

Canonical Quantization of Scalar Field

Mode expansion of scalar field

The scalar field lagrangian is given by

$$\underline{\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}$$

whose eqⁿ of motion is,

$$(\partial_\mu \partial^\mu + m^2) \phi = 0$$

The dispersion relation is given by the following.

$$\text{Let } \phi(x) = A e^{-i p_\mu x^\mu}$$

It satisfies $(\partial_\mu \partial^\mu + m^2) \phi = 0$
provided, $(-p_\mu p^\mu + m^2) = 0$

$$\Rightarrow p_\mu p^\mu = m^2 \rightarrow \underline{(p^0)^2 = |\vec{p}|^2 + m^2}$$

Thus $p_\mu p^\mu = m^2$ is the dispersion relation which should be satisfied.

Moreover if $E_p^2 = |\vec{p}|^2 + m^2$ is the Total energy of quantized particle. Then $p_\mu p^\mu = m^2$ is equivalent to needing $\underline{(p^0)^2 = E_p^2}$

Also we need only +ve energy only $\Rightarrow E_p > 0$ or $p^0 > 0$ is need.

So the mode expansion of $\phi(x)$ is,

$$\hat{\phi}(x) = \int \frac{d^4 p}{(2\pi)^{3/2}} \underbrace{\delta(p^\mu p_\mu - m^2) \Theta(p^0) \hat{A}(p)}_{\text{Fourier transform}} e^{-i p_\mu x^\mu}$$

$\delta(p^\mu p_\mu - m^2) = \delta[(p^0)^2 - E_p^2]$ is to satisfy the dispersion relation.

& $\Theta(p^0)$ is to have only +ve energy where Θ is the heaviside function.

$$\hat{\phi}(x) = \int \frac{d^4 p}{(2\pi)^{3/2}} \delta(p^\mu p_\mu - m^2) \Theta(p^0) \hat{A}(p) e^{-i p_\mu x^\mu}$$

We shall treat $\hat{\phi}(x)$ as operator which is hermitian.

$$\Rightarrow \hat{\phi}^\dagger(x) = \hat{\phi}(x)$$

$$\Rightarrow \hat{A}^\dagger(p) e^{i p_\mu x^\mu} = A(p) e^{-i p_\mu x^\mu}$$

$$\Rightarrow \underline{\hat{A}^\dagger(p) = \hat{A}(-p)}$$

$$\hat{\phi}(x) = \int \frac{d^4 p}{(2\pi)^{3/2}} \delta(p^\mu p_\mu - m^2) \Theta(p^0) \hat{A}(p) e^{-i p_\mu x^\mu}$$

$$= \int \frac{d^4 p}{(2\pi)^{3/2}} \delta(p^\mu p_\mu - m^2) \Theta(p^0) \left[\hat{A}(p) e^{-i p_\mu x^\mu} + \hat{A}(-p) e^{+i p_\mu x^\mu} \right]$$

$$\delta(p^\mu p_\mu - m^2) = \delta[(p^0)^2 - E_p^2]$$

$$\delta[f(x)] = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}, \quad \forall x_i: f(x_i)=0$$

$$\Rightarrow \delta[(p^0)^2 - E_p^2] = \frac{\delta(p^0 - E_p)}{2E_p} + \frac{\delta(p^0 + E_p)}{2E_p}$$

$$\begin{aligned} \Rightarrow \hat{\phi}(x) &= \int \frac{d^4 p}{(2\pi)^{3/2} 2E_p} \delta(p^0 - E_p) \left[\hat{A}(p) e^{-ip_\mu x^\mu} + \hat{A}(-p) e^{+ip_\mu x^\mu} \right] \\ &= \int \frac{d^3 p}{(2\pi)^{3/2} (2E_p)} \left[\hat{A}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{A}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} \right] \end{aligned}$$

$$\underline{p^\mu = (E_p, \vec{p}), \quad p^0 = E_p \text{ is fixed}}$$

$$\text{Let } \hat{a}(\vec{p}) = \frac{\hat{A}(\vec{p})}{\sqrt{2E_p}} \quad \& \quad \hat{a}^\dagger(\vec{p}) = \frac{\hat{A}^\dagger(\vec{p})}{\sqrt{2E_p}}$$

$$\Rightarrow \hat{\phi}(x) = \int \frac{d^3 p}{\sqrt{(2\pi)^3 (2E_p)}} \left[\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} \right]$$

is the final mode expansion of scalar field.

$\hat{\phi}(x) \longrightarrow \hat{\phi}(t, \vec{x})$ is a time dependent operator. So we are working in Heisenberg picture.

$$\hat{a}(p) \longrightarrow \hat{a}(E_p, \vec{p}) \equiv \hat{a}(\vec{p})$$

$$\& \hat{a}^\dagger(p) \longrightarrow \hat{a}^\dagger(E_p, \vec{p}) \equiv \hat{a}^\dagger(\vec{p})$$

$\hat{a}^\dagger(\vec{p})|\Omega\rangle = |\vec{p}\rangle$ is a 3-momentum state.

$\hat{a}^\dagger(\vec{p}, t)$ & $\hat{a}(\vec{p}, t)$ are the creation & annihilation operators in Heisenberg picture which are given by,

$$\hat{a}(\vec{p}, t) = e^{-iE_p t} \hat{a}(\vec{p})$$

$$\& \hat{a}^\dagger(\vec{p}, t) = e^{iE_p t} \hat{a}^\dagger(\vec{p})$$

where $\hat{a}(\vec{p})$ & $\hat{a}^\dagger(\vec{p})$ are in the Schrodinger picture.

Normalizing factor

d^4p is a lorentz invariant measure.

$$\begin{aligned}(d^4p) \delta[(p^0)^2 - E_p^2] \Theta(p^0) &= \frac{d^4p \delta(p^0 - E_p) \Theta(p^0)}{2E_p} \\ &= \frac{d^3p}{2E_p}\end{aligned}$$

is also a lorentz invariant measure.

$\rightarrow \frac{d^3p}{(2\pi)^3 (2E_p)}$ is lorentz invariant.

Consider the 3-momentum states $\{|\vec{p}\rangle\}$

where $\int d^3p |\vec{p}\rangle \langle \vec{p}| = 1$

Such that $\langle \vec{p} | \vec{q} \rangle = \delta^3(\vec{p} - \vec{q})$

This is not lorentz invariant.

But we can define lorentz invariant 4-momentum states $\{|p\rangle\}$ as follows.

$$|p\rangle = (2\pi)^{3/2} \sqrt{2E_p} |\vec{p}\rangle$$

$$\text{where } E_p = \sqrt{|\vec{p}|^2 + m^2}$$

$$\Rightarrow \langle q | p \rangle = (2\pi)^3 (2E_p) \delta(\vec{p} - \vec{q})$$

$$\oint \int \frac{d^3p}{(2\pi)^3 (2E_p)} |p\rangle \langle p| = 1$$

This is Lorentz invariant eqⁿ.

We can define ladder operators $\hat{\alpha}^\dagger(\vec{p})$ which create a 4-momentum state $|p\rangle$. Similarly, $\hat{a}^\dagger(\vec{p})$ will create a 3-momentum state $|\vec{p}\rangle$

$$\begin{aligned} \rightarrow \hat{\alpha}^\dagger(p) |\Omega\rangle &= |p\rangle = (2\pi)^{3/2} \sqrt{2E_p} |\vec{p}\rangle \\ &= (2\pi)^{3/2} \sqrt{2E_p} \hat{a}^\dagger(\vec{p}) |\Omega\rangle \end{aligned}$$

$$\Rightarrow \underline{\hat{\alpha}^\dagger(p) = (2\pi)^{3/2} \sqrt{2E_p} \hat{a}^\dagger(\vec{p})}$$

In terms of $\hat{\alpha}(p)$ & $\hat{\alpha}^\dagger(p)$,

$$\hat{\phi}(x) = \frac{\int d^3p [\hat{a}(\vec{p}) e^{-i p_\mu x^\mu} + \hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$\Rightarrow$

$$\hat{\phi}(x) = \frac{\int d^3p [\hat{\alpha}(p) e^{-i p_\mu x^\mu} + \hat{\alpha}^\dagger(p) e^{+i p_\mu x^\mu}]}{(2\pi)^3 (2E_p)}$$

This just inverse fourier transform using the lorentz invariant measure.

$$\begin{aligned} \text{Observe: } \hat{\phi}(x) |\Omega\rangle &= \frac{\int d^3p e^{+i p_\mu x^\mu} |\vec{p}\rangle}{\sqrt{(2\pi)^3 (2E_p)}} \\ &= \frac{\int d^3p e^{-i \vec{p} \cdot \vec{x}} e^{i E_p t} |\vec{p}\rangle}{\sqrt{(2\pi)^3 (2E_p)}} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \langle q | \hat{\phi}(x) | \Omega \rangle &= (2\pi)^{3/2} \sqrt{2E_q} \langle \vec{q} | \hat{\phi}(x) | \Omega \rangle \\
&= (2\pi)^{3/2} \sqrt{2E_q} \int \frac{d^3p \, e^{-i\vec{p} \cdot \vec{x}} e^{iE_p t} \langle \vec{q} | \vec{p} \rangle}{\sqrt{(2\pi)^3 (2E_p)}} \\
&= (2\pi)^{3/2} \sqrt{2E_q} \int \frac{d^3p \, e^{-i\vec{p} \cdot \vec{x}} e^{iE_p t} \delta(\vec{q} - \vec{p})}{\sqrt{(2\pi)^3 (2E_p)}} \\
&= \frac{(2\pi)^{3/2} \sqrt{2E_q} \, e^{-i\vec{q} \cdot \vec{x}} e^{iE_q t}}{\sqrt{(2\pi)^3 (2E_q)}} = e^{iq_\mu x^\mu}
\end{aligned}$$

$$\Rightarrow \langle q | \hat{\phi}(x) | \Omega \rangle = e^{iq_\mu x^\mu}$$

$$\Rightarrow \langle \vec{q} | \hat{\phi}(x) | \Omega \rangle = \frac{e^{iq_\mu x^\mu}}{(2\pi)^{3/2} \sqrt{2E_q}}$$

$$\Rightarrow \hat{\phi}(x) = \int d^3p \left[\hat{a}(\vec{p}) \langle \Omega | \hat{\phi}(x) | \vec{p} \rangle + \hat{a}^\dagger(\vec{p}) \langle \vec{p} | \hat{\phi}(x) | \Omega \rangle \right]$$

Solving for $\hat{a}(\vec{p})$ & $\hat{a}^\dagger(\vec{p})$

Start from

$$\hat{\phi}(t, \vec{x}) = \int \frac{d^3p [\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu}]}{(2\pi)^{3/2} \sqrt{2E_p}}$$

Multiply by $\frac{e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} d^3x$ throughout
& integrate over entire $\vec{x}$.

$$\Rightarrow \int \hat{\phi}(t, \vec{x}) \frac{e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} d^3x$$

$$= \int \frac{d^3p}{\sqrt{2E_p}} \left[\hat{a}(\vec{p}) e^{-iE_p t} \int \frac{d^3x}{(2\pi)^3} e^{i(\vec{p}-\vec{q}) \cdot \vec{x}} \right. \\ \left. + \hat{a}^\dagger(\vec{p}) e^{iE_p t} \int \frac{d^3x}{(2\pi)^3} e^{-i(\vec{p}+\vec{q}) \cdot \vec{x}} \right]$$

$$= \int \frac{d^3p}{\sqrt{2E_p}} \left[\hat{a}(\vec{p}) e^{-iE_p t} \delta(\vec{p}-\vec{q}) \right. \\ \left. + \hat{a}^\dagger(\vec{p}) e^{iE_p t} \delta(\vec{p}+\vec{q}) \right]$$

→

$$\int \frac{d^3x \hat{\phi}(t, \vec{x}) e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} = \frac{\hat{a}(\vec{q}) e^{-iE_q t} + \hat{a}^\dagger(-\vec{q}) e^{iE_q t}}{\sqrt{2E_q}}$$

Then, the Canonical momenta for Scalar field is given by,

$$\Pi_0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = (\partial_0 \phi)$$

$$\Rightarrow \hat{\Pi}_0(t, \vec{x}) = \frac{\int d^3p (iE_p) [\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} - \hat{a}(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

Multiply by $\frac{e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} d^3x$ throughout & integrate over entire $\vec{x}$.

$$\begin{aligned} \Rightarrow \int \hat{\Pi}_0(t, \vec{x}) \frac{e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} d^3x \\ = \frac{iE_q}{\sqrt{2E_q}} [\hat{a}^\dagger(-\vec{q}) e^{iE_q t} - \hat{a}(\vec{q}) e^{-iE_q t}] \end{aligned}$$

$$\Rightarrow \int \frac{d^3x \Pi_0(t, \vec{x}) e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} = \frac{iE_q [\hat{a}^\dagger(-\vec{q}) e^{iE_q t} - \hat{a}(\vec{q}) e^{-iE_q t}]}{\sqrt{2E_q}}$$

Thus we got two relations,

$$\int \frac{d^3x \hat{\phi}(t, \vec{x}) e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} = \frac{\hat{a}^\dagger(-\vec{q}) e^{iE_q t} + \hat{a}(\vec{q}) e^{-iE_q t}}{\sqrt{2E_q}}$$

$$\& \int \frac{d^3x \hat{\Pi}_0(t, \vec{x}) e^{-i\vec{q} \cdot \vec{x}}}{(2\pi)^{3/2}} = iE_q \left[\frac{\hat{a}^\dagger(-\vec{q}) e^{iE_q t} - \hat{a}(\vec{q}) e^{-iE_q t}}{\sqrt{2E_q}} \right]$$

$$\Rightarrow \int \frac{d^3x}{(2\pi)^{3/2}} [E_q \hat{\phi}(t, \vec{x}) - i \hat{\Pi}_0(t, \vec{x})] e^{-i\vec{q} \cdot \vec{x}}$$

$$= \hat{a}^\dagger(-\vec{q}) e^{iE_q t} \sqrt{2E_q}$$

$$\Rightarrow \hat{a}^\dagger(-\vec{q}) = \int \frac{d^3x}{(2\pi)^{3/2} \sqrt{2E_q}} \left[E_q \hat{\phi}(t, \vec{x}) - i \hat{\Pi}_0(t, \vec{x}) \right] e^{-iE_q t - i\vec{q} \cdot \vec{x}}$$

$$\Rightarrow \hat{a}^\dagger(\vec{p}) = \int \frac{d^3x}{(2\pi)^{3/2} \sqrt{2E_p}} \left[E_p \hat{\phi}(x) - i \hat{\Pi}_0(x) \right] e^{-ip_\mu x^\mu}$$

Similarly, we get,

$$\hat{a}(\vec{p}) = \int \frac{d^3x}{(2\pi)^{3/2} \sqrt{2E_p}} \left[E_p \hat{\phi}(x) + i \hat{\Pi}_0(x) \right] e^{ip_\mu x^\mu}$$

The results we obtained until now are all classical. But the quantization of the scalar field occurs only when we define the commutation relations between the operators. Thus the commutation relations are the heart of Second quantization of fields.

Commutator relations & Quantization

We impose the bosonic commutation relations for the scalar field. Implying that scalar field describe bosons.

$$\Rightarrow [\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\Pi}_0(x), \hat{\Pi}_0(y)] = 0$$

$$\& \underline{[\hat{\phi}(t, \vec{x}), \hat{\Pi}_0(t, \vec{y})] = i\delta(\vec{x} - \vec{y})}$$

x & y are two space-time points.

Note that the commutators are taken at the same time. Hence they are called equal time commutation relations.

Also the above relations are just upgraded poisson bracket version from the classical field theory.

$$[,]_{\text{commutator}} = i\{, \}_{\text{Poisson}}$$

Using the commutations of field & momentum density operators, we can find commutation relation between the mode operators $\hat{a}^\dagger(\vec{p})$ & $\hat{a}(\vec{p})$.

$$\hat{a}(\vec{p}) = \int \frac{d^3x [E_p \hat{\phi}(x) + i \hat{\Pi}_0(x)] e^{i p_\mu x^\mu}}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\hat{a}^\dagger(\vec{p}) = \int \frac{d^3x [E_p \hat{\phi}(x) - i \hat{\Pi}_0(x)] e^{-i p_\mu x^\mu}}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$[\hat{a}(\vec{p}), \hat{a}(\vec{q})] =$$

$$= \int \frac{d^3x d^3y e^{i p_\mu x^\mu} e^{i q_\mu y^\mu}}{\sqrt{(2\pi)^3} \sqrt{2E_p} \sqrt{2E_q}} \left[\begin{matrix} (E_p \hat{\phi}(x) + i \hat{\Pi}_0(x)), \\ (E_q \hat{\phi}(y) + i \hat{\Pi}_0(y)) \end{matrix} \right]$$

$$= \int \frac{d^3x d^3y e^{i p_\mu x^\mu} e^{i q_\mu y^\mu}}{\sqrt{(2\pi)^3} \sqrt{2E_p} \sqrt{2E_q}} \left(\begin{matrix} i E_p [\hat{\phi}(x), \hat{\Pi}_0(y)] \\ + i E_q [\hat{\Pi}_0(x), \hat{\phi}(y)] \end{matrix} \right)$$

$$= \int \frac{d^3x d^3y e^{i p_\mu x^\mu} e^{i q_\mu y^\mu}}{\sqrt{(2\pi)^3} \sqrt{2E_p} \sqrt{2E_q}} \left(\begin{matrix} i E_p [\hat{\phi}(x), \hat{\Pi}_0(y)] \\ + i E_q [\hat{\Pi}_0(x), \hat{\phi}(y)] \end{matrix} \right)$$

$$\Rightarrow [\hat{a}(\vec{p}), \hat{a}(\vec{q})]$$

$$= \int \frac{d^3x d^3y e^{i\vec{p}_\mu x^\mu} e^{i\vec{q}_\mu y^\mu}}{(2\pi)^3 \sqrt{2E_p} \sqrt{2E_q}} \left(-E_p \delta(\vec{x} - \vec{y}) + E_q \delta(\vec{x} - \vec{y}) \right)$$

$$= \int \frac{d^3x e^{i(\vec{p} + \vec{q})x^\mu}}{(2\pi)^3 \sqrt{2E_p} \sqrt{2E_q}} (E_q - E_p)$$

$$= \frac{\delta(\vec{p} + \vec{q}) e^{i(E_p + E_q)t} (E_q - E_p)}{2\sqrt{E_p E_q}} = 0$$

(Since when $\vec{p} = -\vec{q}$, $E_q = E_p$)

$$\Rightarrow \underline{[\hat{a}(\vec{p}), \hat{a}(\vec{q})] = 0}$$

Similarly we can get the relation

$$\Rightarrow \underline{[\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{q})] = 0}$$

Consider $[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] =$

$$= \int \frac{d^3x d^3y e^{i p_\mu x^\mu} e^{-i q_\mu y^\mu}}{(2\pi)^3 \sqrt{2E_p} \sqrt{2E_q}} \begin{bmatrix} (E_p \hat{\phi}(x) + i \hat{\Pi}_0(x)), \\ (E_q \hat{\phi}(y) - i \hat{\Pi}_0(y)) \end{bmatrix}$$

$$= \int \frac{d^3x d^3y e^{i p_\mu x^\mu} e^{-i q_\mu y^\mu}}{(2\pi)^3 \sqrt{2E_p} \sqrt{2E_q}} \left(-i E_p [\hat{\phi}(x), \hat{\Pi}_0(y)] + i E_q [\hat{\Pi}_0(x), \hat{\phi}(y)] \right)$$

$$= \int \frac{d^3x d^3y e^{i p_\mu x^\mu} e^{-i q_\mu y^\mu}}{(2\pi)^3 \sqrt{2E_p} \sqrt{2E_q}} \left(E_p \delta(\vec{x} - \vec{y}) + E_q \delta(\vec{x} - \vec{y}) \right)$$

$$= \int \frac{d^3x e^{i(p_\mu - q_\mu)x^\mu}}{(2\pi)^3 \sqrt{2E_p} \sqrt{2E_q}} (E_p + E_q)$$

$$= \underbrace{\frac{e^{i(E_p - E_q)t} (E_p + E_q)}{2 \sqrt{E_p E_q}}}_1 \underbrace{\int \frac{d^3x e^{i(\vec{p} - \vec{q}) \cdot \vec{x}}}{(2\pi)^3}}_{\delta(\vec{p} - \vec{q})}$$

(when $\vec{p} = \vec{q}$)

$$\Rightarrow \underline{[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] = \delta(\vec{p} - \vec{q})}$$

Thus the commutation relations b/w the operators of scalar field theory can be written together as,

- $[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\Pi}_0(x), \hat{\Pi}_0(y)] = 0$
- $[\hat{\phi}(t, \vec{x}), \hat{\Pi}_0(t, \vec{y})] = i\delta(\vec{x} - \vec{y})$
- $[\hat{a}(\vec{p}), \hat{a}(\vec{q})] = [\hat{a}^\dagger(\vec{p}), \hat{a}^\dagger(\vec{q})] = 0$
- $[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{q})] = \delta(\vec{p} - \vec{q})$

All the above commutation relations are taken at equal time.

Since we impose bosonic commutation relations, the quantized excitations are bosonic particles which will become evident when we find the Hamiltonian of the scalar field and its action on vacuum state in Fock space.

Hamiltonian of scalar field in terms of $\hat{a}(\vec{p})$ & $\hat{a}^\dagger(\vec{p})$

The hamiltonian density is given by,

$$\mathcal{H} = \dot{\phi} \pi_0 - \mathcal{L}$$

$$\Rightarrow \hat{\mathcal{H}} = \frac{\hat{\pi}_0^2}{2} + \frac{1}{2} (\vec{\nabla} \hat{\phi})^2 + \frac{1}{2} m^2 \hat{\phi}^2$$

Then the hamiltonian is given by,

$$\hat{H} = \int d^3x \hat{\mathcal{H}}$$

$$\Rightarrow \hat{H} = \int \frac{d^3x}{2} \left(\hat{\pi}_0^2 + (\vec{\nabla} \hat{\phi})^2 + m^2 \hat{\phi}^2 \right)$$

The momentum density $\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}$

$$\pi^\mu(x) = \partial^\mu \phi \Rightarrow \hat{\pi}_\mu(x) = (\partial_\mu \hat{\phi})$$

$$\hat{\Pi}_\mu(x) = \frac{\int d^3p (i p_\mu) [\hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu} - \hat{a}(\vec{p}) e^{-i p_\mu x^\mu}]}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\rightarrow \vec{\nabla} \phi = -\hat{\vec{\Pi}}(x) = \hat{\Pi}^i(x)$$

$$\rightarrow \vec{\nabla} \phi = \frac{\int d^3p (-i \vec{p}) [\hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu} - \hat{a}(\vec{p}) e^{-i p_\mu x^\mu}]}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\Rightarrow \hat{H} = \int \frac{d^3x d^3p d^3q}{4(2\pi)^3 \sqrt{E_p E_q}} \left[(i^2 E_p E_q + i^2 \vec{p} \cdot \vec{q}) \right.$$

$$\times [\hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu} - \hat{a}(\vec{p}) e^{-i p_\mu x^\mu}] [\hat{a}^\dagger(\vec{q}) e^{+i q_\mu x^\mu} - \hat{a}(\vec{q}) e^{-i q_\mu x^\mu}]$$

$$\left. + m^2 [\hat{a}(\vec{p}) e^{-i p_\mu x^\mu} + \hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu}] \times [\hat{a}(\vec{q}) e^{-i q_\mu x^\mu} + \hat{a}^\dagger(\vec{q}) e^{+i q_\mu x^\mu}] \right]$$

Integrate it first over d^3x ,
 then we get dirac delta functions of
 the form $\delta(\vec{p} \pm \vec{q})$. & then perform
 the integral over d^3q

Then the expression becomes,

$$\hat{H} = \int \frac{d^3p}{4E_p} (E_p^2 + p^2 + m^2) [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})] \\ + \int \frac{d^3p}{4E_p} (-E_p^2 + p^2 + m^2) \left[\hat{a}^\dagger(\vec{p}) \hat{a}^\dagger(-\vec{p}) e^{2iE_p t} \right. \\ \left. + \hat{a}(\vec{p}) \hat{a}(-\vec{p}) e^{-2iE_p t} \right]$$

$$\text{But } E_p^2 = p^2 + m^2.$$

$$\Rightarrow \hat{H} = \frac{1}{2} \int d^3p E_p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})]$$

$$\text{But we can use, } [\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{p})] = \delta(0)$$

$$\Rightarrow \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) = \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \delta(0)$$

$$\Rightarrow \hat{H} = \int d^3p E_p \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \frac{\delta(0)}{2} \right]$$

Notice that this Hamiltonian resembles that of uncoupled harmonic oscillators.

Consider the commutation of the Hamiltonian with $\hat{a}(\vec{p}) \neq \hat{a}(\vec{p})$.

$$[\hat{H}, \hat{a}(\vec{q})] = \left[\int d^3p E_p \left(\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \frac{\mathcal{Z}(0)}{2} \right), \hat{a}(\vec{q}) \right]$$

$$= \int d^3p E_p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}), \hat{a}(\vec{q})]$$

$$= \int d^3p E_p [\hat{a}^\dagger(\vec{p}), \hat{a}(\vec{q})] \hat{a}(\vec{p})$$

$$= - \int d^3p E_p \mathcal{Z}(\vec{p} - \vec{q}) \hat{a}(\vec{p})$$

$$= -E_q \hat{a}(\vec{q})$$

$$\Rightarrow \underline{[\hat{H}, \hat{a}(\vec{q})] = -E_q \hat{a}(\vec{q})}$$

Similarly,

$$\underline{[\hat{H}, \hat{a}^\dagger(\vec{q})] = E_q \hat{a}^\dagger(\vec{q})}$$

Thus the results are exactly similar to that of infinitely many uncoupled quantum harmonic oscillators.

For every value of momentum mode " $\vec{p}$ ", there exists a quantum oscillator, whose excitations are quantized with values $E_p = \sqrt{|\vec{p}|^2 + m^2}$. An excitation in this mode represents a scalar field particle of energy E_p .

This can be described using a Fock Space & any state in this space is given by, $|\{n_{\vec{p}}\}\rangle$, where $n_{\vec{p}}$ represents the number of particles with momentum $\vec{p}$ & energy E_p .

The action of $\hat{a}^+(\vec{p})$ on vacuum $|\Omega\rangle$ is to create a single particle with energy E_p & momentum $\vec{p}$ denoting it as:
 $\hat{a}^+(\vec{p})|\Omega\rangle = |\vec{p}\rangle$ with $\langle\vec{q}|\vec{p}\rangle = \delta(\vec{p}-\vec{q})$

$$\hat{a}(\vec{p})|\Omega\rangle = 0 \neq \hat{a}(\vec{p})|\vec{p}\rangle = |\Omega\rangle$$

Thus $\hat{a}^\dagger(\vec{p})$ acts as creation operator
 $\neq \hat{a}(\vec{p})$ acts as annihilation operator.

Consider the expectation value of $\hat{H}$
in ground state.

$$\langle\Omega|\hat{H}|\Omega\rangle = \int d^3p E_p \left[\underbrace{\langle\Omega|\hat{a}^\dagger(\vec{p})\hat{a}(\vec{p})|\Omega\rangle}_0 + \frac{\mathcal{Z}(0)}{2} \right]$$

$$\begin{aligned} \rightarrow \langle\Omega|\hat{H}|\Omega\rangle &= \int d^3p E_p \frac{\mathcal{Z}(0)}{2} \\ &= \frac{\mathcal{Z}(0)}{2} \underbrace{\int d^3p E_p}_{\infty} = \infty \end{aligned}$$

Thus we get the absolute value of the
ground state expectation of Hamiltonian
to be infinity. This is because the
zero point energy of infinite number
of oscillators add up to infinity.

Normal Ordering

To remove the unnecessary ∞ 's that arise in the expectation values we measure the quantities on the Normal ordered operators instead of the original operator products.

When we encounter a product of any number of $\hat{a}(\vec{p})$ & $\hat{a}^\dagger(\vec{p})$ operators we define normal ordered product by moving all creation operators to the left and all annihilation operators to the right without adding any extra term due to commutation.

For Bosonic operators we can just rearrange them during normal ordering. But for fermionic operators we need to multiply by a $(-1)^S$, where 'S' is the total number of permutations done.

Eg: For bosonic operators

$$\bullet \mathcal{N}[\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)] = \hat{a}^\dagger(\vec{p}_2)\hat{a}(\vec{p}_1)$$

$$\bullet \mathcal{N}[\hat{a}^\dagger(\vec{p}_1)\hat{a}(\vec{p}_2)] = \hat{a}^\dagger(\vec{p}_1)\hat{a}(\vec{p}_2)$$

$$\bullet \mathcal{N}[\hat{a}^\dagger(\vec{p}_1)\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_3)\hat{a}^\dagger(\vec{p}_4)] = \hat{a}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_4)\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_3)$$

$$\bullet \mathcal{N}[\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{a}(\vec{p}_3)\hat{a}^\dagger(\vec{p}_4)] = \hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_4)\hat{a}(\vec{p}_1)\hat{a}(\vec{p}_3)$$

For Fermionic operators

$$\bullet \mathcal{N}[\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)] = -\hat{a}^\dagger(\vec{p}_2)\hat{a}(\vec{p}_1)$$

$$\bullet \mathcal{N}[\hat{a}^\dagger(\vec{p}_1)\hat{a}(\vec{p}_2)] = \hat{a}^\dagger(\vec{p}_1)\hat{a}(\vec{p}_2)$$

$$\bullet \mathcal{N}[\hat{a}(\vec{p}_1)\hat{a}(\vec{p}_2)\hat{a}^\dagger(\vec{p}_3)\hat{a}^\dagger(\vec{p}_4)] = \hat{a}^\dagger(\vec{p}_1)\hat{a}^\dagger(\vec{p}_4)\hat{a}(\vec{p}_2)\hat{a}(\vec{p}_3)$$

$$\bullet \mathcal{N}[\hat{a}(\vec{p}_1)\hat{a}^\dagger(\vec{p}_2)\hat{a}(\vec{p}_3)\hat{a}^\dagger(\vec{p}_4)] = -\hat{a}^\dagger(\vec{p}_2)\hat{a}^\dagger(\vec{p}_4)\hat{a}(\vec{p}_1)\hat{a}(\vec{p}_3)$$

Now we can get the normal ordered expectation value of the scalar field Hamiltonian in ground state.

The Hamiltonian for scalar field is

$$\hat{H} = \frac{1}{2} \int d^3p E_p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})]$$

$$\Rightarrow \mathcal{N}[\hat{H}] = \int d^3p E_p \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})$$

Defining $\hat{n}(\vec{p}) = \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})$ as the number operator, we see that they give information about the number of excitations / Particles with momentum of $\vec{p}$ & energy E_p .

& $\mathcal{N}[\hat{H}]$ gives the energy of the particles or the quantized excitations of the field.

$$\Rightarrow \langle \Omega | \mathcal{N}[\hat{H}] | \Omega \rangle = 0$$

Meaning of Mode Expansion

Consider the mode expansion for the free scalar field theory,

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^{3/2} \sqrt{2E_p}} \left[\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} + \hat{a}(\vec{p}) e^{-ip_\mu x^\mu} \right]$$

This can be written as,

$$\hat{\phi}(x) = \hat{\phi}^+(x) + \hat{\phi}^-(x)$$

$$\text{where } \hat{\phi}^+(x) = \int \frac{d^3p \hat{a}^\dagger(\vec{p})}{(2\pi)^{3/2} \sqrt{2E_p}} e^{+ip_\mu x^\mu}$$

$$\& \hat{\phi}^-(x) = \int \frac{d^3p \hat{a}(\vec{p})}{(2\pi)^{3/2} \sqrt{2E_p}} e^{-ip_\mu x^\mu}$$

$\hat{\phi}^+(x) \rightarrow$ Positive E_p mode

$\hat{\phi}^-(x) \rightarrow$ Negative E_p mode

$e^{-ip_\mu x^\mu}$ tells that the particles are incoming plane waves and also gets absorbed into the system. This implies that the particles are annihilated.

$e^{ip_\mu x^\mu}$ tells that the particles are outgoing plane waves and also are released by the system. This implies that the particles are created.

$\Rightarrow \hat{\phi}^+(x) \rightarrow$ Particle Creation part

$\hat{\phi}^-(x) \rightarrow$ Particle Annihilation part

$$\Rightarrow \hat{\phi}(x) = \sum_{\vec{k}} \left(\begin{array}{c} \text{Incoming} \\ +ve E_p \\ \text{Particle} \\ \text{Created} \end{array} \right) + \sum_{\vec{k}} \left(\begin{array}{c} \text{Outgoing} \\ +ve E_p \\ \text{Particle} \\ \text{annihilated} \end{array} \right)$$

Energy Momentum Tensor of Scalar field

$$T^{\mu\nu} = \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right] (\partial^\nu \phi) - \eta^{\mu\nu} \mathcal{L}$$

$$\text{where } \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\Rightarrow \hat{T}^{\mu\nu} = (\partial^\mu \hat{\phi})(\partial^\nu \hat{\phi}) - \eta^{\mu\nu} \hat{\mathcal{L}}$$

The four momentum operator is given by,

$$\hat{P}^\mu = \int d^3x \hat{T}^{0\mu}$$

$\hat{P}^0 = \hat{H}$ is the Hamiltonian operator

$$\hat{P}^0 = \hat{H} = \frac{1}{2} \int d^3p E_p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})]$$

$$\& \mathcal{N}[\hat{P}^0] = \int d^3p E_p \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})$$

$$\hat{P}^i = \int d^3x \hat{T}^{0i} = \int d^3x (\partial^0 \hat{\phi}) (\partial^i \hat{\phi})$$

But $\partial^0 \hat{\phi} = \hat{\Pi}^0 = \hat{\Pi}_0$, $\partial^i \hat{\phi} = -\partial_i \hat{\phi}$

$$\Rightarrow \hat{\vec{P}} = -\int d^3x [\hat{\Pi}_0(x) \vec{\nabla} \hat{\phi}(x)]$$

$$\hat{\Pi}_0(x) = \frac{\int d^3q (iE_q) [\hat{a}^\dagger(\vec{q}) e^{+iq_\mu x^\mu} - \hat{a}(\vec{q}) e^{-iq_\mu x^\mu}]}{\int \sqrt{(2\pi)^3 (2E_q)}}$$

$$\vec{\nabla} \hat{\phi}(x) = \frac{\int d^3p (-i\vec{p}) [\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} - \hat{a}(\vec{p}) e^{-ip_\mu x^\mu}]}{\int (2\pi)^{3/2} \sqrt{(2E_p)}}$$

$$\begin{aligned} \Rightarrow \hat{\vec{P}} = & \frac{\int d^3x d^3q d^3p E_q \vec{p}}{\int (2\pi)^3 2\sqrt{E_p E_q}} \left[\hat{a}^\dagger(\vec{q}) \hat{a}(\vec{p}) e^{+i(q_\mu - p_\mu)x^\mu} \right. \\ & + \hat{a}(\vec{q}) \hat{a}^\dagger(\vec{p}) e^{-i(q_\mu - p_\mu)x^\mu} - \hat{a}^\dagger(\vec{q}) \hat{a}^\dagger(\vec{p}) e^{+i(q_\mu + p_\mu)x^\mu} \\ & \left. - \hat{a}(\vec{q}) \hat{a}(\vec{p}) e^{-i(q_\mu + p_\mu)x^\mu} \right] \end{aligned}$$

Performing the d^3x integral first, we get dirac delta functions.

$$\begin{aligned} \rightarrow \hat{\vec{P}} = & \int \frac{d^3q d^3p E_q \vec{p}}{2\sqrt{E_p E_q}} \left[\hat{a}^\dagger(\vec{q}) \hat{a}(\vec{p}) e^{i(E_q - E_p)t} \delta(\vec{q} - \vec{p}) \right. \\ & + \hat{a}(\vec{q}) \hat{a}^\dagger(\vec{p}) e^{i(E_q - E_p)t} \delta(\vec{q} - \vec{p}) \\ & - \hat{a}^\dagger(\vec{q}) \hat{a}^\dagger(\vec{p}) e^{i(E_q + E_p)t} \delta(\vec{q} + \vec{p}) \\ & \left. - \hat{a}(\vec{q}) \hat{a}(\vec{p}) e^{-i(E_q + E_p)t} \delta(\vec{q} + \vec{p}) \right] \end{aligned}$$

Performing the d^3q integral & since $E_p = E_{-p}$, we get,

$$\hat{\vec{P}} = \int \frac{d^3p \vec{p}}{2} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) \right. \\ \left. - \hat{a}^\dagger(-\vec{p}) \hat{a}^\dagger(\vec{p}) e^{2iE_p t} \right. \\ \left. - \hat{a}(-\vec{p}) \hat{a}(\vec{p}) e^{-2iE_p t} \right]$$

Consider, $I_1 = \int \frac{d^3p \vec{p}}{2} \hat{a}^\dagger(-\vec{p}) \hat{a}^\dagger(\vec{p}) e^{2iE_p t}$

Upon $\vec{p} \rightarrow -\vec{p}$, $I_1 = -I_1 \Rightarrow I_1 = 0$

$$\text{Similarly, } I_2 = \int \frac{d^3p}{2} \vec{p} \hat{a}(-\vec{p}) \hat{a}(\vec{p}) e^{-2iE_p t} = 0$$

$$\Rightarrow \hat{\vec{P}} = \int \frac{d^3p}{2} \vec{p} [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})]$$

$$\Rightarrow \boxed{\mathcal{N}[\hat{\vec{P}}] = \int d^3p \vec{p} [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})]}$$

Thus in general, if we define,

$\hat{P}^\mu = (\hat{H}, \hat{\vec{P}})$ as the four momentum operator & $p^\mu = (E_p, \vec{p})$ as the four momentum eigen value, then

$$\boxed{\hat{P}^\mu = \int \frac{d^3p}{2} p^\mu [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})]}$$

$$\& \boxed{\mathcal{N}[\hat{P}^\mu] = \int d^3p p^\mu [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})]}$$

$\hat{P}^\mu$: generator of space-time translations

Consider $[\hat{\phi}(x), \hat{P}^\mu]$ where

$$\hat{\phi}(x) = \int \frac{d^3q}{(2\pi)^{3/2} \sqrt{2E_q}} \left[\hat{a}^\dagger(\vec{q}) e^{+iq_\mu x^\mu} + \hat{a}(\vec{q}) e^{-iq_\mu x^\mu} \right]$$

$$\hat{P}^\mu = \int \frac{d^3p}{2} p^\mu \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) \right]$$

$$[\hat{\phi}(x), \hat{P}^\mu] = \int \frac{d^3q d^3p}{2(2\pi)^{3/2} \sqrt{2E_q}} p^\mu$$

$$\times \left[\begin{aligned} & [\hat{a}^\dagger(\vec{q}), \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})] e^{+iq_\mu x^\mu} \\ & + [\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})] e^{-iq_\mu x^\mu} \\ & + [\hat{a}^\dagger(\vec{q}), \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})] e^{+iq_\mu x^\mu} \\ & + [\hat{a}(\vec{q}), \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p})] e^{-iq_\mu x^\mu} \end{aligned} \right]$$

$$\rightarrow [\hat{\phi}(x), \hat{p}^\mu] = \int \frac{d^3q d^3p p^\mu}{2(2\pi)^{3/2} \sqrt{2E_q}}$$

$$\times \left[\begin{aligned} & -\hat{a}^\dagger(\vec{p}) \delta(\vec{q} - \vec{p}) e^{+iq_\mu x^\mu} \\ & \quad + \delta(\vec{q} - \vec{p}) \hat{a}(\vec{p}) e^{-iq_\mu x^\mu} \\ & -\delta(\vec{q} - \vec{p}) \hat{a}^\dagger(\vec{p}) e^{+iq_\mu x^\mu} \\ & \quad + \hat{a}(\vec{p}) \delta(\vec{q} - \vec{p}) e^{-iq_\mu x^\mu} \end{aligned} \right]$$

Perform the d^3q integral,

$$\rightarrow [\hat{\phi}(x), \hat{p}^\mu] = \int \frac{d^3p p^\mu}{2(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\times \left[\begin{aligned} & -\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} + \hat{a}(\vec{p}) e^{-ip_\mu x^\mu} - \hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} \\ & \quad + \hat{a}(\vec{p}) e^{-ip_\mu x^\mu} \end{aligned} \right]$$

$$\rightarrow [\hat{\phi}(x), \hat{p}^\mu] = \int \frac{d^3p p^\mu}{(2\pi)^{3/2} \sqrt{2E_p}} [\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} - \hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu}]$$

But the four momentum density operator is,

$$\hat{T}^\mu(x) = \frac{\int d^3p (ip^\mu) [\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} - \hat{a}(\vec{p}) e^{-ip_\mu x^\mu}]}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\rightarrow i\hat{T}^\mu(x) = \frac{\int d^3p p^\mu [\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} - \hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu}]}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\rightarrow [\hat{\phi}(x), \hat{P}^\mu] = i\hat{T}^\mu(x)$$

But for scalar field theory, we get

$$\underline{\hat{T}^\mu = \partial^\mu \hat{\phi}(x)}$$

$$\underline{\rightarrow [\hat{\phi}(x), \hat{P}_\mu] = i \partial_\mu \hat{\phi}(x)}$$

Thus $\hat{T} = e^{-ia_\mu \hat{P}^\mu}$ is the translational operator which gives

$$\underline{\hat{T}^{-1} \hat{\phi}(x^\mu) \hat{T} = \hat{\phi}(x^\mu + a^\mu)}$$

This is because $\hat{T}^{-1} \hat{\phi}(x^\mu) \hat{T}$ is

$$= e^{ia_\mu \hat{P}^\mu} \hat{\phi}(x^\mu) e^{-ia_\mu \hat{P}^\mu}$$

(By BCH formula)

$$= \hat{\phi}(x^\mu) + [ia_\mu \hat{P}^\mu, \hat{\phi}(x^\mu)] \\ + \frac{1}{2!} [ia_\mu \hat{P}^\mu, [ia_\mu \hat{P}^\mu, \hat{\phi}(x^\mu)]] + \dots$$

$$= \hat{\phi}(x^\mu) + a_\mu \partial^\mu \hat{\phi}(x) + \frac{1}{2!} [ia_\mu \hat{P}^\mu, a_\nu \partial^\nu \hat{\phi}(x)] \\ + \dots$$

$$= \hat{\phi}(x^\mu) + a_\mu \partial^\mu \hat{\phi}(x) + \frac{a_\mu a_\nu \partial^\mu \partial^\nu \hat{\phi}(x)}{2!} + \dots$$

$$= \hat{\phi}(x^\mu) + a^\mu \partial_\mu \hat{\phi}(x) + \frac{a^\mu a^\nu \partial_\mu \partial_\nu \hat{\phi}(x)}{2!} + \dots$$

$$\rightarrow e^{ia_\mu \hat{P}^\mu} \hat{\phi}(x^\mu) e^{-ia_\mu \hat{P}^\mu} = \hat{\phi}(x^\mu + a^\mu)$$

Thus, $\hat{P}^\mu$ is the generator of translation along x^μ direction.

Generator of Lorentz transform in scalar field theory

We know that the scalar field theory is Lorentz invariant. It implies that under L.T., the Lagrangian is symmetric. So, by using Noether's theorem we can get the conserved Noether charges and also relate them to the generators of these transformations. L.T. is given by,

$$x^\mu \xrightarrow{\text{L.T.}} \Lambda^\mu{}_\nu x^\nu$$

where $\Lambda^\mu{}_\nu$ satisfies $\eta^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta}$

Consider the infinitesimal L.T. given by,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \underbrace{\omega^\mu{}_\nu}_{\text{(some parameter)}}, \text{ where } \mathcal{O}(\omega^2) = 0$$

$$\rightarrow (\delta^\mu{}_\alpha - \omega^\mu{}_\alpha)(\delta^\nu{}_\beta - \omega^\nu{}_\beta)\eta^{\alpha\beta} = \eta^{\mu\nu}$$

$$\rightarrow \partial_{\alpha}^{\mu} \eta^{\nu\alpha} - \partial_{\alpha}^{\mu} \omega^{\nu\alpha} - \underbrace{\omega_{\alpha}^{\mu} \partial^{\nu\alpha}}_0 + \omega_{\alpha}^{\mu} \omega^{\nu\alpha} = \eta^{\mu\nu}$$

$$\rightarrow \eta^{\nu\mu} - \omega^{\nu\mu} - \omega^{\mu\nu} = \eta^{\mu\nu}$$

Since $\eta^{\mu\nu} = \eta^{\nu\mu}$, $\rightarrow \underline{\omega^{\mu\nu} = -\omega^{\nu\mu}}$

Thus the infinitesimal L.T. requires a six parametric anti-symmetric tensor given by ' $\omega^{\mu\nu}$ '. Out of these six parameters, three correspond to the Spatial rotation, and other three will correspond to Lorentz boosts.

$$\rightarrow \omega^{\mu\nu} = \begin{bmatrix} 0 & -\vartheta_1 & -\vartheta_2 & -\vartheta_3 \\ \vartheta_1 & 0 & \theta_3 & -\theta_2 \\ \vartheta_2 & -\theta_3 & 0 & \theta_1 \\ \vartheta_3 & \theta_2 & -\theta_1 & 0 \end{bmatrix}$$

$\vec{\theta}$ gives rotation axis & angle

$\vec{\vartheta}$ gives boost direction & rapidity

$$\rightarrow x^\mu \xrightarrow{L.T} (\delta^\mu_\nu - \omega^\mu_\nu) x^\nu$$

$$\rightarrow \underline{\delta x^\mu = -\omega^\mu_\nu x^\nu}$$

Let $\mathcal{L}(\phi, \partial_\mu \phi)$ be the lagrangian of the ϕ field which has eqⁿ of motion

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) = 0$$

The field $\phi(x)$ under L.T changes by

$$\delta \phi = (\partial_\mu \phi) \delta x^\mu = -(\partial_\mu \phi) \omega^\mu_\alpha x^\alpha$$

$$\rightarrow \delta \phi = -(\partial_\mu \phi) \omega^\mu_\alpha x^\alpha$$

$$\neq \underbrace{\delta(\partial_\nu \phi)}_{(\delta \partial = \partial \delta)} = \partial_\nu (\delta \phi) = -\omega^\mu_\alpha \partial_\nu [(\partial_\mu \phi) x^\alpha]$$

$$\rightarrow \delta(\partial_\nu \phi) = -\omega^\mu_\alpha [(\partial_\nu \partial_\mu \phi) x^\alpha + (\partial_\mu \phi) \delta^\alpha_\nu]$$

Under this transform the change in Lagrangian density is given by,

$$\delta \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) \delta \phi + \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta (\partial_\mu \phi)$$

$$\left[\text{denote } \pi^\mu = \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right]$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) (\partial_\mu \phi) (-\omega^\mu_\alpha) x^\alpha$$

$$+ \pi^\mu (-\omega^\nu_\alpha) [(\partial_\mu \partial_\nu \phi) x^\alpha + (\partial_\nu \phi) \delta^\alpha_\mu]$$

$$= \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} \right) (\partial_\nu \phi) + \pi^\mu (\partial_\mu \partial_\nu \phi) \right] (-\omega^\nu_\alpha) x^\alpha$$

$\underbrace{\hspace{10em}}_{\left(\frac{\partial \mathcal{L}}{\partial x^\nu} \right)} \quad + \pi^\mu (\partial_\nu \phi) (-\omega^\nu_\mu)$

$$= \left(\frac{\partial \mathcal{L}}{\partial x^\nu} \right) (-\omega^\nu_\alpha) x^\alpha + \pi^\mu (\partial_\nu \phi) (-\omega^\nu_\mu)$$

$$\Rightarrow \partial \mathcal{L} = -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha) + \underbrace{\mathcal{L} \omega^\nu_\alpha \partial^\alpha}_{0} + \Pi^\mu (\partial_\nu \phi) (-\omega^\nu_\mu)$$

$$= -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha) + \Pi^\mu (\partial^\nu \phi) (-\omega_{\nu\mu})$$

Since $\omega_{\nu\mu} = -\omega_{\mu\nu} \rightarrow \omega_{\nu\mu} = \frac{\omega_{\nu\mu} - \omega_{\mu\nu}}{2}$

$$\Rightarrow \partial \mathcal{L} = -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha) - \Pi^\mu (\partial^\nu \phi) \frac{(\omega_{\nu\mu} - \omega_{\mu\nu})}{2}$$

$$= -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha) - \frac{1}{2} \Pi^\mu (\partial^\nu \phi) \omega_{\nu\mu} + \frac{1}{2} \Pi^\nu (\partial^\mu \phi) \omega_{\nu\mu}$$

$$= -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha) + [\Pi^\nu (\partial^\mu \phi) - \Pi^\mu (\partial^\nu \phi)] \frac{\omega_{\nu\mu}}{2}$$

Now, $T^{\mu\nu} = \Pi^\mu (\partial^\nu \phi) - \eta^{\mu\nu} \mathcal{L}$

$$\rightarrow \Pi^\mu (\partial^\nu \phi) = T^{\mu\nu} + \eta^{\mu\nu} \mathcal{L}$$

$$\& \Pi^\mu (\partial^\nu \phi) - \Pi^\nu (\partial^\mu \phi) = T^{\mu\nu} - T^{\nu\mu}$$

$$\rightarrow \delta \mathcal{L} = -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha) + (T^{\mu\nu} - T^{\nu\mu}) \frac{\omega_{\mu\nu}}{2}$$

For a scalar field $T^{\mu\nu} = T^{\nu\mu}$.

But for multicomponent field ϕ^A , the energy-momentum tensor $T^{\mu\nu}$, is not symmetric. There, we can perform the complete $L \cdot T$ for both ϕ -indices & x^μ together. And then by the Bilinear method we get $T^{\mu\nu}$ symmetric.

For symmetric $T^{\mu\nu}$

$$\rightarrow \delta \mathcal{L} = -\partial_\nu (\mathcal{L} \omega^\nu_\alpha x^\alpha)$$

$$\text{But } \delta \mathcal{L} = \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} \right) \delta \phi + \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta (\partial_\mu \phi)}_{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) \delta \phi} \rightarrow \text{Eq. of motion}$$

$$\rightarrow \delta \mathcal{L} = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right]$$

$$(\text{use } \delta \phi = -(\partial_\mu \phi) \omega^\mu_\alpha x^\alpha)$$

$$\rightarrow \partial_\mu \left[\frac{\partial \mathcal{L}(\partial_\nu \phi)}{\partial (\partial_\mu \phi)} \omega^\nu_\alpha x^\alpha \right] = \partial_\mu (\mathcal{L} \omega^\mu_\alpha x^\alpha)$$

$$\rightarrow \partial_\mu \left[\frac{\partial \mathcal{L}(\partial_\nu \phi)}{\partial (\partial_\mu \phi)} \omega^\nu_\alpha x^\alpha - \mathcal{L} \omega^\mu_\alpha x^\alpha \right] = 0$$

$$\Rightarrow \partial_\mu \left[\left(\frac{\partial \mathcal{L}(\partial_\nu \phi)}{\partial (\partial_\mu \phi)} - \delta^\mu_\nu \mathcal{L} \right) x^\alpha \right] \omega^\nu_\alpha = 0$$

$$\Rightarrow \partial_\mu \left[\left(\frac{\partial \mathcal{L}(\partial_\nu \phi)}{\partial (\partial_\mu \phi)} - \eta^{\mu\nu} \mathcal{L} \right) x^\alpha \right] \omega_{\nu\alpha} = 0$$

$$\rightarrow \partial_\mu [T^{\mu\nu} x^\alpha] \omega_{\nu\alpha} = 0$$

Thus only the antisymmetric part of $T^{\mu\nu} x^\alpha$ with 2nd & 3rd indices vanishes.

$$\rightarrow \partial_\mu [T^{\mu\nu} x^\alpha] \frac{(\omega_{\nu\alpha} - \omega_{\alpha\nu})}{2} = 0$$

$$\rightarrow \partial_\mu [T^{\mu\nu} x^\alpha] \omega_{\nu\alpha} - \partial_\mu [T^{\mu\alpha} x^\nu] \omega_{\nu\alpha} = 0$$

$$\Rightarrow \left(\partial_\mu [T^{\mu\nu} x^\alpha - T^{\mu\alpha} x^\nu] \right) \omega_{\nu\alpha} = 0$$

$$\text{Let } M^{\mu\nu\alpha} = T^{\mu\nu} x^\alpha - T^{\mu\alpha} x^\nu$$

$$\Rightarrow (\partial_\mu M^{\mu\nu\alpha}) \omega_{\nu\alpha} = 0$$

But 'M' is ~~asymmetric~~ symmetric in 2nd & 3rd index

$$M^{\mu\alpha\nu} = -M^{\mu\nu\alpha}$$

$$\Rightarrow \underline{\partial_\mu M^{\mu\nu\alpha} = 0}$$

Thus 'M^{μν^α}' is the conserved Noether current corresponding to the L.T, given by,

$$M^{\mu\nu\alpha} = T^{\mu\nu} x^\alpha - T^{\mu\alpha} x^\nu$$

$$\text{where } \underline{\partial_\mu M^{\mu\nu\alpha} = 0}$$

Then the conserved Noether charges corresponding to L.T is given by,

$$L^{\mu\nu} = \int d^3x M^{0\mu\nu}, \quad \Rightarrow \underline{\frac{dL^{\mu\nu}}{dt} = 0}$$

$$\rightarrow L^{\mu\nu} = \int d^3x (T^{0\mu} x^\nu - T^{0\nu} x^\mu)$$

Observe, $L^{\mu\nu} = -L^{\nu\mu}$

In terms of Π 's, we have

$$T^{\alpha\beta} = \Pi^\alpha \Pi^\beta - \eta^{\alpha\beta} \mathcal{L}$$

$$\begin{aligned} \rightarrow L^{\mu\nu} &= \int d^3x (\Pi^0 \Pi^\mu - \delta^{0\mu} \mathcal{L}) x^\nu \\ &\quad - \int d^3x (\Pi^0 \Pi^\nu - \delta^{0\nu} \mathcal{L}) x^\mu \end{aligned}$$

$$\Rightarrow L^{\mu\nu} = \int d^3x \left[\Pi^0 (\Pi^\mu x^\nu - \Pi^\nu x^\mu) - \mathcal{L} (\delta^{0\mu} x^\nu - \delta^{0\nu} x^\mu) \right]$$

Consider L^{ij} (Spatial component)

$$\Rightarrow L^{ij} = \int d^3x \Pi^0 (\Pi^i x^j - \Pi^j x^i)$$

This generator corresponds to the angular momentum. Denoting

$$L^{12} \rightarrow L_x, L^{23} \rightarrow L_y, \& L^{31} \rightarrow L_z$$

$$\Rightarrow L^k = \int d^3x \pi^i x^j \varepsilon^{ijk}$$

$$\text{if } \vec{L} = (L_x, L_y, L_z) \& \vec{\pi} = (\pi_x, \pi_y, \pi_z)$$

$$\Rightarrow \boxed{\vec{L} = \int d^3x \pi^0 [\vec{\pi} \times \vec{x}]}$$

Then, consider L^{j0} components, they generate lorentz boosts,

$$\Rightarrow L^{j0} = \int d^3x \left[\pi^0 (\pi^j x^0 - \pi^0 x^j) - \mathcal{L} (\delta^{0j} x^0 - \delta^{00} x^j) \right]$$

$$\Rightarrow L^{j0} = \int d^3x \left(\pi^0 \pi^j x^0 - [(\pi^0)^2 - \mathcal{L}] x^j \right)$$

But $\mathcal{H} = (\Pi^0)^2 - \mathcal{L}$ & denoting $L^{j0} = K^j$ as generators of L. boosts,

$$\rightarrow \boxed{\vec{K} = \int d^3x (x^0 \Pi^0 \vec{\Pi} - \mathcal{H} \vec{x})}$$

Thus we can combine the $\vec{L}$ & $\vec{K}$ as

$$\rightarrow L^{\mu\nu} = \begin{bmatrix} 0 & -K_x & -K_y & -K_z \\ K_x & 0 & L_z & -L_y \\ K_y & -L_z & 0 & L_x \\ K_z & L_y & -L_x & 0 \end{bmatrix}$$

$\downarrow$
 (Generator of L.T.)

Note that these results are the same even in Second quantization picture. Thus the L.T. & its generators are operators which will give its own representation of $SO(1,3)$ in Scalar field theory.

$$\left. \begin{array}{l} \text{They transform} \\ \text{field operators} \\ \hat{\phi} \text{ \& } \hat{\Pi} \end{array} \right\} \begin{array}{l} L_{\mu\nu} \xrightarrow[2^{\text{nd}}]{\text{quantization}} \hat{L}_{\mu\nu} \\ \Lambda \xrightarrow[2^{\text{nd}}]{\text{quantization}} \hat{\Lambda} \end{array}$$

Representation of Lorentz transformations in scalar field theory

The generators of rotations are given by,

$$\Rightarrow \vec{L} = \int d^3x \pi^0 [\vec{\pi} \times \vec{x}]$$

In momentum space $\vec{x} = i \vec{\nabla}_{\vec{p}}$ (gradient w.r.t momentum) & $\vec{\pi}$ gives $\vec{p}$ factor. Thus, on going to $\vec{p}$ space,

$$\hat{\vec{L}} = i \int d^3p \hat{a}^\dagger(\vec{p}) (\vec{p} \times \vec{\nabla}_{\vec{p}}) \hat{a}(\vec{p})$$

& the generator of Lorentz boost is,

$$\hat{\vec{K}} = \int d^3x (x^0 \pi^0 \vec{\pi} - \vec{x} \hat{H})$$

$$\Rightarrow \hat{\vec{K}} = i \int d^3p \hat{a}^\dagger(\vec{p}) E_p \vec{\nabla}_{\vec{p}} \hat{a}(\vec{p})$$

The commutators b/w $\hat{L}^{\mu\nu}$ & $\hat{\phi}$ is given by,

$$[\hat{\phi}(x), \hat{L}^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \hat{\phi}(x)$$

Thus $\hat{L}^{\mu\nu}$ generates L.T on ' $\hat{\phi}$ '. The representation of lorentz transform " Λ^μ_ν " in the scalar field theory is given by,

$$\hat{\Lambda}([\Lambda^\mu_\nu]) = e^{\frac{-i\omega_{\mu\nu}}{2} \hat{L}^{\mu\nu}}$$

where $\omega_{\mu\nu}$ is the six parameteric anti-symmetric tensor that define the lorentz transform (3 rot. + 3 boost)

& $\Lambda^\mu_\nu(\omega) \rightarrow 4 \times 4$ lorentz matrix that transforms x^μ .

i.e. $x'^\mu = \Lambda^\mu_\nu x^\nu \rightarrow$ lorentz transform

& $\hat{\Lambda}([\Lambda^\mu_\nu])$ is the representation of L.T. as operator that acts on $\hat{\phi}$ & states.

Using,

$$[\hat{\phi}(x), \hat{L}^{\mu\nu}] = i(x^\mu \partial^\nu - x^\nu \partial^\mu) \hat{\phi}(x)$$

we can show that,

$$\underline{\rightarrow \hat{\Lambda}(\Lambda^\mu_\nu) \hat{\phi}(x) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu) = \hat{\phi}(\Lambda^\mu_\nu x^\nu)}$$

Now,

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3 (2E_p)} [\hat{\alpha}^\dagger(p^\mu) e^{+ipx} + \hat{\alpha}(p^\mu) e^{-ipx}]$$

$$\text{where } \hat{\alpha}^\dagger(p^\mu) = (2\pi)^{3/2} \sqrt{2E_p} \hat{a}^\dagger(\vec{p})$$

$|p^\mu\rangle = \hat{\alpha}^\dagger(p^\mu) |\Omega\rangle$ is the on shell one particle four vector state.

Then, $\hat{\Lambda}(\Lambda^\mu_\nu) \hat{\phi}(x) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu) = \hat{\phi}(\Lambda^\mu_\nu x^\nu)$ gives how $\hat{\alpha}^\dagger(p^\mu) \neq \hat{\alpha}(p^\mu)$ transforms under L.T.

$$\hat{\phi}(\Lambda x) = \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left[\hat{\alpha}^\dagger(p) e^{+ip(\Lambda x)} + \hat{\alpha}(p) e^{-ip(\Lambda x)} \right]$$

$$\text{Now } p(\Lambda x) = p_\nu (\Lambda^\nu_\mu x^\mu) = \Lambda_\nu^\mu p^\nu x_\mu \\ = (\Lambda^{-1} p) x$$

$$\rightarrow p(\Lambda x) = (\Lambda^{-1} p) x$$

$$\rightarrow \hat{\phi}(\Lambda x) = \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left[\hat{\alpha}^\dagger(p) e^{+i(\Lambda^{-1} p) x} + \hat{\alpha}(p) e^{-i(\Lambda^{-1} p) x} \right]$$

$$\text{Put } \Lambda^{-1} p = q \text{ (change of variable)}$$

$$\rightarrow \hat{\phi}(\Lambda x) = \int \underbrace{\frac{d^3 q}{(2\pi)^3 (2E_q)}}_{\text{Lorentz invariant measure}} \left[\hat{\alpha}^\dagger(\Lambda q) e^{+ipx} + \hat{\alpha}(\Lambda q) e^{-iqx} \right]$$

$$\text{But } \hat{\phi}(\Lambda x) = \hat{\Lambda}(\Lambda^\mu_\nu) \hat{\phi}(x) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu)$$

$$\Rightarrow \hat{\Lambda}(\Lambda^\mu_\nu) \hat{\alpha}^\dagger(p^\mu) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu) = \hat{\alpha}^\dagger(\Lambda^\mu_\nu p^\nu)$$

$$\Rightarrow \hat{\Lambda}(\Lambda^\mu_\nu) \hat{\alpha}(p^\mu) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu) = \hat{\alpha}(\Lambda^\mu_\nu p^\nu)$$

Consider,

$$\hat{\Lambda}(\Lambda^\mu_\nu) \hat{\alpha}^\dagger(p^\mu) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu) = \hat{\alpha}^\dagger(\Lambda^\mu_\nu p^\nu)$$

Act it on the Ground state $|\Omega\rangle$

$$\rightarrow \hat{\Lambda}(\Lambda^\mu_\nu) \hat{\alpha}^\dagger(p^\mu) \hat{\Lambda}^\dagger(\Lambda^\mu_\nu) |\Omega\rangle = \hat{\alpha}^\dagger(\Lambda^\mu_\nu p^\nu) |\Omega\rangle$$

(Provided $|\Omega\rangle$ is Lorentz invariant)

$$\rightarrow \hat{\Lambda}(\Lambda^\mu_\nu) \hat{\alpha}^\dagger(p^\mu) |\Omega\rangle = \hat{\alpha}^\dagger(\Lambda^\mu_\nu p^\nu) |\Omega\rangle$$

$$\rightarrow \underline{\underline{\hat{\Lambda}(\Lambda^\mu_\nu) |p^\mu\rangle = |\Lambda^\mu_\nu p^\nu\rangle}}$$

Thus the four vector state $\hat{\alpha}^\dagger(p^\mu) |\Omega\rangle$ transform as a Spin zero particle. There is no need of any Spin matrices needed to describe completely the total L.T & as well as the translation transformation.

' $\hat{\alpha}^\dagger(p^\mu)$, $\hat{\alpha}(p^\mu)$ & $\hat{\phi}(x)$ ' transform as as a scalar under L.T. Thus they describe Spin 0 and chargeless particles.

Finally, it can be shown that, $\hat{P}^\mu$ & $\hat{L}^{\mu\nu}$ together generates the full Poincare algebra of Scalar field Theory.

They satisfy the following commutations.

- $[\hat{P}_\mu, \hat{P}_\nu] = 0$

- $[\hat{L}_{\mu\nu}, \hat{L}_{\sigma\rho}] = -i \left(\eta_{\mu\rho} \hat{L}_{\nu\sigma} + \eta_{\nu\sigma} \hat{L}_{\mu\rho} - \eta_{\nu\rho} \hat{L}_{\mu\sigma} - \eta_{\mu\sigma} \hat{L}_{\nu\rho} \right)$

- $[\hat{P}_\rho, \hat{L}_{\mu\nu}] = i (\eta_{\rho\mu} \hat{P}_\nu - \eta_{\rho\nu} \hat{P}_\mu)$

Thus we get the Poincare algebra repⁿ in Scalar field Theory in terms of the ladder operators.

This repⁿ of Poincare group results in the description of Spin 0 (Scalar) & chargeless particles.

Quantization of Complex Scalar Field

The lagrangian is given by,

$$\mathcal{L}_{\text{complex}} = (\partial^\mu \psi^\dagger)(\partial_\mu \psi) - m^2 \psi^\dagger \psi$$

where $\psi^\dagger \neq \psi$ (non Hermitian field)

Then the equation of motion becomes,

$$(\partial_\mu \partial^\mu + m^2)\psi = 0 \quad \& \quad (\partial_\mu \partial^\mu + m^2)\psi^\dagger = 0$$

The dispersion relation for both ψ and $\psi^\dagger$ field is given by,

$$p_\mu p^\mu = m^2 \Rightarrow (p^0)^2 = |\vec{p}|^2 + m^2$$

Since $\psi \neq \psi^\dagger$ we can decompose them to real & imaginary parts.

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad \& \quad \psi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

where ϕ_1 & ϕ_2 are real fields.

$$\Rightarrow \mathcal{L}_{\text{complex}} = \frac{1}{2} (\partial^\mu \phi_1) (\partial_\mu \phi_1) + \frac{1}{2} (\partial^\mu \phi_2) (\partial_\mu \phi_2) - \frac{1}{2} m^2 \phi_1^2 - \frac{1}{2} m^2 \phi_2^2$$

$$\text{Or } \mathcal{L}_{\text{complex}} = \sum_{A=\{1,2\}} \frac{1}{2} (\partial^\mu \phi_A) (\partial_\mu \phi_A) - \frac{1}{2} m^2 \phi_A^2$$

This is the sum of two different scalar field lagrangians corresponding to real fields $\phi_1(x)$ & $\phi_2(x)$ which are non-interacting & are independent.

$$\Rightarrow \underline{\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2}$$

The equations of motions for $\{\phi_A\}$ are $(\partial_\mu \partial^\mu + m^2) \phi_1 = 0$ & $(\partial_\mu \partial^\mu + m^2) \phi_2 = 0$ which are 2 independent Klein-Gordon eq;

The dispersion relation for both $\{\phi_A\}$ is $p_\mu p^\mu = m^2 \Rightarrow (p^0)^2 = |\vec{p}|^2 + m^2 = E_p$ where we interpret $p^0 = E_p$ as the total energy of the particles described by complex scalar field.

Mode expansion of Complex Scalar field

Similarly how we wrote the mode expansion of a single scalar field we can write for $\hat{\phi}_1 \neq \hat{\phi}_2$ as,

$$\hat{\phi}_A(x) = \frac{\int d^3p [\hat{a}_A(\vec{p}) e^{-ip_\mu x^\mu} + \hat{a}_A^\dagger(\vec{p}) e^{+ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

where $\{\hat{a}_1, \hat{a}_2\}$ are two annihilation operators & $\{\hat{a}_1^\dagger, \hat{a}_2^\dagger\}$ are creation operators of independent scalar fields.

Note: Subscript $(A, B, C, \dots) = \{1, 2\}$ is the field index.

The canonical momentum density is given by, $\Pi_A^\circ = \frac{\partial \mathcal{L}}{\partial(\partial_\circ \phi_A)} = \partial^\circ \phi_A$

$$\hat{\Pi}_A^\circ(x) = \frac{\int d^3p (iE_p) [\hat{a}_A^\dagger(\vec{p}) e^{+ip_\mu x^\mu} - \hat{a}_A(\vec{p}) e^{-ip_\mu x^\mu}]}{(2\pi)^{3/2} \sqrt{(2E_p)}}$$

The expression for $\{\hat{a}_A(\vec{p})\}$ & $\{\hat{a}_A^\dagger(\vec{p})\}$ in terms of $\{\phi_A\}$ & $\{\hat{\Pi}_A^\circ(x)\}$ is same as in free scalar field theory.

$$\hat{a}_A^\dagger(\vec{p}) = \int \frac{d^3x \left[E_p \hat{\phi}_A(x) - i \hat{\Pi}_A^\circ(x) \right] e^{-i p_\mu x^\mu}}{(2\pi)^{3/2} \sqrt{2E_p}}$$

$$\hat{a}_A(\vec{p}) = \int \frac{d^3x \left[E_p \hat{\phi}_A(x) + i \hat{\Pi}_A^\circ(x) \right] e^{i p_\mu x^\mu}}{(2\pi)^{3/2} \sqrt{2E_p}}$$

Commutation relations

We impose the usual Equal time commutation relations for $\{\phi_A, \hat{\Pi}_A^\circ(x)\}$:

$$\left[\hat{\phi}_A(x), \hat{\phi}_B(y) \right]_{(x^0=y^0)} = \left[\hat{\Pi}_A^\circ(x), \hat{\Pi}_B^\circ(y) \right]_{(x^0=y^0)} = 0$$

$$\left[\hat{\phi}_A(x), \hat{\Pi}_B^\circ(y) \right]_{(x^0=y^0)} = \delta_{AB} i \delta(\vec{x} - \vec{y})$$

This results in bosonic commutation relations between the ladder operators

$$[\hat{a}_A(\vec{p}), \hat{a}_B(\vec{q})] = [\hat{a}_A^+(\vec{p}), \hat{a}_B^+(\vec{q})] = 0$$

$$[\hat{a}_A(\vec{p}), \hat{a}_B^+(\vec{q})] = \delta_{AB} \delta(\vec{p} - \vec{q})$$

Quantized excitations & particle picture

The Hamiltonian density for complex field is given by,

$$\hat{H}_{\text{complex}} = \hat{H}_1 + \hat{H}_2$$

So in terms of ladder operators,

$$\hat{H}_{\text{complex}} = \int \frac{d^3p E_p}{2} \sum_{A=\{1,2\}} [\hat{a}_A^+(\vec{p}) \hat{a}_A(\vec{p}) + \hat{a}_A(\vec{p}) \hat{a}_A^+(\vec{p})]$$

&

$$\mathcal{N}[\hat{H}_{\text{complex}}] = \int d^3p E_p [\hat{a}_1^+(\vec{p}) \hat{a}_1(\vec{p}) + \hat{a}_2^+(\vec{p}) \hat{a}_2(\vec{p})]$$

Thus we see that there are two types of particles in our theory.

$\hat{a}_1^\dagger(\vec{p}) \rightarrow$ Creates particle of type 1

$\hat{a}_2^\dagger(\vec{p}) \rightarrow$ Creates particle of type 2

$\hat{a}_1(\vec{p}) \rightarrow$ Annihilates particle of type 1

$\hat{a}_2(\vec{p}) \rightarrow$ Annihilates particle of type 2

Since the creation & annihilation operators of particle type 1 & 2 obey bosonic commutation relations, the particle type 1 & 2 are bosons.

Consider an orthonormal combination of the ladder operators defined by,

$$\hat{a}(\vec{p}) = \frac{1}{\sqrt{2}} [\hat{a}_1(\vec{p}) + i\hat{a}_2(\vec{p})]$$

$$\hat{b}(\vec{p}) = \frac{1}{\sqrt{2}} [\hat{a}_1(\vec{p}) - i\hat{a}_2(\vec{p})]$$

$$\hat{a}^+(\vec{p}) = \frac{1}{\sqrt{2}} [\hat{a}_1^+(\vec{p}) - i\hat{a}_2^+(\vec{p})]$$

$$\hat{b}^+(\vec{p}) = \frac{1}{\sqrt{2}} [\hat{a}_1^+(\vec{p}) + i\hat{a}_2^+(\vec{p})]$$

This is a unitary transformation on bosonic operators $\{\hat{a}_1(\vec{p}), \hat{a}_2(\vec{p})\}$, thus the operators $\{\hat{a}(\vec{p}), \hat{b}(\vec{p})\}$ also follow the bosonic commutation relations.

$$\Rightarrow [\hat{a}(\vec{p}), \hat{a}^+(\vec{q})] = \delta(\vec{p} - \vec{q})$$

$$\& [\hat{b}(\vec{p}), \hat{b}^+(\vec{q})] = \delta(\vec{p} - \vec{q})$$

and all other combinations are zero.

We can write $\{\hat{a}_A(\vec{p})\}$ in terms of $\hat{a}$ & $\hat{b}$ as:

$$\hat{a}_1(\vec{p}) = \frac{1}{\sqrt{2}} [\hat{a}(\vec{p}) + \hat{b}(\vec{p})]$$

$$\hat{a}_2(\vec{p}) = \frac{-i}{\sqrt{2}} [\hat{a}(\vec{p}) - \hat{b}(\vec{p})]$$

$$\& \hat{a}_1^+(\vec{p}) = \frac{1}{\sqrt{2}} [\hat{a}^+(\vec{p}) + \hat{b}^+(\vec{p})]$$

$$\hat{a}_2^+(\vec{p}) = \frac{i}{\sqrt{2}} [\hat{a}^+(\vec{p}) - \hat{b}^+(\vec{p})]$$

$$\text{Then, } \hat{\psi}(x) = \frac{1}{\sqrt{2}} [\hat{\phi}_1(x) + i\hat{\phi}_2(x)]$$

$$\& \hat{\psi}^\dagger(x) = \frac{1}{\sqrt{2}} [\hat{\phi}_1(x) - i\hat{\phi}_2(x)]$$

On substituting the mode expansion of $\hat{\phi}_1(x)$ & $\hat{\phi}_1(x)^\dagger$ we get

$$\hat{\psi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \left(\frac{[\hat{a}_1(\vec{p}) + i\hat{a}_2(\vec{p})]}{\sqrt{2}} e^{-ik_\mu x^\mu} + \frac{[\hat{a}_1^+(\vec{p}) + i\hat{a}_2^+(\vec{p})]}{\sqrt{2}} e^{+ik_\mu x^\mu} \right)$$

Then using defⁿ of $\hat{a}(\vec{p})$ & $\hat{b}^+(\vec{p})$

$$\boxed{\hat{\psi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} [\hat{a}(\vec{p}) e^{-ik_\mu x^\mu} + \hat{b}^+(\vec{p}) e^{+ik_\mu x^\mu}]}$$

and similarly for $\hat{\psi}^+(x)$ we get

$$\hat{\psi}^+(x) = \frac{\int d^3p [\hat{a}^+(\vec{p}) e^{+ip_\mu x^\mu} + \hat{b}(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

Then $\Pi_\psi^0(x)$ & $\Pi_{\psi^+}^0(x)$ are the canonical momenta of $\hat{\psi}(x)$ & $\hat{\psi}^+(x)$ respectively.

$$\Rightarrow \Pi_\psi^0(x) = \partial_0 \psi^+(x) \text{ \& } \Pi_{\psi^+}^0(x) = \partial_0 \psi(x)$$

$$\Pi_\psi^0(x) = \frac{\int d^3p (iE_p) [\hat{a}^+(\vec{p}) e^{+ip_\mu x^\mu} - \hat{b}(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

&

$$\Pi_{\psi^+}^0(x) = \frac{\int d^3p (iE_p) [\hat{b}^+(\vec{p}) e^{+ip_\mu x^\mu} - \hat{a}(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

Thus we finally got the mode expansion for even the complex fields $\psi(x)$ and $\psi^+(x)$.

The Hamiltonian density in terms of ψ & $\psi^\dagger$ is given by,

$$\hat{\mathcal{H}} = \pi_\psi^\circ (\partial_0 \hat{\psi}) + \pi_{\psi^\dagger}^\circ (\partial_0 \hat{\psi}^\dagger) - \mathcal{L}$$

$$= \pi_\psi^\circ \pi_{\psi^\dagger}^\circ + \pi_{\psi^\dagger}^\circ \pi_\psi^\circ - \pi_\psi^\circ \pi_{\psi^\dagger}^\circ + (\vec{\nabla} \psi^\dagger) \cdot (\vec{\nabla} \psi) + m^2 \psi^\dagger \psi$$

$$\Rightarrow \underline{\mathcal{H} = \pi_{\psi^\dagger}^\circ \pi_\psi^\circ + (\vec{\nabla} \psi^\dagger) \cdot (\vec{\nabla} \psi) + m^2 \psi^\dagger \psi}$$

Note that the commutators for the fields $\hat{\psi}(x)$, $\hat{\psi}^\dagger(x)$, $\pi_\psi^\circ(x)$, $\pi_{\psi^\dagger}^\circ(x)$ are

$$[\hat{\psi}(x), \hat{\psi}^\dagger(y)] = [\pi_\psi^\circ(x), \pi_{\psi^\dagger}^\circ(y)] = 0$$

$$[\hat{\psi}(x), \pi_{\psi^\dagger}^\circ(y)] = [\hat{\psi}^\dagger(x), \pi_\psi^\circ(y)] = 0$$

$$[\hat{\psi}(x), \pi_\psi^\circ(y)] = [\hat{\psi}^\dagger(x), \pi_{\psi^\dagger}^\circ(y)] = i\delta(\vec{x} - \vec{y})$$

$(x^0 = y^0)$
 $(x^0 = y^0)$

The hamiltonian in terms of the momentum modes of $\hat{\psi}$ and $\hat{\psi}^\dagger$ is given by

$$\hat{H}_{\text{complex}} = \int \frac{d^3p E_p}{2} \sum_{A=\{1,2\}} \left[\hat{a}_A^\dagger(\vec{p}) \hat{a}_A(\vec{p}) + \hat{a}_A(\vec{p}) \hat{a}_A^\dagger(\vec{p}) \right]$$

Substitute $\{\hat{a}_A(\vec{p})\}$ in terms of the operators $\hat{a}(\vec{p})$ & $\hat{b}(\vec{p})$.

$$\hat{a}_1^\dagger \hat{a}_1 = \frac{1}{2} [\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + \hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b}]$$

$$\hat{a}_2^\dagger \hat{a}_2 = \frac{1}{2} [\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}]$$

$$\hat{a}_1 \hat{a}_1^\dagger = \frac{1}{2} [\hat{a} \hat{a}^\dagger + \hat{b} \hat{b}^\dagger + \hat{b} \hat{a}^\dagger + \hat{a} \hat{b}^\dagger]$$

$$\hat{a}_2 \hat{a}_2^\dagger = \frac{1}{2} [\hat{a} \hat{a}^\dagger + \hat{b} \hat{b}^\dagger - \hat{b} \hat{a}^\dagger - \hat{a} \hat{b}^\dagger]$$

$$\rightarrow \hat{H}_{\text{complex}} = \int \frac{d^3p E_p}{2} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right]$$

Thus we see that even in the representation of $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$ we see two kinds of particles of type-a & type-b.

$$\mathcal{N}[\hat{H}_{\text{complex}}] = \int d^3p E_p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p})]$$

$\hat{a}^\dagger(\vec{p}) \rightarrow$ Creates particle of type-a

$\hat{b}^\dagger(\vec{p}) \rightarrow$ Creates particle of type-b

$\hat{a}(\vec{p}) \rightarrow$ Annihilates particle of type-a

$\hat{b}(\vec{p}) \rightarrow$ Annihilates particle of type-b

Type 'a' & 'b' at a particular momentum mode $\vec{p}$ have same mass 'm' & energy E_p .

Since the ladder operators obey the bosonic commutation relation we can say that type-a & type-b particles are bosons.

U(1) Symmetry & Conserved quantities

The lagrangian has a global U(1) Symmetry. i.e. if $\psi \rightarrow e^{-i\alpha}\psi$,
& $\psi^\dagger \rightarrow e^{+i\alpha}\psi^\dagger$

$$\text{But } \mathcal{L} \rightarrow \mathcal{L} \Rightarrow \delta\mathcal{L} = 0.$$

Thus there is a conserved quantity according to Noether's theorem.

Consider the infinitesimal global change of phase by an amount ' $\delta\alpha$ '.

$$\begin{aligned} \Rightarrow \psi &\rightarrow (1 - i\delta\alpha)\psi \Rightarrow \delta\psi = -i\psi\delta\alpha \\ &\& \psi^\dagger \rightarrow (1 + i\delta\alpha)\psi^\dagger \Rightarrow \delta\psi^\dagger = +i\psi^\dagger\delta\alpha \end{aligned}$$

$\Rightarrow$ The conserved noether current is given by $J_N^\mu = \sum_{A=\{\psi, \psi^\dagger\}} \pi_A^\mu \delta\psi_A$,
with $\delta\alpha = c$.

$$\Rightarrow \partial_\mu \hat{J}_N^\mu = 0$$

where, $\hat{J}_N^\mu = \hat{\Pi}_\psi^\mu (-ie\hat{\psi}) + \hat{\Pi}_{\psi^\dagger}^\mu (ie\hat{\psi}^\dagger)$

But $\hat{\Pi}_\psi^\mu = \partial^\mu \hat{\psi}^\dagger \neq \hat{\Pi}_{\psi^\dagger}^\mu = \partial^\mu \hat{\psi}$

$$\Rightarrow \hat{J}^\mu = -ie \left[(\partial^\mu \hat{\psi}^\dagger) \hat{\psi} - (\partial^\mu \hat{\psi}) \hat{\psi}^\dagger \right]$$

$e \rightarrow$ appears as the phase of the infinitesimal $U(1)$ transformation

This is the form of noether current operator which is conserved: $\partial_\mu \hat{J}_N^\mu = 0$

Now we can find $\int d^3x \hat{J}_N^\mu$ in terms of ladder operators.

$$\hat{\psi}(x) = \frac{\int d^3p \left[\hat{a}(\vec{p}) e^{-ik_\mu x^\mu} + \hat{b}^\dagger(\vec{p}) e^{+ik_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\psi}^\dagger(x) = \frac{\int d^3p \left[\hat{a}^\dagger(\vec{p}) e^{+ik_\mu x^\mu} + \hat{b}(\vec{p}) e^{-ik_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\Pi}_\psi^\mu = \frac{\int d^3p (i p^\mu) [\hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu} - \hat{b}(\vec{p}) e^{-i p_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\Pi_{\psi^\dagger}^\mu(x) = \frac{\int d^3p (i p^\mu) [\hat{b}^\dagger(\vec{p}) e^{+i p_\mu x^\mu} - \hat{a}(\vec{p}) e^{-i p_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$\Rightarrow \int d^3x i(\partial^\mu \hat{\psi}^\dagger) \hat{\psi}$ is given by

$$\begin{aligned} & \int \frac{d^3x d^3p d^3q (-p^\mu)}{(2\pi)^3 2\sqrt{E_p E_q}} \left([\hat{a}^\dagger(\vec{p}) e^{+i p_\mu x^\mu} - \hat{b}(\vec{p}) e^{-i p_\mu x^\mu}] \right. \\ & \quad \left. \times [\hat{a}(\vec{q}) e^{-i q_\mu x^\mu} + \hat{b}^\dagger(\vec{q}) e^{+i q_\mu x^\mu}] \right) \\ &= \int \frac{d^3x d^3p d^3q (-p^\mu)}{(2\pi)^3 2\sqrt{E_p E_q}} \left(\begin{aligned} & \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{q}) e^{+i(p_\mu - q_\mu)x^\mu} \\ & - \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{q}) e^{-i(p_\mu - q_\mu)x^\mu} \\ & + \hat{a}^\dagger(\vec{p}) \hat{b}^\dagger(\vec{q}) e^{+i(p_\mu + q_\mu)x^\mu} \\ & - \hat{b}(\vec{p}) \hat{a}(\vec{q}) e^{-i(p_\mu + q_\mu)x^\mu} \end{aligned} \right) \end{aligned}$$

Performing integration over d^3x we get dirac delta functions & then doing the d^3q integral we get,

$$\int d^3x i(\partial^\mu \hat{\psi}^+) \psi$$

$$= \int \frac{d^3p (p^\mu)}{2E_p} \left[\hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) - \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) \right. \\ \left. + \hat{b}(\vec{p}) \hat{a}(-\vec{p}) e^{-2iE_p t} \right. \\ \left. - \hat{a}^\dagger(\vec{p}) \hat{b}^\dagger(-\vec{p}) e^{2iE_p t} \right]$$

Similarly $\int d^3x (-i\partial^\mu \hat{\psi}) \hat{\psi}^+$ is given by

$$\int \frac{d^3x d^3p d^3q (p^\mu)}{(2\pi)^3 2\sqrt{E_p E_q}} \left(\left[\hat{b}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} - \hat{a}(\vec{p}) e^{-ip_\mu x^\mu} \right] \right. \\ \left. \times \left[\hat{a}^\dagger(\vec{q}) e^{+iq_\mu x^\mu} + \hat{b}(\vec{q}) e^{-iq_\mu x^\mu} \right] \right)$$

$$= \int \frac{d^3x d^3p d^3q (p^\mu)}{(2\pi)^3 2\sqrt{E_p E_q}} \left(\hat{b}^\dagger(\vec{p}) \hat{b}(\vec{q}) e^{+i(p_\mu - q_\mu) x^\mu} \right. \\ \left. - \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{q}) e^{-i(p_\mu - q_\mu) x^\mu} \right. \\ \left. + \hat{b}^\dagger(\vec{p}) \hat{a}^\dagger(\vec{q}) e^{+i(p_\mu + q_\mu) x^\mu} \right. \\ \left. - \hat{a}(\vec{p}) \hat{b}(\vec{q}) e^{-i(p_\mu + q_\mu) x^\mu} \right)$$

$$= \int \frac{d^3p (p^\mu)}{2E_p} \left(\hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) - \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) \right. \\ \left. + \hat{b}^\dagger(\vec{p}) \hat{a}^\dagger(-\vec{p}) e^{2iE_p t} \right. \\ \left. - \hat{a}(\vec{p}) \hat{b}(-\vec{p}) e^{-2iE_p t} \right)$$

$\Rightarrow \int d^3x J_N^\mu$ is given by,

$$= \int \frac{d^3p (e p^\mu)}{2E_p} \left[\begin{aligned} & \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) - \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \\ & + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) - \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \\ & - [\hat{b}^\dagger(\vec{p}) \hat{a}^\dagger(-\vec{p}) - \hat{a}^\dagger(\vec{p}) \hat{b}^\dagger(-\vec{p})] e^{2iE_p t} \\ & + [\hat{a}(\vec{p}) \hat{b}(-\vec{p}) - \hat{b}(\vec{p}) \hat{a}(-\vec{p})] e^{-2iE_p t} \end{aligned} \right]$$

Consider $\int d^3x \vec{J}_N$

$$= \int \frac{d^3p (e \vec{p})}{2E_p} \left[\begin{aligned} & \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) - \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \\ & + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) - \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \\ & - [\hat{b}^\dagger(\vec{p}) \hat{a}^\dagger(-\vec{p}) - \hat{a}^\dagger(\vec{p}) \hat{b}^\dagger(-\vec{p})] e^{2iE_p t} \\ & + [\hat{a}(\vec{p}) \hat{b}(-\vec{p}) - \hat{b}(\vec{p}) \hat{a}(-\vec{p})] e^{-2iE_p t} \end{aligned} \right]$$

Now Consider the first and second term under $\vec{p} \rightarrow -\vec{p}$, they are odd under this, Hence their integration = 0

But 3rd & 4th term are even under $\vec{p} \rightarrow -\vec{p}$.

$\Rightarrow \int d^3x \vec{J}_N$ is given by,

$$= \int \frac{d^3p (e^{\vec{p}})}{2E_p} \left[\left[\hat{a}^\dagger(\vec{p}) \hat{b}^\dagger(-\vec{p}) - \hat{b}^\dagger(\vec{p}) \hat{a}^\dagger(-\vec{p}) \right] e^{2iE_p t} + \left[\hat{a}(\vec{p}) \hat{b}(-\vec{p}) - \hat{b}(\vec{p}) \hat{a}(-\vec{p}) \right] e^{-2iE_p t} \right]$$

which is not diagonal in fock basis.

But the Noether charge, given by,

$$\hat{Q} = \int d^3x \mathcal{J}_N^0 = -ie \int d^3x \left[(\partial^0 \hat{\psi}^\dagger) \hat{\psi} - (\partial^0 \hat{\psi}) \hat{\psi}^\dagger \right]$$

$$= e \int \frac{d^3p}{2} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) - \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) - \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \left[\hat{a}^\dagger(\vec{p}) \hat{b}^\dagger(-\vec{p}) - \hat{b}^\dagger(\vec{p}) \hat{a}^\dagger(-\vec{p}) \right] e^{2iE_p t} + \left[\hat{a}(\vec{p}) \hat{b}(-\vec{p}) - \hat{b}(\vec{p}) \hat{a}(-\vec{p}) \right] e^{-2iE_p t} \right]$$

Now Consider the first and second term under $\vec{p} \rightarrow -\vec{p}$, they are even under this transform. But 3rd & 4th term are odd under $\vec{p} \rightarrow -\vec{p}$.

$$\Rightarrow \hat{Q} = e \int \frac{d^3 p}{2} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) \right] - \left[\hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \right]$$

This is a conserved quantity unlike $\int d^3 x \vec{T}$. Thus it commutes with Hamiltonian operator & hence it is diagonal in fock basis.

$$\Rightarrow \boxed{\mathcal{N}[\hat{Q}] = e \int d^3 p [\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) - \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p})]}$$

Let $N_a(\vec{p}) = \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p})$ & $N_b(\vec{p}) = \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p})$ be the number operators of 'a' & 'b' type particles in a particular $\vec{p}$. Then the total number operators are,

$$\hat{N}_a = \int d^3 p \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) \quad \& \quad \hat{N}_b = \int d^3 p \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p})$$

⇒ Normal ordered Charge operator
$$N[\hat{Q}] = e\hat{N}_a - e\hat{N}_b$$

Thus total charge is given by the difference b/w the total a-type charge & b-type charge, which is a conserved quantity.

Particle - Antiparticle Pair

So 'a' & 'b' particles have opposite type of charges. If a-particles have +ve charge, then b-particles have -ve charge, but have the same mass. Now, observe,

$$\hat{\psi}^+(x) = \frac{\int d^3p [\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} + \hat{b}(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

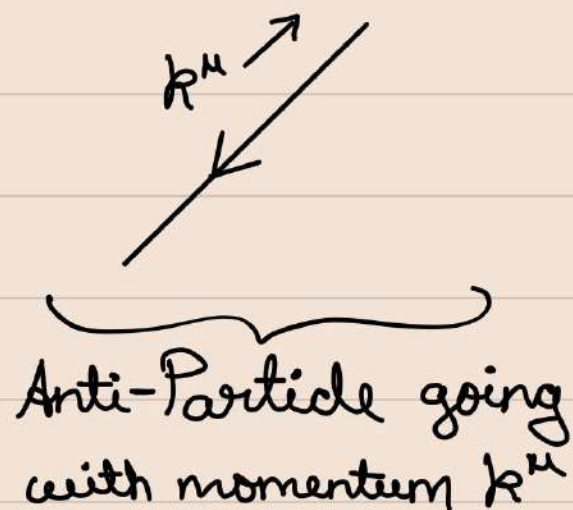
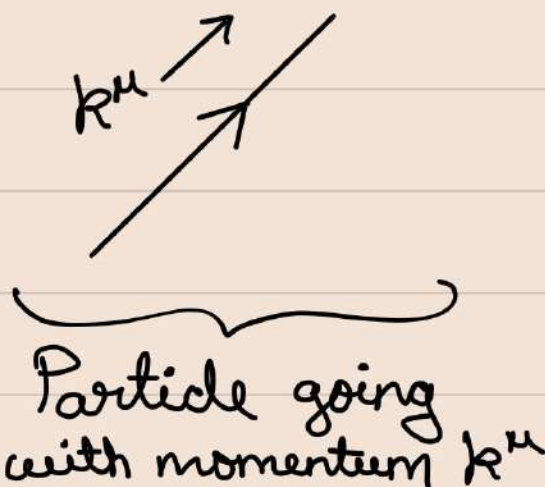
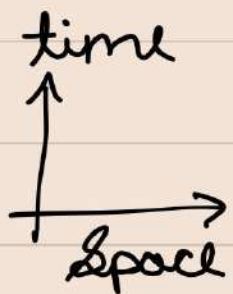
&

$$\hat{\psi}(x) = \frac{\int d^3p [\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{b}^\dagger(\vec{p}) e^{+ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$\hat{\psi}^+$ creates an a-type particle which is outgoing and annihilates b-type which is incoming.

Similarly, $\hat{\psi}$ creates b-type particle which is outgoing and annihilates a-type which is incoming.

By Feynmann interpretation, 'a' is particle with +ve energy, moving forward in time. And 'b' has +ve energy, but a time reversed state of the a-type particle & thus b-type particle is considered as the anti-particle of 'a'.



Energy Momentum Tensor of Complex Scalar field

$$T^{\mu\nu} = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right] (\partial^\nu \psi) + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^\dagger)} \right] (\partial^\nu \psi^\dagger) - \eta^{\mu\nu} \mathcal{L}$$

$$\rightarrow \hat{T}^{\mu\nu} = (\partial^\mu \hat{\psi}^\dagger)(\partial^\nu \hat{\psi}) + (\partial^\mu \hat{\psi})(\partial^\nu \hat{\psi}^\dagger) - \eta^{\mu\nu} \hat{\mathcal{L}}$$

The four momentum operator is given by,

$$\hat{P}^\mu = \int d^3x \hat{T}^{0\mu}$$

$\hat{P}^0 = \hat{H}$ is the Hamiltonian operator
complex

$$\rightarrow \hat{H}_{\text{complex}} = \int \frac{d^3p E_p}{2} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right]$$

$$\hat{P}^i = \int d^3x \hat{T}^{0i} = \int d^3x \left[(\partial^0 \hat{\psi}^\dagger)(\partial^i \hat{\psi}) + (\partial^0 \hat{\psi})(\partial^i \hat{\psi}^\dagger) \right]$$

$$\text{But } \hat{\Pi}_\psi = \partial^0 \hat{\psi}^\dagger \quad \& \quad \hat{\Pi}_{\psi^\dagger} = \partial^0 \hat{\psi}$$

$$\& \quad \partial^i = -\partial_i$$

$$\rightarrow \hat{\vec{P}} = -\int d^3x \left[\hat{\Pi}_\psi \vec{\nabla} \psi + \hat{\Pi}_{\psi^\dagger} \vec{\nabla} \psi^\dagger \right]$$

We will get it to be,

$$\Rightarrow \hat{\vec{P}} = \int \frac{d^3p}{2} \vec{p} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right]$$

$$\Rightarrow \boxed{\mathcal{N}[\hat{\vec{P}}] = \int d^3p \vec{p} \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \right]}$$

Thus in general, if we define,

$\hat{P}^\mu = (\hat{H}_{\text{complex}}, \hat{\vec{P}})$ as the four momentum operator & $p^\mu = (E_p, \vec{p})$ as the four momentum eigen value, then

$$\boxed{\hat{P}^\mu = \int \frac{d^3p}{2} p^\mu \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right]}$$

$$\& \boxed{\mathcal{N}[\hat{P}^\mu] = \int d^3p p^\mu \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) \right]}$$

$\hat{P}^\mu$: generator of space-time translations

Consider $[\hat{\psi}(x), \hat{P}^\mu]$ where

$$\hat{\psi}(x) = \int \frac{d^3q}{(2\pi)^{3/2} \sqrt{2E_q}} \left[\hat{a}(\vec{q}) e^{-iqx} + \hat{b}^\dagger(\vec{q}) e^{+iqx} \right]$$

$$\hat{P}^\mu = \int \frac{d^3p}{2} p^\mu \left[\hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right]$$

So, we need to evaluate the commutator,

$$\left[\hat{a}(\vec{q}) e^{-iqx} + \hat{b}^\dagger(\vec{q}) e^{+iqx}, \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) + \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right]$$

$$= \left[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \hat{a}^\dagger(\vec{p}) \right] e^{-iqx} + \left[\hat{b}^\dagger(\vec{q}), \hat{b}^\dagger(\vec{p}) \hat{b}(\vec{p}) + \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) \right] e^{+iqx}$$

$$= \left(\left[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{p}) \right] \hat{a}(\vec{p}) + \hat{a}(\vec{p}) \left[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{p}) \right] \right) e^{-iqx} + \left(\hat{b}^\dagger(\vec{p}) \left[\hat{b}^\dagger(\vec{q}), \hat{b}(\vec{p}) \right] + \left[\hat{b}^\dagger(\vec{q}), \hat{b}(\vec{p}) \right] \hat{b}^\dagger(\vec{p}) \right) e^{+iqx}$$

$$= 2 \hat{a}(\vec{p}) \delta(\vec{q} - \vec{p}) e^{-iqx} - 2 \hat{b}^\dagger(\vec{p}) \delta(\vec{q} - \vec{p}) e^{+iqx}$$

$$\begin{aligned}
 \rightarrow [\hat{\psi}(x), \hat{p}^\mu] &= \int \frac{d^3q d^3p p^\mu \delta(\vec{q} - \vec{p})}{(2\pi)^{3/2} \sqrt{2E_q}} \\
 &\quad \times (\hat{a}(\vec{p}) e^{-iqx} - \hat{b}^\dagger(\vec{p}) e^{+iqx}) \\
 &= \int \frac{d^3p p^\mu [\hat{a}(\vec{p}) e^{-ipx} - \hat{b}^\dagger(\vec{p}) e^{+ipx}]}{(2\pi)^{3/2} \sqrt{2E_p}} \\
 &\quad \underbrace{\hspace{10em}}_{i\hat{T}_{\psi}^\mu}
 \end{aligned}$$

$$\rightarrow \underline{[\hat{\psi}(x), \hat{p}^\mu] = i\hat{T}_{\psi}^\mu = i\partial^\mu \hat{\psi}}$$

Thus $\hat{T} = e^{-ia_\mu \hat{p}^\mu}$ is the translational operator which gives

$$\underline{\hat{T}^{-1} \hat{\psi}(x^\mu) \hat{T} = \hat{\psi}(x^\mu + a^\mu)}$$

Can be proved using BCH formula.

$$\text{Similarly, } [\hat{\psi}^\dagger(x), \hat{p}^\mu] = i\hat{T}_{\psi}^\mu = i\partial^\mu \hat{\psi}^\dagger$$

$$\rightarrow \underline{\hat{T}^{-1} \hat{\psi}^\dagger(x^\mu) \hat{T} = \hat{\psi}^\dagger(x^\mu + a^\mu)}$$

Many component scalar field

Now we shall discuss a scalar field theory with many components.

Consider a universe of a set of 'N' scalar fields given by

$$\{\phi_1(x), \phi_2(x), \phi_3(x), \dots, \phi_N(x)\}$$

We can combine them all to form a column vector given by,

$$\vec{\phi}(x) = \begin{bmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \vdots \\ \phi_N(x) \end{bmatrix}$$

Any component of $\vec{\phi}(x)$ is denoted by $\phi_A(x)$. (A, B, C, ...) denotes field index.

The lagrangian density is given by,

$$\mathcal{L} = \sum_{A=1}^N \mathcal{L}_A = \sum_{A=1}^N \frac{1}{2} (\partial^\mu \phi_A) (\partial_\mu \phi_A) - \frac{1}{2} m^2 \phi_A^2$$

We can compactify the notations by using the vector field notation $\vec{\phi}$.

$$\rightarrow \mathcal{L} = \frac{1}{2} (\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) - \frac{1}{2} m^2 \vec{\phi} \cdot \vec{\phi}$$

$$\vec{\phi} \cdot \vec{\phi} = \phi_1 \phi_1 + \phi_2 \phi_2 + \phi_3 \phi_3 + \dots + \phi_N \phi_N$$

$$\begin{aligned} (\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) &= (\partial^\mu \phi_1) (\partial_\mu \phi_1) + (\partial^\mu \phi_2) (\partial_\mu \phi_2) \\ &\quad + (\partial^\mu \phi_3) (\partial_\mu \phi_3) + \dots + (\partial^\mu \phi_N) (\partial_\mu \phi_N) \end{aligned}$$

Observe that this theory has $SO(N)$ symmetry, i.e. $\forall R \in SO(N)$, Let $\vec{\phi}' = R\vec{\phi}$ still the lagrangian will remain invariant under the action of $SO(N)$.

$$\mathcal{L}' = \frac{1}{2} (\partial^\mu \vec{\phi}') \cdot (\partial_\mu \vec{\phi}') - \frac{1}{2} m^2 \vec{\phi}' \cdot \vec{\phi}' = \mathcal{L}$$

The equations of motions for $\{\phi_A\}$ are $(\partial_\mu \partial^\mu + m^2) \phi_A = 0$, $A = \{1, 2, \dots, N\}$ which are N independent Klein-Gordon eq;

The dispersion relation for all $\{\phi_A\}$ is $p_\mu p^\mu = m^2 \Rightarrow (p^0)^2 = |\vec{p}|^2 + m^2 = E_p$

where we interpret $p^0 = E_p$ as the total energy of the particles for all ϕ_A .

Mode expansion

Similarly how we wrote the mode expansion of a single scalar field we can write for $\hat{\phi}_A$ as,

$$\hat{\phi}_A(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 (2E_p)}} \left[\hat{a}_A(\vec{p}) e^{-ip_\mu x^\mu} + \hat{a}_A^\dagger(\vec{p}) e^{+ip_\mu x^\mu} \right]$$

$\{\hat{a}_1, \hat{a}_2, \hat{a}_3, \dots, \hat{a}_N\}$ are N annihilation operators & $\{\hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger, \dots, \hat{a}_N^\dagger\}$ are N creation operators of independent scalar fields $\{\hat{\phi}_A\}$

In terms of vector $\vec{\phi}$ notation,

$$\vec{\hat{\phi}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \begin{bmatrix} \hat{a}_1(\vec{p})e^{-ip_\mu x^\mu} + \hat{a}_1^\dagger(\vec{p})e^{+ip_\mu x^\mu} \\ \hat{a}_2(\vec{p})e^{-ip_\mu x^\mu} + \hat{a}_2^\dagger(\vec{p})e^{+ip_\mu x^\mu} \\ \hat{a}_3(\vec{p})e^{-ip_\mu x^\mu} + \hat{a}_3^\dagger(\vec{p})e^{+ip_\mu x^\mu} \\ \vdots \\ \hat{a}_N(\vec{p})e^{-ip_\mu x^\mu} + \hat{a}_N^\dagger(\vec{p})e^{+ip_\mu x^\mu} \end{bmatrix}$$

Consider the basis vectors $\{\vec{h}_A\}$ for the internal vector space where $\vec{\hat{\phi}}(x)$ lives.

$$\vec{h}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{h}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{h}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \dots \vec{h}_N = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$\{\vec{h}_A\}$ are vectors that tell us about the polarization of $\vec{\hat{\phi}}(x)$ in its internal space. & we can write

$$\vec{\hat{\phi}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \sum_{A=1}^N \vec{h}_A \left[\hat{a}_A(\vec{p})e^{-ip_\mu x^\mu} + \hat{a}_A^\dagger(\vec{p})e^{+ip_\mu x^\mu} \right]$$

The canonical momentum density is given by, $\hat{\Pi}_A^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_A)} = \partial^0 \hat{\phi}_A$

$$\hat{\Pi}_A^0(x) = \frac{\int d^3p (iE_p) [\hat{a}_A^\dagger(\vec{p}) e^{+ik_\mu x^\mu} - \hat{a}_A(\vec{p}) e^{-ik_\mu x^\mu}]}{\sqrt{(2\pi)^3} \sqrt{(2E_p)}}$$

In terms of vector $\vec{\Pi}_0$ notation,

$$\vec{\Pi}_0(x) = \frac{\int d^3p (iE_p)}{\sqrt{(2\pi)^3} \sqrt{(2E_p)}} \begin{bmatrix} \hat{a}_1^\dagger(\vec{p}) e^{+ik_\mu x^\mu} - \hat{a}_1(\vec{p}) e^{-ik_\mu x^\mu} \\ \hat{a}_2^\dagger(\vec{p}) e^{+ik_\mu x^\mu} - \hat{a}_2(\vec{p}) e^{-ik_\mu x^\mu} \\ \hat{a}_3^\dagger(\vec{p}) e^{+ik_\mu x^\mu} - \hat{a}_3(\vec{p}) e^{-ik_\mu x^\mu} \\ \vdots \\ \hat{a}_N^\dagger(\vec{p}) e^{+ik_\mu x^\mu} - \hat{a}_N(\vec{p}) e^{-ik_\mu x^\mu} \end{bmatrix}$$

And in terms of polarization vectors $\{\vec{h}_A\}$

$$\vec{\Pi}_0(x) = \frac{\int d^3p (iE_p) \sum_{A=1}^N \vec{h}_A [\hat{a}_A^\dagger(\vec{p}) e^{+ik_\mu x^\mu} - \hat{a}_A(\vec{p}) e^{-ik_\mu x^\mu}]}{\sqrt{(2\pi)^3} \sqrt{(2E_p)}}$$

The expression for $\{\hat{a}_A(\vec{p})\}$ & $\{\hat{a}_A^\dagger(\vec{p})\}$ in terms of $\{\hat{\phi}_A\}$ & $\{\hat{\Pi}_A^\circ(x)\}$ is same as in free scalar field theory.

$$\hat{a}_A^\dagger(\vec{p}) = \int \frac{d^3x}{(2\pi)^{3/2} \sqrt{2E_p}} [E_p \hat{\phi}_A(x) - i \hat{\Pi}_A^\circ(x)] e^{-ip_\mu x^\mu}$$

$$\hat{a}_A(\vec{p}) = \int \frac{d^3x}{(2\pi)^{3/2} \sqrt{2E_p}} [E_p \hat{\phi}_A(x) + i \hat{\Pi}_A^\circ(x)] e^{ip_\mu x^\mu}$$

Commutation relations

We impose the usual Equal time commutation relations for $\{\hat{\phi}_A, \hat{\Pi}_A^\circ(x)\}$:

$$[\hat{\phi}_A(x), \hat{\phi}_B(y)]_{(x^0=y^0)} = [\hat{\Pi}_A^\circ(x), \hat{\Pi}_B^\circ(y)]_{(x^0=y^0)} = 0$$

$$[\hat{\phi}_A(x), \hat{\Pi}_B^\circ(y)]_{(x^0=y^0)} = \delta_{AB} i \delta(\vec{x} - \vec{y})$$

This results in bosonic commutation relations between the ladder operators

$$[\hat{a}_A(\vec{p}), \hat{a}_B(\vec{q})] = [\hat{a}_A^\dagger(\vec{p}), \hat{a}_B^\dagger(\vec{q})] = 0$$

$$[\hat{a}_A(\vec{p}), \hat{a}_B^\dagger(\vec{q})] = \delta_{AB} \delta(\vec{p} - \vec{q})$$

Quantized excitations & particle picture

The total Hamiltonian density for all fields $\{\hat{\phi}_A\}$ is given by,

$$\hat{H} = \sum_{A=1}^N \hat{H}_N$$

So in terms of Ladder operators,

$$\hat{H} = \int \frac{d^3p}{2} E_p \sum_{A=1}^N [\hat{a}_A^\dagger(\vec{p}) \hat{a}_A(\vec{p}) + \hat{a}_A(\vec{p}) \hat{a}_A^\dagger(\vec{p})]$$

&

$$\mathcal{N}[\hat{H}] = \int d^3p E_p \sum_{A=1}^N \hat{a}_A^\dagger(\vec{p}) \hat{a}_A(\vec{p})$$

Thus we see that there are N types of particles in our theory.

$\hat{a}_A^\dagger(\vec{p}) \rightarrow$ Creates particle of type A

$\hat{a}_A(\vec{p}) \rightarrow$ Annihilates particle of type A

Since the creation and annihilation operators of any particle type A obey bosonic commutation relations, the particles are non-interacting bosons.

$SO(N)$ Symmetry & Conserved quantities

According to Noether's theorem, every symmetry of the system has a corresponding conserved current and conserved charge. N -non interacting scalar fields have $SO(N)$ symmetry.

The internal vector space of $\vec{\Phi}$ has the basis vectors $\{\vec{h}_A\}$ for every axis denoted by ' A ' = $\{1, 2, 3, \dots, N\}$.

$SO(N)$ has $\frac{N(N-1)}{2}$ generators which correspond to the rotations in $\frac{N(N-1)}{2}$ mutually orthogonal planes formed by the mutually orthogonal N -number of axis. Any plane formed by the axis 'A' & 'B' is given by (A,B). All the possible such planes are given by:

$$\begin{aligned} \{ & (1,2), (1,3), (1,4) \dots (1,N-1), (1,N), \\ & (2,3), (2,4) \dots (2,N-1), (2,N), \\ & (3,4) \dots (3,N-1), (3,N), \\ & \vdots \\ & (N-2,N-1), (N-2,N), \\ & (N-1,N) \} \end{aligned}$$

The infinitesimal $SO(N)$ rotation about a plane (a,b) has a generator J^{ab} which transforms

$$\phi_A \longrightarrow \phi_A - e \underbrace{(\epsilon^{jk})}_{\text{Small rot}^n} \sum_B (J^{jk})_{AB} \phi_B$$

$$\text{where } (J^{jk})_{AB} = \delta_A^j \delta_B^k - \delta_B^j \delta_A^k$$

This is exactly the generalization to what we have seen in the study of $SO(3)$ or $SO(4)$ groups.

Observe that we have found a property of the internal $\vec{\phi}$ space which is analogous to angular momentum in N -dimensional Euclidean space. This quantity is called as the Isospin angular momentum.

The conserved isospin current is given by, $\hat{J}_{(AB)}^\mu = -e \left[\hat{\phi}_A (\partial^\mu \hat{\phi}_B) - \hat{\phi}_B (\partial^\mu \hat{\phi}_A) \right]$

Then we get the conserved charges

$$\hat{Q}_{AB} = \int d^3x \hat{J}_{AB}^\mu$$

$$\rightarrow \hat{Q}_{AB} = -e \int d^3x \left[\hat{\phi}_A (\partial^\mu \hat{\phi}_B) - \hat{\phi}_B (\partial^\mu \hat{\phi}_A) \right]$$

In terms of ladder operators,

$$N[\hat{Q}_{(AB)}] = ie \int d^3p [\hat{a}_A^\dagger(\vec{p}) \hat{a}_B(\vec{p}) - \hat{a}_B^\dagger(\vec{p}) \hat{a}_A(\vec{p})]$$

There are $\frac{N(N-1)}{2}$, Isospin currents & charges corresponding every generator.

This has angular momentum structure. Consider $N=3$. Then we can define

$$\vec{\hat{Q}} = \begin{bmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ \hat{Q}_3 \end{bmatrix} \quad \text{such that} \quad \hat{Q}_j = \epsilon_{jkl} \hat{Q}(kl)$$

$$\text{or } \vec{\hat{Q}} = -e \int d^3x \left[\underbrace{\vec{\hat{\phi}} \times \vec{\hat{\pi}}_0}_{\text{Angular momentum structure}} \right]$$

Angular momentum structure

Thus isospin is like an angular momentum in internal space of fields & is analogous to angular momentum in euclidian space.

System of two Complex fields

Now we consider the case where we have two complex fields ψ_1 & ψ_2 .

$$\psi_1^+ \neq \psi_1 \text{ \& } \psi_2^+ \neq \psi_2.$$

Then the lagrangian is given by,

$$\mathcal{L} = \sum_{A=\{1,2\}} \left[(\partial^\mu \psi_A^+) (\partial_\mu \psi_A) - m^2 \psi_A^+ \psi_A \right]$$

$$\vec{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \text{ is a vector of complex fields.}$$
$$\text{ \& } \vec{\psi}^+ = [\psi_1^+ \quad \psi_2^+]$$

$$\rightarrow \mathcal{L} = (\partial^\mu \vec{\psi}^+) \cdot (\partial_\mu \vec{\psi}) - m^2 \vec{\psi}^+ \cdot \vec{\psi}$$

Let $\vec{\psi} \longrightarrow U \vec{\psi}$ where U is a 2×2 unitary matrix. $U \in U(2)$

$$\text{Then } \vec{\psi}^+ \longrightarrow \vec{\psi}^+ U^+$$

$$\Rightarrow \vec{\psi}^\dagger \cdot \vec{\psi} \longrightarrow (\vec{\psi}^\dagger U^\dagger) \cdot (U \vec{\psi})$$

$$\rightarrow \vec{\psi}^\dagger \cdot \vec{\psi} \longrightarrow \vec{\psi}^\dagger \cdot \vec{\psi}$$

Similarly $(\partial^\mu \vec{\psi}^\dagger) \cdot (\partial_\mu \vec{\psi})$ is also invariant under $U(2)$ transformation.

The elements of $U(2)$ have complex determinant but of unit modulus.

$$\rightarrow \det(U) = e^{i\theta} \text{ for some real } \theta$$

$$\text{Now } U(1) = \{e^{i\theta} / \theta \in \mathbb{R}\}$$

$$\& SU(2) = \{U \in U(2) / \det = 1\}$$

$$\Rightarrow U(2) = SU(2) \otimes U(1)$$

The $SU(2)$ has three generators which correspond to angular momentum.

& $U(1)$ has one generator which correspond to electric charge.

The representation of the generators of $SU(2)$ for a two component complex vector field space is given by Pauli matrices ' $\vec{\sigma}$ ' as $\frac{\vec{\sigma}}{2}$. This represent the Spin degree of freedom.

Then the $U(1)$ part has trivial generator of '1'.

Then the four independent Noether currents corresponding to each one is given by,

$$J^\mu_{[U(1)]} = -ie \left[(\partial^\mu \hat{\psi}^+) \hat{\psi} - (\partial^\mu \hat{\psi}) \hat{\psi}^+ \right]$$

The $SU(2)$ transformation is given by the operator $e^{-i\theta \frac{\vec{\sigma}}{2}}$,

Infinitesimally, it is $1 - i \frac{\vec{\sigma}}{2} (\partial\theta)$

$$\rightarrow \vec{J}^\mu_{[SU(2)]} = (\vec{\pi}^\mu)^\dagger \left(\frac{-i\vec{\sigma}}{2} \right) \vec{\psi}$$

$$\vec{J}_{SU(2)}^\mu = \frac{-i}{2} (\partial^\mu \vec{\psi})^\dagger \cdot (\vec{\sigma} \vec{\psi})$$

Then the Noether charge is,

$$\vec{S}_{SU(2)} = \int \vec{J}_{SU(2)}^0 d^3x$$

$$\vec{S}_{SU(2)} = \frac{1}{2} \int d^3x (-i \partial_0 \vec{\psi})^\dagger \cdot \vec{\sigma} (\vec{\psi})$$

In terms of Creation & annihilation operators we get after normal order,

$$\vec{S}_{SU(2)} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3 2E_p} [\hat{\vec{a}}^\dagger(p) \vec{\sigma} \hat{\vec{a}}(p) - \hat{\vec{b}}^\dagger(p) \vec{\sigma} \hat{\vec{b}}(p)]$$

$$\text{where } \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Similarly for the System of 'N' complex fields which are not interacting, it will have $SU(N)$ and $U(1)$ Symmetry.

$U(1)$ Symmetry has 1 generator analogous to electric charge. And $SU(N)$ has $N^2 - 1$ generators analogous to some higher level of charges. For example, if $N=3$, then the $3^2 - 1 = 8$ generators will be analogous to the color charge of the gluons which also have $SU(3)$ symmetry.