_{\mu\nu} L^\nu_0 = 1$$

$$\Rightarrow \underline{(L^0_0)^2 - (L^1_0)^2 - (L^2_0)^2 - (L^3_0)^2 = 1}$$

Thus $L^\mu_0 = (L^0_0, L^1_0, L^2_0, L^3_0)$ act as a time-like normalized four vector.

For $\sigma = j, \rho = k$ ($j, k = \{1, 2, 3\}$)

$$\underline{L^\mu_j \eta_{\mu\nu} L^\nu_k = \eta_{jk} = -\delta_{jk}}$$

If we take L^μ_j as a four vector labelled by $j = \{1, 2, 3\}$, they are space-like orthonormal.

$$\Rightarrow L = \begin{bmatrix} L_0^0 & L_1^0 & L_2^0 & L_3^0 \\ L_0^1 & L_1^1 & L_2^1 & L_3^1 \\ L_0^2 & L_1^2 & L_2^2 & L_3^2 \\ L_0^3 & L_1^3 & L_2^3 & L_3^3 \end{bmatrix}$$

Time-like normalized
Space like normalized

Each column of 'L' is minkowski orthogonal to other columns of L.

Since $L^T \eta L = \eta = L \eta L^T$
 we get $L^T L = L L^T = \mathbb{I}$

Thus every row of L is also minkowski orthonormal to each other.

1st row is time like normalized
 2nd, 3rd & 4th rows are space like normal

And every row is minkowski orthogonal to every other row.

The number of independent real parameters to determine lorentz transformation is given by,

$$\underbrace{16}_{\text{total no: of entries}} - \underbrace{4}_{\text{column normality conditions}} - \underbrace{6}_{\text{column orthogonality conditions}} = 6$$

Thus there are 6 independent real parameters to determine lorentz transformation matrix.

Thus the set of all possible matrices that obey $L^T \eta L = \eta$, will form a orthogonal group called lorentz group $O(1,3)$ where '1' represents no: of time like components & '3' represents number of space-like components.

Types of Lorentz transformations (LT)

a) Based on determinant

$$\text{Since } \det(L)^2 = 1 \Rightarrow \det(L) = \pm 1$$

- Proper LT: $\det(L) = +1$
- Improper LT: $\det(L) = -1$

b) Based on sign of L^0_0

$$\text{Since } (L^0_0)^2 - \sum_j (L^j_0)^2 = 1$$

$$\Rightarrow L^0_0 \geq 1 \text{ or } L^0_0 \leq -1$$

- Orthochronous LT: $L^0_0 \geq 1$
 $\Rightarrow \text{Sign}(x^0) = \text{Sign}(\bar{x}^0)$, $\bar{x} = Lx$
It has no time reversality
- Anti-chronous LT: $L^0_0 \leq -1$
 $\Rightarrow \text{Sign}(x^0) = -\text{Sign}(\bar{x}^0)$, $\bar{x} = Lx$
It has time-reversality.

Special Lorentz Group $SO(1,3)$

The lorentz transformations whose determinant is '1' form Proper/Special lorentz group denoted $SO(1,3)$. It preserves the orientation of temporal component with spatial component.

i.e. if $\xi^0 = \text{sign}(x^0)$ & $\xi^i = \text{sign}(x^i)$
and $\bar{\xi}^0 = \text{sign}(\bar{x}^0)$ & $\bar{\xi}^i = \text{sign}(\bar{x}^i)$
where $\bar{x} = (Lx)$. Then,

- $\xi^0 = \bar{\xi}^0 \Rightarrow \xi^1 \xi^2 \xi^3 = \bar{\xi}^1 \bar{\xi}^2 \bar{\xi}^3$
- $\xi^0 = -\bar{\xi}^0 \Rightarrow \xi^1 \xi^2 \xi^3 = -\bar{\xi}^1 \bar{\xi}^2 \bar{\xi}^3$

The change of sign of all components of x^μ is allowed in $SO(1,3)$, as even $L = -I_4 \in SO(1,3)$ such that $\det(-I_4) = +1$. So here if there is time reversal, then it also includes parity reversal.

Restricted Lorentz Group $SO^+(1,3)$

The restricted Lorentz group is the identity component of $O(1,3)$. i.e., every element in $SO^+(1,3)$ can be connected to Identity I_4 , by a continuous change of parameters.

$SO^+(1,3)$ is connected, i.e. all elements are connected to I_4 . Where as $SO(1,3)$ is disconnected, i.e. Some elements are connected to I_4 & some are connected to $-I_4$ & I_4 & $-I_4$ are disconnected.

Thus $SO^+(1,3)$ is a normal and connected subgroup of $O(1,3)$, where as $SO(1,3)$ is just normal subgroup of $O(1,3)$.

$$\underline{SO^+(1,3) = \{L \in O(1,3) : \det(L) = 1, L^0_0 \geq +1\}}$$

$SO^-(1,3)$ is the set of all elements of $SO(1,3)$ that are connected to -ve identity $(-\mathbb{I}_4)$.

$$\Rightarrow SO^-(1,3) = \underbrace{-\mathbb{I}_4}_{\text{coset}} SO^+(1,3)$$

$$\& \underline{SO(1,3) = SO^+(1,3) \cup SO^-(1,3)}$$

Similarly the Lorentz transforms whose determinant is -1 , can be decomposed to orthochronous & anti-chronous, which are disconnected. One set connects to 'P' & other set connects to T, where $P \neq T$ are disconnected.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \& \quad T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Parity reversal LT) (Time reversal LT)

$$\rightarrow O_{\det(-1)}(1,3) = \underline{[P SO^+(1,3)] \cup [T SO^+(1,3)]}$$

$\Rightarrow O(1,3)$, the full Lorentz group has totally 4 disconnected components:

$$O(1,3) = [SO^+(1,3)] \cup [PT \cdot SO^+(1,3)] \\ \cup [P \cdot SO^+(1,3)] \cup [T \cdot SO^+(1,3)]$$

- $SO^+(1,3)$: Proper - Orthochronous
 $\forall L \in SO^+(1,3), \det(L) = 1, L^0_0 \geq +1$.
It forms restricted Lorentz group.

- $SO^-(1,3)$: Proper - Antichronous
 $\forall L \in SO^-(1,3), \det(L) = 1, L^0_0 \leq -1$.
where $SO^-(1,3) = PT \cdot SO^+(1,3), PT = -\mathbb{I}_4$

$SO(1,3) = [SO^+(1,3)] \cup [PT \cdot SO^+(1,3)]$ forms the Special Lorentz group.

- $P \cdot SO^+(1,3)$: Improper - Orthochronous
 $\forall L \in P \cdot SO^+(1,3), \det(L) = -1, L^0_0 \geq +1$.

- $T \cdot SO^+(1,3)$: Improper - Antichronous
 $\forall L \in T \cdot SO^+(1,3), \det(L) = -1, L^0_0 \leq -1$.

The restricted lorentz group $SO^+(1,3)$ has six independent parameters & hence its lie algebra consists of six generators. Three ordinary spatial rotations (Pure rotations) that doesn't mix time with space and three lorentz boosts or Hyperbolic rotations that mix space & time.

The pure rotations in 3D space are given by the matrices $\begin{bmatrix} 1 & 0 \\ 0 & R_{3 \times 3} \end{bmatrix}_{4 \times 4}$

Where $R \in SO(3)$. Thus $SO(3)$ is a subgroup of $SO(4)$. Since $SO(3)$ is generated by three generators J_x, J_y & J_z , even the pure rotations of lorentz group are generated by J_x, J_y & J_z .

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, J_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, J_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Lorentz Boosts

Lorentz Boosts are hyperbolic rotations in time direction and boost axis direction. For example, the Lorentz boost along x^1 -direction is given by,

$$\Lambda = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{v}{c}$$

$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu$ is given by

$$\bar{x}^0 = \gamma (x^0 - \beta x^1)$$

$$\bar{x}^1 = \gamma (x^1 - \beta x^0)$$

$$\bar{x}^2 = x^2$$

$$\bar{x}^3 = x^3$$

} Boost only along x^1 direction. x^2 & x^3 remain unchanged

But notice that if we define the rapidity parameter as,

$$\underline{\psi = \tanh^{-1}(\beta)} \quad \text{or} \quad \beta = \tanh(\psi)$$

Then by writing $\gamma = \cosh(\psi)$
and $\beta\gamma = \sinh(\psi)$

We can write the Lorentz matrix with boost only along x^1 -axis as,

$$\Lambda_x(\psi) = \begin{bmatrix} \cosh(\psi) & -\sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Set of all matrices of the type $\Lambda_x(\psi)$ form an abelian group.

$$\underline{\Lambda_x(\psi_1) \Lambda_x(\psi_2) = \Lambda_x(\psi_1 + \psi_2)}$$

$\{\Lambda_x(\psi)\}$ is a subgroup of $SO^+(1,3)$

Similarly, we can define $\Lambda_y(\psi)$ & $\Lambda_z(\psi)$ as follows,

$$\Lambda_y(\psi) = \begin{bmatrix} \cosh(\psi) & 0 & -\sinh(\psi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh(\psi) & 0 & \cosh(\psi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Lambda_z(\psi) = \begin{bmatrix} \cosh(\psi) & 0 & 0 & -\sinh(\psi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\psi) & 0 & 0 & \cosh(\psi) \end{bmatrix}$$

Thus $\{\Lambda_y(\psi)\}$ & $\{\Lambda_z(\psi)\}$, along with $\{\Lambda_x(\psi)\}$ form abelian subgroups of $SO^+(1,3)$ which are all isomorphic to each other.

$\{\Lambda_x(\psi)\} \rightarrow$ Pure boost along $\hat{x}$
 $\{\Lambda_y(\psi)\} \rightarrow$ Pure boost along $\hat{y}$
 $\{\Lambda_z(\psi)\} \rightarrow$ Pure boost along $\hat{z}$

A pure boost along an $\hat{n}$ direction with rapidity ψ is given as follows.

Let $\hat{n} = (n_x, n_y, n_z) \rightarrow$ direction of boost, Let $B = v/c \Rightarrow$ boost velocity along $\hat{n}$ & $\psi = \tanh^{-1}(B)$ with

$$\gamma = \frac{1}{\sqrt{1-B^2}}$$

Thus, pure boost $\in SO^+(1,3)$ is given by, $\Lambda_{\hat{n}}(\psi)$ as,

$$\begin{bmatrix} \gamma & -\gamma B n_x & -\gamma B n_y & -\gamma B n_z \\ -\gamma B n_x & 1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z \\ -\gamma B n_y & (\gamma-1)n_y n_x & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z \\ -\gamma B n_z & (\gamma-1)n_z n_x & (\gamma-1)n_z n_y & 1+(\gamma-1)n_z^2 \end{bmatrix}$$

$\hat{n} = (\theta, \phi)$ in polar coordinates & ψ determines, B & γ . Thus, totally three real parameters are needed to specify pure boost ' $\Lambda_{\hat{n}}(\psi)$ '.

Note that set of all $\{\Lambda_{\hat{n}}(\psi)\}$ don't form a group. Since multiplying two non-collinear pure boosts is not a pure boost, rather it is composition of some boost along with rotation.

Any two non collinear boosts given by $\vec{u}$ & $\vec{v}$ velocities where $|\vec{u}| = |\vec{v}| \Rightarrow \psi_u = \psi_v$, but $\vec{u} \times \vec{v} \neq 0$, can be related as follows:

$$\Lambda_{\vec{u}}(\psi) = R^{-1} \Lambda_{\vec{v}}(\psi) R$$

where $\underline{R(\vec{v}) = \vec{u}} \Rightarrow |\vec{v}| = |\vec{u}|$

Thus different lorentz boosts but with same rapidity are conjugate to each other.

$$\Lambda_{\vec{u}}(\psi) \simeq \Lambda_{\vec{v}}(\psi)$$

Generators of Lorentz boosts

The three basic Lorentz boosts are given by $\Lambda_{\hat{x}}(\psi)$, $\Lambda_{\hat{y}}(\psi)$, $\Lambda_{\hat{z}}(\psi)$.

The infinitesimal boosts are then given by,

$$\Lambda_{\hat{x}}(\psi) = \begin{bmatrix} \cosh(\psi) & -\sinh(\psi) & 0 & 0 \\ -\sinh(\psi) & \cosh(\psi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cosh(\psi) = 1 + \underbrace{\frac{(\psi)^2}{2!} + \dots}_{\text{neglect}}$$

$$\sinh(\psi) = \psi + \underbrace{\frac{(\psi)^3}{3!} + \dots}_{\text{neglect}}$$

By keeping terms only till 1st order in (ψ) we get,

$$\Lambda_{\hat{x}}(2\psi) = \begin{bmatrix} 1 & -2\psi & 0 & 0 \\ -2\psi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \hat{\mathbb{I}}_4 - 2\psi \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \hat{\mathbb{I}}_4 + i^2 2\psi \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \hat{\mathbb{I}}_4 - i 2\psi \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Let } K_x = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \Lambda_{\hat{x}}(\delta\psi) = \hat{\mathbb{I}}_4 - i(\delta\psi)K_x$$

Similarly $\Lambda_{\hat{y}}(\delta\psi) = \hat{\mathbb{I}}_4 - i(\delta\psi)K_y$

& $\Lambda_{\hat{z}}(\delta\psi) = \hat{\mathbb{I}}_4 - i(\delta\psi)K_z$

$$K_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

Thus K_x, K_y & K_z form the generators for Lorentz boosts. Note that K_j are anti-Hermitian $\Rightarrow iK_j$ is Hermitian.
So ' Λ ' are not unitary matrices.

Similarly pure infinitesimal rotations can be written as following,

$$R_x(\delta\psi) = \mathbb{I}_4 - i(\delta\psi)J_x$$

$$R_y(\delta\psi) = \mathbb{I}_4 - i(\delta\psi)J_y$$

$$R_z(\delta\psi) = \mathbb{I}_4 - i(\delta\psi)J_z$$

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$J_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus J_x , J_y & J_z form the generators for pure rotations.

The commutation relation b/w the matrices (J_x, J_y, J_z) & (K_x, K_y, K_z) are:

$$\left. \begin{aligned} [J_j, J_k] &= i \varepsilon_{jkl} J_l \\ [K_j, K_k] &= -i \varepsilon_{jkl} J_l \\ [K_j, J_k] &= i \varepsilon_{jkl} K_l \end{aligned} \right\} \text{so}(1,3) \text{ algebra}$$

This resembles the $so(4)$ algebra but except for the extra -ve sign in commutation $[K_j, K_k] = -i \epsilon_{jkl} J_l$.

This change is because not all elements of $SO^+(1,3)$ are unitary. But if $K \rightarrow iK$, then the algebra reduces to $so(4)$ algebra.

Now consider the $so(1,3)$ algebra. If we define two sets of vector operators,

$$\vec{N} = \frac{\vec{J} + i\vec{K}}{2}, \quad \vec{M} = \frac{\vec{J} - i\vec{K}}{2},$$

then the resulting commutations are:

- $[N_j, M_k] = 0$
 - $[N_j, N_k] = i \epsilon_{jkl} N_l$
 - $[M_j, M_k] = i \epsilon_{jkl} M_l$
- } Two non-overlapping $SU(2)$ algebras

Exponentiating $so(1,3)$ gives $SO^+(1,3)$

Any element of $SO^+(1,3)$ can be got by exponentiating some algebra element which is got by spanning $\vec{J} \neq \vec{K}$.

$$\Rightarrow L = \mathbb{I}_4 - i(\partial\phi)_k J_k - i(\partial\psi)_j K_j$$

(near identity)

We can define the generators more compactly by two index notation.

First of all any spatial rotation in the (m,n) -plane can be interpreted as the rotation about an axis 'k' perpendicular to (m,n) -plane.

$$\Rightarrow \text{Let } \left. \begin{aligned} \partial\phi_1 &= \partial\omega^{23} \\ \partial\phi_2 &= \partial\omega^{31} \\ \partial\phi_3 &= \partial\omega^{12} \end{aligned} \right\} \partial\omega^{jk} = \epsilon_{jkl} \phi_l$$

where (jk) represents the plane.

$$\Rightarrow \underbrace{\partial \omega^{jk}}_{\text{anti-symmetric tensor}} = \begin{bmatrix} 0 & (\partial \phi)_3 & -(\partial \phi)_2 \\ -(\partial \phi)_3 & 0 & (\partial \phi)_1 \\ (\partial \phi)_2 & -(\partial \phi)_1 & 0 \end{bmatrix}$$

$$\text{Similarly } \begin{aligned} J_x &\rightarrow J_{23} \\ J_y &\rightarrow J_{31} \\ J_z &\rightarrow J_{12} \end{aligned}$$

$$\rightarrow J_{mn} = \epsilon_{mnk} J_k$$

$$\begin{aligned} \text{Consider } (\partial \omega)^{mn} J_{mn} &= \epsilon_{mnl} (\partial \phi)_l \epsilon_{mnk} J_k \\ &= (\partial \phi)_l J_k (\delta_{nn} \delta_{lk} - \delta_{nk} \delta_{nl}) \\ &= (\partial \phi)_l J_k (3 \delta_{lk} - \delta_{lk}) \\ &= 2 (\partial \phi)_k J_k \end{aligned}$$

$$\rightarrow (\partial \omega)^{mn} J_{mn} = 2 (\partial \phi)_k J_k$$

Since $R = \mathbb{I} - i(\partial\phi_k)J_k$

$$\Rightarrow R = \mathbb{I} - i \frac{(\partial\omega)^{mn}}{2} J_{mn}$$

Thus $\frac{(\partial\omega)^{mn}}{2} J_{mn}$ spans all the possible generators of rotations

Similarly for generators of Lorentz boosts along x, y, z we can define,

$$(\partial\psi)_k = (\partial\omega)^{k0} = (-\partial\omega)^{0k}$$

$$\& K_j = J_{j0} = -J_{0j}$$

then $\underbrace{(\partial\psi)_j K_j}_{\text{spans all possible Lorentz boosts.}} = (\partial\omega)^{j0} J_{j0} = (\partial\omega)^{0j} J_{j0}$

spans all possible Lorentz boosts.

Thus if we define $\omega^{\mu\nu}$ as a six parameter anti-symmetric rank-2 tensor, similarly $J_{\mu\nu}$ as the corresponding generators, then,

$$\frac{(\omega^{\mu\nu}) J_{\mu\nu}}{2} = (\phi)_k J_k + (\psi)_j K_j$$

will span the entire $so(1,3)$ algebra. And hence any infinitesimal $SO^+(1,3)$ element can be written as,

$$\underline{L = \mathbb{I}_4 - i \frac{\omega^{\mu\nu} J_{\mu\nu}}{2}}$$

And finite Lorentz transformations can be written as depending on six parameter anti symmetric rank-2 tensor as,

$$L \in SO^+(1,3) \Rightarrow \boxed{L(\omega) = e^{-i \frac{\omega^{\mu\nu} J_{\mu\nu}}{2}}}$$

$$\omega^{\mu\nu} = \begin{bmatrix} 0 & -\psi_1 & -\psi_2 & -\psi_3 \\ \psi_1 & 0 & \phi_3 & -\phi_2 \\ \psi_2 & -\phi_3 & 0 & \phi_1 \\ \psi_3 & \phi_2 & -\phi_1 & 0 \end{bmatrix}$$

$$J_{\mu\nu} = \begin{bmatrix} 0 & -K_x & -K_y & -K_z \\ K_x & 0 & J_z & -J_y \\ K_y & -J_z & 0 & J_x \\ K_z & J_y & -J_x & 0 \end{bmatrix}$$

In this notation the commutators

$$[J_{\mu\nu}, J_{\sigma\rho}] = -i \left(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} \right)$$

Let $L(\omega) = e^{-i\frac{\omega^{\mu\nu}}{2}J_{\mu\nu}}$ be some LT in $SO^+(1,3)$, & Λ be any other lorentz transform, then,

$$\Lambda L(\omega) \Lambda^{-1} = L(\bar{\omega})$$

where $\bar{\omega}^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\rho \omega^{\sigma\rho}$

& $\Lambda J_{\mu\nu} \Lambda^{-1} = J_{\sigma\rho} \Lambda^\sigma_\mu \Lambda^\rho_\nu$

Thus under lorentz transform, $J_{\mu\nu}$ behaves as the components of a second rank tensor.

Note that: $\Lambda L(\omega) \Lambda^{-1} = L(\bar{\omega})$ doesn't mean that all lorentz transformations are conjugate since $L(\omega)$ & $L(\bar{\omega})$ are not same.

This relation is called as the adjoint action of lorentz group on its own algebra.

Homomorphism b/w $SL(2, \mathbb{C})$ & $SO^+(1, 3)$ & double cover

Consider a matrix from $SL(2, \mathbb{C})$.
It can be of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{l} a, b, c, d \in \mathbb{C} \\ \& \det(A) = 1 \end{array}$$

$$a = a_1 + ia_2$$

$$b = b_1 + ib_2$$

$$c = c_1 + ic_2$$

$$d = d_1 + id_2$$

$$\det(A) = ad - bc = 1$$

$$\Rightarrow (a_1 + ia_2)(d_1 + id_2) - (b_1 + ib_2)(c_1 + ic_2) = 1$$

$$\Rightarrow a_1d_1 - a_2d_2 - b_1c_1 + b_2c_2 = 1$$

$$\& a_1d_2 + a_2d_1 - b_1c_2 - c_1b_2 = 0$$

Thus out of 8 entries, there are only 6 independent real parameters.

Also consider the one to one mapping of a four vector $x^\mu = (x^0, x^1, x^2, x^3)$ to a 2×2 hermitian matrix using the pauli basis.

$$\text{Let } \sigma^\mu = (\mathbb{I}_2, \sigma_x, \sigma_y, \sigma_z)$$

$$\& \bar{\sigma}^\mu = (\mathbb{I}_2, -\sigma_x, -\sigma_y, -\sigma_z)$$

$$\sigma_\mu = \eta_{\mu\nu} \sigma^\nu = (\mathbb{I}_2, -\sigma_x, -\sigma_y, -\sigma_z)$$

$$\bar{\sigma}_\mu = \eta_{\mu\nu} \bar{\sigma}^\nu = (\mathbb{I}_2, \sigma_x, \sigma_y, \sigma_z)$$

$$\text{Note } \{\sigma^\mu\} = \{\bar{\sigma}_\mu\} \text{ and } \{\sigma_\mu\} = \{\bar{\sigma}^\mu\}$$

$$\Rightarrow x^\mu \bar{\sigma}_\mu = x^0 \mathbb{I}_2 + x^1 \sigma_x + x^2 \sigma_y + x^3 \sigma_z$$

$$\rightarrow \chi = x^\mu \bar{\sigma}_\mu = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}$$

general 2×2 Hermitian matrix

Thus $\forall x^\mu \in \mathbb{R}^4$, $\exists$ 2×2 hermitian matrix $X = x^\mu \bar{\sigma}_\mu$ which is a bijective mapping. i.e. any 2×2 Hermitian matrix (X) can also be mapped to a unique 4-vector.

$$x^0 = \frac{1}{2} \text{trace}(X) \text{ \& } x^j = \frac{1}{2} \text{trace}(\sigma^j X)$$

$$\Rightarrow \underline{x^\mu = \frac{1}{2} \text{trace}(\sigma^\mu X)}$$

Then, determinant of $X = x^\mu \bar{\sigma}_\mu$ is given by,

$$\det(X) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

$\Rightarrow \det(x^\mu \bar{\sigma}_\mu)$ corresponds to the norm of x^μ ,

$$\text{i.e. } \underline{\det(x^\mu \bar{\sigma}_\mu) = x^\nu x_\nu = x^\mu x^\nu \eta_{\mu\nu}}$$

Now consider a matrix $A \in SL(2, \mathbb{C})$
& the congruent action of A on χ .

$$\Rightarrow \bar{\chi} = A^\dagger \chi A$$

$$\text{now } (\bar{\chi})^\dagger = A^\dagger \chi^\dagger (A^\dagger)^\dagger$$

$$\Rightarrow \bar{\chi}^\dagger = A^\dagger \chi A = \bar{\chi}.$$

$\Rightarrow \bar{\chi}$ corresponds to some other
4-vector $\bar{\chi}^\mu = \frac{1}{2} \text{trace}(\sigma^\mu \bar{\chi})$

$$\begin{aligned} \text{Now } \det(\bar{\chi}) &= \det(A^\dagger \chi A) \\ &= \underbrace{\det(A^\dagger)}_1 \det(\chi) \underbrace{\det(A)}_1 \end{aligned}$$

Since $A, A^\dagger \in SL(2, \mathbb{C}) \Rightarrow \text{determinant} = 1$.

$$\Rightarrow \det(\bar{\chi}) = \det(\chi)$$

$$\Rightarrow \underline{\bar{\chi}^\mu \bar{\chi}_\mu = \chi^\nu \chi_\nu}$$

$$\Rightarrow (\bar{\chi}^0)^2 - (\bar{\chi}^1)^2 - (\bar{\chi}^2)^2 - (\bar{\chi}^3)^2 = (\chi^0)^2 - (\chi^1)^2 - (\chi^2)^2 - (\chi^3)^2$$

⇒ Even though $x^\mu \rightarrow \bar{x}^\mu$, i.e. the four vector has transformed, the Minkowski norm of four vector is conserved, under the congruent action of $A \in SL(2, \mathbb{C})$ on X .

Thus $A \in SL(2, \mathbb{C})$, induces a Lorentz transformation on x^μ . This defines a homomorphism map from $SL(2, \mathbb{C})$ to $SO(1, 3)$.

Consider $X = x^\mu \bar{\sigma}_\mu$ & $A \in SL(2, \mathbb{C})$

$$\& \bar{X} = A^\dagger X A$$

$$\Rightarrow \bar{x}^\mu \bar{\sigma}_\mu = A^\dagger x^\nu \bar{\sigma}_\nu A$$

$$\Rightarrow \bar{x}^\mu \bar{\sigma}_\mu = x^\nu A^\dagger \bar{\sigma}_\nu A$$

$$\text{But } \bar{x}^\mu = L^\mu{}_\nu x^\nu$$

$$\rightarrow L^\mu{}_\nu x^\nu \bar{\sigma}_\mu = x^\nu A^\dagger \bar{\sigma}_\nu A$$

$$\Rightarrow \underline{L^\mu{}_\nu \bar{\sigma}_\mu = A^\dagger \bar{\sigma}_\nu A}$$

$$\begin{bmatrix} L^0_\nu + L^3_\nu & L^1_\nu - iL^2_\nu \\ L^1_\nu + iL^2_\nu & L^0_\nu - L^3_\nu \end{bmatrix} = A^\dagger \sigma_\nu A$$

$$\Rightarrow L^0_\nu = \frac{1}{2} \text{trace}(L^\mu_\nu \overline{\sigma}_\mu)$$

$$\Rightarrow \underline{L^0_\nu = \frac{1}{2} \text{trace}(A^\dagger \overline{\sigma}_\nu A)}$$

$$\text{Similarly, } L^\mu_\nu = \frac{1}{2} \text{trace}(\sigma^\mu L^\rho_\nu \overline{\sigma}_\rho)$$

$$\Rightarrow \underline{L^\mu_\nu = \frac{1}{2} \text{trace}(\sigma^\mu A^\dagger \overline{\sigma}_\nu A)}$$

$$L^0_0 = \frac{1}{2} \text{trace}(A^\dagger A)$$

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow L^0_0 = \frac{1}{2} (|a|^2 + |b|^2 + |c|^2 + |d|^2)$$

$$\Rightarrow L^0_0 \text{ is always +ve.}$$

Thus L is orthochronous.

Since every L is mapped to some $A \in SL(2, \mathbb{C})$ whose determinant is always $+1$, we can say that the matrix $[L^\mu_\nu]$ has $\det = +1$.

$\Rightarrow$ The Lorentz transform $L(A)$ is both special ($\det = 1$) & orthochronous.

$$\Rightarrow L(A) \in SO^+(1, 3)$$

Also there exists a Homomorphism
b/w $A \in SL(2, \mathbb{C})$ & $L(A) \in SO^+(1, 3)$

i.e. $L(AB) = L(A)L(B)$
 $A, B \in SL(2, \mathbb{C})$.

Since $[L(A)]^\mu_\nu \bar{\sigma}_\mu = A^\dagger \bar{\sigma}_\nu A$

& $[L(B)]^\mu_\nu \bar{\sigma}_\mu = B^\dagger \bar{\sigma}_\nu B$

$$\Rightarrow [L(AB)]^\mu_\nu \bar{\sigma}_\mu = (AB)^\dagger \bar{\sigma}_\nu (AB)$$

$$\begin{aligned}
\Rightarrow [\mathcal{L}(AB)]_{\nu}^{\mu} \overline{\sigma}_{\mu} &= B^{\dagger} A^{\dagger} \overline{\sigma}_{\nu} A B \\
&= B^{\dagger} (A^{\dagger} \overline{\sigma}_{\nu} A) B \\
&= B^{\dagger} [\mathcal{L}(A)]_{\nu}^{\rho} \overline{\sigma}_{\rho} B \\
&= [\mathcal{L}(A)]_{\nu}^{\rho} (B^{\dagger} \overline{\sigma}_{\rho} B) \\
&= [\mathcal{L}(A)]_{\nu}^{\rho} [\mathcal{L}(B)]_{\rho}^{\mu} \overline{\sigma}_{\mu}
\end{aligned}$$

$$\Rightarrow [\mathcal{L}(AB)]_{\nu}^{\mu} \overline{\sigma}_{\mu} = [\mathcal{L}(A)]_{\nu}^{\rho} [\mathcal{L}(B)]_{\rho}^{\mu} \overline{\sigma}_{\mu}$$

Thus by comparing,

$$[\mathcal{L}(AB)]_{\nu}^{\mu} = \underbrace{[\mathcal{L}(A)]_{\nu}^{\rho} [\mathcal{L}(B)]_{\rho}^{\mu}}_{\text{matrix multiplication}}$$

$$\Rightarrow \underline{\mathcal{L}(AB) = \mathcal{L}(A) \mathcal{L}(B)}$$

Thus $\mathcal{L}: SL(2, \mathbb{C}) \longrightarrow SO^+(1, 3)$
is a Homomorphism.

Thus we can summarize that,

- The Lorentz transform $L^\mu_\nu(A)$ is both special ($\det = 1$) and orthochronous. $\Rightarrow L^\mu_\nu(A) \in SO^+(1, 3)$
- $L^\mu_\nu(A) = \frac{1}{2} \text{trace}(\sigma^\mu A^\dagger \bar{\sigma}_\nu A)$ is a 2 to 1 homomorphism from $SL(2, \mathbb{C}) \longrightarrow SO^+(1, 3)$.
Since both A and $-A \in SL(2, \mathbb{C})$ maps to the same $L(A)$.
- Hence $SL(2, \mathbb{C})$ double covers $SO^+(1, 3)$.

$$\Rightarrow \frac{SL(2, \mathbb{C})}{\mathbb{Z}_2} \cong SO^+(1, 3)$$

$\mathbb{Z}_2 = \{1, -1\}$ $\searrow$ quotient group

i.e. the quotient group $\frac{SL(2, \mathbb{C})}{\mathbb{Z}_2}$ is isomorphic to $SO^+(1, 3)$.

Representations of $SO^+(1,3)$ group

Consider the generators $\vec{J}$ & $\vec{K}$

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$J_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_x = \begin{bmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_y = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_z = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$\vec{J}$ is Hermitian which generates the $SU(2)$ group which is unitary in finite dimensional representations. But $\vec{K}$ is anti-Hermitian, thus the matrices generated by $\vec{K}$ are not unitary.

The commutation relation b/w the matrices $(J_x, J_y, J_z) \neq (K_x, K_y, K_z)$ are:

$$\left. \begin{aligned} [J_j, J_k] &= i \varepsilon_{jkl} J_l \\ [K_j, J_k] &= i \varepsilon_{jkl} K_l \\ [K_j, K_k] &= -i \varepsilon_{jkl} J_l \end{aligned} \right\} \text{so}(1,3) \text{ algebra}$$

If we define two sets of vector operators,

$$\vec{N} = \frac{\vec{J} + i\vec{K}}{2}, \quad \vec{M} = \frac{\vec{J} - i\vec{K}}{2},$$

Which is got by complexification of the $\text{so}(1,3)$ algebra, then the commutations are:

$$\left. \begin{aligned} \bullet [N_j, M_k] &= 0 \\ \bullet [N_j, N_k] &= i \varepsilon_{jkl} N_l \\ \bullet [M_j, M_k] &= i \varepsilon_{jkl} M_l \end{aligned} \right\} \begin{array}{l} \text{Two non-} \\ \text{overlapping} \\ \text{SU}(2) \\ \text{algebras} \\ \text{SU}(2) \oplus \text{SU}(2) \end{array}$$

Now an $SU(2)$ algebra is labelled by integers/half-integers $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ which results in $2j+1$ dim. repⁿ.

Since the complexification of $so(1,3)$ algebra results in the direct sum algebra $SU(2) \oplus SU(2)$, we can get the complexified repⁿ labelled by the doublet (j_N, j_M) which is a repⁿ. in $(2j_N+1)(2j_M+1)$ dimensions.

Eg: $(0,0): 1 \text{ dim} \rightarrow \text{Identity rep}^n$
 $(\frac{1}{2}, 0): 2 \text{ dim} \rightarrow \text{Left handed Weyl rep}^n$
 $(0, \frac{1}{2}): 2 \text{ dim} \rightarrow \text{Right handed Weyl rep}^n$
 $(0, 1): 3 \text{ dim}$
 $(1, 0): 3 \text{ dim}$
 $(\frac{1}{2}, \frac{1}{2}): 4 \text{ dim} \rightarrow 4 \text{ vector rep}^n$
 $(\frac{3}{2}, 0): 4 \text{ dim}$
 $(0, \frac{3}{2}): 4 \text{ dim}$
 $(2, 0): 5 \text{ dim}$
 $(0, 2): 5 \text{ dim}$

Since $SO(3)$ is a natural subgroup of $SO(1,3)$ we should be able to get representations of pure rotations in $SO(1,3)$. This can be got by the concept of addition of angular momentum $j_N \neq j_M$.

Since $\vec{J} = \vec{N} + \vec{M}$, where $\vec{N} \neq \vec{M}$ have their own angular momentum algebra that are not overlapping, then for a given (j_N, j_M) if we do a direct sum of rep^n given by $j_N + j_M, j_N + j_M - 1, j_N + j_M - 2$, so on till $|j_N - j_M|$, then we get a resulting rep^n of only the $SO(3)$ subgroup of $SO(1,3)$.

$$(j_N, j_M) \rightarrow \underbrace{(j_N + j_M) \oplus (j_N + j_M - 1) \oplus \dots \oplus (|j_N - j_M|)}$$

Addition of angular momentum of j_N & j_M restricts the rep^n to only $SO(3)$.

Weyl Spinor repⁿ of $SO^+(1,3)$

Here we consider the representation space to be the 2-dimensional Spinor Space. And try to repⁿ the $SO(1,3)$ algebra and $SO^+(1,3)$ group elements as the operators acting on the elements of Spinor Space as matrices.

Consider the two dimensional repⁿ of $SO^+(1,3)$, where (j_n, j_m) is given by either $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. The objects being transformed are two component objects called Spinors.

In the repⁿ $(\frac{1}{2}, 0)$, the Spinor is denoted by,
$$\psi_L = \begin{bmatrix} \psi_L^{(1)} \\ \psi_L^{(2)} \end{bmatrix}$$

Now we would wish to see the action of elements of $SO^+(1,3)$ on the spinors ψ_L & see how ψ_L changes under Lorentz transformation.

Here we will denote the generators of rotations as $\vec{S} = (S_x, S_y, S_z)$. & the generators of Lorentz boost as $\vec{K} = (K_x, K_y, K_z)$, which satisfy,

$$\left. \begin{aligned} [S_j, S_k] &= i \epsilon_{jkl} S_l \\ [K_j, S_k] &= i \epsilon_{jkl} K_l \\ [K_j, K_k] &= -i \epsilon_{jkl} S_l \end{aligned} \right\} \begin{array}{l} \text{SO}(1,3) \text{ algebra} \\ \text{in spinor space} \end{array}$$

And since $j_N = \frac{1}{2}$ & $j_m = 0$, this implies, $\vec{N} = \frac{\vec{S} + i\vec{K}}{2}$ acting on ψ_L

is represented by $\frac{\vec{\sigma}}{2}$ (Pauli matrix)

And $\vec{M} = \frac{\vec{S} - i\vec{K}}{2}$ acting on ψ_L is represented by 0.

$$\Rightarrow \frac{\vec{S} + i\vec{K}}{2} = \frac{\vec{\sigma}}{2} \text{ \& } \frac{\vec{S} - i\vec{K}}{2} = 0$$

$$\Rightarrow \vec{S} + i\vec{K} = \vec{\sigma} \text{ \& } i\vec{K} = -\vec{S}$$

$$\Rightarrow \underline{\vec{S} = \frac{\vec{\sigma}}{2} = i\vec{K}}$$

Under pure rotations given by $e^{-i\vec{\phi} \cdot \vec{S}}$
 $\vec{\phi} = \phi \hat{n}$, $\phi \rightarrow$ angle of rotⁿ. & $\hat{n}$
 gives axis of rotation, the spinor
 ψ_L is transformed as follows,

$$\psi_L \xrightarrow[\text{rotations}]{\text{Pure}} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

Under pure boosts given by $e^{-i\vec{C} \cdot \vec{K}}$
 $\vec{C} = C \hat{n}$, $C \rightarrow$ rapidity of boost & $\hat{n}$
 gives boost direction, the spinor
 ψ_L is transformed as follows,

$$\psi_L \xrightarrow[\text{boost}]{\text{Pure}} e^{-\vec{C} \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

For both boost + rotation

$$\psi_L \xrightarrow{\text{Rot}^n + \text{boost}} e^{\frac{-i\vec{\phi} \cdot \vec{\sigma} - \vec{E}_3 \cdot \vec{\sigma}}{2}} \psi_L$$

If $\vec{E}_3 = \vec{\phi} - i\vec{E}$ then

$$\psi_L \xrightarrow{\text{rot}^n + \text{boost}} e^{-i\vec{E}_3 \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

$$\Rightarrow \psi^\dagger \xrightarrow{\text{rot}^n + \text{boost}} \psi_L^\dagger e^{i\vec{E}_3^* \cdot \frac{\vec{\sigma}}{2}}$$

Thus ψ_L transforms as in $SU(2)$ but with complex angles $\vec{E}_3$, whose real part gives rot^n & imaginary part gives boost.

$e^{-i\vec{E}_3 \cdot \frac{\vec{\sigma}}{2}}$ represents $SO^+(1,3)$

in 2 dimensions which is Special ($\det=1$) but not unitary. Moreover

$e^{-i\vec{E}_3 \cdot \frac{\vec{\sigma}}{2}}$ is an element of $SL(2, \mathbb{C})$ as expected.

In the repⁿ $(0, \frac{1}{2})$, the spinor is denoted by ψ_R

$$\psi_R = \begin{bmatrix} \psi_R^{(1)} \\ \psi_R^{(2)} \end{bmatrix}$$

And since $j_N = 0$ & $j_m = \frac{1}{2}$, this implies, $\vec{N} = \frac{\vec{S} + i\vec{K}}{2}$ acting on ψ_L

is represented by 0. And $\vec{M} = \frac{\vec{S} - i\vec{K}}{2}$ acting on ψ_L is represented by Pauli matrices $\frac{\vec{\sigma}}{2}$.

$$\Rightarrow \frac{\vec{S} + i\vec{K}}{2} = 0 \text{ \& } \frac{\vec{S} - i\vec{K}}{2} = \frac{\vec{\sigma}}{2}$$

$$\Rightarrow \vec{S} = -i\vec{K} \text{ \& } \vec{S} - i\vec{K} = \vec{\sigma}$$

$$\Rightarrow \underline{\vec{S} = \frac{\vec{\sigma}}{2} = -i\vec{K}}$$

Under pure rotations given by $e^{-i\vec{\phi} \cdot \vec{S}}$, the spinor ψ_R is transformed as,

$$\psi_R \xrightarrow[\text{rotations}]{\text{Pure}} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \psi_R$$

Under pure boosts given by $e^{-i\vec{\xi} \cdot \vec{K}}$, the spinor ψ_R is transformed as,

$$\psi_R \xrightarrow[\text{boost}]{\text{Pure}} e^{\vec{\xi} \cdot \frac{\vec{\sigma}}{2}} \psi_R$$

For both boost + rotation

$$\psi_R \xrightarrow{\text{Rot}^n + \text{boost}} e^{\frac{-i(\vec{\phi} + i\vec{\xi}) \cdot \vec{\sigma}}{2}} \psi_R$$

If $\vec{\xi} = \vec{\phi} + i\vec{\xi}$ then

$$\psi_R \xrightarrow{\text{rot}^n + \text{boost}} e^{-i\vec{\xi} \cdot \frac{\vec{\sigma}}{2}} \psi_R$$

$$\Rightarrow \psi_R^\dagger \xrightarrow{\text{rot}^n + \text{boost}} \psi_R^\dagger e^{i\vec{\xi}^* \cdot \frac{\vec{\sigma}}{2}}$$

Consider $\tilde{\psi}_L = i\sigma_y \psi_L^*$ ($\tilde{\psi}_L \Rightarrow$ Time reversed version of ψ_L)

$$\tilde{\psi}_L = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \psi_L^{*(1)} \\ \psi_L^{*(2)} \end{bmatrix} = \begin{bmatrix} \psi_L^{*(2)} \\ -\psi_L^{*(1)} \end{bmatrix}$$

Under Lorentz transformation, we get

$$\psi_L \xrightarrow{\text{rot}^n + \text{boost}} e^{-i(\vec{\phi} - i\vec{t}_3) \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

$$\psi_L^* \xrightarrow{\text{rot}^n + \text{boost}} e^{i(\vec{\phi} + i\vec{t}_3) \cdot \frac{\vec{\sigma}}{2}} \psi_L^*$$

$$\underbrace{i\sigma_y \psi_L^*}_{\tilde{\psi}_L} \xrightarrow{\text{rot}^n + \text{boost}} i\sigma_y e^{i(\vec{\phi} + i\vec{t}_3) \cdot \frac{\vec{\sigma}}{2}} \psi_L^*$$

$$\tilde{\psi}_L \xrightarrow{\text{rot}^n + \text{boost}} \sigma_y e^{i(\vec{\phi} + i\vec{t}_3) \cdot \frac{\vec{\sigma}}{2}} \sigma_y \underbrace{i\sigma_y \psi_L^*}_{\tilde{\psi}_L}$$

$$\text{Also } \sigma_y \vec{\sigma} \sigma_y = -\vec{\sigma}$$

$$\Rightarrow \tilde{\psi}_L \xrightarrow{\text{rot}^n + \text{boost}} e^{-i(\vec{\phi} + i\vec{t}_3) \cdot \frac{\vec{\sigma}}{2}} \tilde{\psi}_L$$

Thus $\tilde{\psi}_L$ transforms like a ψ_R .
So $\tilde{\psi}_L$ transforms like repⁿ of $(0, \frac{1}{2})$.

Consider $\tilde{\psi}_R = i\sigma_y \psi_R^*$ ($\tilde{\psi}_R \Rightarrow$ Time reversed version of ψ_R)

$$\tilde{\psi}_R = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \psi_R^{*(1)} \\ \psi_R^{*(2)} \end{bmatrix} = \begin{bmatrix} \psi_R^{*(2)} \\ -\psi_R^{*(1)} \end{bmatrix}$$

Under Lorentz transformation, we get

$$\psi_R \xrightarrow{\text{rot}^n + \text{boost}} e^{-i(\vec{\phi} + i\vec{E}) \cdot \frac{\vec{\sigma}}{2}} \psi_R$$

$$\psi_R^* \xrightarrow{\text{rot}^n + \text{boost}} e^{i(\vec{\phi} - i\vec{E}) \cdot \frac{\vec{\sigma}}{2}} \psi_R^*$$

$$\underbrace{i\sigma_y \psi_R^*}_{\tilde{\psi}_R} \xrightarrow{\text{rot}^n + \text{boost}} i\sigma_y e^{i(\vec{\phi} - i\vec{E}) \cdot \frac{\vec{\sigma}}{2}} \psi_R^*$$

$$\tilde{\psi}_R \xrightarrow{\text{rot}^n + \text{boost}} \sigma_y e^{i(\vec{\phi} - i\vec{E}) \cdot \frac{\vec{\sigma}}{2}} \sigma_y i\sigma_y \psi_R^*$$

$$\text{Also } \sigma_y \vec{\sigma} \sigma_y = -\vec{\sigma}$$

$$\Rightarrow \tilde{\psi}_R \xrightarrow{\text{rot}^n + \text{boost}} e^{-i(\vec{\phi} - i\vec{E}) \cdot \frac{\vec{\sigma}}{2}} \tilde{\psi}_R$$

Thus $\tilde{\psi}_R$ transforms like a ψ_L .
So $\tilde{\psi}_R$ transforms like repⁿ of $(\frac{1}{2}, 0)$.

Thus the time reversal of ψ_L results in a spinor of ψ_R type & time reversal of ψ_R results in a spinor of ψ_L type.

$$\text{rep}^n. \left(\frac{1}{2}, 0\right) \xleftrightarrow{\text{time reversal}} \left(0, \frac{1}{2}\right)$$

Also under parity reversal, $\vec{x} \rightarrow -\vec{x}$ we see that $\vec{J} \rightarrow \vec{J}$ & $\vec{K} \rightarrow -\vec{K}$. So, $SO(1,3)$ algebra remains invariant under parity reversal. This makes $\vec{N} \leftrightarrow \vec{M}$ interchange. Thus,

$$\rightarrow \left(\frac{1}{2}, 0\right) \xleftrightarrow{\text{Parity reversal}} \left(0, \frac{1}{2}\right)$$

Thus to describe e^- (Spin $\frac{1}{2}$ Particle) that has Parity & time reversal Symmetry, we need to use both $\left(\frac{1}{2}, 0\right)$ & $\left(0, \frac{1}{2}\right)$ repⁿ in the form of direct sum $\text{rep}^n \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ which constitutes a Dirac spinor $\text{rep}^n. \psi_{\text{Dirac}} = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$.

Transformation of $\psi_L^\dagger \psi_L$ & $\psi_L^\dagger \vec{\sigma} \psi_L$

Consider the Weyl Spinor ψ_L which transforms according to the representation $(\frac{1}{2}, 0)$.

$$\psi_L \xrightarrow[L \cdot T]{\text{rot}^n + \text{boost}} e^{-i(\vec{\phi} - i\vec{c}) \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

$$\& \psi_L^\dagger \xrightarrow[L \cdot T]{\text{rot}^n + \text{boost}} \psi_L^\dagger e^{i(\vec{\phi} + i\vec{c}) \cdot \frac{\vec{\sigma}}{2}}$$

Consider pure rotation of $\psi_L^\dagger \psi_L$ & $\psi_L^\dagger \vec{\sigma} \psi_L$

$$\text{a) } \psi_L^\dagger \psi_L \xrightarrow{\text{rot}^n} \psi_L^\dagger e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

(rotⁿ is unitary) inverse of each other

$$\Rightarrow \underline{\psi_L^\dagger \psi_L \xrightarrow{\text{rot}^n} \psi_L^\dagger \psi_L}$$

$$\text{b) } \psi_L^\dagger \vec{\sigma} \psi_L \xrightarrow{\text{rot}^n} \psi_L^\dagger e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \vec{\sigma} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \psi_L$$

$$e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} = \cos\left(\frac{\phi}{2}\right) \mathbb{I}_2 + i(\hat{n} \cdot \vec{\sigma}) \sin\left(\frac{\phi}{2}\right)$$

where $\phi = |\vec{\phi}|$ & $\vec{\phi} = \phi \hat{n}$.

Similarly,

$$e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} = \cos\left(\frac{\phi}{2}\right) \mathbb{I}_2 - i(\hat{n} \cdot \vec{\sigma}) \sin\left(\frac{\phi}{2}\right)$$

$$\Rightarrow e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \vec{\sigma} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}}$$

$$= \left[\cos\left(\frac{\phi}{2}\right) \mathbb{I}_2 + i(\hat{n} \cdot \vec{\sigma}) \sin\left(\frac{\phi}{2}\right) \right]$$

$$\times \left[\cos\left(\frac{\phi}{2}\right) \vec{\sigma} - i \vec{\sigma} (\hat{n} \cdot \vec{\sigma}) \sin\left(\frac{\phi}{2}\right) \right]$$

$$= \cos^2\left(\frac{\phi}{2}\right) \vec{\sigma} + (\hat{n} \cdot \vec{\sigma}) \vec{\sigma} (\hat{n} \cdot \vec{\sigma}) \sin^2\left(\frac{\phi}{2}\right)$$

$$+ i \sin\left(\frac{\phi}{2}\right) \cos\left(\frac{\phi}{2}\right) [\hat{n} \cdot \vec{\sigma}, \vec{\sigma}]$$

we can find,

$$\begin{aligned} [n_j \sigma_j, \sigma_k] &= n_j [\sigma_j, \sigma_k] = 2i n_j \epsilon_{jkl} \sigma_l \\ &= 2i \epsilon_{ljk} \sigma_l n_j = 2i (\vec{\sigma} \times \hat{n})_k \end{aligned}$$

$$\Rightarrow [\hat{n} \cdot \vec{\sigma}, \vec{\sigma}] = 2i (\vec{\sigma} \times \hat{n})$$

$(\hat{n} \cdot \vec{\sigma}) \vec{\sigma} (\hat{n} \cdot \vec{\sigma})$ in component form is

$$(n_j \sigma_j) \sigma_k (n_l \sigma_l) = n_j n_l \sigma_j \sigma_k \sigma_l \\ = n_j n_l \sigma_j [\delta_{kl} + i \epsilon_{klm} \sigma_m]$$

$$= n_j n_k \sigma_j + i n_j n_l \sigma_j \epsilon_{klm} \sigma_m \\ = n_j n_k \sigma_j + i n_j n_l \epsilon_{klm} (\delta_{jm} + i \epsilon_{jmq} \sigma_q) \\ = n_j n_k \sigma_j + i n_j n_l \underbrace{\epsilon_{klj}}_{\delta_{kl}} - n_j n_l \epsilon_{klm} \epsilon_{qjm} \sigma_q \\ = (\hat{n} \cdot \vec{\sigma}) n_k - n_j n_l (\delta_{qk} \delta_{lj} - \delta_{lq} \delta_{kj}) \sigma_q \\ = (\hat{n} \cdot \vec{\sigma}) n_k - \underbrace{n_j n_j}_{1} \sigma_k + n_k n_q \sigma_q \\ = 2(\hat{n} \cdot \vec{\sigma}) n_k - \sigma_k$$

$$\rightarrow \underline{(\hat{n} \cdot \vec{\sigma}) \vec{\sigma} (\hat{n} \cdot \vec{\sigma}) = 2(\hat{n} \cdot \vec{\sigma}) \hat{n} - \vec{\sigma}}$$

$$\Rightarrow e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \vec{\sigma} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \\ = \cos^2\left(\frac{\phi}{2}\right) \vec{\sigma} + [2(\hat{n} \cdot \vec{\sigma}) \hat{n} - \vec{\sigma}] \sin^2\left(\frac{\phi}{2}\right) \\ + i \sin\left(\frac{\phi}{2}\right) \cos\left(\frac{\phi}{2}\right) [2i(\vec{\sigma} \times \hat{n})]$$

$$\Rightarrow e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \vec{\sigma} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}}$$

$$= \cos(\phi) \vec{\sigma} + (\hat{n} \cdot \vec{\sigma}) \hat{n} \left[2 \sin^2\left(\frac{\phi}{2}\right) \right] - \sin(\phi) (\vec{\sigma} \times \hat{n})$$

$$\Rightarrow e^{i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}} \vec{\sigma} e^{-i\vec{\phi} \cdot \frac{\vec{\sigma}}{2}}$$

$$= \vec{\sigma} \cos(\phi) + (\hat{n} \times \vec{\sigma}) \sin(\phi) + \hat{n} (\hat{n} \cdot \vec{\sigma}) [1 - \cos(\phi)]$$

This is exactly the Rodrigues rotation formula of a vector operator $\vec{\sigma}$ about $\hat{n}$ by angle ϕ .

$$\begin{aligned} \psi_L^\dagger \vec{\sigma} \psi_L \xrightarrow{\text{rot}^n} \psi_L^\dagger \vec{\sigma} \psi_L \cos(\phi) \\ + (\hat{n} \times \psi_L^\dagger \vec{\sigma} \psi_L) \sin(\phi) \\ + \hat{n} [\hat{n} \cdot (\psi_L^\dagger \vec{\sigma} \psi_L)] [1 - \cos(\phi)] \end{aligned}$$

$$\Rightarrow \psi_L^\dagger \vec{\sigma} \psi_L \text{ transforms like vector.}$$

Consider pure boost of $\psi_L^\dagger \psi_L$ & $\psi_L^\dagger \vec{\sigma} \psi_L$

$$\psi_L^\dagger \psi_L \xrightarrow{\text{boost}} \psi_L^\dagger e^{-\frac{\vec{E} \cdot \vec{\sigma}}{2}} e^{-\frac{\vec{E} \cdot \vec{\sigma}}{2}} \psi_L$$
$$= \psi_L^\dagger e^{-\vec{E} \cdot \vec{\sigma}} \psi_L$$

$$e^{-\vec{E} \cdot \vec{\sigma}} = \cosh(E) \mathbb{I}_2 - \sinh(E) (\vec{\sigma} \cdot \hat{n})$$

(where $\vec{E} = E \hat{n}$)

So,

$$\psi_L^\dagger \psi_L \xrightarrow{\text{boost}} \cosh(E) \psi_L^\dagger \psi_L - \sinh(E) (\psi_L^\dagger \vec{\sigma} \psi_L) \cdot \hat{n}$$

Thus under a pure boost, $\psi_L^\dagger \psi_L$ transforms as 0th component of four vector $(\psi_L^\dagger \psi_L, \psi_L^\dagger \vec{\sigma} \psi_L)$

Similarly consider boost of $\psi_L^\dagger \vec{\sigma} \psi_L$

$$\psi_L^\dagger \vec{\sigma} \psi_L \xrightarrow{\text{boost}} \psi_L^\dagger e^{-\frac{\vec{E} \cdot \vec{\sigma}}{2}} \vec{\sigma} e^{-\frac{\vec{E} \cdot \vec{\sigma}}{2}} \psi_L$$

We should find $e^{-\frac{\vec{E} \cdot \vec{\sigma}}{2}} \vec{\sigma} e^{-\frac{\vec{E} \cdot \vec{\sigma}}{2}}$

$$e^{-\vec{G} \cdot \vec{\sigma} / 2} \vec{\sigma} e^{-\vec{G} \cdot \vec{\sigma} / 2}$$

$$= \left[\cosh\left(\frac{G}{2}\right) I_2 - \sinh\left(\frac{G}{2}\right) (\vec{\sigma} \cdot \hat{n}) \right]$$

$$\times \left[\cosh\left(\frac{G}{2}\right) \vec{\sigma} - \sinh\left(\frac{G}{2}\right) \vec{\sigma} (\vec{\sigma} \cdot \hat{n}) \right]$$

$$= \cosh^2\left(\frac{G}{2}\right) \vec{\sigma} + \sinh^2\left(\frac{G}{2}\right) (\vec{\sigma} \cdot \hat{n}) \vec{\sigma} (\vec{\sigma} \cdot \hat{n})$$

$$- \cosh\left(\frac{G}{2}\right) \sinh\left(\frac{G}{2}\right) \{ \vec{\sigma}, (\vec{\sigma} \cdot \hat{n}) \}$$

$$\{ \vec{\sigma}, (\vec{\sigma} \cdot \hat{n}) \} = 2 \hat{n} I_2$$

$$\& (\hat{n} \cdot \vec{\sigma}) \vec{\sigma} (\hat{n} \cdot \vec{\sigma}) = 2(\hat{n} \cdot \vec{\sigma}) \hat{n} - \vec{\sigma}$$

$$\rightarrow e^{-\vec{G} \cdot \vec{\sigma} / 2} \vec{\sigma} e^{-\vec{G} \cdot \vec{\sigma} / 2}$$

$$= \cosh^2\left(\frac{G}{2}\right) \vec{\sigma} - \sinh^2\left(\frac{G}{2}\right) \vec{\sigma}$$

$$+ \hat{n} (\hat{n} \cdot \vec{\sigma}) \left[2 \sinh^2\left(\frac{G}{2}\right) \right]$$

$$- 2 \cosh\left(\frac{G}{2}\right) \sinh\left(\frac{G}{2}\right) \hat{n} I_2$$

$$\rightarrow e^{-\vec{\mathcal{E}} \cdot \frac{\vec{\sigma}}{2}} \vec{\sigma} e^{-\vec{\mathcal{E}} \cdot \frac{\vec{\sigma}}{2}}$$

$$= \vec{\sigma} + \hat{n} (\hat{n} \cdot \vec{\sigma}) [\cosh(\mathcal{E}) - 1] - \sinh(\mathcal{E}) \hat{n} I_2$$

$\Rightarrow$ Under a boost $\psi_L^+ \Rightarrow \psi_L$ becomes

$$\psi_L^+ \Rightarrow \psi_L \xrightarrow{\text{boost}} \psi_L^+ \Rightarrow \psi_L - \sinh(\mathcal{E}) \hat{n} \psi_L^+ \psi_L + \hat{n} [\hat{n} \cdot (\psi_L^+ \Rightarrow \psi_L)] [\cosh(\mathcal{E}) - 1]$$

Any four vector $(x^0, \vec{x})$ under Lorentz boost transforms into,

$$\vec{x} \rightarrow \vec{x} + \hat{n} (\hat{n} \cdot \vec{x}) (\gamma - 1) - \gamma \beta \hat{n} x^0$$

$$\text{But } \gamma = \cosh(\mathcal{E}), \quad \gamma \beta = \sinh(\mathcal{E})$$

Thus we see that $(\psi^+ \psi_L, \psi_L^+ \Rightarrow \psi_L)$ behaves exactly like a 4-vector.

Let $\mathbb{I}_2 = \sigma^0$, $\sigma^\mu = (\mathbb{I}_2, \vec{\sigma})$
 & $\bar{\sigma}^\mu = (\mathbb{I}_2, -\vec{\sigma})$

then $\psi_L^\dagger \sigma^\mu \psi_L$ is a 4-vector
 & transforms like a four vector.

Also since ∂_μ transforms as
 a covariant four vector, we
 can say that $\partial_\mu (\psi_L^\dagger \sigma^\mu \psi_L)$ is
 a lorentz scalar.

Note: Same like how $\psi_L^\dagger \sigma^\mu \psi_L$ is
 a four vector, the similar term
 with ψ_R , given by $\psi_R^\dagger \bar{\sigma}^\mu \psi_R$,

i.e. $\psi_R^\dagger \bar{\sigma}^\mu \psi_R = (\psi_R^\dagger \psi_R, -\psi_R^\dagger \vec{\sigma} \psi_R)$
 also transforms as contravariant
 four vector. The -ve sign is
 because in $(1/2, 0) \rightarrow \vec{S} = \frac{\vec{\sigma}}{2} = i\vec{K}$

but in $(0, 1/2) \rightarrow \vec{S} = \frac{\vec{\sigma}}{2} = -i\vec{K}$
 thus the -ve sign is carried over.

Then the Weyl Lagrangian is given by

$$\mathcal{L} = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L$$

which yields the Weyl eqⁿ,

$$\sigma^\mu (i\partial_\mu \psi_L) = 0$$

Thus Weyl Spinor should satisfy the above eqⁿ.

Quantum mechanically, we get

$$i \frac{\partial \psi_L}{\partial t} - \underbrace{\vec{\sigma} \cdot \vec{p}}_{\text{momentum operator}} \psi_L = 0$$

$$\text{or } \hat{H} \psi_L = (\vec{\sigma} \cdot \vec{p}) \psi_L \Rightarrow \text{Weyl Hamiltonian}$$

& the dispersion relation is given by $E^2 = |\vec{p}|^2$.

The conserved 4-current of Weyl Spinor becomes,

$$j_L^\mu = \psi_L^\dagger \sigma^\mu \psi_L \quad \text{where} \quad \partial_\mu j_L^\mu = 0.$$

Thus we previously saw that $\psi_L^\dagger \sigma^\mu \psi_L$ transforms as components of a four vector, which actually represents 4-probability current density vector of Weyl Spinor.

Similarly for ψ_R ,

$$\mathcal{L} = i \psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R$$

$$\psi_R \text{ satisfies: } \bar{\sigma}^\mu \partial_\mu \psi_R = 0$$

& the conserved four current is

$$j_R^\mu = \psi_R^\dagger \bar{\sigma}^\mu \psi_R, \quad \underline{\partial_\mu j_R^\mu = 0}$$

Dirac Spinor repⁿ of $SO^+(1,3)$

Dirac Spinor repⁿ is the direct sum repⁿ of the left handed and the right handed Weyl spinor repⁿ.

$$\Rightarrow \text{Dirac Spinor rep}^n = \left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$$

$$\Rightarrow \psi_{\text{Dirac}} = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} \text{ is a possible bi-spinor.}$$

We know that the transformation of ψ_L & ψ_R is given by,

$$\psi_L \longrightarrow e^{-i[\vec{\phi} - i\vec{t}]\cdot\frac{\vec{\sigma}}{2}} \psi_L$$

$$\& \psi_R \longrightarrow e^{-i[\vec{\phi} + i\vec{t}]\cdot\frac{\vec{\sigma}}{2}} \psi_R$$

So ψ_L & ψ_R transforms oppositely under boosts but similarly under rotations.

$$\Rightarrow \psi_L^\dagger \psi_R \longrightarrow \psi_L^\dagger \underbrace{e^{+i[\vec{\phi} + i\vec{E}] \cdot \frac{\vec{\sigma}}{2}} e^{-i[\vec{\phi} + i\vec{E}] \cdot \frac{\vec{\sigma}}{2}}}_{\text{inverse of each other}} \psi_R$$

$$\Rightarrow \underline{\psi_L^\dagger \psi_R} \longrightarrow \psi_L^\dagger \psi_R \text{ (Invariant)}$$

Similarly $\psi_R^\dagger \psi_L \longrightarrow \psi_R^\dagger \psi_L$
is also an invariant.

Then we can construct a Lagrangian which is Lorentz invariant by adding the two Weyl Lagrangians one for ψ_L & other for ψ_R , & also with the invariant terms $\psi_L^\dagger \psi_R$ & $\psi_R^\dagger \psi_L$. Thus we get,

$$\mathcal{L} = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + i\psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

$m \rightarrow$ coupling term b/w ψ_L & ψ_R .

Then the Euler lagrange eqⁿ. for ψ_L & ψ_R is,

$$i\sigma^\mu \partial_\mu \psi_L - m\psi_R = 0$$
$$\Rightarrow \underline{i\sigma^\mu \partial_\mu \psi_L = m\psi_R}$$

Similarly, $i\bar{\sigma}^\mu \partial_\mu \psi_R - m\psi_L = 0$

$$\Rightarrow \underline{i\bar{\sigma}^\mu \partial_\mu \psi_R = m\psi_L}$$

Quantum mechanically $i\partial_\mu = \hat{p}_\mu$

$$\Rightarrow \sigma^\mu p_\mu \psi_L = m\psi_R$$
$$\Rightarrow (E - \vec{\sigma} \cdot \vec{p}) \psi_L = m\psi_R$$

$$\neq \bar{\sigma}^\mu p_\mu \psi_R = m\psi_L$$
$$\Rightarrow (E + \vec{\sigma} \cdot \vec{p}) \psi_R = m\psi_L$$

Acting by $(E + \vec{\sigma} \cdot \vec{p})$ on 1st eqⁿ we get,

$$(E + \vec{\sigma} \cdot \vec{p})(E - \vec{\sigma} \cdot \vec{p}) \psi_L = m(E + \vec{\sigma} \cdot \vec{p}) \psi_R$$
$$\Rightarrow (E^2 - |\vec{p}|^2) \psi_L = m^2 \psi_L$$

$$\Rightarrow \underline{(E^2 - |\vec{p}|^2 - m^2) \psi_L = 0}$$

Thus for non-zero ψ_L we need $\underline{E^2 = |\vec{p}|^2 + m^2}$
 $\hookrightarrow$ Energy-momentum relation

Thus the coupling constant b/w ψ_L & ψ_R given by 'm' is nothing but the mass.

Getting the Dirac eqⁿ.

The two eqⁿ for ψ_L & ψ_R are,

$$i\bar{\sigma}^\mu \partial_\mu \psi_R = m\psi_L$$

$$i\sigma^\mu \partial_\mu \psi_L = m\psi_R$$

$$\Rightarrow \begin{bmatrix} 0 & i\bar{\sigma}^\mu \partial_\mu \\ i\sigma^\mu \partial_\mu & 0 \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = m \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

Defining $\gamma^\mu = \begin{bmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{bmatrix}$ & $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$

we get, $i(\gamma^\mu \partial_\mu - m)\psi = 0$

This is the Dirac - eqⁿ.

The 4×4 matrices γ^μ are called as the gamma matrices which have the following properties.

- $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}_4$

This is the defining property of the gamma matrices (γ^μ) which generates a clifford-Algebra. This property is more fundamental than the numerical entries of γ^μ .

- $\gamma_\mu = \eta_{\mu\nu} \gamma^\nu$

$\Rightarrow (\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3)$

- $\gamma^0 \Rightarrow$ Hermitian, eigen vals = ± 1 .
- $\gamma^i \Rightarrow$ Anti-Hermitian, $(\gamma^i)^2 = -1$
 $\rightarrow$ eigen values of $\gamma^i = \pm i$
- $(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$
- $\text{trace}(\gamma^\mu) = 0$
- $\gamma^\mu \gamma_\mu = 4 \mathbb{I}_4$
- $\text{trace}(\gamma^\mu \gamma^\nu) = 4 \eta^{\mu\nu}$

• Let a_μ & b_μ be any two four vectors. Then,

$$\begin{aligned} (a_\mu \gamma^\mu)(b_\nu \gamma^\nu) &= \frac{(a_\mu \gamma^\mu)(b_\nu \gamma^\nu) + (a_\nu \gamma^\nu)(b_\mu \gamma^\mu)}{2} \\ &= \frac{a_\mu b_\nu \gamma^\mu \gamma^\nu + a_\mu b_\nu \gamma^\nu \gamma^\mu}{2} \end{aligned}$$

$$= \frac{a_\mu b_\nu}{2} \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2\eta^{\mu\nu} \mathbb{I}_4} = a_\mu b_\nu \eta^{\mu\nu} \mathbb{I}_4$$

$$\Rightarrow \underline{(a_\mu \gamma^\mu)(b_\nu \gamma^\nu) = a_\mu b^\mu \mathbb{I}_4}$$

Change of Basis of γ -matrices

Consider a 4×4 invertible matrix 'A' & conjugate action of A on γ^μ .

Then, $\tilde{\gamma}^\mu = A \gamma^\mu A^{-1}$ also satisfies the Clifford algebra.

$$\begin{aligned}\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} &= A \gamma^\mu A^{-1} A \gamma^\nu A^{-1} + A \gamma^\nu A^{-1} A \gamma^\mu A^{-1} \\&= A \gamma^\mu \gamma^\nu A^{-1} + A \gamma^\nu \gamma^\mu A^{-1} \\&= A (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) A^{-1} \\&= A 2 \eta^{\mu\nu} A^{-1} \\&= 2 \eta^{\mu\nu} \mathbb{I}_4\end{aligned}$$

$$\Rightarrow \underline{\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2 \eta^{\mu\nu} \mathbb{I}_4}$$

If ψ satisfies the Dirac eqⁿ with the matrices γ^μ , then $\psi' = A \psi$ satisfies the Dirac eqⁿ with matrices $A \gamma^\mu A^{-1}$.

$$\text{Let, } (i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\Rightarrow (i\gamma^\mu A^{-1} A \partial_\mu \psi - m A^{-1} A \psi) = 0$$

$$\Rightarrow (i\gamma^\mu A^{-1} \partial_\mu - m A^{-1}) A \psi = 0$$

$$\Rightarrow S(i\gamma^\mu A^{-1} \partial_\mu - m A^{-1}) A \psi = 0$$

$$\Rightarrow [i(A\gamma^\mu A^{-1})\partial_\mu - m] (A\psi) = 0$$

$$\text{Let } \tilde{\gamma}^\mu = A\gamma^\mu A^{-1}, \psi' = A\psi$$

$$\rightarrow [i\tilde{\gamma}^\mu \partial_\mu - m]\psi' = 0$$

Weyl Basis

The form of γ^μ matrices we obtained was $\gamma^\mu = \begin{bmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{bmatrix}$, in this

basis we can decouple the left & right handed spinors ψ_L & ψ_R .

Given a reducible repⁿ $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ of $SO^+(1,3)$, we can try to obtain the two irreducible repⁿ of ψ_L & ψ_R can be obtained as follows.

Consider $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$

$$\Rightarrow \gamma^5 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}$$

it can be shown that, $\{\gamma^5, \gamma^\mu\} = 0$.

Also $(\gamma^5)^2 = 1$, thus we can form two projection operators

$$P_L = \frac{1}{2}(1 + \gamma^5) \quad \& \quad P_R = \frac{1}{2}(1 - \gamma^5)$$

$$\Rightarrow P_L^2 = P_L \quad \& \quad P_R^2 = P_R \quad \& \quad P_L P_R = 0$$

$$P_L \psi = \frac{1}{2}[1 + \gamma^5] \psi = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}$$

$$P_R \psi = \frac{1}{2}[1 - \gamma^5] \psi = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}$$

$$\Rightarrow P_L \psi = \psi_L = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} \quad \& \quad P_R \psi = \psi_R = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}$$

$$\& \quad \psi = \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \psi_R \end{bmatrix} = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

Thus we were able to define the 2 component Weyl Spinors ψ_L & ψ_R initially, obtain the transformation rules for them. Then define the Dirac Spinor by stacking the ψ_L & ψ_R into a 4-component bispinor. Then, finally we were able to decompose it back to two Weyl spinors by defining the projection operators, P_L & P_R .

Acting P_L or P_R on the complete Dirac eqⁿ we can get the corresponding Weyl equation.

$$i\sigma^\mu \psi_L = m \psi_R$$

$$\& \quad i\bar{\sigma}^\mu \psi_R = m \psi_L$$

Generators $S^{\mu\nu}$ of Lorentz transform in the Dirac spinor repⁿ.

Any general Lorentz transformation on a four vector x^μ is written as,

$$L(\omega) = e^{-i \frac{\omega^{\mu\nu}}{2} J_{\mu\nu}}$$

where $\omega^{\mu\nu}$ is the six parameter anti-symmetric tensor containing the rotation angles (ϕ_i) & boost angles (ξ_i). And $J_{\mu\nu}$ are the six generators.

$$\omega^{\mu\nu} = \begin{bmatrix} 0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & 0 & \phi_3 & -\phi_2 \\ \xi_2 & -\phi_3 & 0 & \phi_1 \\ \xi_3 & \phi_2 & -\phi_1 & 0 \end{bmatrix}, \omega^{\mu\nu} = -\omega^{\nu\mu}$$

$$J_{\mu\nu} = \begin{bmatrix} 0 & -K_x & -K_y & -K_z \\ K_x & 0 & J_z & -J_y \\ K_y & -J_z & 0 & J_x \\ K_z & J_y & -J_x & 0 \end{bmatrix}, J_{\mu\nu} = -J_{\nu\mu}$$

But in Dirac-Spinor representation, the corresponding Lorentz transform on a Dirac spinor ψ can be written as a matrix $\Lambda(L)$ acting on spinor ψ . Then our goal is to find the representation of the generators here denoted by $S_{\mu\nu}$ in the Dirac spinor representation.

Let $\Lambda(L) = e^{\frac{-i\omega_{\mu\nu}\sigma^{\mu\nu}}{4}}$ where we need to determine $\sigma^{\mu\nu}$ such that $S^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2}$ is the repⁿ of the generators² in Dirac spinor repⁿ.

Note that Dirac spinor has four components, so $\Lambda(L)$ has to be a 4×4 matrix acting on ψ .

There are at most 16 linearly independent matrices to generate any 4×4 matrix.

Now, we need to find a suitable basis to generate any 4×4 matrix.

They are given by the 16 matrices given below:

$\mathbb{I}_4$: One in number

γ^μ : Four in number

$[\gamma^\mu, \gamma^\nu]$: Six in number (Non-zero)

$\gamma^\mu \gamma^5$: Four in number

γ^5 : One in number

Denoting $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$, we have the 16 linearly independent 4×4 matrices as

$$\{\mathbb{I}_4, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma^5, \gamma^5\}$$

$$(1 + 4 + 6 + 4 + 1 = 16)$$

Any 4×4 matrix is linear combination of the above set.

In the Weyl basis,

$$\mathbb{I}_4 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}$$

$$\gamma^5 \gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{bmatrix}$$

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}$$

$$\gamma^5 \gamma^1 = \begin{bmatrix} 0 & -\sigma_x \\ -\sigma_x & 0 \end{bmatrix}$$

$$\gamma^1 = \begin{bmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{bmatrix}$$

$$\gamma^5 \gamma^2 = \begin{bmatrix} 0 & -\sigma_y \\ -\sigma_y & 0 \end{bmatrix}$$

$$\gamma^2 = \begin{bmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{bmatrix}$$

$$\gamma^5 \gamma^3 = \begin{bmatrix} 0 & -\sigma_z \\ -\sigma_z & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{bmatrix}$$

$$\sigma^{01} = i \begin{bmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{bmatrix}$$

$$\gamma^5 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}$$

$$\sigma^{02} = i \begin{bmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{bmatrix}$$

$$\sigma^{03} = i \begin{bmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{bmatrix}$$

$$\sigma^{12} = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}$$

$$\sigma^{23} = \begin{bmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{bmatrix}$$

$$\sigma^{31} = \begin{bmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{bmatrix}$$

The matrices σ^{ij} are similar to the block diagonal $\vec{\sigma} \oplus \vec{\sigma}$ matrices.

Since $\frac{\vec{\sigma}}{2}$ were the generators of rotation in $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ repⁿ, Naturally, in the Dirac repⁿ, that is just the direct sum repⁿ given by $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, we would expect that that $\frac{\vec{\sigma} \oplus \vec{\sigma}}{2}$ to generate rotations.

$$\Rightarrow \frac{\sigma_k \oplus \sigma_k}{2} = \frac{1}{2} \epsilon_{ijk} \sigma^{ij} = S_k$$

S_k given here represents the generator of rotation about the k -th axis.

Similarly, we would expect that σ^{0j} to generate the lorentz boost along the j^{th} axis.

Thus the dirac representation of lorentz transformation acting on the dirac spinor is given by $\Lambda(L)$ as,

$$\underline{\Lambda(L) = e^{-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}}}$$

$$\text{where } \underline{\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]}$$

• $\frac{\sigma^{0j}}{2} = \frac{i}{2} \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix} \equiv K_j$ generates lorentz boost along the j^{th} axis.

• $\frac{1}{2} \epsilon_{ijk} \sigma^{ij} = S_k = \frac{1}{2} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix}$ generates rotation about the k^{th} axis.

Thus in the Dirac Spinor repⁿ.

$$S_x = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K_x = \frac{1}{2} \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix}$$

$$S_y = \frac{1}{2} \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$K_y = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$S_z = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$K_z = \frac{1}{2} \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

$[S_x, S_y] = iS_z$	$[K_x, K_y] = -iS_z$	$[K_x, S_y] = iK_z$
$[S_y, S_z] = iS_x$	$[K_y, K_z] = -iS_x$	$[K_y, S_z] = iK_x$
$[S_z, S_x] = iS_y$	$[K_z, K_x] = -iS_y$	$[K_z, S_x] = iK_y$

Thus we got the $SO^+(1,3)$ algebra in dirac spinor representation.

$$[S_j, S_k] = i\epsilon_{jkl} S_l \quad \} \text{Generates } SO(3)$$

$$[K_j, S_k] = i\epsilon_{jkl} K_l \quad \} \vec{K} \text{ behaves as vector operator}$$

$$[K_j, K_k] = -i\epsilon_{jkl} S_l \quad \} SO(1,3) \text{ algebra is a closed algebra.}$$

Lorentz Covariance of Dirac eqⁿ

The lorentz transformation of dirac spinor $\psi(x^\mu)$ is given by,

$$\psi'(x') = \Lambda(L) \psi(x) \quad (L^\mu_\nu \Rightarrow \text{lorentz matrix})$$

where $\Lambda(L) = e^{-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}}$

Now $\psi'(x')$ satisfies the dirac eqⁿ

Lorentz transformed dirac equation $\Rightarrow (i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0$

inverse lorentz matrix

where $\partial'_\mu = L_\mu^\nu \partial_\nu \Rightarrow$ lorentz transform of ∂_μ .

Consider, $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

Under lorentz transform,

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0$$

$$\Rightarrow (i\gamma^\mu L_\mu^\nu \partial_\nu - m)\Lambda(L)\psi(x) = 0$$

$$\Rightarrow (i\gamma^\mu \Lambda(L) L_\mu^\nu \partial_\nu - m\Lambda(L))\psi(x) = 0$$

act $\Lambda^{-1}(L)$ from left,

$$\Rightarrow [i\Lambda^{-1}(L)\gamma^\mu \Lambda(L) L_\mu^\nu \partial_\nu - m]\psi(x) = 0$$

Now $\Lambda^{-1}(L)\gamma^\mu \Lambda(L) = L^\mu_\rho \gamma^\rho$ is the lorentz transform of the γ^μ matrices.

$$\Rightarrow [i\gamma^\rho L^\mu_\rho L_\mu^\nu \partial_\nu - m]\psi(x) = 0$$

$$(\text{use } L^\mu_\rho L_\mu^\nu = \delta^\nu_\rho)$$

$$\Rightarrow [i\gamma^\nu \partial_\nu - m]\psi(x) = 0$$

Thus dirac's eqⁿ is lorentz invariant.

The Dirac Lagrangian

The coupling of the two weyl equations using the mass term gave us the dirac lagrangian as,

$$\mathcal{L} = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + i\psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R - m[\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L]$$

where $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ is the lorentz invariant term. We can use the γ^0 matrix to combine $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$

as

$$\psi^\dagger \gamma^0 \psi = \begin{bmatrix} \psi_L^\dagger & \psi_R^\dagger \end{bmatrix} \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

$$\Rightarrow \psi^\dagger \gamma^0 \psi = \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \quad \left[\begin{array}{l} \text{Is a lorentz} \\ \text{scalar} \end{array} \right]$$

Defining $\bar{\psi} = \psi^\dagger \gamma^0$ we can say that $\bar{\psi} \psi$ is a lorentz scalar.

Similarly, $i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L + i\psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R$
can be written as $i\bar{\psi} \gamma^\mu \partial_\mu \psi$

$$\text{Since, } i\bar{\psi} \gamma^\mu \partial_\mu \psi = i\psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi$$

$$= i \begin{bmatrix} \psi_L^\dagger & \psi_R^\dagger \end{bmatrix} \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & \bar{\sigma}^\mu \partial_\mu \\ \sigma^\mu \partial_\mu & 0 \end{bmatrix} \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

$$= i \begin{bmatrix} \psi_R^\dagger & \psi_L^\dagger \end{bmatrix} \begin{bmatrix} \bar{\sigma}^\mu \partial_\mu \psi_R \\ \sigma^\mu \partial_\mu \psi_L \end{bmatrix}$$

$$= i\psi_R^\dagger \bar{\sigma}^\mu \partial_\mu \psi_R + i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L$$

Thus we can write the dirac lagrangian compactly as,

$$\boxed{\mathcal{L} = \bar{\psi} [i\gamma^\mu \partial_\mu - m] \psi}$$

This also gives rise to the dirac equation & its conjugate eqⁿ as required.

Dirac Bilinears

The set of 16 linearly independent 4×4 matrices denoted by Γ are:

$$\Gamma = \{\mathbb{I}_4, \gamma^4, \sigma^{\mu\nu}, \gamma^\mu \gamma^5, \gamma^5\}$$

Then the 16 objects denoted by ' $\bar{\Psi} \Gamma \Psi$ ' are called as Dirac Bilinears

$\bar{\Psi} \Psi \Rightarrow$ Lorentz Scalar

$\bar{\Psi} \gamma^\mu \Psi \Rightarrow$ Transforms as 4-vector

$\bar{\Psi} \sigma^{\mu\nu} \Psi \Rightarrow$ Transforms as the anti-symmetric tensor

$\bar{\Psi} \gamma^\mu \gamma^5 \Psi \Rightarrow$ Transforms as an Axial 4-Vector

$\bar{\Psi} \gamma^5 \Psi \Rightarrow$ Transforms as Pseudo Scalar

Translation group

A translation is a geometric transform that moves every point of a figure, shape or space by the same distance in a given direction.

In space-time the translation means both space translation $\vec{x} \rightarrow \vec{x} + \vec{a}$ and time translation $x^0 \rightarrow x^0 + a^0$.
Then $x^\mu \rightarrow x^\mu + a^\mu \Rightarrow 4D$ Space-time translation can be represented as $T(a)x^\mu$
 $\Rightarrow x^\mu \rightarrow T(a)x^\mu = x^\mu + a^\mu$

Two consecutive translations can be

$$\begin{aligned} T(a_1)T(a_2)x^\mu &= x^\mu + a_1^\mu + a_2^\mu \\ \Rightarrow T(a_1)T(a_2) &= T(a_1 + a_2) = T(a_2)T(a_1) \\ \underline{T(0)x^\mu} &= x^\mu, \quad \underline{T^{-1}(a) = T(-a)} \end{aligned}$$

The set of all translations form an abelian group called Translation Group.

Thus if we need to represent the action of translation as matrices then we need to use a 5×5 matrix repⁿ

$$\text{Let } T(a) = \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix}$$

(Greek indices like $\mu, \nu, \text{etc} = \{0, 1, 2, 3, 4\}$)

Then under matrix multiplication

$$\begin{aligned} T(a)x^\mu &= \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix} = \begin{bmatrix} x^0 + a^0 \\ x^1 + a^1 \\ x^2 + a^2 \\ x^3 + a^3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix} + \begin{bmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \\ 1 \end{bmatrix} = x^\mu + a^\mu \end{aligned}$$

$$\rightarrow \underline{T(a)x^\mu = x^\mu + a^\mu}$$

So we can write

$$T(a) = \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \mathbb{I}_5 + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & a^0 \\ 0 & 0 & 0 & 0 & a^1 \\ 0 & 0 & 0 & 0 & a^2 \\ 0 & 0 & 0 & 0 & a^3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{Generator}}$$

Let the Generator of translation be $a^\mu P_\mu$

$$P_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Define $P^\mu = \eta^{\mu\nu} P_\nu \Rightarrow P^0 = P_0$ & $P^j = -P_j$

$$\Rightarrow \underline{T(a) = \mathbb{I}_5 - i a^\mu P_\mu = e^{-i a^\mu P_\mu}}$$

The commutation relation between the generators of translation are

$$[P_\mu, P_\nu] = 0 \Rightarrow \text{Abelian Lie algebra}$$

The translation group is a non-compact group since the parameters a^μ of the group elements are unbounded.

Thus the finite dimensional repⁿ of a non-compact is non-unitary and generators need not be Hermitian. But to have a unitary representation the repⁿ has to be infinite dimensional & generators will be Hermitian.

Unitary representation of Translation group

We will use the Hilbert Space $L^2(\mathbb{R}^4)$ which is the set of all square integrable functions on $\mathbb{R}^4$.

The elements of $\mathcal{A} = L^2(\mathbb{R}^4)$ are denoted by 'f' which is a function of $x^\mu \in \mathbb{R}^4$

$$f \longrightarrow f(x^\mu) = f(x^0, x^1, x^2, x^3)$$

$$\forall f \in \mathcal{A}, \quad \int_{\mathbb{R}^4} d^4x |f(x^\mu)|^2 \in \mathbb{R}$$

$\longrightarrow$ finite

The set of all such 'f' forms $L^2(\mathbb{R}^4)$.

Representation of Translation in $L^2(\mathbb{R}^4)$

Consider an infinitesimal translation
 $x^\mu \longrightarrow x^\mu + \delta x^\mu$

$$\Rightarrow f(x^\mu + \delta x^\mu) = f(x^\mu) + \delta x^\mu \partial_\mu f$$

$$= [1 + \delta x^\mu \partial_\mu] f(x^\mu) = [1 - i \delta x^\mu (i \partial_\mu)] f(x^\mu)$$

$$\Rightarrow \hat{T}(\delta x^\mu) f(x^\mu) = [\hat{\mathbb{I}} - i \delta x^\mu P_\mu] f(x^\mu)$$

Thus the infinitesimal translations are represented by

$$\hat{T}(\delta x^\mu) = \hat{I} - i \delta x^\mu \hat{P}_\mu$$

where $\hat{P}_\mu = i \partial_\mu$ is the repⁿ of generators of translation in the infinite dimensional $L^2(\mathbb{R}^4)$ space.

Then the finite translation is given by

$$\underline{\hat{T}(a^\mu) = e^{-i a^\mu \hat{P}_\mu} = e^{a^\mu \partial_\mu}}$$

$$\begin{aligned} \rightarrow e^{a^\mu \partial_\mu} f(x^\nu) &= 1 + a^\mu \partial_\mu f + a^\mu a^\nu \partial_\mu \partial_\nu f \\ &\quad + a^\mu a^\nu a^\rho \partial_\mu \partial_\nu \partial_\rho f + \dots \\ &= f(x^\nu + a^\nu) \end{aligned}$$

$$\rightarrow \underline{e^{-i a^\mu \hat{P}_\mu} f(x^\nu) = f(x^\nu + a^\nu)}$$

The translation generators $\hat{P}_\mu$ are such that $[\hat{P}_\mu, \hat{P}_\nu] = i^2 [\partial_\mu, \partial_\nu] = 0$. Thus it forms an abelian algebra.

$$P_\mu = i\partial_\mu = i\left[\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right]$$

$$\Rightarrow P_\mu = i\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$$\text{Define } P^\mu = \eta^{\mu\nu} P_\nu = i\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$$

$$\Rightarrow \underline{P^0 = i\frac{\partial}{\partial t}, \quad \vec{P} = -i\vec{\nabla}}$$

$$a_\mu = \eta_{\mu\nu} a^\nu \Rightarrow \begin{matrix} a^\mu = (a^0, \vec{a}) \\ a_\mu = (a^0, -\vec{a}) \end{matrix}$$

$$a_\mu P^\mu = a^\mu P_\mu = i\left[a^0 \frac{\partial}{\partial t} + \vec{a} \cdot \vec{\nabla}\right]$$

$$\& \underline{T(a) = e^{-ia_\mu P^\mu} = e^{-i\left[a^0 \frac{\partial}{\partial t} + \vec{a} \cdot \vec{\nabla}\right]}}$$

$$\text{Since } \underline{T(a^\nu) f(x^\mu) = f(x^\mu + a^\mu)}$$

$$\Rightarrow \underline{e^{-i\left[a^0 \frac{\partial}{\partial t} + \vec{a} \cdot \vec{\nabla}\right]} f(x^\mu) = f(x^\mu + a^\mu)}$$

Poincare Group & Poincare Algebra

A poincare transformation is actually Lorentz transformation + Translation.

$$x^\mu \xrightarrow{\text{Poincare transform}} L^\mu{}_\nu x^\nu + a^\mu$$

Thus any poincare transformation will need the information about 3 rotation angles ($\vec{\phi}$), 3 lorentz boost angles ($\vec{\xi}$) and 4 translation parameters (a^μ).

So we can denote a Poincare transformation by $g(L, a)$

$$\Rightarrow g(L, a)x^\mu = L^\mu{}_\nu x^\nu + a^\mu$$

Action of two consecutive Poincare transformation is $g(L_2, a_2)g(L_1, a_1)$, now we need to know the group Binary operation.

$$g(L_2, a_2)g(L_1, a_1)x^\mu = g(L_2, a_2)[L_1^\mu{}_\nu x^\nu + a_1^\mu] \\ = L_2^\mu{}_\rho L_1^\rho{}_\nu x^\nu + L_2^\mu{}_\rho a_1^\rho + a_2^\mu$$

$$\Rightarrow g(L_2, a_2)g(L_1, a_1) = g(L_2 L_1, L_2 a_1 + a_2)$$

Thus if we need to represent the action of Poincare Transformation as matrices, then, we need to use a 5×5 matrix representation to embed both Lorentz transformation & Space-time translation.

$$\text{Let } g(L, a) = \begin{bmatrix} \boxed{L^\mu{}_\nu} & \begin{matrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{matrix} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix}$$

$\Rightarrow g(L, a)x^\mu = L^\mu{}_\nu x^\nu + a^\mu$
according to matrix multiplication.
which obeys the group operation.

$$g(L_2, a_2)g(L_1, a_1) = g(L_2 L_1, L_2 a_1 + a_2)$$

Lorentz Group as a Subgroup of Poincare group

Performing just lorentz transformation on the four vector x^μ , can be represented as usual using 5×5 matrices & 5×1 column vector repⁿ.

$$\text{where } L = \begin{bmatrix} \boxed{L^\mu_\nu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ \& } x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix}$$

$\Rightarrow L^\mu_\nu x^\nu = x^\mu$ is still satisfied.
along with preserving $\eta^{\mu\nu}$, where
 $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1, 0) = \eta_{\mu\nu}$

The Set of all such 'L' forms the lorentz group with exact isomorphism between the 4×4 representation & the 5×5 representation.

Translation group as the Invariant Subgroup of Poincare group

The 4D Space-time translations given by $x^\mu \rightarrow T(a)x^\mu = x^\mu + a^\mu$, where

$$T(a) = \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ 1 \end{bmatrix}$$

will form an invariant Subgroup.

Consider $L = \begin{bmatrix} \boxed{L^\mu_\nu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \&$

$$L^{-1} = \begin{bmatrix} \boxed{L_\mu^\nu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$\underline{L^\mu_\sigma L^\sigma_\nu = \delta^\mu_\nu}$$

$L T(a) L^{-1}$ is given by

$$\begin{bmatrix} \boxed{L^\mu_\nu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{L_\mu^\nu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{L^\mu_\nu} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boxed{L_\mu^\nu} & a^0 \\ & a^1 \\ & a^2 \\ & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\mathbb{I}_4} & \begin{matrix} L^0_\nu a^\nu \\ L^1_\nu a^\nu \\ L^2_\nu a^\nu \\ L^3_\nu a^\nu \end{matrix} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = T(L^\mu_\nu a^\nu)$$

$$\rightarrow \underline{L T(a) L^{-1} = T(L a)}$$

Thus $\forall T(a) \in \text{Translation}$, $L T(a) L^{-1}$ also belong to Translation. Thus the translations form an invariant Subgroup.

Generators of Poincare Lie algebra

The Poincare group has 10 generators which can be listed as follows:

- 4 for space-time translation $\{P^\mu\}$
- 3 for spatial rotation $\{\vec{J}\}$
- 3 for Lorentz boosts $\{\vec{K}\}$

Let $P^\mu = (P^0, P^1, P^2, P^3) = (H, \vec{P})$
where $P^0 = H$, $\vec{P} = (P_x, P_y, P_z)$, and
we can write $P_\mu = \eta_{\mu\nu} P^\nu = (P^0, -\vec{P})$.
Later we shall see that P^μ
will transform as a four vector.

Then $\vec{J} = (J_x, J_y, J_z)$ & $\vec{K} = (K_x, K_y, K_z)$
where we can combine them as
a 4×4 anti symmetric tensor,
 $J^{\mu\nu} = -J^{\nu\mu}$, where
 $J_k = \epsilon_{ijk} J^{ij} = (J_x, J_y, J_z)$ &
 $K_j = J^{0j} = (K_x, K_y, K_z)$

In the 5×5 matrix repⁿ we can represent the 10 generators as:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_y = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_z = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_x = \begin{bmatrix} 0 & -i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_y = \begin{bmatrix} 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_z = \begin{bmatrix} 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Poincare lie algebra is given by:

$$[P^\mu, P^\nu] = 0 \quad \text{Translation lie algebra}$$

$$[J_j, J_k] = i\epsilon_{jkl} J_l$$

$$[K_j, J_k] = i\epsilon_{jkl} K_l$$

$$[K_j, K_k] = i\epsilon_{jkl} J_l$$

} so(1,3)
algebra

$$[H, \vec{J}] = 0 \rightarrow H \text{ is a scalar under rot.} \\ \& \vec{J} \text{ is invariant under time translation.}$$

$$[P_j, J_k] = i\epsilon_{jkl} P_l \Rightarrow \vec{P} \text{ is vector operator}$$

$$[H, \vec{K}] = -\vec{P}$$

$$[P^k, K_j] = -i\delta_j^k P^0$$

} P^μ transforms
as a
four vector

If we use the 2nd rank repⁿ of Lorentz algebra generators, then,

$$[J_{\mu\nu}] = \begin{bmatrix} 0 & -K_x & -K_y & -K_z \\ K_x & 0 & J_z & -J_y \\ K_y & -J_z & 0 & J_x \\ K_z & J_y & -J_x & 0 \end{bmatrix}$$

Then we get the usual $so(1,3)$ algebra

$$[J_{\mu\nu}, J_{\sigma\rho}] = -i \left(\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} \right)$$

$$\& [P^\mu, P^\nu] = 0 \} \text{ Translation Lie algebra}$$

$$\& [P_\rho, J_{\mu\nu}] = i \left(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu \right)$$

This is how a four vector operator commutes with generators of Lorentz transformation

To show P^μ transforms as the 4-vector under Lorentz transform

We have already seen that

$$L T(a) L^{-1} = T(L a)$$

$$\Rightarrow L (1 - a^\mu P_\mu) L^{-1} = L [1 - L^\mu{}_\nu a^\nu P_\mu] L^{-1}$$

$$\Rightarrow a^\nu L P_\nu L^{-1} = a^\nu L (L^\mu{}_\nu P_\mu) L^{-1}$$

$$\Rightarrow (L P_\nu L^{-1}) = L (L^\mu{}_\nu P_\mu) L^{-1}$$

$$\Rightarrow \underline{P_\nu \xrightarrow{\text{Lorentz transform}} L^\mu{}_\nu P_\mu}$$

P_ν transforms as covariant 4-vector.
Similarly, P^μ transforms as a 4-vector which is contravariant.

Representation of Poincare group in $L^2(\mathbb{R}^4)$

In $L^2(\mathbb{R}^4)$, the generators of the translations is given by $\hat{P}_\mu = i\partial_\mu$ as seen before. Since translations form a subgroup of Poincare group, even here

$$\underline{\hat{P}_\mu = i\partial_\mu}, \text{ where, } \underline{x^\mu = (t, x, y, z)}$$

$$\Rightarrow [\hat{P}_\mu, \hat{P}_\nu] = 0 \text{ is observed}$$

Similarly the generators of $\hat{\vec{L}}$ spatial rotations are given by $\hat{\vec{L}}$ as follows, $\hat{\vec{L}} = \vec{x} \times \vec{P}$

where $\vec{x} = (x, y, z)$ & $\vec{P} = -i(\partial_x, \partial_y, \partial_z)$

$$\Rightarrow \hat{L}_x = -i[y\partial_z - z\partial_y]$$

$$\hat{L}_y = -i[z\partial_x - x\partial_z]$$

$$\& \hat{L}_z = -i[x\partial_y - y\partial_x]$$

Note that in $L^2(\mathbb{R}^4)$ space,

$$[x^j, \hat{p}^k] = i\delta^{jk}$$

This gives rise to the commutation relations,

$$\underline{[\hat{L}_j, \hat{L}_k] = i\epsilon_{jkl} \hat{L}_l}$$

Thus we recovered $SO(3)$ algebra which is a sub algebra of Poincare.

Similarly the lorentz boost generators are as follows,

$$\hat{K}_x = -i(t\partial_x + x\partial_t)$$

$$\hat{K}_y = -i(t\partial_y + y\partial_t)$$

$$\hat{K}_z = -i(t\partial_z + z\partial_t)$$

Then, by observing $\hat{H} = i\partial_t$

$$\& [t, \hat{H}] = -i \Rightarrow \underline{[x^\mu, \hat{p}^\nu] = -i\eta^{\mu\nu}}$$

we get $[\hat{K}_j, \hat{L}_k] = i\varepsilon_{jkl}K_l$

$$\& \underline{[\hat{K}_j, \hat{K}_k] = -i\varepsilon_{jkl}L_l}$$

Thus $\hat{\vec{L}}$ & $\hat{\vec{K}}$ together forms the $so(1,3)$ lorentz algebra.

We can write the lorentz algebra generators more compactly as:

$$\underline{\hat{L}_{\mu\nu} = x_\mu \hat{p}_\nu - x_\nu \hat{p}_\mu = i[x_\mu \partial_\nu - x_\nu \partial_\mu]}$$

$$\hat{L}_{j0} = i[x_j \partial_0 - x_0 \partial_j]$$

$$= -i(x^j \partial_0 + t \partial_j) = \hat{K}_j = \hat{L}_{0j}$$

$$\Rightarrow \underline{\hat{L}_{j0} = \hat{K}_j = -\hat{L}_{0j}}$$

$$\& \hat{L}_{jk} = x_j \hat{p}_k - x_k \hat{p}_j = \varepsilon_{jkl} \hat{L}_l$$

$$\Rightarrow \underline{\hat{L}_{jk} = \varepsilon_{jkl} \hat{L}_l}$$

$$\Rightarrow [\hat{L}_{\mu\nu}] = \begin{bmatrix} 0 & -\hat{K}_x & -\hat{K}_y & -\hat{K}_z \\ \hat{K}_x & 0 & \hat{L}_z & -\hat{L}_y \\ \hat{K}_y & -\hat{L}_z & 0 & \hat{L}_x \\ \hat{K}_z & \hat{L}_y & -\hat{L}_x & 0 \end{bmatrix}$$

& thus $\hat{L}_{\mu\nu}$ also satisfies

$$[\hat{L}_{\mu\nu}, \hat{L}_{\sigma\rho}] = -i \left(\eta_{\mu\rho} \hat{L}_{\nu\sigma} + \eta_{\nu\sigma} \hat{L}_{\mu\rho} - \eta_{\nu\rho} \hat{L}_{\mu\sigma} - \eta_{\mu\sigma} \hat{L}_{\nu\rho} \right)$$

This defines $so(1,3)$ algebra.

Using $[x_\mu, \hat{P}_\nu] = -i\eta_{\mu\nu}$, we get

$$\begin{aligned} [\hat{P}_\rho, \hat{L}_{\mu\nu}] &= [\hat{P}_\rho, x_\mu \hat{P}_\nu - x_\nu \hat{P}_\mu] \\ &= i(\eta_{\rho\mu} \hat{P}_\nu - \eta_{\rho\nu} \hat{P}_\mu) \end{aligned}$$

$$\Rightarrow [\hat{P}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu} \hat{P}_\nu - \eta_{\rho\nu} \hat{P}_\mu)$$

Thus we get the Poincare algebra repⁿ in $L^2(\mathbb{R}^4)$ hilbert space over $\mathbb{C}$.

Poincare Algebra repⁿ on Dirac Spinors

The Spinor Space is a direct sum of the 4-component Hilbert Space & the $L^2(\mathbb{R}^4)$ Space. So any element is represented as

$$\Psi(x^\mu) = \begin{bmatrix} \psi^{(1)}(x^\mu) \\ \psi^{(2)}(x^\mu) \\ \psi^{(3)}(x^\mu) \\ \psi^{(4)}(x^\mu) \end{bmatrix}$$

Here $\psi^{(\alpha)}(x^\mu)$ is taken from the $L^2(\mathbb{R}^4)$ Space.

The translation group/algebra elements act directly on the functions $\psi^{(\alpha)}(x^\mu)$ of the space $L^2(\mathbb{R}^4)$.

$$\Rightarrow \hat{P}^\mu \Psi(x^\mu) = \begin{bmatrix} \hat{P}^\mu \psi^{(1)}(x^\mu) \\ \hat{P}^\mu \psi^{(2)}(x^\mu) \\ \hat{P}^\mu \psi^{(3)}(x^\mu) \\ \hat{P}^\mu \psi^{(4)}(x^\mu) \end{bmatrix}$$

As the representation of $\hat{P}_\mu = i\partial_\mu$ in $L^2(\mathbb{R}^4)$ space, we can write

$$\hat{P}_\mu \psi(x^\mu) = \begin{bmatrix} \hat{P}_\mu \psi^{(1)}(x^\mu) \\ \hat{P}_\mu \psi^{(2)}(x^\mu) \\ \hat{P}_\mu \psi^{(3)}(x^\mu) \\ \hat{P}_\mu \psi^{(4)}(x^\mu) \end{bmatrix} = \begin{bmatrix} i\partial_\mu \psi^{(1)}(x^\mu) \\ i\partial_\mu \psi^{(2)}(x^\mu) \\ i\partial_\mu \psi^{(3)}(x^\mu) \\ i\partial_\mu \psi^{(4)}(x^\mu) \end{bmatrix}$$

$$\Rightarrow \underbrace{\hat{T}(a^\mu)}_{\text{Action of translation}} \psi(x^\mu) = \begin{bmatrix} \psi^{(1)}(x^\mu + a^\mu) \\ \psi^{(2)}(x^\mu + a^\mu) \\ \psi^{(3)}(x^\mu + a^\mu) \\ \psi^{(4)}(x^\mu + a^\mu) \end{bmatrix}$$

But now we have two types of representation of Lorentz group.

a) Orbital Lorentz: $\hat{L}_{\mu\nu} = x_\mu \hat{P}_\nu - x_\nu \hat{P}_\mu$
 They directly act on the spinor components $\psi^{(\alpha)}(x^\mu)$ like the operators of the $L^2(\mathbb{R}^4)$ space. Similar to the action of $\hat{P}_\mu$ on $\psi(x^\mu)$.

b) Spinor Lorentz ($S^{\mu\nu}$): They are 4×4 matrices that act on the Dirac Spinor which is represented as a 4×1 column matrix. They correspond to how Spinor transform under Lorentz transformation which will mix different Spinor components.

$$S^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$$

(γ^μ) are defined by Clifford algebra through the anti-commutators

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{I}_4$$

And the corresponding 4×4 matrix of Lorentz transformation in Dirac Spinor repⁿ is

$$\Lambda(\mathcal{L}) = e^{\frac{-i\omega_{\mu\nu}\sigma^{\mu\nu}}{4}}$$

In Weyl basis $\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}$, $\vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}$

The total Lorentz generators are

$$\hat{J}^{\mu\nu} = \hat{L}^{\mu\nu} \otimes \mathbb{I}_4 + \hat{\mathbb{I}} \otimes S^{\mu\nu}$$

& under clear interpretation, we can write it in short,

$$\underline{\hat{J}^{\mu\nu} = \hat{L}^{\mu\nu} + \hat{S}^{\mu\nu}}$$

The total angular momentum operator is then given by, $\underline{\hat{J} = \hat{L} + \hat{S}}$

$$\hat{L} = \vec{x} \times \hat{P} \quad \text{and} \quad \hat{S} = \frac{1}{2} \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

Thus the generators $\hat{J}_{\mu\nu}$ & $\hat{P}_\mu$ will satisfy the Poincare algebra.

$$[\hat{J}_{\mu\nu}, \hat{J}_{\sigma\rho}] = -i \left(\eta_{\mu\rho} \hat{J}_{\nu\sigma} + \eta_{\nu\sigma} \hat{J}_{\mu\rho} - \eta_{\nu\rho} \hat{J}_{\mu\sigma} - \eta_{\mu\sigma} \hat{J}_{\nu\rho} \right)$$

$$[\hat{P}^\mu, \hat{P}^\nu] = 0$$

$$[\hat{P}_\rho, \hat{J}_{\mu\nu}] = i(\eta_{\mu\rho} \hat{P}_\nu - \eta_{\nu\rho} \hat{P}_\mu)$$