Dirac Equation

Dirac eqn in natural units ($c = \hbar = 1$) is given as,

$$(i\not{\partial} - m)\psi = 0$$

- $\not{\partial} = \gamma^\mu \partial_\mu$

- γ^μ are called Gamma matrices which satisfy the anti-commutation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4$$

& $$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

These are the only defining properties of γ-matrices.

- The γ-matrices have to be atleast 4×4. So 'ψ' should be a 4 component quantity represented as a 4×1 column vector.

- $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$

$\rightarrow (\gamma_0, \gamma_1, \gamma_2, \gamma_3) = (\gamma^0, -\gamma^1, -\gamma^2, -\gamma^3)$

- $\gamma^0 \Rightarrow$ Hermitian & eigenvals $= \pm 1$.
- $\gamma^i \Rightarrow$ Anti-Hermitian, $(\gamma^i)^2 = -1$

$\rightarrow$ eigenvalues of $\gamma^i = \pm i$

- trace $(\gamma^\mu) = 0$
- $\gamma^\mu \gamma_\mu = 4\mathbb{I}_4$
- trace $(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu}$

- Hermitian adjoint of Dirac eqn is

$$\overline{\psi}(i\overleftarrow{\not\partial} + m) = 0 \quad \text{where } \overline{\psi} = \psi^\dagger \gamma^0$$

- Let a_μ & b_μ be any two four vectors. Then,

$$(a_\mu \gamma^\mu)(b_\nu \gamma^\nu) = a_\mu b^\mu \mathbb{I}_4$$

$$\Longrightarrow \underline{\not a \not b = a_\mu b^\mu \mathbb{I}_4}$$

- The representation of lorentz transformation acting on the dirac spinor is given by $\Lambda(L)$ as,

$$\Lambda(L) = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$\rightarrow \frac{\sigma^{\mu\nu}}{2}$ are the generators

$\omega_{\mu\nu}$ = 6 parameter anti-Symmetric rank-2 tensor which gives three rotation angles & three boost angles.

- $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is called the 5^{th} gamma matrise which anti-commutes with all γ^μ.

$$\{\gamma^5, \gamma^\mu\} = 0 \quad , \quad (\gamma^5)^\dagger = \gamma^5$$

$$(\gamma^5)^2 = 1 \quad , \quad \text{Trace}(\gamma^5) = 0$$

- Let $\Gamma = \{\mathbb{I}_4, \gamma^4, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$

Then the 16 objects denoted by $\bar{\psi}\Gamma\psi$ are called as Dirac Bilinears

$\bar{\psi}\psi \Rightarrow$ Lorentz Scalar

$\bar{\psi}\gamma^\mu\psi \Rightarrow$ Transforms as 4-vector

$\bar{\psi}\sigma^{\mu\nu}\psi \Rightarrow$ Transforms as the anti-Symmetric tensor

$\bar{\psi}\gamma^\mu\gamma^5\psi \Rightarrow$ Transforms as an Axial 4-Vector

$\bar{\psi}\gamma^5\psi \Rightarrow$ Transforms as Pseudo-Scalar

- <u>In Chiral / Weyl basis</u>

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}$$

or $$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$$

Let $\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$

The Chirality operator is γ^5,

$$\gamma^5 = \begin{bmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}$$

$$\gamma^5 \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} = - \begin{bmatrix} \psi_L \\ 0 \end{bmatrix} \quad \& \quad \gamma^5 \begin{bmatrix} 0 \\ \psi_R \end{bmatrix} = \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}$$

$\Rightarrow$ Dirac equation predicts two types of particles based on Chirality.

- Left handed particles $\rightarrow \begin{bmatrix} \psi_L \\ 0 \end{bmatrix}$ which gives eigen value -1 for γ^5.
- Right handed particles $\rightarrow \begin{bmatrix} 0 \\ \psi_R \end{bmatrix}$ which gives eigen value $+1$ for γ^5.

The Chirality Projection operators are

$$\hat{P}_L = \frac{\mathbb{I}_4 - \gamma^5}{2} = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & 0 \end{bmatrix}$$

& $$\hat{P}_R = \frac{\mathbb{I}_4 + \gamma^5}{2} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I}_2 \end{bmatrix}$$

which extracts ψ_L & ψ_R part of ψ.

In Weyl basis dirac eq^n becomes,

$$(i\partial\!\!\!/ - m)\psi = 0$$

$$\Rightarrow \quad \gamma^\mu \hat{P}_\mu \psi = m\psi$$

$$\rightarrow [\gamma^0 \hat{P}_0 - \vec{\gamma}.\hat{\vec{p}}]\psi = m\psi$$

$$\rightarrow \left(\begin{bmatrix} 0 & \hat{P}_0 \\ \hat{P}_0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \vec{\sigma}.\hat{\vec{p}} \\ -\vec{\sigma}.\hat{\vec{p}} & 0 \end{bmatrix} \right) \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix} = m \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}$$

$$\Rightarrow \quad (\hat{P}_0 - \vec{\sigma}.\hat{\vec{p}})\psi_R = m\psi_L$$

$$(\hat{P}_0 + \vec{\sigma}.\hat{\vec{p}})\psi_L = m\psi_R$$

Thus we see that mass of the particle couples the left & right handed property of particles.

Eg:- For $\vec{p}=0$, $m\neq 0$ particle

$\hat{P}_0 \psi_R = m\psi_L$ & $\hat{P}_0 \psi_L = m\psi_R$

$\Rightarrow \partial_t \psi_R = -im\psi_L$ & $\partial_t \psi_L = -im\psi_R$

$\rightarrow \partial_t^2 \psi_R = -m^2\psi_R$ & $\partial_t^2 \psi_L = -m^2\psi_L$

Thus ψ_L & ψ_R oscillate with a phase difference of $\frac{\pi}{2}$. So here the massive particle at rest will oscillates periodically from being left handed to right handed and vice versa whose frequency of oscillation is proportional to mass.

Thus mass couples ψ_L & ψ_R.

If $m=0$, $\Rightarrow$ $\sigma^\mu \hat{p}_\mu \psi_R = 0$

or $\underline{i\sigma^\mu \partial_\mu \psi_R = 0}$

Similarly, $\underline{i\bar{\sigma}^\mu \partial_\mu \psi_L = 0}$

Thus we get Weyl eqn for the mass-less particles. There are two types of mass-less weyl particles given by ψ_R & ψ_L.

For massless particles $E_p = |\vec{p}|$

& $p_0 = E_p \rightarrow \underline{\underline{p_0 = E_p}}$

$\rightarrow$ In a given $\vec{p}$ mode, the ψ_L & ψ_R satisfies,

$$(\hat{p}_0 - \vec{\sigma}\cdot\hat{\vec{p}})\psi_R = 0 \rightarrow \left[|\vec{p}| - (\vec{\sigma}\cdot\hat{\vec{p}})\right]\psi_R = 0$$

$$\rightarrow \underline{\frac{(\vec{\sigma}\cdot\hat{\vec{p}})}{|\vec{p}|}\psi_R = \psi_R}$$

Similarly,

$$(\hat{P}_0 + \vec{\sigma}\cdot\hat{\vec{P}})\psi_L = 0 \Rightarrow \frac{(\vec{\sigma}\cdot\hat{\vec{P}})\psi_L}{|\vec{P}|} = -\psi_L$$

Since $\vec{\sigma} \equiv 2\hat{\vec{S}} \Rightarrow$ Spin operator in weyl rep^n (2 dim rep^n of lorentz group)

For $\hat{h} = \frac{\vec{S}\cdot\hat{\vec{P}}}{|\vec{P}|}$ (Helicity operator)

$$\hat{h}\,\psi_R = \frac{\hbar}{2}\psi_R \quad \& \quad \hat{h}\,\psi_L = \frac{-\hbar}{2}\psi_L$$

→ Thus for mass-less dirac particles, $\psi_R \rightarrow$ +ve helicity
& $\psi_L \rightarrow$ -ve helicity.

Thus a free and massless dirac particle will have a definite chirality & also will be eigen states of Helicity operator.

Particle and Anti-particle Solutions

Let the solution for Dirac eqⁿ be of the form

$$\psi(x) = W(p)\, e^{-ip_\mu x^\mu}$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m) W(p) e^{-ip_\mu x^\mu} = 0$$

$$\Rightarrow (-i^2 \gamma^\mu p_\mu - m) W(p) = 0$$

$$\Rightarrow (\not{p} - m) W(p) = 0$$

multiply throughout by $(\not{p} + m)$

$$\rightarrow \quad (\not{p}\not{p} - m^2) W(p) = 0$$

$$\not{p}\not{p} = p^\mu p_\mu \Longrightarrow (p^\mu p_\mu - m^2) W(p) = 0$$

Thus if $W(p) \neq 0 \Rightarrow \underline{p^\mu p_\mu = m^2}$

But $p^\mu p_\mu = (p^0)^2 - |\vec{p}|^2$

$$\Rightarrow \quad \underline{(p^0)^2 = |\vec{p}|^2 + m^2}$$

But $E_p = \sqrt{|\vec{p}|^2 + m^2} \qquad (E_p > 0)$

$$\Rightarrow \underline{(p^0) = \pm E_p} \text{ is needed.}$$

& Hence the dispersion relation is given by, $\underline{(p^0)^2 = E_p^2 = |\vec{p}|^2 + m^2}$

→ We have +ve & -ve (p^0) solutions predicted by Dirac eqn.

Let $W(E_p, \vec{p}) = u(p) \rightarrow$ +ve p^0

& $W(-E_p, -\vec{p}) = v(p) \rightarrow$ -ve p^0

$$\Rightarrow \quad (\not{p} - m)\, u(p) = 0$$

& $$(\not{p} + m)\, v(p) = 0$$

$$\Rightarrow \quad \psi_+(x) = u(p)\, e^{-ip_\mu x^\mu} \qquad (p_0 > 0)$$

& $$\psi_-(x) = v(p)\, e^{ip_\mu x^\mu} \qquad (p_0 < 0)$$

Particle Sol^n : $p^0 > 0$

Goal: To find $u(p) = \begin{bmatrix} u_L(p) \\ u_R(p) \end{bmatrix}$

Case - 1 : $\vec{p} = 0$, $p^\mu = (m, 0)$

$$\rightarrow (\gamma^0 m - m) u(p^0) = 0$$

$$\rightarrow m \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] u(p^0) = 0$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_L(p^0) \\ u_R(p^0) \end{bmatrix} = 0$$

$$\rightarrow u_L(p^0) = u_R(p^0) = \sqrt{m}\, \xi_s$$

$\xi_s \rightarrow$ any 2-component spinor.
with $\xi_s^+ \xi_s = 1 \rightarrow$ normalization condition.

$\sqrt{m}$ is inserted for convinience.

$\Rightarrow u(p^0) = \sqrt{m} \begin{bmatrix} \xi_s \\ \xi_s \end{bmatrix}$, where $\xi_s \rightarrow$ 2 spinor

ξ_s varies as a usual 2-spinor under 2-dim lorentz transform. Therefore determines the spin of the dirac solution.

A particle with spin up along +ve x^3 axis has $\xi_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & a particle with spin down along +ve x^3 axis has $\xi_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (will be shown later)

Case-2 : $\vec{p} \neq 0$, $p^\mu = (m, \vec{p})$

Consider boosting $u(p^0)$ along the +ve x^3 direction to result in a spinor $u(p^\mu)$ such that $p^\mu = (p^0, 0, 0, p^3)$. The lorentz transformation matrix is,

$$\Lambda(L) = e^{\frac{-i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}}$$

where $\omega_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & \zeta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\zeta & 0 & 0 & 0 \end{bmatrix}$

$\zeta \rightarrow$ rapidity factor of boost

$$\Rightarrow \omega_{\mu\nu}\sigma^{\mu\nu} = \left(\omega_{03}\sigma^{03} + \omega_{30}\sigma^{30}\right)$$

$$= 2\left(-\zeta\sigma^{30}\right)$$

$$\sigma^{30} = \frac{i}{2}\left[\gamma^3, \gamma^0\right] = i\begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix}$$

$$\Rightarrow \Lambda(L) = e^{\frac{i}{2}\zeta\sigma^{30}} = e^{\left(\frac{-\zeta}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}\right)}$$

$$\Rightarrow u(p) = \exp\left(\frac{-\zeta}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix}\right) u(p^0)$$

$\Lambda(L) = \exp\left(\frac{-\zeta}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix}\right)$ is given by

$$\cosh\left(\frac{\zeta}{2}\right)\mathbb{I}_4 - \sinh\left(\frac{\zeta}{2}\right)\begin{bmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\zeta/2}\left(\frac{I_2-\sigma^3}{2}\right) + e^{-\zeta/2}\left(\frac{I_2+\sigma^3}{2}\right) & 0 \\ 0 & e^{\zeta/2}\left(\frac{I_2+\sigma^3}{2}\right) + e^{-\zeta/2}\left(\frac{I_2-\sigma^3}{2}\right) \end{bmatrix}$$

$$\Rightarrow u(p) = \sqrt{m} \begin{pmatrix} \left[e^{\zeta/2}\left(\frac{I_2-\sigma^3}{2}\right) + e^{-\zeta/2}\left(\frac{I_2+\sigma^3}{2}\right)\right]\xi_s \\ \left[e^{\zeta/2}\left(\frac{I_2+\sigma^3}{2}\right) + e^{-\zeta/2}\left(\frac{I_2-\sigma^3}{2}\right)\right]\xi_s \end{pmatrix}$$

Now $E = \gamma m$, $|\vec{p}| = \gamma m v$

and rapidity ζ is such that

$$\tanh(\zeta) = (v) = \frac{\gamma m v}{\gamma m}$$

$$\rightarrow \tanh(\zeta) = \frac{|\vec{p}|}{E}$$

But $|\vec{p}| = p^3$, $E = E_p$.

$$\Rightarrow \tanh(\zeta) = \frac{p^3}{E_p}$$

Also $\cosh(\xi) = \gamma$

$$\longrightarrow m\cosh(\xi) = \gamma m = E_p$$

& $\sinh(\xi) = \gamma v$

$$\longrightarrow m\sinh(\xi) = p^3$$

$$\longrightarrow m e^{\xi} = m\left[\cosh(\xi) + \sinh(\xi)\right]$$

$$\longrightarrow m e^{\xi} = (E_p + p^3)$$

$$\Rightarrow e^{\xi/2} = \sqrt{\frac{E_p + p^3}{m}}$$

& $$e^{-\xi/2} = \sqrt{\frac{E_p - p^3}{m}}$$

$$\Rightarrow u(p) = \begin{pmatrix} \left[\sqrt{E_p+p^3}\left(\frac{I_2-\sigma^3}{2}\right) + \sqrt{E_p-p^3}\left(\frac{I_2+\sigma^3}{2}\right)\right]\xi_s \\ \left[\sqrt{E_p+p^3}\left(\frac{I_2+\sigma^3}{2}\right) + \sqrt{E_p-p^3}\left(\frac{I_2-\sigma^3}{2}\right)\right]\xi_s \end{pmatrix}$$

$$\frac{I_2-\sigma^3}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \& \quad \frac{I_2+\sigma^3}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow u(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{pmatrix} \xi_s \\ \begin{pmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{pmatrix} \xi_s \end{bmatrix}$$

Consider $p_\mu \sigma^\mu$ in our case

$$p_0 \sigma^0 + p_3 \sigma^3 = E_p I_2 - p^3 \sigma^3$$

$$= \begin{bmatrix} E_p - p^3 & 0 \\ 0 & E_p + p^3 \end{bmatrix}$$

$$\Rightarrow \sqrt{p_\mu \sigma^\mu} = \begin{bmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{bmatrix}$$

Similarly $p_\mu \bar{\sigma}^\mu$ is $p_0 \sigma^0 + p_3 \bar{\sigma}^3$

$$\Rightarrow p_\mu \bar{\sigma}^\mu = \begin{bmatrix} E_p - p_3 & 0 \\ 0 & E_p + p_3 \end{bmatrix}$$

$$p_3 = -p^3$$

$$\Rightarrow \sqrt{p_\mu \bar{\sigma}^\mu} = \begin{bmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{bmatrix}$$

$$\Rightarrow \quad u(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \end{bmatrix}$$

This expression is general, and also works for arbitrary $\vec{p}$ or boost direction, where we interpret $\sqrt{p_\mu \sigma^\mu}$ & $\sqrt{p_\mu \bar{\sigma}^\mu}$ as taking the positive root of eigen values of $p_\mu \sigma^\mu$ & $p_\mu \bar{\sigma}^\mu$ respectively.

Now consider $p^\mu = (E_p, 0, 0, p^3)$ and $\xi_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (Spin up along x^3-axis)

$$\Rightarrow u(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{pmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\Rightarrow u(p) = \begin{bmatrix} \sqrt{E_p - p^3} \\ 0 \\ \sqrt{E_p + p^3} \\ 0 \end{bmatrix}$$

If the boost given is too large i.e. $p^3 \sim E_p$ $(\xi \to \infty)$, then

$$\Rightarrow u(p) = \begin{bmatrix} \sqrt{E_p - p^3} \\ 0 \\ \sqrt{E_p + p^3} \\ 0 \end{bmatrix} \xrightarrow[\text{Boost}]{\text{Big}} \sqrt{2E_p} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{Purely Right handed}}$$

Thus ultra relativistic particle is either purely left/Right handed.

Now consider $p^\mu = (E_p, 0, 0, p^3)$

and $\xi_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (Spin down along x^3-axis)

$$\Rightarrow u(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{pmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$\Rightarrow u(p) = \begin{bmatrix} 0 \\ \sqrt{E_p + p^3} \\ 0 \\ \sqrt{E_p - p^3} \end{bmatrix} \xrightarrow[\text{Boost}]{\text{Big}} \sqrt{2E_p} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

Purely left handed

Now we shall find the Helicity of the resultant states.

Helicity operator is $\hat{h} = \frac{\hat{\vec{p}} \cdot \hat{\vec{S}}}{|\vec{p}|}$

where $\hat{\vec{S}} = \frac{\hbar}{2} \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$ (In Weyl basis)

For $p^\mu = (E_p, 0, 0, p^3)$

$$\hat{h} = \frac{\hbar}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\begin{bmatrix} \sqrt{E_p - p^3} \\ 0 \\ \sqrt{E_p + p^3} \\ 0 \end{bmatrix}$ has +ve helicity which implies that it represent a Right handed Particle. And also +ve energy state. Note that this state was got by $\xi_s = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, So if helicity is +ve with $\vec{p} = (0, 0, p^3)$, then this represents a Spin up particle which has $p^0 > 0$ (Particle Solution).

$\begin{bmatrix} 0 \\ \sqrt{E_p + p^3} \\ 0 \\ \sqrt{E_p - p^3} \end{bmatrix}$ has −ve helicity which implies that it represent a Left handed Particle. And also +ve energy state.

Note that this state was got by $\xi_s = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, So if helicity is $-ve$ with $\vec{p} = (0, 0, p^3)$, then this represents a Spin down particle which has $p^0 > 0$ (Particle Solution).

Thus in ultra relativistic limit, a $+ve$ Helicity particle is purely right Handed with up Spin & a $-ve$ Helicity particle purely left handed with down Spin.

Note: If a particle has mass, then the helicity of a particle is not lorentz invariant. Since we can always lorentz transform $\vec{p} \rightarrow -\vec{p}$, but Spin is unchanged. So we can effectively reverse the helicity based on frame of reference. But for a massless particle, it always travels at the speed of light. So, we cannot perform Such a boost.

To find Normalization condition for

$$u(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \end{bmatrix}$$. So, Consider

$$u^\dagger u = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi_s^\dagger & \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s^\dagger \end{bmatrix} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s \end{bmatrix}$$

$$= \left[(p_\mu \sigma^\mu) + (p_\mu \bar{\sigma}^\mu) \right] \xi_s^\dagger \xi_s$$

$$= \left[(p_0 \mathbb{I}_2 - \vec{p} \cdot \vec{\sigma}) + (p_0 \mathbb{I}_2 + \vec{p} \cdot \vec{\sigma}) \right] \xi_s^\dagger \xi_s$$

$$\rightarrow \quad u^\dagger u = (2p_0)\, \xi_s^\dagger \xi_s$$

Even if $\xi_s^\dagger \xi_s = 1$

$u^\dagger u = 2p_0$ is not Lorentz invariant.

Consider $\bar{u} u$ where $\bar{u} = u^\dagger \gamma^0$

& $\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}$

$$\Rightarrow \bar{u} = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi_s^+ & \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s^+ \end{bmatrix} \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}$$

$$\Rightarrow \bar{u} = \begin{bmatrix} \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s^+ & \sqrt{p_\mu \sigma^\mu}\,\xi_s^+ \end{bmatrix} \quad \&$$

$$\bar{u}u = \begin{bmatrix} \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s^+ & \sqrt{p_\mu \sigma^\mu}\,\xi_s^+ \end{bmatrix} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi_s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s \end{bmatrix}$$

$$= \left[\sqrt{(p_\mu \bar{\sigma}^\mu p_\nu \sigma^\nu)} + \sqrt{(p_\nu \sigma^\nu p_\mu \bar{\sigma}^\mu)} \right] \xi_s^+ \xi_s$$

$$= \left(\sqrt{(p_0 \mathbb{I}_2 + \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{I}_2 - \vec{p}\cdot\vec{\sigma})} + \sqrt{(p_0 \mathbb{I}_2 - \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{I}_2 + \vec{p}\cdot\vec{\sigma})} \right) \xi_s^+ \xi_s$$

$$= 2\sqrt{(p_0 \mathbb{I}_2)^2 - (\vec{p}\cdot\vec{\sigma})^2}\;\; \xi_s^+ \xi_s$$

$$= 2\sqrt{\underbrace{(p_0^2 - |\vec{p}|^2)}_{p_\mu p^\mu}}\; \xi_s^+ \xi_s = 2m\, \xi_s^+ \xi_s$$

$$\Rightarrow \underline{\bar{u}u = 2m\, \xi_s^+ \xi_s}$$

If the two component Spinor ξ_s is normalized to 1 ($\xi_s^+ \xi_s = 1$), then we can choose the basis Spinors as

$$\xi_s^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \& \quad \xi_s^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\rightarrow \quad u^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi_s^s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s^s \end{bmatrix} \qquad s = \{1, 2\}$$

are the two linearly independent Solution of $u(p) \rightarrow$ Particle Sol^n. whose $p_0 > 0$. They are normalized according to

$$\underline{\bar{u}^\alpha(p)\, u^s(p) = 2m\, \delta^{\alpha s}}$$

$$u^1(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \\ 0 \\ \sqrt{p_\mu \bar{\sigma}^\mu} \\ 0 \end{bmatrix} \quad \& \quad u^2(p) = \begin{bmatrix} 0 \\ \sqrt{p_\mu \sigma^\mu} \\ 0 \\ \sqrt{p_\mu \bar{\sigma}^\mu} \end{bmatrix}$$

Anti-Particle Sol^n

To solve for $v(p) = \begin{bmatrix} v_L(p) \\ v_R(p) \end{bmatrix}$,

where $(\not{p} + m)\, v(p)\, e^{ip_\mu x^\mu} = 0$

Case - 1 : $\vec{p} = 0$, $p^\mu = (m, 0)$

$$\rightarrow (\gamma^0 m + m)\, v(p^0) = 0$$

$$\rightarrow m \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] v(p^0) = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_L(p^0) \\ v_R(p^0) \end{bmatrix} = 0$$

$$\rightarrow \quad v_L(p^0) = -v_R(p^0) = \sqrt{m}\ \eta$$

$\eta \rightarrow$ any 2-component Spinor with $\eta^+ \eta = 1 \Rightarrow$ Normalization.

$$\rightarrow v(p^0) = \sqrt{m} \begin{bmatrix} \eta \\ -\eta \end{bmatrix} \text{ where } \eta \rightarrow \text{2 Spinor}$$

$\sqrt{m}$ → chosen for convinience for later calculations.

η varies as a usual 2-spinor under 2-dim lorentz transform. Therefore determines the spin of the dirac solution.

Opposite to particles we choose,

An anti-particle with spin up along +ve x^3 axis to have $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

And an anti-particle with spin down along +ve x^3 axis to have $\eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now consider boosting $v(p^0)$ along the +ve x^3 direction to result in a spinor $v(p^\mu)$ such that $p^\mu = (p^0, 0, 0, p^3)$. The lorentz transformation matrix is,

$$\Lambda(\zeta) = \exp\left(-\frac{\zeta}{2}\begin{bmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{bmatrix}\right)$$

$$= \cosh\left(\frac{\zeta}{2}\right) I_4 - \sinh\left(\frac{\zeta}{2}\right)\begin{bmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{bmatrix}$$

$$= \begin{bmatrix} e^{\zeta/2}\left(\frac{I_2-\sigma^3}{2}\right) + e^{-\zeta/2}\left(\frac{I_2+\sigma^3}{2}\right) & 0 \\ 0 & e^{\zeta/2}\left(\frac{I_2+\sigma^3}{2}\right) + e^{-\zeta/2}\left(\frac{I_2-\sigma^3}{2}\right)\end{bmatrix}$$

$$= \begin{bmatrix} e^{-\zeta/2} & 0 & 0 & 0\\ 0 & e^{\zeta/2} & 0 & 0\\ 0 & 0 & e^{\zeta/2} & 0\\ 0 & 0 & 0 & e^{-\zeta/2}\end{bmatrix}$$

$$\Rightarrow v(p^\mu) = \Lambda(\zeta)\, v(p^0) = \Lambda(\zeta)\,\sqrt{m}\begin{bmatrix}\eta\\ -\eta\end{bmatrix}$$

using,

$$e^{\zeta/2} = \sqrt{\frac{E_p + p^3}{m}}$$

& $$e^{-\zeta/2} = \sqrt{\frac{E_p - p^3}{m}}$$

$$\Rightarrow v(p^\mu) = \begin{bmatrix} \begin{pmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{pmatrix} \eta \\ \begin{pmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{pmatrix} (-\eta) \end{bmatrix}$$

$$\rightarrow \quad v(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\ \eta \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\ \eta \end{bmatrix}$$

This espression is general, and also works for arbitrary $\vec{p}$.

Now consider $p^\mu = (E_p, 0, 0, p^3)$ and $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (Spin up along x^3-axis)

$$\Rightarrow v(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{pmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{pmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$\Longrightarrow \underset{\text{up-spin}}{v(p)} = \begin{bmatrix} 0 \\ \sqrt{E_p + p^3} \\ 0 \\ -\sqrt{E_p - p^3} \end{bmatrix} \quad \left(\begin{matrix}\text{Physical} \\ \text{Spin up}\end{matrix}\right)$$

Now consider $p^\mu = (E_p, 0, 0, p^3)$

and $\eta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (Spin down along x^3-axis)

$$\Longrightarrow v(p) = \begin{bmatrix} \begin{pmatrix} \sqrt{E_p - p^3} & 0 \\ 0 & \sqrt{E_p + p^3} \end{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{pmatrix} \sqrt{E_p + p^3} & 0 \\ 0 & \sqrt{E_p - p^3} \end{pmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$\Longrightarrow \underset{\text{down spin}}{v(p)} = \begin{bmatrix} \sqrt{E_p - p^3} \\ 0 \\ -\sqrt{E_p + p^3} \\ 0 \end{bmatrix} \quad \left(\begin{matrix}\text{Physical} \\ \text{Spin down}\end{matrix}\right)$$

Now we shall find the Helicity of the resultant states.

Important: Helicity operator for anti-particles is interpreted oppositely.

$$\hat{h} = \frac{\hat{\vec{p}} \cdot \hat{\vec{S}}}{|\vec{p}|} \quad \text{where} \quad \hat{\vec{S}} = \frac{\hbar}{2}\begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix}$$

For $p^\mu = (E_p, 0, 0, p^3)$

$$\hat{h} = \frac{\hbar}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Consider, $\underset{\text{rep-spin}}{\vartheta(p)}$

$$\hat{h}\begin{bmatrix} 0 \\ \sqrt{E_p + p^3} \\ 0 \\ -\sqrt{E_p - p^3} \end{bmatrix} = -\frac{\hbar}{2}\begin{bmatrix} 0 \\ \sqrt{E_p + p^3} \\ 0 \\ -\sqrt{E_p - p^3} \end{bmatrix}$$

$$\Rightarrow \hat{h}\left(\underset{\text{rep-spin}}{\vartheta(p)}\right) = -\frac{\hbar}{2}\left(\underset{\text{rep-spin}}{\vartheta(p)}\right)$$

$\Rightarrow$ It has +ve helicity.

(Interpreted oppositely to that of particles)

Since this state was got by $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we interpret this to represent a Spin up anti-particle. This is because it has +ve momentum along x^3 & it results in +ve helicity, thus Spin state has to be up.

If we apply ultra-relativistic boost on $\underset{\text{up-Spin}}{v(p)}$, i.e. $p^3 \sim E_p$,

$$\Longrightarrow \underset{\text{up-Spin}}{v(p)} = \begin{bmatrix} 0 \\ \sqrt{E_p + p^3} \\ 0 \\ -\sqrt{E_p - p^3} \end{bmatrix} \xrightarrow[\text{Boost}]{\text{Big}} \sqrt{2E_p} \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{Purely Left handed}}$$

Thus ultra relativistic anti-particle which has +ve helicity is Left handed, with zero right handedness.

Consider, $v(p)_{\text{down-spin}}$

$$\hat{h}\begin{bmatrix}\sqrt{E_p-p^3}\\0\\-\sqrt{E_p+p^3}\\0\end{bmatrix}=\frac{+}{2}\begin{bmatrix}\sqrt{E_p-p^3}\\0\\-\sqrt{E_p+p^3}\\0\end{bmatrix}$$

$$\Rightarrow \hat{h}\left(v(p)_{\text{down-spin}}\right)=\frac{+}{2}\left(v(p)_{\text{down-spin}}\right)$$

$\Rightarrow$ It has $-ve$ helicity.

(Interpreted oppositely to that of particles)

Since this state was got by $\eta=\begin{bmatrix}1\\0\end{bmatrix}$, we interpret this to represent a spin down anti-particle. This is because it has $+ve$ momentum along x^3 & it results in $-ve$ helicity, thus spin state has to be down.

If we apply ultra-relativistic boost on $v(p)_{\text{down-spin}}$, i.e. $p^3 \sim E_p$, then we get,

$$\Rightarrow \underset{\text{down-spin}}{v(p)} = \begin{bmatrix} \sqrt{E_p - p^3} \\ 0 \\ -\sqrt{E_p + p^3} \\ 0 \end{bmatrix} \xrightarrow[\text{Boost}]{\text{Big}} \sqrt{2E_p} \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{\text{Purely Right handed}}$$

Thus ultra relativistic anti-particle which has -ve helicity is Right handed, with zero left handedness.

To find Normalization condition for $v(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\ \eta \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\ \eta \end{bmatrix}$. So, Consider

$$v^\dagger v = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\eta^\dagger & -\sqrt{p_\mu \bar{\sigma}^\mu}\,\eta^\dagger \end{bmatrix} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\ \eta \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\ \eta \end{bmatrix}$$

$$= \left[(p_\mu \sigma^\mu) + (p_\mu \bar{\sigma}^\mu) \right] \eta^\dagger \eta$$

$$= \left[(p_0 I_2 - \vec{p} \cdot \vec{\sigma}) + (p_0 I_2 + \vec{p} \cdot \vec{\sigma}) \right] \eta^\dagger \eta$$

$$\rightarrow v^\dagger v = (2p_0)\eta^\dagger\eta$$

Even if $\eta^\dagger\eta = 1$

$v^\dagger v = 2p_0$ is not Lorentz invariant.

Consider $\bar{v}v$ where $\bar{v} = v^\dagger\gamma^0$

& $\gamma^0 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}$

$$\rightarrow \bar{v} = \begin{bmatrix} \sqrt{p_\mu\bar{\sigma}^\mu}\eta^\dagger & -\sqrt{p_\mu\bar{\sigma}^\mu}\eta^\dagger \end{bmatrix}\begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}$$

$$\rightarrow \bar{v} = \begin{bmatrix} -\sqrt{p_\mu\bar{\sigma}^\mu}\eta^\dagger & \sqrt{p_\mu\sigma^\mu}\eta^\dagger \end{bmatrix} \quad \&$$

$$\bar{v}v = \begin{bmatrix} -\sqrt{p_\mu\bar{\sigma}^\mu}\eta^\dagger & \sqrt{p_\mu\sigma^\mu}\eta^\dagger \end{bmatrix}\begin{bmatrix} \sqrt{p_\mu\sigma^\mu}\,\eta \\ -\sqrt{p_\mu\bar{\sigma}^\mu}\,\eta \end{bmatrix}$$

$$= -\left[\sqrt{(p_\mu\bar{\sigma}^\mu p_\nu\sigma^\nu)} + \sqrt{(p_\nu\sigma^\nu p_\mu\bar{\sigma}^\mu)}\right]\eta^\dagger\eta$$

$$= -\left(\sqrt{(p_0 \mathbb{I}_2 + \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{I}_2 - \vec{p}\cdot\vec{\sigma})} + \sqrt{(p_0 \mathbb{I}_2 - \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{I}_2 + \vec{p}\cdot\vec{\sigma})} \right) \eta^\dagger \eta$$

$$= -2\sqrt{(p_0 \mathbb{I}_2)^2 - (\vec{p}\cdot\vec{\sigma})^2}\ \eta^\dagger \eta$$

$$= -2\sqrt{\underbrace{(p_0^2 - |\vec{p}|^2)}_{p_\mu p^\mu}}\ \eta^\dagger \eta = -2m\, \eta^\dagger \eta$$

$$\Longrightarrow \underline{\bar{v}\, v = -2m\, \eta^\dagger \eta}$$

If the two component Spinor η is normalized to 1 ($\eta^\dagger \eta = 1$), then we can choose the basis Spinors as

$$\eta^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \& \quad \eta^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Longrightarrow v^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\ \eta^s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\ \eta^s \end{bmatrix} \qquad s = \{1, 2\}$$

$\{v^s(p)\}$ are the two linearly independent solution of $v(p)$ for anti-particles. They are normalized according to

$$\overline{v}^r(p)\, v^s(p) = -2m\delta^{rs}$$

$$v^1(p) = \begin{bmatrix} 0 \\ \sqrt{p_\mu \sigma^\mu} \\ 0 \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \end{bmatrix} \quad \& \quad v^2(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \\ 0 \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \\ 0 \end{bmatrix}$$

Summary

The information present in dirac spinor are a) Spin state through ξ_s or η which has two degrees of freedom b) If we know the ξ_s or η & p^μ, then all four components of $u(p)$ or $v(p)$ will be known.

Orthogonality b/w $\{u^s(p)\}$ & $\{v^s(p)\}$

- $\bar{u}^s(p)\, v^r(p)$ is given by

$$= \begin{bmatrix} \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s^{s\dagger} & \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi_s^{s\dagger} \end{bmatrix} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\, \eta^r \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\, \eta^r \end{bmatrix}$$

$$= \left(\sqrt{(p_\mu \bar{\sigma}^\mu)(p_\nu \sigma^\nu)} - \sqrt{(p_\mu \sigma^\mu)(p_\nu \bar{\sigma}^\nu)} \right) \xi_s^{s\dagger} \eta^r$$

$$= \begin{bmatrix} \sqrt{(p_0 \mathbb{I}_2 + \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{I}_2 - \vec{p}\cdot\vec{\sigma})} \\ -\sqrt{(p_0 \mathbb{I}_2 - \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{I}_2 + \vec{p}\cdot\vec{\sigma})} \end{bmatrix} \xi_s^{s\dagger} \eta^r$$

$$= \left(\sqrt{(p_0)^2 - |\vec{p}|^2} - \sqrt{(p_0)^2 - |\vec{p}|^2} \right) \xi_s^{s\dagger} \eta^r$$

$$= \underline{0}$$

$\longrightarrow \{u^s(p)\}$ & $\{v^r(p)\}$ are always orthogonal w.r.t the inner product of the form

$$\underline{\bar{u}^s(p)\, v^r(p) = 0 = \bar{v}^r(p)\, u^s(p)}$$

- $u^{r\dagger}(p)v^{s}(p) \neq 0$

$u^{r\dagger}(p)v^{s}(p)$ is given by,

$$= \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi_s^{s\dagger} & \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi_s^{s\dagger} \end{bmatrix} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\eta^r \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\,\eta^r \end{bmatrix}$$

$$= \left[(p_\mu \sigma^\mu) - (p_\mu \bar{\sigma}^\mu) \right] \xi_s^{s\dagger} \eta^r$$

$$= \left[(p_0 I_2 - \vec{p}\cdot\vec{\sigma}) - (p_0 I_2 + \vec{p}\cdot\vec{\sigma}) \right] \xi_s^{s\dagger} \eta^r$$

$$= -2(\vec{p}\cdot\vec{\sigma})\,\xi_s^{s\dagger}\eta^r \neq 0 \quad \text{in general.}$$

But, $u^{r\dagger}(p_0,-\vec{p})v^{s}(p_0,\vec{p}) = 0$

$u^{r\dagger}(p_0,\vec{p})v^{s}(p_0,-\vec{p}) = 0$

Similarly, $v^{s\dagger}(p_0,\vec{p})u^{r}(p_0,-\vec{p}) = 0$

$v^{s\dagger}(p_0,-\vec{p})u^{r}(p_0,\vec{p}) = 0$

- Spin Sums

a) $$\sum_{s=\{1,2\}} u^s(p)\bar{u}^s(p) = \gamma^\mu p_\mu + m I_4$$

b) $$\sum_{s=\{1,2\}} v^s(p)\bar{v}^s(p) = \gamma^\mu p_\mu - m I_4$$

Proof a) $\sum_{s=\{1,2\}} u^s(p)\bar{u}^s(p)$ is given by,

$$= \sum_{s=\{1,2\}} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi^s \end{bmatrix} \begin{bmatrix} \sqrt{p_\mu \bar{\sigma}^\mu}\,\xi^{s\dagger} & \sqrt{p_\mu \sigma^\mu}\,\xi^{s\dagger} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} & \sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \sigma^\nu} \\ \sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} & \sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\nu \sigma^\nu} \end{bmatrix} \underbrace{\sum_{s=\{1,2\}} \xi^s \xi^{s\dagger}}_{\mathbb{I}_2}$$

Completeness relation

$$= \begin{bmatrix} \sqrt{p_\mu p^\mu} & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & \sqrt{p_\mu p^\mu} \end{bmatrix} = \gamma^\mu p_\mu + m\mathbb{I}_2$$

$$\rightarrow \sum_{s=\{1,2\}} u^s(p)\bar{u}^s(p) = \not{p} + m\mathbb{I}_2$$

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \Rightarrow \text{In chiral basis}$$

$$\not{p} = \gamma^\mu p_\mu$$

Proof b) $\sum_{s=\{1,2\}} v^s(p)\bar{v}^s(p)$ is given by,

$$= \sum_{s=\{1,2\}} \begin{bmatrix} \sqrt{p_\mu \sigma^\mu}\,\eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\,\eta^s \end{bmatrix} \begin{bmatrix} -\sqrt{p_\mu \bar{\sigma}^\mu}\,\eta^{s\dagger} & \sqrt{p_\mu \sigma^\mu}\,\eta^{s\dagger} \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} & \sqrt{p_\mu \sigma^\mu}\sqrt{p_\nu \sigma^\nu} \\ \sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\nu \bar{\sigma}^\nu} & -\sqrt{p_\mu \bar{\sigma}^\mu}\sqrt{p_\nu \sigma^\nu} \end{bmatrix} \underbrace{\sum_{s=\{1,2\}} \eta^s \eta^{s\dagger}}_{\mathbb{I}_2}$$

Completeness relation

$$= \begin{bmatrix} -\sqrt{p_\mu p^\mu} & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -\sqrt{p_\mu p^\mu} \end{bmatrix} = \gamma^\mu p_\mu - m\mathbb{I}_2$$

$$\rightarrow \sum_{s=\{1,2\}} v^s(p)\bar{v}^s(p) = \not{p} - m\mathbb{I}_2$$

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix} \Rightarrow \text{In chiral basis}$$

$$\not{p} = \gamma^\mu p_\mu$$

Quantization of Dirac field

Now we will look in Quantum fields that obey Dirac's eqⁿ as their equations of motion.

The lagrangian of Dirac field, whose eqⁿ of motion results in Dirac's eqⁿ is,

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

It should be kept in mind that, ψ is a spinor field.

$$\psi = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}$$

, ψ_A is A^{th} component of the spinor

And $\bar{\psi} = \psi^+ \gamma^0$ is the Dirac adjoint. In trying to second quantize the dirac field, we treat ψ & $\bar{\psi}$ as independent Spinor field operators.

The lagrangian density is Lorentz invariant it is lorentz scalar.

The conjugate momentum fields for ψ & $\overline{\psi}$ are,

$$\Pi^0_\psi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\overline{\psi}\gamma^0 = i\psi^\dagger$$

& $$\Pi^0_{\overline{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \overline{\psi})} = 0$$

Then the Hamiltonian density is given by,

$$\mathcal{H} = \Pi^0_\psi \partial_0 \psi + \Pi^0_{\overline{\psi}} \partial_0 \overline{\psi} - \mathcal{L}$$

$$= i\psi^\dagger(\partial_0 \psi) - \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$= i\overline{\psi}\gamma^0(\partial_0 \psi) - i\overline{\psi}\gamma^0(\partial_0 \psi) - i\overline{\psi}(\vec{\gamma}\cdot\vec{\nabla})\psi + m\overline{\psi}\psi$$

$$\rightarrow \mathcal{H} = -i\,\overline{\psi}(\vec{\gamma}\cdot\vec{\nabla})\psi + m\overline{\psi}\psi$$

$$= \overline{\psi}[-i\vec{\gamma}\cdot\vec{\nabla} + m]\psi$$

Now 'ψ' satisfies dirac eq^n,

$$\rightarrow \quad i\gamma^\mu\partial_\mu\psi - m\psi = 0$$

$$\rightarrow \quad i\gamma^0\partial_0\psi + i\vec{\gamma}\cdot\vec{\nabla}\psi - m\psi = 0$$

$$\Rightarrow \quad i\gamma^0\partial_0\psi = -i\vec{\gamma}\cdot\vec{\nabla}\psi + m\psi$$

$$\Rightarrow i\partial_0\psi = \underbrace{[\gamma^0\vec{\gamma}\cdot(-i\vec{\nabla}) + m\gamma^0]}\psi$$

$\hat{H}_D \rightarrow$ Dirac hamiltonian

$$\rightarrow i\partial_0 = \hat{H}_D = \gamma^0\vec{\gamma}\cdot(-i\vec{\nabla}) + m\gamma^0$$

$$\rightarrow \mathcal{H} = i\overline{\psi}\gamma^0\partial_0\psi = i\psi^\dagger\partial_0\psi$$

$$\rightarrow \boxed{\mathcal{H} = \psi^\dagger(i\partial_0\psi)}$$

Mode expansion of Dirac field

The dispersion relation that the Dirac field satisfies is given by,

$$(p^0)^2 = E_p^2 = |\vec{p}|^2 + m^2$$

And since ψ_A is not Hermitian & is complex valued, in similarity with complex scalar field & many scalar field theories, we can write the mode expansion of $\hat{\psi}$ & $\hat{\overline{\psi}}$ as

$$\hat{\psi}(x) = \int \frac{d^3p \sum_{s=1}^{2} \left[u^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} + v^s(p) \hat{b}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\overline{\psi}}(x) = \int \frac{d^3p \sum_{s=1}^{2} \left[\overline{u}^s(p) \hat{a}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu} + \overline{v}^s(p) \hat{b}_s(\vec{p}) e^{-ip_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_p)}}$$

The creation and annihilation operators are given as follows,

$\hat{a}_s$ & $\hat{a}_s^+$ → particles (+ve energy sol^n)

$\hat{b}_s$ & $\hat{b}_s^+$ → Anti-particles (-ve energy sol^n)

$\hat{a}_1^+(\vec{p})$ → creates a Dirac particle with 3-momentum $\vec{p}$, energy E_p, with Spin up along $\vec{p}$ axis & hence has +ve helicity.

$\hat{a}_2^+(\vec{p})$ → creates a Dirac particle with 3-momentum $\vec{p}$, energy E_p, with Spin down along $\vec{p}$ axis & hence has -ve helicity.

$\hat{b}_1^+(\vec{p})$ → creates a Dirac anti-particle with 3-momentum $\vec{p}$, energy E_p, with Spin up along $\vec{p}$ axis & hence has +ve helicity.

$\hat{b}_2^+(\vec{p})$ → creates a Dirac anti-particle with 3-momentum $\vec{p}$, energy E_p, with Spin down along $\vec{p}$ axis & hence has -ve helicity.

2nd quantized Dirac field Hamiltonian

The Hamiltonian density for Dirac field is $\hat{\mathcal{H}} = \hat{\psi}^{\dagger}(i\partial_0\hat{\psi})$

$$\Rightarrow \hat{H} = \int d^3x\, \hat{\mathcal{H}} = \int d^3x\, \hat{\psi}^{\dagger}(i\partial_0\hat{\psi})$$

Using the mode expansion of $\hat{\psi}$ & $\hat{\psi}^{\dagger}$, we can express $\hat{H}$ in terms of ladder operators and get the 2nd quantized Dirac field Hamiltonian.

$$i\partial_0\hat{\psi}(x) = \int \frac{d^3p\, E_p \sum_{s=1}^{2}\left[u^s(p)\hat{a}_s(\vec{p})e^{-ip_\mu x^\mu} - v^s(p)\hat{b}_s^{\dagger}(\vec{p})e^{ip_\mu x^\mu}\right]}{\sqrt{(2\pi)^3(2E_p)}}$$

$$\hat{\psi}^{\dagger}(x) = \int \frac{d^3q \sum_{\gamma=1}^{2}\left[u^{\dagger\gamma}(q)\hat{a}_\gamma^{\dagger}(\vec{q})e^{iq_\mu x^\mu} + v^{\dagger\gamma}(q)\hat{b}_\gamma(\vec{q})e^{-iq_\mu x^\mu}\right]}{\sqrt{(2\pi)^3(2E_q)}}$$

$$\hat{H} = \int d^3x\, \hat{\psi}^{\dagger}(i\partial_0\hat{\psi})$$

$$\hat{H} = \int \frac{d^3x\, d^3q\, d^3p\, E_p}{(2\pi)^3 2\sqrt{E_p E_q}}$$

$$\times \sum_{s,r} \Big[\big[u^{\dagger r}(q) \hat{a}^{\dagger}_r(\vec{q}) e^{iq_\mu x^\mu} + v^{\dagger r}(q) \hat{b}_r(\vec{q}) e^{-iq_\mu x^\mu} \big] \times \big[u^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} - v^s(p) \hat{b}^{\dagger}_s(\vec{p}) e^{ip_\mu x^\mu} \big] \Big]$$

$$= \int \frac{d^3x\, d^3q\, d^3p\, E_p}{(2\pi)^3 2\sqrt{E_p E_q}} \sum_{s,r} \Big[u^{\dagger r}(q) u^s(p) \hat{a}^{\dagger}_r(\vec{q}) \hat{a}_s(\vec{p}) e^{i(q_\mu - p_\mu)x^\mu} - v^{\dagger r}(q) v^s(p) \hat{b}_r(\vec{q}) \hat{b}^{\dagger}_s(\vec{p}) e^{i(p_\mu - q_\mu)x^\mu} + v^{\dagger r}(q) u^s(p) \hat{b}_r(\vec{q}) \hat{a}_s(\vec{p}) e^{-i(p_\mu + q_\mu)x^\mu} - u^{\dagger r}(q) v^s(p) \hat{a}^{\dagger}_r(\vec{q}) \hat{b}^{\dagger}_s(\vec{p}) e^{i(p_\mu + q_\mu)x^\mu} \Big]$$

Perform integral over d^3x to get dirac delta functions, & then do the integral over d^3q. This makes $\vec{q} \to \pm\vec{p}$ & $E_p = E_{-p}$, so we get,

$$= \int \frac{d^3p}{2} \sum_{s,r} \Big[u^{\dagger r}(p) u^s(p) \hat{a}^{\dagger}_r(\vec{p}) \hat{a}_s(\vec{p}) - v^{\dagger r}(p) v^s(p) \hat{b}_r(\vec{p}) \hat{b}^{\dagger}_s(\vec{p}) + v^{\dagger r}(E_p, -\vec{p}) u^s(p) \hat{b}_r(-\vec{p}) \hat{a}_s(\vec{p}) e^{-2iE_p t} - u^{\dagger r}(E_p, -\vec{p}) v^s(p) \hat{a}^{\dagger}_r(-\vec{p}) \hat{b}^{\dagger}_s(\vec{p}) e^{2iE_p t} \Big]$$

The orthonormalization for spinor basis $\{u^r\}$ & $\{v^s\}$ can be used to get

$$u^{r\dagger}(p)\,u^s(p) = (2p_0)\,\delta^{rs}$$

$$v^{r\dagger}(p)\,v^s(p) = (2p_0)\,\delta^{rs}$$

$$v^{\dagger r}(E_p, -\vec{p})\,u^s(p) = u^{\dagger r}(E_p, -\vec{p})\,v^s(p) = 0$$

$$\longrightarrow \hat{H} = \int d^3p\, E_p \sum_s \left[\hat{a}_s^\dagger(\vec{p})\,\hat{a}_s(\vec{p}) - \hat{b}_s(\vec{p})\,\hat{b}_s^\dagger(\vec{p})\right]$$

Notice that we haven't defined or used any commutation or anti-commutation relations yet in this theory. If in case we treat the particles as bosons, then on normal ordering, the Hamiltonian becomes the difference in particle & anti-particle energy. And by having an infinite number of anti-particles, we can lower the energy indefinitely. This doesn't make sense. Thus we cannot treat Dirac particles as bosons.

Anti-commutation relation for Dirac field Operators

The dirac eqn describes particles of spin 1/2 which are fermions. So to have meaningful interpretations, we need to impose the equal time anti-commutation relations b/w the field operators. They are given by,

$$\{\hat{\psi}_A(x), \hat{\psi}_B(y)\} = \{\Pi^0_A(x), \Pi^0_B(y)\} = 0$$

$$\{\hat{\psi}_A(t,\vec{x}), \Pi^0_B(t,\vec{y})\} = i\delta_{AB}\delta(\vec{x}-\vec{y})$$

But note that $\Pi^0_\psi = i\psi^\dagger$. Thus the anti-commutation relations become,

$$\{\hat{\psi}_A(x), \hat{\psi}_B(y)\} = \{\hat{\psi}^\dagger_A(x), \hat{\psi}^\dagger_B(y)\} = 0$$

$$\{\hat{\psi}_A(t,\vec{x}), \hat{\psi}^\dagger_B(t,\vec{y})\} = \delta_{AB}\delta(\vec{x}-\vec{y})$$

This results in the following anti-commutation relations for ladder operators.

$$\{\hat{a}_s(\vec{p}), \hat{a}_\gamma^+(\vec{q})\} = \delta_{s\gamma}\delta(\vec{p}-\vec{q})$$

$$\{\hat{b}_s(\vec{p}), \hat{b}_\gamma^+(\vec{q})\} = \delta_{s\gamma}\delta(\vec{p}-\vec{q})$$

And all other anti-commutations relations are zero.

Normal ordering for fermionic operators

Here along with rearrangement of creation & annihilation operators, we should multiply by '-1' for every every permutation / exchange.

Example: $\mathcal{N}[\hat{a}\hat{a}^+] = -\hat{a}^+\hat{a}$

$\mathcal{N}[\hat{a}^+\hat{a}] = \hat{a}^+\hat{a}$, $\mathcal{N}[\hat{a}\hat{a}\hat{a}^+\hat{a}^+] = \hat{a}^+\hat{a}^+\hat{a}\hat{a}$

$\mathcal{N}[\hat{a}\hat{a}^+\hat{a}\hat{a}^+] = -\hat{a}^+\hat{a}^+\hat{a}\hat{a}$

Quantized Dirac field

Resuming from the expression of Dirac field Hamiltonian, we have

$$\hat{H} = \int d^3p\, E_p \sum_s \left[\hat{a}_s^+(\vec{p})\hat{a}_s(\vec{p}) - \hat{b}_s(\vec{p})\hat{b}_s^+(\vec{p})\right]$$

We shall normal order this Hamiltonian in terms of Anti-commuting ladder operators. Then we finally get,

$$N[\hat{H}] = \int d^3p\, E_p \sum_s \left[\hat{a}_s^+(\vec{p})\hat{a}_s(\vec{p}) + \hat{b}_s^+(\vec{p})\hat{b}_s(\vec{p})\right]$$

Thus we can observe that vacuum state is the lowest energy state.

Creating either a Dirac particle or anti-particle will result in the energy of E_p.

Energy Momentum tensor for Dirac Field

The dirac lagrangian is given by

$$\hat{\mathcal{L}} = \hat{\bar{\psi}}(i\gamma^\mu \partial_\mu \hat{\psi}) - m\hat{\bar{\psi}}\hat{\psi}$$

The energy-momentum tensor is

$$\hat{\Theta}^{\mu\nu} = \hat{\Pi}^\mu_\psi(\partial^\nu\hat{\psi}) + \hat{\Pi}^\mu_{\bar{\psi}}(\partial^\nu\hat{\bar{\psi}}) - \eta^{\mu\nu}\hat{\mathcal{L}}$$

$$\hat{\Pi}^\mu_\psi = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\hat{\psi})} = i\hat{\bar{\psi}}\gamma^\mu \quad , \quad \hat{\Pi}^\mu_{\bar{\psi}} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\hat{\bar{\psi}})} = 0$$

$$\rightarrow \underline{\hat{\Theta}^{\mu\nu} = i\hat{\bar{\psi}}\gamma^\mu(\partial^\nu\hat{\psi}) - \eta^{\mu\nu}\hat{\mathcal{L}}}$$

$$\hat{\Theta}^{00} = \underbrace{i\hat{\bar{\psi}}\gamma^0(\partial^0\hat{\psi}) - \hat{\mathcal{L}}}_{\text{Hamiltonian Density}} = \hat{\mathcal{H}}$$

$$\hat{\Theta}^{0j} = i\hat{\bar{\psi}}\gamma^0(\partial^j\hat{\psi}) - \underbrace{\eta^{0i}}_{0}\mathcal{L}$$

$$\rightarrow \hat{\Theta}^{0j} = \hat{\bar{\psi}}\gamma^0(-i\partial_j\hat{\psi}) = \hat{\psi}^\dagger(-i\partial_j\psi)$$

Defining the momentum operator as

$$\hat{P}^j = \int d^3x \, \hat{\mathcal{H}}^{0j}$$

$$\longrightarrow \boxed{\hat{\vec{P}} = \int d^3x \, \hat{\psi}^\dagger (-i \vec{\nabla} \hat{\psi})}$$

Thus the four momentum operator is given by,

$$\hat{P}^\mu = (\hat{H}, \hat{\vec{P}}) = \int d^3x \, \hat{\psi}^\dagger (i \partial^\mu \hat{\psi})$$

This can be espressed in terms of ladder operators in exact same calculations as before.

$$\hat{\psi}^\dagger(x) = \int \frac{d^3q \sum_{s=1}^{2} \left[u^{s\dagger}(q) \hat{a}^\dagger_s(\vec{q}) e^{iq_\mu x^\mu} + v^{s\dagger}(q) \hat{b}_s(\vec{q}) e^{-iq_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_q)}}$$

$$i\partial^\mu \hat{\psi}(x) = \int \frac{d^3p \, p^\mu \sum_{s=1}^{2} \left[u^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} - v^s(p) \hat{b}^\dagger_s(\vec{p}) e^{ip_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{P}^\mu = \int \frac{d^3x\, d^3q\, d^3p}{(2\pi)^3 2\sqrt{E_p E_q}}\, p^\mu$$

$$\times \sum_{s,r} \Big[\big[u^r(q)\hat{a}^\dagger_r(\vec{q}) e^{iq_\mu x^\mu} + v^r(q)\hat{b}_r(\vec{q}) e^{-iq_\mu x^\mu} \big] \times \big[u^s(p)\hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} - v^s(p)\hat{b}^\dagger_s(\vec{p}) e^{ip_\mu x^\mu} \big] \Big]$$

$$= \int \frac{d^3x\, d^3q\, d^3p\, p^\mu}{(2\pi)^3 2\sqrt{E_p E_q}} \sum_{s,r} \Big[u^{\dagger r}(q) u^s(p) \hat{a}^\dagger_r(\vec{q}) \hat{a}_s(\vec{p}) e^{i(q_\mu - p_\mu)x^\mu} - v^{\dagger r}(q) v^s(p) \hat{b}_r(\vec{q}) \hat{b}^\dagger_s(\vec{p}) e^{i(p_\mu - q_\mu)x^\mu} + v^{\dagger r}(q) u^s(p) \hat{b}_r(\vec{q}) \hat{a}_s(\vec{p}) e^{-i(p_\mu + q_\mu)x^\mu} - u^{\dagger r}(q) v^s(p) \hat{a}^\dagger_r(\vec{q}) \hat{b}^\dagger_s(\vec{p}) e^{i(p_\mu + q_\mu)x^\mu} \Big]$$

Perform integral over d^3x to get dirac delta functions, & then do the integral over d^3q. This makes $\vec{q} \to \pm\vec{p}$ & $E_p = E_{-p}$, so we get,

$$= \int \frac{d^3p}{2E_p}\, p^\mu \sum_{s,r} \Big[u^{\dagger r}(p) u^s(p) \hat{a}^\dagger_r(\vec{p}) \hat{a}_s(\vec{p}) - v^{\dagger r}(p) v^s(p) \hat{b}_r(\vec{p}) \hat{b}^\dagger_s(\vec{p}) + v^{\dagger r}(E_p, -\vec{p}) u^s(p) \hat{b}_r(-\vec{p}) \hat{a}_s(\vec{p}) e^{-2iE_p t} - u^{\dagger r}(E_p, -\vec{p}) v^s(p) \hat{a}^\dagger_r(-\vec{p}) \hat{b}^\dagger_s(\vec{p}) e^{2iE_p t} \Big]$$

The orthonormalization for spinor basis $\{u^r\}$ & $\{v^s\}$ can be used to get

$$u^{r\dagger}(p)\,u^s(p) = (2E_p)\,\delta^{rs}$$

$$v^{r\dagger}(p)\,v^s(p) = (2E_p)\,\delta^{rs}$$

$$v^{\dagger r}(E_p, -\vec{p})\,u^s(p) = u^{\dagger r}(E_p, -\vec{p})\,v^s(p) = 0$$

$$\rightarrow \hat{P}^\mu = \int d^3p\; p^\mu \sum_s \left[\hat{a}_s^\dagger(\vec{p})\,\hat{a}_s(\vec{p}) - \hat{b}_s(\vec{p})\,\hat{b}_s^\dagger(\vec{p})\right]$$

On normal ordering, we get

$$\boxed{N\left[\hat{P}^\mu\right] = \int d^3p\; p^\mu \sum_s \left[\hat{a}_s^\dagger(\vec{p})\,\hat{a}_s(\vec{p}) + \hat{b}_s^\dagger(\vec{p})\,\hat{b}_s(\vec{p})\right]}$$

Thus $\hat{a}_s^\dagger$ & $\hat{b}_s^\dagger$ will create Particles & anti-particles with the same energy E_p & momentum $\vec{p}$.

Consider the Normalization factors for the one-particle states created by the action of ladder operators.

The one particle state created by $\hat{a}_s^+(\vec{p})$ acting on $|\Omega\rangle$ will now carry the information of 4 momentum $(E_p, \vec{p})$ & as well as the spin state (s).

$$\Longrightarrow \hat{a}_s^+(\vec{p})|\Omega\rangle = |\vec{p}, s\rangle \quad \text{where,}$$

$$\langle \vec{q}, \gamma | \vec{p}, s\rangle = \langle \Omega | \hat{a}_\gamma(\vec{q}) \hat{a}_s^+(\vec{p}) |\Omega\rangle = \delta(\vec{p}-\vec{q})\delta_{\gamma s}$$

But consider $\hat{\alpha}_s^+(p^\mu) = (2\pi)^{3/2}\sqrt{2E_p}\,\hat{a}_s^+(\vec{p})$

$$\Rightarrow \hat{\alpha}_s^+(p^\mu)|\Omega\rangle = |p^\mu, s\rangle, \text{ such that}$$

$$\langle q^\mu, \gamma | p^\mu, s\rangle = \langle \Omega | \hat{\alpha}_\gamma(q^\mu)\hat{\alpha}_s^+(p^\mu)|\Omega\rangle$$

$$\Rightarrow \langle q^\mu, \gamma | p^\mu, s\rangle = 2E_p(2\pi)^3 \delta(\vec{p}-\vec{q})\delta_{\gamma s}$$

is lorentz invariant inner product.

In terms of $\{\hat{\alpha}_s(p^\mu)\}$ and $\{\hat{\beta}_s(p^\mu)\}$ where, $\hat{\beta}_s^+(p^\mu) = (2\pi)^{3/2}\sqrt{2E_p}\,\hat{b}_s^+(\vec{p})$

$$\hat{\psi}(x) = \int \frac{d^3p \sum_{s=1}^{2} \left[u^s(p)\hat{\alpha}_s(p^\mu)e^{-ip_\mu x^\mu} + v^s(p)\hat{\beta}_s^+(p^\mu)e^{ip_\mu x^\mu}\right]}{(2\pi)^3(2E_p)}$$

Representation of Translational algebra of Dirac Field Theory

The infinitesimal translation in space-time is given by,

$$x^\mu \longrightarrow x'^\mu = x^\mu + \delta x^\mu$$

$$\rightarrow \hat{\psi}(x) \longrightarrow \hat{\psi}(x^\mu + \delta x^\mu)$$

$$\rightarrow \hat{\psi}(x) \longrightarrow \hat{\psi}(x) + \delta x^\mu [\partial_\mu \hat{\psi}(x)]$$

$$\rightarrow \delta\hat{\psi}(x) = \delta x^\mu [\partial_\mu] \hat{\psi}(x)$$
$$= -i\, \delta x_\mu (i\partial^\mu) \hat{\psi}(x)$$

$$\rightarrow \delta\hat{\psi}(x) = -i\, \delta x_\mu (i\partial^\mu) \hat{\psi}(x)$$

$$\rightarrow \Delta\hat{\psi} \equiv -\frac{\delta\hat{\psi}(x)}{\delta x_\mu} = \partial^\mu \hat{\psi}(x)$$

Then the corresponding Noether charge that is conserved is the generator of Space-time translation. It is:

$$\int d^3x \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}\right] \Delta\hat{\psi}$$

$$= \int d^3x \, (i\hat{\psi}^+)(\partial^\mu \hat{\psi})$$

$$= \int d^3x \, \hat{\psi}^+ (i\partial^\mu)\hat{\psi}$$

But $\hat{P}^\mu = \int d^3x \, \hat{\psi}^+ (i\partial^\mu)\hat{\psi}$

where,

$$\hat{P}^\mu = \int d^3p \, p^\mu \sum_s \left[\hat{a}_s^+(\vec{p})\hat{a}_s(\vec{p}) - \hat{b}_s^+(\vec{p})\hat{b}_s(\vec{p})\right]$$

$\rightarrow \hat{P}^\mu$ is the generator of the Space-time translations given in the Dirac field Theory representation.

Consider the 4-momentum operator

$$\hat{P}^\mu = \int d^3x \, \hat{\psi}^\dagger (i \partial^\mu \hat{\psi})$$

$$\{\hat{P}^\mu, \hat{\psi}(y)\} = \int d^3x \{\hat{\psi}^\dagger(x)[i\partial^\mu \hat{\psi}(x)], \hat{\psi}(y)\}$$

$$= \int d^3x \{\hat{\psi}^\dagger(x), \hat{\psi}(y)\}[i\partial^\mu \hat{\psi}(x)]$$

$$= \int d^3x \, \delta(\vec{x}-\vec{y})[i\partial^\mu \hat{\psi}(x)]$$

$$= [i\partial^\mu \hat{\psi}(y)]$$

$$\longrightarrow \underline{\{\hat{P}^\mu, \hat{\psi}(x)\} = i\partial^\mu \psi(x)}$$

$\Longrightarrow$ Thus $\hat{T} = e^{-ia_\mu \hat{P}^\mu}$ is the translational operator which gives

$$\underline{\hat{T}^{-1}\hat{\psi}(x^\mu)\hat{T} = \hat{\psi}(x^\mu + a^\mu)}$$

$\hat{T} \rightarrow$ element of translational group

group Algebra

Representation of Rotation algebra of Dirac Field Theory

We still haven't shown that the Dirac particles are spin-half particles or this 'spin' has anything to do with angular momentum. Thus we shall try to write the representation of angular Momentum for Dirac Spinors.

We can find the angular momentum representation as a consequence of the rotational invariance.

$$\text{Let, } x^\mu \xrightarrow[x^3\text{- axis}]{\text{Rot. about}} x'^\mu = \Lambda^\mu_{\ \nu}\, x^\nu$$

$$\longrightarrow \hat{\psi}(x) \xrightarrow[x^3\text{- axis}]{\text{Rot. about}} \hat{\psi}'(x) = \Lambda_{\frac{1}{2}}\, \hat{\psi}(\Lambda^\mu_{\ \nu}, x^\nu)$$

Consider an infinitesimal rotation about x^3-axis by an angle $\delta\phi$. Then the coordinate & Spinor rep^n only upto 1^{st} order in $\delta\phi$ is:

$$x^1 \longrightarrow x^1 + x^2 \delta\phi$$
$$x^2 \longrightarrow x^2 - x^1 \delta\phi$$
$$x^3 \longrightarrow x^3$$
} Coordinate rep^n

$$\Lambda_{\frac{1}{2}} \approx \mathbb{I}_4 - i\,\delta\phi\, S_3$$

where $S_3 = \frac{1}{2}\Sigma_3$

$$\Rightarrow \delta\hat{\psi} = \Lambda_{\frac{1}{2}} \hat{\psi}(\Lambda^\mu_\nu x^\nu) - \hat{\psi}(x)$$

$$= \left[\mathbb{I}_4 - i\,\delta\phi\, S_z\right] \hat{\psi}(\Lambda^\mu_\nu x^\nu) - \hat{\psi}(x)$$

$$= \hat{\psi}(\Lambda^\mu_\nu x^\nu) - \hat{\psi}(x) - i\,\delta\phi\, S_z \hat{\psi}(\Lambda^\mu_\nu x^\nu)$$

$$= \hat{\psi}(x^1 + x^2\delta\phi, x^2 - x^1\delta\phi) - \hat{\psi}(x)$$
$$- \frac{i\,\delta\phi}{2} \Sigma_3 \hat{\psi}(x^1 + x^2\delta\phi, x^2 - x^1\delta\phi)$$

$$= \delta\phi\,(x^2\partial_1 - x^1\partial_2)\hat{\psi}(x)$$
$$- \frac{i\,\delta\phi}{2}\Sigma_3 \left[\hat{\psi}(x) + \underbrace{\delta\phi\,(x^2\partial_1 - x^1\partial_2)}_{\text{gives } 2^{nd} \text{ order term.}} \hat{\psi}(x)\right]$$

$$\Rightarrow \delta\hat{\psi} = \delta\phi\left[(x^2\partial_1 - x^1\partial_2) - \frac{i}{2}\Sigma_3\right]\hat{\psi}(x)$$

$$\rightarrow \Delta\hat{\psi} \equiv \frac{\delta\hat{\psi}(x)}{\delta\phi}$$

$$\rightarrow \Delta\hat{\psi} = \left[(x^2\partial_1 - x^1\partial_2) - \frac{i}{2}\Sigma_3\right]\hat{\psi}(x)$$

Then the corresponding Noether charge that is conserved is the generator of rotation about x^3-axis which is the 3rd component of Angular momentum.

$$\hat{J}_3 = \int d^3x \left[\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}\right]\Delta\hat{\psi}$$

$$= \int d^3x \left(i\hat{\psi}^+\right)\left[(x^2\partial_1 - x^1\partial_2) - \frac{i}{2}\Sigma_3\right]\hat{\psi}(x)$$

$$= \int d^3x\, \hat{\psi}^+\left[-i(x^1\partial_2 - x^2\partial_1) + \frac{1}{2}\Sigma_3\right]\hat{\psi}(x)$$

$\vec{J} = (J_1, J_2, J_3)$ form the basis of space of generators of rotations.

It is given by,

$$\hat{\vec{J}} = \int d^3x\, \hat{\psi}^{+} \left[\vec{x} \times (-i\vec{\nabla}) + \frac{1}{2}\vec{\Sigma} \right] \hat{\psi}(x)$$

Then, $\hat{\vec{L}} = \int d^3x\, \hat{\psi}^{+} \left[\vec{x} \times (-i\vec{\nabla}) \right] \hat{\psi}(x)$

is the generator of coordinate rotation called as orbital angular momentum.

& $\hat{\vec{S}} = \int d^3x\, \hat{\psi}^{+} \left(\frac{1}{2}\vec{\Sigma} \right) \hat{\psi}(x)$ is the generator of rotation in spinor space, called as the spin angular momentum.

$\rightarrow \hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ is the total angular momentum, which includes both orbital & spin angular momentum.

Relation b/w the Spinor state & Spin eigen value for Dirac particle & anti particle

Here we shall prove that the Dirac particles are of Spin 1/2. And also map the 2-Spinors $\begin{bmatrix}1\\0\end{bmatrix}$ & $\begin{bmatrix}0\\1\end{bmatrix}$ to Spin eigen value of Dirac particle & anti particle.

Consider a Dirac particle at rest got by acting $\hat{a}_s^\dagger(0)$ on vacuum $|-\Omega\rangle$. We shall show that $\hat{a}_s^\dagger(0)|-\Omega\rangle$ is an eigen state of $\hat{S}_z$ & then find the eigen value.

where $\hat{S}_z = \int d^3x\, \hat{\psi}^\dagger \left(\frac{1}{2}\Sigma_z\right)\hat{\psi}(x)$

$$= \int \frac{d^3x\, d^3q\, d^3p}{(2\pi)^3 2\sqrt{E_p E_q}}$$

$$\times \sum_{s=1}^{2}\left[u^{\dagger s}(q)\hat{a}_s^\dagger(\vec{q})e^{iq_\mu x^\mu} + v^{\dagger s}(q)\hat{b}_s(\vec{q})e^{-iq_\mu x^\mu}\right]\left(\frac{1}{2}\Sigma_z\right)$$

$$\times \sum_{s=1}^{2}\left[u^s(p)\hat{a}_s(\vec{p})e^{-ip_\mu x^\mu} + v^s(p)\hat{b}_s^\dagger(\vec{p})e^{ip_\mu x^\mu}\right]$$

$$= \int \frac{d^3x\, d^3q\, d^3p \sum_{r,s}}{(2\pi)^3 4\sqrt{E_p E_q}} \Big[u^{\dagger r}(q) \sum_z u^s(p) \hat{a}^\dagger_r(\vec{q}) \hat{a}_s(\vec{p}) e^{i(q_\mu - p_\mu)x^\mu} + v^{\dagger r}(q) \sum_z v^s(p) \hat{b}_r(\vec{q}) \hat{b}^\dagger_s(\vec{p}) e^{i(p_\mu - q_\mu)x^\mu} + u^{\dagger r}(q) \sum_z v^s(p) \hat{a}^\dagger_r(\vec{q}) \hat{b}^\dagger_s(\vec{p}) e^{i(p_\mu + q_\mu)x^\mu} + v^{\dagger r}(q) \sum_z u^s(p) \hat{b}_r(\vec{q}) \hat{a}_s(\vec{p}) e^{-i(p_\mu + q_\mu)x^\mu} \Big]$$

Performing first d^3x integral to get dirac delta function, & then perform d^3q integral, & use $E_p = E_{-p}$, we get

$$\hat{S}_z = \int \frac{d^3p \sum_{r,s}}{4E_p} \Big[u^{\dagger r}(p) \sum_z u^s(p) \hat{a}^\dagger_r(\vec{p}) \hat{a}_s(\vec{p}) + v^{\dagger r}(p) \sum_z v^s(p) \hat{b}_r(\vec{p}) \hat{b}^\dagger_s(\vec{p}) + u^{\dagger r}(E_p, -\vec{p}) \sum_z v^s(p) \hat{a}^\dagger_r(-\vec{p}) \hat{b}^\dagger_s(\vec{p}) e^{2iE_p t} + v^{\dagger r}(E_p, -\vec{p}) \sum_z u^s(p) \hat{b}_r(-\vec{p}) \hat{a}_s(\vec{p}) e^{-2iE_p t} \Big]$$

Now we shall use a simple trick, Since vacuum has no particles or anti-particles, $\Longrightarrow \hat{S}_z |\Omega\rangle = 0$

$$\Longrightarrow \hat{S}_z \hat{a}^\dagger_l(0) |\Omega\rangle = \left(\hat{S}_z \hat{a}^\dagger_l(0) - \hat{a}^\dagger_l(0) \hat{S}_z \right) |\Omega\rangle$$

$$\Longrightarrow \hat{S}_z \hat{a}^\dagger_l(0) |\Omega\rangle = \left[\hat{S}_z, \hat{a}^\dagger_l(0) \right] |\Omega\rangle$$

Now we shall find $\left[\hat{S}_z, \hat{a}^+_\ell(0)\right]$

$$= \int \frac{d^3p}{4E_p} \sum_{r,s} \Big[u^{\dagger r}(p) \sum_z u^s(p) \left[\hat{a}^+_r(\vec{p})\hat{a}_s(\vec{p}), \hat{a}^+_\ell(0)\right] + v^{\dagger r}(p) \sum_z v^s(p) \left[\hat{b}_r(\vec{p})\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right] + u^{\dagger r}(E_p, -\vec{p}) \sum_z v^s(p) \left[\hat{a}^+_r(-\vec{p})\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right] e^{2iE_p t} + v^{\dagger r}(E_p, -\vec{p}) \sum_z u^s(p) \left[\hat{b}_r(-\vec{p})\hat{a}_s(\vec{p}), \hat{a}^+_\ell(0)\right] e^{-2iE_p t} \Big]$$

using ladder operator anti-commutation

$$\left[\hat{b}_r(\vec{p})\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right] = \hat{b}_r(\vec{p})\hat{b}^+_s(\vec{p})\hat{a}^+_\ell(0) - \hat{a}^+_\ell(0)\hat{b}_r(\vec{p})\hat{b}^+_s(\vec{p})$$

$$= \hat{b}_r(\vec{p})\hat{b}^+_s(\vec{p})\hat{a}^+_\ell(0) + \hat{b}_r(\vec{p})\hat{a}^+_\ell(0)\hat{b}^+_s(\vec{p})$$

$$= \hat{b}_r(\vec{p})\left(\hat{b}^+_s(\vec{p})\hat{a}^+_\ell(0) + \hat{a}^+_\ell(0)\hat{b}^+_s(\vec{p})\right)$$

$$= \hat{b}_r(\vec{p})\underbrace{\left\{\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right\}}_{0}$$

$$\rightarrow \left[\hat{b}_r(\vec{p})\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right] = 0$$

$$\left[\hat{a}^+_r(-\vec{p})\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right] = \hat{a}^+_r(-\vec{p})\hat{b}^+_s(\vec{p})\hat{a}^+_\ell(0) - \hat{a}^+_\ell(0)\hat{a}^+_r(-\vec{p})\hat{b}^+_s(\vec{p})$$

$$= -\hat{a}^+_r(-\vec{p})\hat{a}^+_\ell(0)\hat{b}^+_s(\vec{p}) - \hat{a}^+_\ell(0)\hat{a}^+_r(-\vec{p})\hat{b}^+_s(\vec{p})$$

$$= -\underbrace{\left\{\hat{a}^+_r(-\vec{p}), \hat{a}^+_\ell(0)\right\}}_{0}\hat{b}^+_s(\vec{p})$$

$$\rightarrow \left[\hat{a}^+_r(-\vec{p})\hat{b}^+_s(\vec{p}), \hat{a}^+_\ell(0)\right] = 0$$

$$\left[\hat{a}^{+}_{\gamma}(\vec{p})\hat{a}_{s}(\vec{p}),\hat{a}^{+}_{\ell}(0)\right]=\hat{a}^{+}_{\gamma}(\vec{p})\hat{a}_{s}(\vec{p})\hat{a}^{+}_{\ell}(0)-\hat{a}^{+}_{\ell}(0)\hat{a}^{+}_{\gamma}(\vec{p})\hat{a}_{s}(\vec{p})$$

$$=\hat{a}^{+}_{\gamma}(\vec{p})\hat{a}_{s}(\vec{p})\hat{a}^{+}_{\ell}(0)+\hat{a}^{+}_{\gamma}(\vec{p})\hat{a}^{+}_{\ell}(0)\hat{a}_{s}(\vec{p})$$

$$=\hat{a}^{+}_{\gamma}(\vec{p})\{\hat{a}_{s}(\vec{p}),\hat{a}^{+}_{\ell}(0)\}$$

$$=\hat{a}^{+}_{\gamma}(\vec{p})\,\delta_{s\ell}\,\delta^{(3)}(\vec{p})$$

$$\rightarrow\left[\hat{a}^{+}_{\gamma}(\vec{p})\hat{a}_{s}(\vec{p}),\hat{a}^{+}_{\ell}(0)\right]=\hat{a}^{+}_{\gamma}(\vec{p})\,\delta_{s\ell}\,\delta^{(3)}(\vec{p})$$

$$\left[\hat{b}_{\gamma}(-\vec{p})\hat{a}_{s}(\vec{p}),\hat{a}^{+}_{\ell}(0)\right]=\hat{b}_{\gamma}(-\vec{p})\hat{a}_{s}(\vec{p})\hat{a}^{+}_{\ell}(0)-\hat{a}^{+}_{\ell}(0)\hat{b}_{\gamma}(-\vec{p})\hat{a}_{s}(\vec{p})$$

$$=\hat{b}_{\gamma}(-\vec{p})\hat{a}_{s}(\vec{p})\hat{a}^{+}_{\ell}(0)+\hat{b}_{\gamma}(-\vec{p})\hat{a}^{+}_{\ell}(0)\hat{a}_{s}(\vec{p})$$

$$=\hat{b}_{\gamma}(-\vec{p})\{\hat{a}_{s}(\vec{p}),\hat{a}^{+}_{\ell}(0)\}$$

$$=\hat{b}_{\gamma}(-\vec{p})\,\delta_{s\ell}\,\delta^{(3)}(\vec{p})$$

$$\rightarrow\left[\hat{b}_{\gamma}(-\vec{p})\hat{a}_{s}(\vec{p}),\hat{a}^{+}_{\ell}(0)\right]=\hat{b}_{\gamma}(-\vec{p})\,\delta_{s\ell}\,\delta^{(3)}(\vec{p})$$

$$\rightarrow\left[\hat{S}_{z},\hat{a}^{+}_{\ell}(0)\right]$$

$$=\int\frac{d^{3}p\sum_{\gamma,s}}{4E_{p}}\left[\begin{array}{l}u^{\dagger\gamma}(p)\sum_{z}u^{s}(p)\,\hat{a}^{+}_{\gamma}(\vec{p})\,\delta_{s\ell}\,\delta^{(3)}(\vec{p})\\+v^{\dagger\gamma}(E_{p},-\vec{p})\sum_{z}u^{s}(p)\,\hat{b}_{\gamma}(-\vec{p})\,\delta_{s\ell}\,\delta^{(3)}(\vec{p})\,e^{-2iE_{p}t}\end{array}\right]$$

note: $E(\vec{p}=0)=m$

$$= \sum_{\substack{\gamma,s \\ 4m}} \begin{bmatrix} u^{\dagger\gamma}(m,0) \sum_z u^s(m,0) \hat{a}^+_\gamma(0) \delta_{sl} \\ + v^{\dagger\gamma}(m,0) \sum_z u^s(m,0) \hat{b}_\gamma(0) \delta_{sl} e^{-2imt} \end{bmatrix}$$

$$\rightarrow \{\hat{S}_z, \hat{a}^+_l(0)\}$$

$$= \sum_{\substack{\gamma \\ 4m}} \begin{bmatrix} u^{\dagger\gamma}(m,0) \sum_z u^l(m,0) \hat{a}^+_\gamma(0) \\ + v^{\dagger\gamma}(m,0) \sum_z u^l(m,0) \hat{b}_\gamma(0) e^{-2imt} \end{bmatrix}$$

Notice that 2nd term annihilates the vacuum. So it wont contribute when $\{\hat{S}_z, \hat{a}^+_l(0)\}$ acts on vacuum.

$$\Rightarrow \hat{S}_z \hat{a}^+_l(0)|\Omega\rangle = \{\hat{S}_z, \hat{a}^+_l(0)\}|\Omega\rangle$$

$$= \frac{1}{4m} \sum_\gamma u^{\dagger\gamma}(m,0) \sum_z u^l(m,0) \hat{a}^+_\gamma(0)|\Omega\rangle$$

we shall use the chiral basis

$$u^s(m,0) = \sqrt{m} \begin{bmatrix} \xi^s \\ \xi^s \end{bmatrix} \& \sum_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}$$

$$\Rightarrow \hat{S}_z \hat{a}^{+}_{\ell}(0)|\Omega\rangle =$$

$$= \frac{1}{4m} \sum_{\gamma} u^{\dagger\gamma}(m,0) \sum_z u^{\ell}(m,0) \hat{a}^{+}_{\gamma}(0)|\Omega\rangle$$

$$= \frac{1}{4} \sum_{\gamma} \begin{bmatrix} \xi^{\gamma +} & \xi^{\gamma +} \end{bmatrix} \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix} \begin{bmatrix} \xi^{\ell} \\ \xi^{\ell} \end{bmatrix} \hat{a}^{+}_{\gamma}(0)|\Omega\rangle$$

$$= \frac{1}{4} \sum_{\gamma} 2 \left(\xi^{\gamma +} \sigma_z \xi^{\ell} \right) \hat{a}^{+}_{\gamma}(0)|\Omega\rangle$$

$$= \sum_{\gamma} \xi^{\gamma +} \left(\frac{\sigma_z}{2} \right) \xi^{\ell} \left(\hat{a}^{+}_{\gamma}(0)|\Omega\rangle \right)$$

$$\Rightarrow \hat{S}_z \hat{a}^{+}_{\ell}(0)|\Omega\rangle$$

$$= \sum_{\gamma} \xi^{\gamma +} \left(\frac{\sigma_z}{2} \right) \xi^{\ell} \left(\hat{a}^{+}_{\gamma}(0)|\Omega\rangle \right)$$

Now we can choose the explict form for the Spinors ξ^{γ} to be basis for σ_z

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \xi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \xi^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{S}_z \hat{a}_l^+(0)|\Omega\rangle = (\xi^1)^+ \frac{\sigma_z}{2} \xi^l \hat{a}_1^+(0)|\Omega\rangle + (\xi^2)^+ \frac{\sigma_z}{2} \xi^l \hat{a}_2^+(0)|\Omega\rangle$$

$$= \left[+\frac{1}{2}(\xi^1)^+ \xi^l \hat{a}_1^+(0) - \frac{1}{2}(\xi^2)^+ \xi^l \hat{a}_2^+(0) \right]|\Omega\rangle$$

$$= \frac{1}{2}\left[\delta_{1l} \hat{a}_1^+(0) - \delta_{2l} \hat{a}_2^+(0) \right]|\Omega\rangle$$

$$\Rightarrow \hat{S}_z \hat{a}_l^+(0)|\Omega\rangle = \frac{1}{2}\left[\delta_{1l} \hat{a}_1^+(0) - \delta_{2l} \hat{a}_2^+(0) \right]|\Omega\rangle$$

$$\rightarrow \hat{S}_z \left(\hat{a}_1(0)|\Omega\rangle \right) = +\frac{1}{2}\left(\hat{a}_1(0)|\Omega\rangle \right)$$

$$\rightarrow \hat{S}_z \left(\hat{a}_2(0)|\Omega\rangle \right) = -\frac{1}{2}\left(\hat{a}_2(0)|\Omega\rangle \right)$$

Thus, $\hat{a}_1^+(0)|\Omega\rangle$ that corresponding to the dirac Spinor $u^1(m,0) = \sqrt{m}\begin{bmatrix} \xi^1 \\ \xi^1 \end{bmatrix}$

where $\xi^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, represents one particle state which has Spin up along z-direction.

And $\hat{a}_2^\dagger(0)|\Omega\rangle$ that corresponding to the dirac Spinor $u^2(m,0) = \sqrt{m}\begin{bmatrix} \xi_s^2 \\ \xi_s^2 \end{bmatrix}$

where $\xi_s^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, represents one particle state which has Spin down along z-direction.

Now we shall do the same procedure & find Spin for anti-particles states. We shall show that $\hat{b}_\ell^\dagger(0)$ creates a single anti-particle at rest & $\hat{b}_\ell^\dagger(0)|\Omega\rangle$ is an eigen state of $\hat{S}_z$ operator.

Consider the Spin operator in terms of ladder operators,

$$\hat{S}_z = \int \frac{d^3p \sum_{\gamma,s}}{4E_p} \Big[u^{\dagger\gamma}(p) \textstyle\sum_z u^s(p) \hat{a}_\gamma^\dagger(\vec{p}) \hat{a}_s(\vec{p}) + v^{\dagger\gamma}(p) \sum_z v^s(p) \hat{b}_\gamma(\vec{p}) \hat{b}_s^\dagger(\vec{p}) + u^{\dagger\gamma}(E_p, -\vec{p}) \sum_z v^s(p) \hat{a}_\gamma^\dagger(-\vec{p}) \hat{b}_s^\dagger(\vec{p}) e^{2iE_p t} + v^{\dagger\gamma}(E_p, -\vec{p}) \sum_z u^s(p) \hat{b}_\gamma(-\vec{p}) \hat{a}_s(\vec{p}) e^{-2iE_p t} \Big]$$

Now we shall use the same trick ∴ Since vacuum has no particles or anti-particles,

$$\longrightarrow \hat{S}_z|\Omega\rangle = 0$$

$$\longrightarrow \hat{S}_z \hat{b}_\ell^+(0)|\Omega\rangle = \left(\hat{S}_z \hat{b}_\ell^+(0) - \hat{b}_\ell^+(0)\hat{S}_z\right)|\Omega\rangle$$

$$\longrightarrow \hat{S}_z \hat{b}_\ell^+(0)|\Omega\rangle = \left[\hat{S}_z, \hat{b}_\ell^+(0)\right]|\Omega\rangle$$

Now we shall find $\left[\hat{S}_z, \hat{b}_\ell^+(0)\right]$

$$= \int \frac{d^3p \sum_{\gamma,s}}{4E_p} \Big[u^{\dagger\gamma}(p) \sum_z u^s(p) \left[\hat{a}_\gamma^+(\vec{p})\hat{a}_s(\vec{p}), \hat{b}_\ell^+(0)\right] + v^{\dagger\gamma}(p) \sum_z v^s(p) \left[\hat{b}_\gamma(\vec{p})\hat{b}_s^+(\vec{p}), \hat{b}_\ell^+(0)\right] + u^{\dagger\gamma}(E_p, -\vec{p}) \sum_z v^s(p) \left[\hat{a}_\gamma^+(-\vec{p})\hat{b}_s^+(\vec{p}), \hat{b}_\ell^+(0)\right] e^{2iE_p t} + v^{\dagger\gamma}(E_p, -\vec{p}) \sum_z u^s(p) \left[\hat{b}_\gamma(-\vec{p})\hat{a}_s(\vec{p}), \hat{b}_\ell^+(0)\right] e^{-2iE_p t} \Big]$$

using ladder operator anti-commutation

$$\left[\hat{a}_\gamma^+(\vec{p})\hat{a}_s(\vec{p}), \hat{b}_\ell^+(0)\right] = \hat{a}_\gamma^+(\vec{p})\hat{a}_s(\vec{p})\hat{b}_\ell^+(0) - \hat{b}_\ell^+(0)\hat{a}_\gamma^+(\vec{p})\hat{a}_s(\vec{p})$$

$$= \hat{a}_\gamma^+(\vec{p})\hat{a}_s(\vec{p})\hat{b}_\ell^+(0) + \hat{a}_\gamma^+(\vec{p})\hat{b}_\ell^+(0)\hat{a}_s(\vec{p})$$

$$= \hat{a}_\gamma^+(\vec{p}) \underbrace{\left\{\hat{a}_s(\vec{p}), \hat{b}_\ell^+(0)\right\}}_{0}$$

$$\longrightarrow \left[\hat{a}_\gamma^+(\vec{p})\hat{a}_s(\vec{p}), \hat{b}_\ell^+(0)\right] = 0$$

$$\left[\hat{a}^+_\gamma(-\vec{p})\hat{b}^+_s(\vec{p}),\hat{b}^+_\ell(0)\right] = \hat{a}^+_\gamma(-\vec{p})\hat{b}^+_s(\vec{p})\hat{b}^+_\ell(0) - \hat{b}^+_\ell(0)\hat{a}^+_\gamma(-\vec{p})\hat{b}^+_s(\vec{p})$$

$$= \hat{a}^+_\gamma(-\vec{p})\hat{b}^+_s(\vec{p})\hat{b}^+_\ell(0) + \hat{a}^+_\gamma(-\vec{p})\hat{b}^+_\ell(0)\hat{b}^+_s(\vec{p})$$

$$= \hat{a}^+_\gamma(-\vec{p})\underbrace{\left\{\hat{b}^+_s(\vec{p}),\hat{b}^+_\ell(0)\right\}}_{0}$$

$$\longrightarrow \left[\hat{a}^+_\gamma(-\vec{p})\hat{b}^+_s(\vec{p}),\hat{b}^+_\ell(0)\right] = 0$$

$$\left[\hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p}),\hat{b}^+_\ell(0)\right] = \hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p})\hat{b}^+_\ell(0) - \hat{b}^+_\ell(0)\hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p})$$

$$= \hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p})\hat{b}^+_\ell(0) - \hat{b}^+_\ell(0)\hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p})$$

$$= -\hat{b}_\gamma(\vec{p})\hat{b}^+_\ell(0)\hat{b}^+_s(\vec{p}) - \hat{b}^+_\ell(0)\hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p})$$

$$= -\left\{\hat{b}_\gamma(\vec{p}),\hat{b}^+_\ell(0)\right\}\hat{b}^+_s(\vec{p})$$

$$\longrightarrow \left[\hat{b}_\gamma(\vec{p})\hat{b}^+_s(\vec{p}),\hat{b}^+_\ell(0)\right] = -\delta_{\gamma\ell}\,\delta^{(3)}(\vec{p})\,\hat{b}^+_s(\vec{p})$$

$$\left[\hat{b}_\gamma(-\vec{p})\hat{a}_s(\vec{p}),\hat{b}^+_\ell(0)\right] = \hat{b}_\gamma(-\vec{p})\hat{a}_s(\vec{p})\hat{b}^+_\ell(0) - \hat{b}^+_\ell(0)\hat{b}_\gamma(-\vec{p})\hat{a}_s(\vec{p})$$

$$= -\hat{a}_s(\vec{p})\hat{b}_\gamma(-\vec{p})\hat{b}^+_\ell(0) - \hat{b}^+_\ell(0)\hat{b}_\gamma(-\vec{p})\hat{a}_s(\vec{p})$$

$$= -\hat{a}_s(\vec{p})\left\{\hat{b}_\gamma(-\vec{p}),\hat{b}^+_\ell(0)\right\}$$

$$\longrightarrow \left[\hat{b}_\gamma(-\vec{p})\hat{a}_s(\vec{p}),\hat{b}^+_\ell(0)\right] = -\hat{a}_s(\vec{p})\,\delta_{\gamma\ell}\,\delta^{(3)}(-\vec{p})$$

$$\rightarrow \left[\hat{S}_z, \hat{b}_\ell^+(0)\right]$$

$$= \int \frac{d^3p \sum_{r,s}}{4E_p} \left[\begin{array}{l} -v^{\dagger r}(p) \sum_z v^s(p)\, \delta_{r\ell}\, \delta^{(3)}(\vec{p})\, \hat{b}_s^+(\vec{p}) \\ -v^{\dagger r}(E_p, -\vec{p}) \sum_z u^s(p)\, \hat{a}_s(\vec{p})\, \delta_{r\ell}\, \delta^{(3)}(-\vec{p})\, e^{-2iE_p t} \end{array} \right]$$

(note: $E(\vec{p}=0) = m$)

$$= \sum_s \frac{}{4m} \left[\begin{array}{l} -v^{\dagger \ell}(m,0) \sum_z v^s(m,0)\, \hat{b}_s^+(0) \\ -v^{\dagger \ell}(m,0) \sum_z u^s(m,0)\, \hat{a}_s(0)\, e^{-2imt} \end{array} \right]$$

$$\rightarrow \hat{S}_z \hat{b}_\ell^+(0)|\Omega\rangle = \{\hat{S}_z, \hat{b}_\ell^+(0)\}|\Omega\rangle$$

$$= \frac{-1}{4m} \sum_s v^{\dagger \ell}(m,0) \sum_z v^s(m,0)\, \hat{b}_s^+(0)\, |\Omega\rangle$$

(since $\hat{a}_s(0)|\Omega\rangle = 0$)

we shall use the chiral basis

$$v^s(m,0) = \sqrt{m} \begin{bmatrix} \eta^s \\ -\eta^s \end{bmatrix} \quad \& \quad \sum_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix}$$

$$\rightarrow \hat{S}_z \hat{b}_\ell^+(0)|\Omega\rangle$$

$$= -\frac{1}{4}\sum_s \begin{bmatrix} \eta^{\ell +} & -\eta^{\ell +} \end{bmatrix} \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix} \begin{bmatrix} \eta^s \\ -\eta^s \end{bmatrix} \hat{b}_s^+(0)|\Omega\rangle$$

$$= -\frac{1}{4}\sum_s \left[2(\eta^\ell)^+ \sigma_z \eta^s\right] \hat{b}_s^+(0)|\Omega\rangle$$

$$= \sum_s (\eta^\ell)^+ \left(\frac{-\sigma_z}{2}\right) \eta^s \hat{b}_s^+(0)|\Omega\rangle$$

Now we can choose the explict form for the spinors η^s to be basis for σ_z

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \eta^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \eta^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rightarrow \hat{S}_z \hat{b}_\ell^+(0)|\Omega\rangle$$

$$= (\eta^\ell)^+ \left(\frac{-\sigma_z}{2}\right) \eta^1 \hat{b}_1^+(0)|\Omega\rangle$$

$$+ (\eta^\ell)^+ \left(\frac{-\sigma_z}{2}\right) \eta^2 \hat{b}_2^+(0)|\Omega\rangle$$

$$= (\eta^{\ell})^{+}\left(\frac{-1}{2}\right)(-\eta^{1})\hat{b}_1^{+}(0)|\Omega\rangle$$

$$+ (\eta^{\ell})^{+}\left(\frac{-1}{2}\right)(+\eta^{2})\hat{b}_2^{+}(0)|\Omega\rangle$$

$$= \frac{1}{2}\left[\delta_{\ell 1}\hat{b}_1^{+}(0) - \delta_{\ell 2}\hat{b}_2^{+}(0)\right]|\Omega\rangle$$

$$\rightarrow \hat{S}_z\left[\hat{b}_1^{+}(0)|\Omega\rangle\right] = \frac{1}{2}\left[\hat{b}_1^{+}(0)|\Omega\rangle\right]$$

$$\& \ \hat{S}_z\left[\hat{b}_2^{+}(0)|\Omega\rangle\right] = \frac{-1}{2}\left[\hat{b}_2^{+}(0)|\Omega\rangle\right]$$

Thus, $\hat{b}_1^{+}(0)|\Omega\rangle$ that corresponding to the dirac Spinor $v^1(m,0) = \sqrt{m}\begin{bmatrix}\eta^1 \\ \eta^1\end{bmatrix}$

where $\eta^1 = \begin{bmatrix}0 \\ 1\end{bmatrix}$, represents an anti-particle state which has Spin up along z-direction.

Note: $\begin{bmatrix}0 \\ 1\end{bmatrix} = \xi_s$ correspond to Spin down for particle.

Thus, $\hat{b}_2^+(0)|\Omega\rangle$ that corresponding to the dirac Spinor $v^2(m,0) = \sqrt{m}\begin{bmatrix}\eta^2\\ \eta^2\end{bmatrix}$

where $\eta^2 = \begin{bmatrix}1\\0\end{bmatrix}$, represents an anti-particle state which has Spin down along z-direction.

Note: $\begin{bmatrix}1\\0\end{bmatrix} = \xi_s$ correspond to Spin up for particle.

Thus the 2-Spinor $\begin{bmatrix}1\\0\end{bmatrix}$ correspond to physical Spin up for Dirac particle & physical Spin down for anti-particle.

And the 2-Spinor $\begin{bmatrix}0\\1\end{bmatrix}$ correspond to physical Spin down for Dirac particle & physical Spin up for anti-particle.

Global U(1) symmetry & conserved Noether current & Noether charge

The lagrangian for Dirac field is given by,

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi$$

It has a global U(1) symmetry. i.e. $\partial_\mu \alpha = 0$. So under this,

$$\psi \longrightarrow e^{-i\alpha}\psi \quad \& \quad \overline{\psi} \longrightarrow e^{i\alpha}\overline{\psi}$$

$$\rightarrow \mathcal{L} \longrightarrow e^{i\alpha}\overline{\psi}(i\not{\partial} - m)e^{-i\alpha}\psi$$
$$= e^{i\alpha}e^{-i\alpha}\overline{\psi}(i\not{\partial} - m)\psi$$

$$\Rightarrow \underline{\delta\mathcal{L} = 0}$$ (Lagrangian is invariant)

$\Rightarrow$ There exists some conserved Noether current & a corresponding charge, which is diagonal in same basis where hamiltonian is diagonal.

Consider an infinitesimal global phase rotation by an angle $\delta\alpha$; $(\delta\alpha)^2 = 0$

$$\rightarrow \hat{\psi} \rightarrow (1 - i\delta\alpha)\hat{\psi} \rightarrow \delta\hat{\psi} = -i(\delta\alpha)\hat{\psi}$$
$$\& \quad \hat{\bar{\psi}} \rightarrow (1 + i\delta\alpha)\hat{\bar{\psi}} \rightarrow \delta\hat{\bar{\psi}} = +i(\delta\alpha)\hat{\bar{\psi}}$$

Let $\delta\alpha = -e \rightarrow$ The conserved noether current is,

$$\hat{j}^\mu = \hat{\Pi}^\mu_{\psi}\,\delta\hat{\psi} + \hat{\Pi}^\mu_{\bar{\psi}}\,\delta\hat{\bar{\psi}}$$

$$\hat{\Pi}^\mu_{\psi} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\psi})} = i\hat{\bar{\psi}}\gamma^\mu$$

$$\& \quad \hat{\Pi}^\mu_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \hat{\bar{\psi}})} = 0$$

$$\rightarrow \hat{j}^\mu = i\hat{\bar{\psi}}\gamma^\mu(-ie\hat{\psi})$$

$$\rightarrow \boxed{\hat{j}^\mu = -e(\hat{\bar{\psi}}\,\gamma^\mu\hat{\psi})}$$

By Noether's theorem, $\underline{\partial_\mu \hat{j}^\mu = 0}$

The conserved Noether charge is

$$\hat{Q} = \int d^3x \; \hat{j}^0 \qquad \text{such that} \quad \frac{d\hat{Q}}{dt} = 0$$

$$\hat{Q} = -\int d^3x \, e \left(\hat{\overline{\psi}} \, \gamma^0 \, \hat{\psi} \right)$$

$$\Longrightarrow \hat{Q} = -e \int d^3x \left(\hat{\psi}^+ \hat{\psi} \right)$$

$$\hat{\psi}(x) = \int \frac{d^3p \sum_{\gamma=1}^{2} \left[u^\gamma(p) \hat{a}_\gamma(\vec{p}) e^{-ip_\mu x^\mu} + v^\gamma(p) \hat{b}^+_\gamma(\vec{p}) e^{ip_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\psi}^+(x) = \int \frac{d^3q \sum_{s=1}^{2} \left[u^+_s(q) \hat{a}^+_s(\vec{q}) e^{iq_\mu x^\mu} + v^+_s(q) \hat{b}_s(\vec{q}) e^{-iq_\mu x^\mu} \right]}{\sqrt{(2\pi)^3 (2E_q)}}$$

$$\Longrightarrow \hat{Q} = -e \int \frac{d^3x \, d^3q \, d^3p}{(2\pi)^3 2\sqrt{E_p E_q}}$$

$$\times \sum_{\gamma,s} \left[u^+_s(q) \hat{a}^+_s(\vec{q}) e^{iq_\mu x^\mu} + v^+_s(q) \hat{b}_s(\vec{q}) e^{-iq_\mu x^\mu} \right]$$

$$\times \left[u^\gamma(p) \hat{a}_\gamma(\vec{p}) e^{-ip_\mu x^\mu} + v^\gamma(p) \hat{b}^+_\gamma(\vec{p}) e^{ip_\mu x^\mu} \right]$$

$$\hat{Q} = -e \int \frac{d^3x\, d^3q\, d^3p \sum_{r,s}}{(2\pi)^3 2\sqrt{E_p E_q}} \Big[u^{+s}(q) u^r(p) \hat{a}_s^+(\vec{q}) \hat{a}_r(\vec{p}) e^{i(q_\mu - p_\mu)x^\mu} + v^{+s}(q) v^r(p) \hat{b}_s(\vec{q}) \hat{b}_r^+(\vec{p}) e^{i(p_\mu - q_\mu)x^\mu} + u^{+s}(q) v^r(p) \hat{a}_s^+(\vec{q}) \hat{b}_r^+(\vec{p}) e^{i(q_\mu + p_\mu)x^\mu} + v^{+s}(q) u^r(p) \hat{b}_s(\vec{q}) \hat{a}_r(\vec{p}) e^{-i(q_\mu + p_\mu)x^\mu} \Big]$$

Perform d^3x integral to get dirac delta's & then do d^3q integral. This makes $\vec{q} \to \pm\vec{p}$ & $E_p = E_{-p}$, so we get,

$$\hat{Q} = -e \int \frac{d^3p}{2E_p} \sum_{s,r} \Big[u^{+s}(p) u^r(p) \hat{a}_s^+(\vec{p}) \hat{a}_r(\vec{p}) + v^{+s}(p) v^r(p) \hat{b}_s(\vec{p}) \hat{b}_r^+(\vec{p}) + u^{+s}(E_p, -\vec{p}) v^r(p) \hat{a}_s^+(-\vec{p}) \hat{b}_r^+(\vec{p}) e^{2iE_p t} + v^{+s}(E_p, -\vec{p}) u^r(p) \hat{b}_s(-\vec{p}) \hat{a}_r(\vec{p}) e^{-2iE_p t} \Big]$$

The orthonormalization for spinor basis $\{u^r\}$ & $\{v^s\}$ can be used to get

$$u^{s\dagger}(p) u^r(p) = (2E_p)\, \delta^{rs}$$

$$v^{s\dagger}(p) v^r(p) = (2E_p)\, \delta^{rs}$$

$$u^{+s}(E_p, -\vec{p}) v^r(p) = v^{+s}(E_p, -\vec{p}) u^r(p) = 0$$

$$\Longrightarrow \hat{Q} = -e\int d^3p \sum_s \left[\hat{a}_s^+(\vec{p})\hat{a}_s(\vec{p}) + \hat{b}_s(\vec{p})\hat{b}_s^+(\vec{p})\right]$$

$$\boxed{N[\hat{Q}] = -e\int d^3p \sum_s \left[\hat{a}_s^+(\vec{p})\hat{a}_s(\vec{p}) - \hat{b}_s^+(\vec{p})\hat{b}_s(\vec{p})\right]}$$

Thus the charge of Dirac particle is $-e$ & charge of anti-particle is $+e$. Since $\frac{dQ}{dt} = 0$ the net charge is conserved.

When we couple Electro-magnetic field to the Dirac field, '$-e$' becomes the electric charge of the dirac particle.

The particle created by $\hat{a}_s^+(\vec{p})$ have energy E_p, 3-momentum $\vec{p}$, charge of $-e$ and $1/2$ spin particles with following polarization.

The 2-spinor $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the particle dirac spinor $\begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \\ 0 \\ \sqrt{p_\mu \bar{\sigma}^\mu} \\ 0 \end{bmatrix}$ which has spin up along $\vec{p}$ direction & hence has +ve helicity.

The 2-Spinor $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives the particle dirac Spinor $\begin{bmatrix} 0 \\ \sqrt{p_\mu \sigma^\mu} \\ 0 \\ \sqrt{p_\mu \bar{\sigma}^\mu} \end{bmatrix}$ which has Spin down along $\vec{p}$ direction & hence has -ve helicity.

Similarly, $\hat{b}^+_s(\vec{p})$ creates anti-particles that have energy E_p, 3-momentum $\vec{p}$, charge of $+e$ & $1/2$ Spin anti-particles with following polarization.

The 2-Spinor $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ gives the anti-particle dirac Spinor $\begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \\ 0 \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \\ 0 \end{bmatrix}$ which has Spin down along $\vec{p}$ direction & has -ve helicity.

The 2-Spinor $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gives the anti-particle dirac Spinor $\begin{bmatrix} 0 \\ \sqrt{p_\mu \sigma^\mu} \\ 0 \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \end{bmatrix}$ which has Spin down along $\vec{p}$ direction & has -ve helicity.