

Covariant Formulation of ED

Electromagnetic theory or Electrodynamics is one of the simplest gauge theory which is built on the 4-vector potential (A^μ) formulation.

The observable fields are the Electric field ($\vec{E}$) & magnetic field ($\vec{B}$) which have three spatial components
$$\vec{E}(x^\mu) = [E_x(x^\mu), E_y(x^\mu), E_z(x^\mu)] = E^i$$

&
$$\vec{B}(x^\mu) = [B_x(x^\mu), B_y(x^\mu), B_z(x^\mu)] = B^i$$

In the usual electro-magnetic theory these fields are derived by the scalar potential (ϕ) & the vector potential ($\vec{A}$) which are functions of (x^μ), $\phi \rightarrow \phi(x^\mu)$

$$\vec{A} \rightarrow \vec{A}(x^\mu) = [A_x(x^\mu), A_y(x^\mu), A_z(x^\mu)] = A^i$$

where
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad \& \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$E^i = -\partial_i \phi - \frac{\partial A^i}{\partial t}, \quad B^i = \varepsilon^{ij}_k \partial_j A^k$$

$$\text{or } \frac{E^i}{c} = -\partial_i (\phi/c) - \frac{\partial A^i}{\partial (ct)}$$

$$\& B^1 = \partial_2 A^3 - \partial_3 A^2$$

$$B^2 = \partial_3 A^1 - \partial_1 A^3$$

$$B^3 = \partial_1 A^2 - \partial_2 A^1$$

If we define a 4-vector potential as $A^\mu = (A^0, A^1, A^2, A^3) = (\phi/c, A_x, A_y, A_z)$

$$\Rightarrow A_\mu = (A_0, A_1, A_2, A_3) = (\phi/c, -A_x, -A_y, -A_z)$$

$$\text{where } A_\mu = \eta_{\mu\nu} A^\nu \Rightarrow A^0 = A_0 \quad \& \quad A^i = -A_i$$

$$\text{Thus } \frac{E^i}{c} = -\partial_i A^0 - \partial_0 A^i = -\partial_i A_0 + \partial_0 A_i$$

$$\Rightarrow \left(\frac{E^i}{c} \right) = \partial_0 A_i - \partial_i A_0$$

$$E_x/c = \partial_0 A_1 - \partial_1 A_0$$

$$E_y/c = \partial_0 A_2 - \partial_2 A_0$$

$$E_z/c = \partial_0 A_3 - \partial_3 A_0$$

$$B_x = \partial_2 A^3 - \partial_3 A^2 = \partial_3 A_2 - \partial_2 A_3$$

$$B_y = \partial_3 A^1 - \partial_1 A^3 = \partial_1 A_3 - \partial_3 A_1$$

$$B_z = \partial_1 A^2 - \partial_2 A^1 = \partial_2 A_1 - \partial_1 A_2$$

Electromagnetic Stress tensor

Thus the object $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$ is called as Electromagnetic Stress tensor where $\vec{E}$ & $\vec{B}$ are its elements.

$$\Rightarrow F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

$$\& F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

$(\partial^\mu A^\nu - \partial^\nu A^\mu)$

Defining $G^{\alpha\sigma} = \frac{1}{2} \epsilon^{\alpha\sigma\mu\nu} F_{\mu\nu}$ as the dual of EM tensor, we get

$$G^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ B_y & -\frac{E_z}{c} & 0 & \frac{E_x}{c} \\ B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{bmatrix} = -G^{\nu\mu}$$

$$G_{\mu\nu} = \eta_{\mu\alpha} \eta_{\nu\beta} G^{\alpha\beta}$$

$$G_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ -B_y & -\frac{E_z}{c} & 0 & E_x/c \\ -B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{bmatrix}$$

The eqⁿ $\partial_\mu G^{\mu\nu} = 0$ gives the Homogenous maxwell equations

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\text{and } \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

Properties of EM tensors (F & G)

$$(i) \quad F_{\mu\nu} F^{\mu\nu} = 2 \left(|\vec{B}|^2 - \frac{|\vec{E}|^2}{c^2} \right)$$

$$(ii) \quad F_{\mu\nu} G^{\mu\nu} = F^{\mu\nu} G_{\mu\nu} = -\frac{4}{c} (\vec{E} \cdot \vec{B})$$

$$(iii) \quad G_{\mu\nu} G^{\mu\nu} = 2 \left(\frac{|\vec{E}|^2}{c^2} - |\vec{B}|^2 \right)$$

$$(iv) \quad \eta_{\mu\nu} F^{\mu\nu} = F^\mu{}_\mu = \text{Trace}(F) = 0$$

$$(v) \quad \text{Det}(F) = \frac{1}{c^2} (\vec{B} \cdot \vec{E})^2$$

Source for $\vec{E}$ & $\vec{B}$ fields

The charge & current densities act as the source for $\vec{E}$ & $\vec{B}$ as shown by the non-homogenous maxwell eqⁿ,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho(x^\mu)}{\epsilon_0}, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\text{where } \mu_0 \epsilon_0 = \frac{1}{c^2}$$

These equations can be incorporated into a tensor eqⁿ of $F^{\mu\nu}$ as,

$$\underline{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$

provided $J^\nu = (c\rho, J_x, J_y, J_z)$

J^ν acts as the source for electric & magnetic fields, where the 4-divergence of EM tensor is proportional to source field called as the 4-current density.

The charge-continuity equation which signifies the conservation of charge is $\partial_\mu J^\mu = 0$. This also follows from the Maxwell's eqⁿ directly.

$$\underbrace{\partial_\nu \partial_\mu}_{\text{Symmetric anti Symmetric}} \underbrace{F^{\mu\nu}}_{\text{Symmetric}} = \mu_0 (\partial_\nu J^\nu) = 0 \Rightarrow \underline{\partial_\nu J^\nu = 0}$$

Hence Continuity eqⁿ is just a resultant of Maxwell's eqⁿ & the definition of F-tensor

Gauge invariance of EM theory

A^μ , i.e., the four vector potential is a gauge field and the theory of A^μ is gauge theory. The EM theory is same if we choose the 4-vector potential to be A^μ or $\tilde{A}^\mu = A^\mu - \partial^\mu \chi$ where ' χ ' is any twice differentiable scalar function.

The transformation, $A^\mu \rightarrow \tilde{A}^\mu = A^\mu - \partial^\mu \chi$ is called as gauge transformation & EM theory is invariant if it is written in terms of either A^μ or $\tilde{A}^\mu = A^\mu - \partial^\mu \chi$. This is because the EM field tensor $F_{\mu\nu}$ & hence $\vec{E}$ & $\vec{B}$ are invariant under the gauge transformation, since

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \tilde{F}_{\mu\nu}$$

Thus $F^{\mu\nu}$, $G^{\mu\nu}$ & also J^μ (sources) are gauge invariants.

Lorenz & Coulomb Gauge

Consider a four potential A^μ that the EM theory is built on. We can perform a gauge transformation $A^\mu \rightarrow A^\mu_{\text{Lorenz}}$ such that $\partial_\mu A^\mu_{\text{Lorenz}} = 0$, then it is called as the Lorenz gauge. This is achieved by performing

$$A^\mu_{\text{Lorenz}} = A^\mu - \partial^\mu \chi$$

where we set $\partial_\mu \partial^\mu \chi = \partial_\mu A^\mu$

Lorenz gauge is a special gauge since the Maxwell's eqⁿ reduce to wave eqⁿ for A^μ components. Consider,

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

$$\Rightarrow \partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \mu_0 J^\nu$$

$$\Rightarrow (\partial_\mu \partial^\mu) A^\nu - \partial^\nu (\partial_\mu A^\mu) = \mu_0 J^\nu$$

If we choose χ such that

$$\partial_\mu \partial^\mu \chi = \partial_\mu A^\mu$$

& then use $A_{\text{lorenz}}^\mu = A^\mu - \partial^\mu \chi$ & if we work with lorentz gauge, then maxwell's eqⁿ reduce to,

$$(\partial_\mu \partial^\mu) A_{\text{lorenz}}^\nu = \mu_0 J^\nu$$

or $\square A_{\text{lorenz}}^\nu = \mu_0 J^\nu$ $\rightarrow$ Maxwell's eqⁿ in Potential form.

In the absence of source, we get,

$$\square A_{\text{lorenz}}^\mu = 0$$

Thus every component of A_{lorenz}^μ will satisfy massless K.G eqⁿ,

$$\frac{1}{c^2} \frac{\partial^2 A_{\text{lorenz}}^\mu}{\partial t^2} = \nabla^2 A_{\text{lorenz}}^\mu$$

$\Rightarrow A_{\text{lorenz}}^\mu$ are waves in 4D space-time.

The condition $\partial_\mu A_{\text{lorenz}}^\mu = 0$ makes the four components of A_{lorenz}^μ to relate by one equation & hence the number of independent components is three.

But still A_{lorenz}^μ is not unique as we can perform another transform on it such that the resultant four potential is again another lorentz gauge $\tilde{A}_{\text{lorenz}}^\mu$.

$$\text{i.e. } \tilde{A}_{\text{lorenz}}^\mu = A_{\text{lorenz}}^\mu - \partial^\mu \epsilon$$

$$\rightarrow \partial_\mu \partial^\mu \epsilon = 0 \text{ is needed.}$$

$$\rightarrow \partial_\mu A_{\text{lorenz}}^\mu = 0 \text{ \& } \partial_\mu \tilde{A}_{\text{lorenz}}^\mu = 0$$

Thus A_{lorenz}^μ is not unique. But if we can choose to set ' ϵ ' such that $\partial^0 \epsilon = A_{\text{lorenz}}^0$, then $\tilde{A}_{\text{lorenz}}^0 = 0$

$$\Rightarrow \tilde{A}_{\text{lorenz}}^\mu \text{ is such that } \underline{\underline{\vec{\nabla} \cdot \vec{A} = 0}}$$

This is Coulomb gauge, which has only two degrees of freedom & is unique. Even though we can second quantize the EM field in coulomb gauge, it is not lorentz covariant & has serious issues in Quantum Electrodynamics.

Lagrangian formulation of EM theory

Thus an appropriate lagrangian for EM field, whose action

$S_{EM} = \int d^4x \mathcal{L}_{EM}$ is gauge invariant provided $J^\mu \rightarrow 0$ at $\pm\infty$, is

$$\mathcal{L}_{EM} = \frac{-1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu$$

The eqⁿ of motion is given by

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -J_\nu$$

$$\& \pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{-1}{\mu_0} F^{\mu\nu}$$

$$\Rightarrow \text{EOM is } \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \pi^{\mu\nu} = 0$$

$$\Rightarrow -J^\nu = \partial_\mu \Pi^{\mu\nu} = -\frac{1}{\mu_0} \partial_\mu F^{\mu\nu}$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$

Thus we retrieved non-homogeneous Maxwell's eqⁿ that govern the EM fields from $\mathcal{L}_{EM}$.

Electromagnetic Energy momentum Tensor

$$\Theta^{\mu\nu} = \Pi_\sigma^\mu (\partial^\nu A^\sigma) - \eta^{\mu\nu} \mathcal{L}$$

$$= \Pi^{\mu\sigma} (\partial^\nu A_\sigma) - \eta^{\mu\nu} \mathcal{L}$$

$$\rightarrow \Theta^{\mu\nu} = -\frac{1}{\mu_0} (F^{\mu\sigma} \partial^\nu A_\sigma) - \eta^{\mu\nu} \mathcal{L}$$

The canonical energy-momentum tensor is not symmetric and also it is unphysical as it is also gauge dependent ($\partial^\nu A_\sigma$ term).

But we can construct a symmetric general energy momentum tensor $T^{\mu\nu}$ in the absence of source ($J^\mu = 0$), where

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\sigma \mathcal{B}^{\sigma\mu\nu}$$

where $\mathcal{B}^{\sigma\mu\nu} = \frac{1}{\mu_0} F^{\mu\sigma} A^\nu$

$$\Rightarrow T^{\mu\nu} = \frac{1}{\mu_0} (F^{\mu\sigma} F_\sigma{}^\nu) - \frac{\eta^{\mu\nu}}{4\mu_0} (F^{\alpha\beta} F_{\alpha\beta})$$

is a symmetric energy momentum tensor which is gauge independent & physically equivalent to $\Theta^{\mu\nu}$, additionally $T^{\mu\nu}$ is conserved, where,

$$\underline{\partial_\mu T^{\mu\nu} = 0}$$

But in the presence of J^μ the energy-momentum tensor $T^{\mu\nu}$ is not conserved as there is exchange of energy-momentum b/w fields & sources in the form of Lorentz-force & work done on charges by EM field.

Hamiltonian density is given by T^{00}

$$T^{00} = \frac{1}{\mu_0} F^{0\sigma} F_{\sigma}^0 - \frac{1}{4\mu_0} (F^{\alpha\beta} F_{\alpha\beta})$$

$$\mathcal{H} = \frac{1}{\mu_0} (F^{01} F_{10} + F^{02} F_{20} + F^{03} F_{30}) - \left[\frac{1}{2} \epsilon_0 |\vec{E}|^2 - \frac{1}{2} \frac{|\vec{B}|^2}{\mu_0} \right]$$

$$= \frac{1}{\mu_0 c^2} (E_x^2 + E_y^2 + E_z^2) - \frac{1}{2} \epsilon_0 |\vec{E}|^2 + \frac{1}{2} \frac{|\vec{B}|^2}{\mu_0}$$

$$= \epsilon_0 |\vec{E}|^2 - \frac{1}{2} \epsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2$$

$$\Rightarrow \boxed{\mathcal{H} = \frac{1}{2} \epsilon_0 |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2}$$

Thus hamiltonian density is nothing but the total energy density of EM fields in the absence of source.

Similarly, Poynting vector gives the 3-momentum density as follows.

$$T^{i0} = \frac{1}{\mu_0} (F^{i\sigma} F_{\sigma 0})$$

$$= \frac{1}{\mu_0} (F^{i1} F_{10} + F^{i2} F_{20} + F^{i3} F_{30})$$

$$\Rightarrow T^{10} = \frac{1}{\mu_0} (F^{12} F_{20} + F^{13} F_{30})$$

$$= \frac{1}{\mu_0} (B_z E_y - B_y E_z)$$

$$T^{10} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})^1$$

$$\text{Similarly } T^{i0} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})^i$$

$$i^{\text{th}} \text{-component of } \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$\vec{S} \Rightarrow$ Poynting vector that gives the momentum density which has the information of energy flux of EM field, it describes the rate at which energy flowing per area per unit time $\perp$ to the direction of propagation.

Canonical Quantization of EM fields

Here we shall proceed with the coulomb gauge where $\vec{\nabla} \cdot \vec{A} = 0$ and in the absence of sources which gives $A^0 = 0$. Since this leads to $\partial_\mu A^\mu = 0$, we get, A^μ to satisfy the wave equations.

$$\partial_\mu \partial^\mu A^\mu = 0$$

Since $A^0 = 0$, we can now work with $A^i = (A_x, A_y, A_z) \equiv \vec{A}$ (3-vector potential) which satisfies, $\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = 0$.

$$(c = \mu_0 = \hbar = 1)$$

Treating $\vec{A}$ as field parameters, the lagrangian density becomes,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (|\vec{E}|^2 - |\vec{B}|^2)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\text{Now } \pi^i = \frac{\partial \mathcal{L}}{\partial (\partial_0 A^i)} = -E^i$$

$$\rightarrow \mathcal{H} = \pi^i (\partial_0 A^i) - \mathcal{L}$$

$$\rightarrow \underline{\mathcal{H} = \frac{1}{2} (|\vec{E}|^2 + |\vec{B}|^2)}$$

Now we need to impose the commutation relations. But we need to first check the allowed polarization states of $\vec{A}$.

Since $\partial_\mu \partial^\mu \vec{A} = 0$, each component of $\vec{A}$ satisfies, massless Klein-gordon eqⁿ whose the dispersion relation is $p_\mu p^\mu = 0$.

$$\rightarrow \underline{E_p^2 - |\vec{p}|^2 = 0} \text{ or choose } \underline{E_p = +|\vec{p}|}$$

So any photon (quantized EM particle) should satisfy the above condition.

If $\vec{A}(x^\mu) = \vec{A}_p e^{-i p_\mu x^\mu}$ & Since $\vec{\nabla} \cdot \vec{A} = 0$
 $\Rightarrow \vec{p} \cdot \vec{A}_p = 0 \Rightarrow$ direction of polarization is always $\perp$ to direction of propagation.

Thus there are only two degrees of freedom for the polarization of the 3-vector field. Thus there are two orthogonal polarization states that can span, any given general polarization.

We can choose the two polarization basis for a given momentum state $(\vec{p})$ as $\{\vec{\epsilon}_1(\vec{p}), \vec{\epsilon}_2(\vec{p})\}$, which are complex, & satisfy, $\vec{p} \cdot \vec{\epsilon}_\alpha(\vec{p}) = 0$, $\vec{p} \cdot \vec{\epsilon}_\alpha^*(\vec{p}) = 0$,

$$\& \vec{\epsilon}_\alpha^*(\vec{p}) \cdot \vec{\epsilon}_\beta(\vec{p}) = \delta_{\alpha\beta}$$

This type of polarization is called as transverse polarization.

Since $\vec{A}$ is a real field, we can perform the mode expansion similar to scalar Klein-gordon field as follows.

$$\hat{\vec{A}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \sum_{\lambda=\{1,2\}} \left[\vec{\epsilon}_\lambda(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} + \vec{\epsilon}_\lambda^*(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]$$

Commutation relations

For the ladder operators $\hat{a}_\alpha(\vec{p})$ & $\hat{a}_\alpha^\dagger(\vec{p})$, we want their commutation relations to follow usually as,

$$\underline{[\hat{a}_\alpha(\vec{p}), \hat{a}_\beta^\dagger(\vec{q})] = \delta(\vec{p} - \vec{q}) \delta_{\alpha\beta}}$$

But this can't be achieved simply by usual commutation of a field component and its conjugate momentum.

Consider, $[\hat{A}^\mu(t, \vec{x}), \hat{\Pi}^\nu(t, \vec{y})] = i \delta(\vec{x} - \vec{y}) \eta^{\mu\nu}$ as the commutation relation. But since $\hat{A}^0 = 0$, & $\hat{\Pi}^j = -\hat{E}^j$ [Electric field]

$$\rightarrow [\hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})] = i \delta(\vec{x} - \vec{y}) \delta^{kj}$$

↳ This doesn't work.

If we take divergence with respect to ' $\vec{x}$ ' both sides, we get,

$$[\partial_k \hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})] = i \partial_k \underline{\delta(\vec{x} - \vec{y})} \delta^{kj}$$

$$= i \partial_k \int \frac{d^3 p}{(2\pi)^3} e^{i \vec{p} \cdot (\vec{x} - \vec{y})} \delta^{kj}$$

$$= i \int \frac{d^3 p}{(2\pi)^3} (i p_k \delta^{kj}) e^{i \vec{p} \cdot (\vec{x} - \vec{y})}$$

$$= - \int \frac{d^3 p}{(2\pi)^3} p^j e^{i \vec{p} \cdot (\vec{x} - \vec{y})}$$

$$\Rightarrow [\partial_k \hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})]$$

$$= - \int \frac{d^3 p}{(2\pi)^3} p^j e^{i \vec{p} \cdot (\vec{x} - \vec{y})}$$

not zero.

But $\partial_k \hat{A}^k = 0$ (Coulomb gauge).

So $[\hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})] = i \delta(\vec{x} - \vec{y}) \delta^{kj}$
is inconsistent in Coulomb gauge.

This is because in defining $\mathcal{Z}(\vec{x}-\vec{y})$ we have allowed all components of $\vec{p}$ to enter the fourier integral. We should ensure that only the transverse components are accounted. So we shall define,

$$[\hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})] = i \mathcal{Z}_{tr}^{kj}(\vec{x}-\vec{y})$$

where $\mathcal{Z}_{tr}^{kj}(\vec{x}-\vec{y}) =$ transverse dirac delta function.

It is given by,

$$\mathcal{Z}_{tr}^{kj}(\vec{x}-\vec{y}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x}-\vec{y})} \left[\delta^{kj} - \frac{p^k p^j}{|\vec{p}|^2} \right]$$

$$\begin{aligned} \text{now } \partial_k \mathcal{Z}_{tr}^{kj}(\vec{x}-\vec{y}) &= \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x}-\vec{y})} i p_k \left[\delta^{kj} - \frac{p^k p^j}{|\vec{p}|^2} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p} \cdot (\vec{x}-\vec{y})} i \underbrace{\left[p^j - \cancel{p_k p^k p^j} \right]}_0 \end{aligned}$$

$$\Rightarrow \underline{\partial_k \mathcal{Z}_{tr}^{kj}(\vec{x}-\vec{y}) = 0}$$

So, $[\hat{A}^k(t, \vec{x}), \hat{E}^j(t, \vec{y})] = i \partial_{\vec{x}}^{kj} (\vec{x} - \vec{y})$
 will result in the bosonic commutation relations for the ladder operators as,

$$[\hat{a}_\alpha(\vec{p}), \hat{a}_\beta(\vec{q})] = [\hat{a}_\alpha(\vec{p}), \hat{a}_\beta^\dagger(\vec{q})] = 0 \quad \neq$$

$$[\hat{a}_\alpha(\vec{p}), \hat{a}_\beta^\dagger(\vec{q})] = \delta(\vec{p} - \vec{q}) \delta_{\alpha\beta}$$

Hamiltonian in terms of ladder operators

$$\hat{H} = \int d^3x \hat{\mathcal{H}} = \int d^3x \frac{1}{2} (|\hat{\vec{E}}|^2 + |\hat{\vec{B}}|^2)$$

$$\hat{\vec{A}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \sum_{\lambda=\{1,2\}} \left[\vec{\epsilon}_\lambda(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} + \vec{\epsilon}_\lambda^*(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]$$

$$\hat{\vec{E}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \sum_{\lambda=\{1,2\}} \left[\vec{\epsilon}_\lambda(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} - \vec{\epsilon}_\lambda^*(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]$$

$$\hat{\vec{B}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \sum_{\lambda=\{1,2\}} i\vec{p} \times \left[\vec{\epsilon}_\lambda(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} - \vec{\epsilon}_\lambda^*(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]$$

Substituting $\hat{\vec{E}}(x)$ & $\hat{\vec{B}}(x)$ in the expression for hamiltonian, and also by

$$\vec{p} \cdot \vec{\epsilon}_\alpha(\vec{p}) = 0, \quad \vec{\epsilon}_\alpha(\vec{p}) \cdot \vec{\epsilon}_\beta(\vec{p}) = 0,$$

we get the following,

$$\int d^3x |\hat{\vec{E}}|^2 = \int d^3x \hat{\vec{E}}(x) \cdot \hat{\vec{E}}(x)$$

$$= \frac{\int d^3x d^3p d^3q (-\hat{E}_p \hat{E}_q)}{(2\pi)^3 2\sqrt{E_p E_q}} \sum_{\alpha, \beta} \left[\begin{aligned} &\vec{\epsilon}_\alpha(\vec{p}) \cdot \vec{\epsilon}_\beta(\vec{q}) \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(\vec{q}) \\ &\quad \times e^{-i(q_\mu + p_\mu)x^\mu} \\ &+ \vec{\epsilon}_\alpha^*(\vec{p}) \cdot \vec{\epsilon}_\beta^*(\vec{q}) \hat{a}_\alpha^+(\vec{p}) \hat{a}_\beta^+(\vec{q}) \\ &\quad \times e^{i(q_\mu + p_\mu)x^\mu} \\ &- \vec{\epsilon}_\alpha(\vec{p}) \cdot \vec{\epsilon}_\beta^*(\vec{q}) \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^+(\vec{q}) \\ &\quad \times e^{i(q_\mu - p_\mu)x^\mu} \\ &- \vec{\epsilon}_\alpha^*(\vec{p}) \cdot \vec{\epsilon}_\beta(\vec{q}) \hat{a}_\alpha^+(\vec{p}) \hat{a}_\beta(\vec{q}) \\ &\quad \times e^{i(p_\mu - q_\mu)x^\mu} \end{aligned} \right]$$

Perform d^3x integral & then we get the dirac delta functions. And then we can perform d^3q integrals, & by noting that $E_p = E_{-p}$, we get,

$$\int d^3x |\hat{\vec{E}}|^2 = \int \frac{d^3p(E_p)}{2} \sum_{\alpha, \beta} \left[\begin{aligned} & \delta_{\alpha\beta} \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^\dagger(\vec{p}) \\ & + \delta_{\alpha\beta} \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta(\vec{p}) \\ & - \vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{\varepsilon}_\beta(-\vec{p}) \\ & \times \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(-\vec{p}) e^{-2iE_p t} \\ & - \vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{\varepsilon}_\beta^*(-\vec{p}) \\ & \times \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta^\dagger(-\vec{p}) e^{2iE_p t} \end{aligned} \right]$$

Similarly,

$$\int d^3x |\hat{\vec{B}}|^2 = \int d^3x \hat{\vec{B}}(x) \cdot \hat{\vec{B}}(x)$$

$$= - \int \frac{d^3x d^3p d^3q}{(2\pi)^3 2\sqrt{E_p E_q}} \sum_{\alpha, \beta} \left[\begin{aligned} & [\vec{p} \times \vec{\varepsilon}_\alpha(\vec{p})] \cdot [\vec{q} \times \vec{\varepsilon}_\beta(\vec{q})] \\ & \times \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(\vec{q}) e^{-i(q_\mu + p_\mu)x^\mu} \\ & + [\vec{p} \times \vec{\varepsilon}_\alpha^*(\vec{p})] \cdot [\vec{q} \times \vec{\varepsilon}_\beta^*(\vec{q})] \\ & \times \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta^\dagger(\vec{q}) e^{-i(q_\mu + p_\mu)x^\mu} \\ & - [\vec{p} \times \vec{\varepsilon}_\alpha(\vec{p})] \cdot [\vec{q} \times \vec{\varepsilon}_\beta^*(\vec{q})] \\ & \times \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^\dagger(\vec{q}) e^{i(q_\mu - p_\mu)x^\mu} \\ & - [\vec{p} \times \vec{\varepsilon}_\alpha^*(\vec{p})] \cdot [\vec{q} \times \vec{\varepsilon}_\beta(\vec{q})] \\ & \times \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta(\vec{q}) e^{i(p_\mu - q_\mu)x^\mu} \end{aligned} \right]$$

$$= \int \frac{d^3p}{2E_p} \sum_{\alpha, \beta} \left[\begin{aligned} & [\vec{p} \times \vec{\epsilon}_\alpha(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta^*(\vec{p})] \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^\dagger(\vec{p}) \\ & + [\vec{p} \times \vec{\epsilon}_\alpha^*(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta(\vec{p})] \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta(\vec{p}) \\ & + [\vec{p} \times \vec{\epsilon}_\alpha(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta(-\vec{p})] \\ & \quad \times \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(-\vec{p}) e^{-2iE_p t} \\ & + [\vec{p} \times \vec{\epsilon}_\alpha^*(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta^*(-\vec{p})] \\ & \quad \times \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta^\dagger(-\vec{p}) e^{2iE_p t} \end{aligned} \right]$$

Using the vector identity,

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

$$[\vec{p} \times \vec{\epsilon}_\alpha(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta^*(\vec{p})] = p^2 \delta_{\alpha\beta}$$

$$[\vec{p} \times \vec{\epsilon}_\alpha^*(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta(\vec{p})] = p^2 \delta_{\alpha\beta}$$

$$[\vec{p} \times \vec{\epsilon}_\alpha(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta(-\vec{p})] = p^2 \vec{\epsilon}_\alpha(\vec{p}) \cdot \vec{\epsilon}_\beta(-\vec{p})$$

$$[\vec{p} \times \vec{\epsilon}_\alpha^*(\vec{p})] \cdot [\vec{p} \times \vec{\epsilon}_\beta^*(-\vec{p})] = p^2 \vec{\epsilon}_\alpha^*(\vec{p}) \cdot \vec{\epsilon}_\beta^*(-\vec{p})$$

& also using $E_p^2 = p^2$, we get,

$$\int d^3x |\hat{\vec{B}}|^2 = \int \frac{d^3p}{2} E_p \sum_{\alpha, \beta} \left[\begin{aligned} &\delta_{\alpha\beta} \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^\dagger(\vec{p}) \\ &+ \delta_{\alpha\beta} \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta(\vec{p}) \\ &+ \vec{\epsilon}_\alpha(\vec{p}) \cdot \vec{\epsilon}_\beta(-\vec{p}) \\ &\quad \times \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(-\vec{p}) e^{-2iE_p t} \\ &+ \vec{\epsilon}_\alpha^*(\vec{p}) \cdot \vec{\epsilon}_\beta^*(-\vec{p}) \\ &\quad \times \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta^\dagger(-\vec{p}) e^{2iE_p t} \end{aligned} \right]$$

$$\Rightarrow \int d^3x \left[\frac{|\hat{\vec{E}}|^2 + |\hat{\vec{B}}|^2}{2} \right] \quad \left(\begin{array}{l} \text{cross terms} \\ \text{cancel out} \end{array} \right)$$

$$= (E_p) \sum_{\alpha, \beta} \left[\begin{aligned} &2 \delta_{\alpha\beta} \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^\dagger(\vec{p}) \\ &+ 2 \delta_{\alpha\beta} \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta(\vec{p}) \end{aligned} \right]$$

$$\Rightarrow \hat{H} = \int d^3p \frac{E_p}{2} \sum_{\lambda} \left[\hat{a}_\lambda^\dagger(\vec{p}) \hat{a}_\lambda(\vec{p}) + \hat{a}_\lambda(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) \right]$$

$$\Rightarrow \mathcal{N}[\hat{H}] = \int d^3p E_p \sum_{\lambda} \hat{a}_\lambda^\dagger(\vec{p}) \hat{a}_\lambda(\vec{p})$$

where $E_p = |\vec{p}|$

Thus we can show that the excitations of EM fields are massless and an excitation from a mode $\vec{p}$ has energy of $E_p = |\vec{p}|$. These excitations are called as photons, and they can have only two transverse polarization states.

Transverse Polarization States of Photons

Consider a photon travelling in the z -direction with a momentum $|\vec{p}|$.

$$\rightarrow p^\mu = (|\vec{p}|, 0, 0, |\vec{p}|)$$

We shall consider the left and right circularly polarized vectors for the above momentum state as the polarization basis.

$$\rightarrow \vec{\mathcal{E}}_L = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

Left - circular Polarization

$$\vec{\mathcal{E}}_R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

Right - circular Polarization

Now, $\vec{E} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is not allowed,
longitudinal $\swarrow$ as $\vec{E} \cdot \vec{p} \neq 0$.
Polarization

And hence not a transverse polarization.

Now the rotation matrix (operator) that can find the spin number of these states are as follows.

Rotation of a 3D vector about the z -axis is,

$$\hat{R}(\hat{z}, \theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{R}(\hat{z}, 2\theta) = \hat{I}_3 + (2\theta) \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \hat{I}_3 - i(2\theta) \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \hat{J}_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \hat{R}(\hat{z}, 2\theta) = \hat{I}_3 - i(2\theta) \hat{J}_z$$

$$\text{or } \underline{\hat{R}(\hat{z}, \theta) = e^{-i\theta \hat{J}_z}}$$

$\rightarrow \hat{J}_z \rightarrow$ generates rotations about the z -axis.

Thus eigen vectors of $\hat{J}_z$ gives the spin state of photon

$$\begin{aligned} \hat{J}_z \vec{E}_L &= \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} \\ &= (-1) \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = (-1) \vec{E}_L \end{aligned}$$

$$\rightarrow \underline{\hat{J}_z \vec{E}_L = (-1) \vec{E}_L}$$

Similarly, $\hat{J}_z \vec{E}_R = (+1) \vec{E}_R$

And the forbidden state $\vec{E}_{\text{long.}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ gives eigen value zero.

Thus even though $\hat{J}_z$ has eigen values $+1, 0, -1$, only those polarizations, whose spin state is ± 1 is allowed.

The Helicity operator is given by,
$$\hat{h} = \frac{\hat{\vec{J}} \cdot \vec{p}}{|\vec{p}|} = \hat{J}_z.$$

Thus the spin projection along the direction of propagation gives the Helicity.

- Right circularly polarized light has Spin of $+1$, and +ve Helicity.
- Left circularly polarized light has Spin of -1 and -ve Helicity.
- Spin 0 state of photon is not allowed.

Linearly polarized light have two orthonormal basis, they come in two flavours,

$$\vec{E}_H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{\vec{E}_R + \vec{E}_L}{\sqrt{2}} \rightarrow \text{Horizontal Polarization}$$

$$\vec{E}_V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{\vec{E}_R - \vec{E}_L}{\sqrt{2}i} \rightarrow \text{Vertical Polarization}$$

Thus linearly polarized light is a equal superposition of LCP & RCP.

So if $\hat{a}_R^\dagger(\vec{p})$ & $\hat{a}_L^\dagger(\vec{p})$ creates RCP & LCP photons with momentum $\vec{p}$, & if $\hat{S}_z$ is spin operator that act on Fock Space (repⁿ of the above $\hat{J}_z$)

$$\rightarrow \hat{S}_z \hat{a}_R^\dagger(\vec{p}) |\Omega\rangle = (+1) \hat{a}_R^\dagger(\vec{p}) |\Omega\rangle$$

$$\& \hat{S}_z \hat{a}_L^\dagger(\vec{p}) |\Omega\rangle = (-1) \hat{a}_L^\dagger(\vec{p}) |\Omega\rangle$$

$$\text{Then, } [\hat{S}_z, \hat{a}_R^+(\vec{p})] = \hat{a}_R^+(\vec{p})$$

$$[\hat{S}_z, \hat{a}_L^+(\vec{p})] = -\hat{a}_L^+(\vec{p})$$

So $\hat{a}_R^+(\vec{p})$ will increase spin value by $+1$ & $\hat{a}_L^+(\vec{p})$ will decrease the spin value by -1 .

We can define two operators,

$$\hat{b}_H^+(\vec{p}) = \frac{\hat{a}_R^+(\vec{p}) + \hat{a}_L^+(\vec{p})}{\sqrt{2}}$$

$$\hat{b}_V^+(\vec{p}) = \frac{\hat{a}_R^+(\vec{p}) - \hat{a}_L^+(\vec{p})}{\sqrt{2}i}$$

- $\hat{b}_H^+(\vec{p})$ creates a linearly polarized photon with horizontal polarization.
- $\hat{b}_V^+(\vec{p})$ creates a linearly polarized photon with vertical polarization.

Photon momentum in terms of ladder operators

The Poynting vector carries the momentum density of the EM field.

$$\rightarrow \hat{\vec{P}} = \int d^3x \hat{\vec{E}}(x) \times \hat{\vec{B}}(x)$$

$$\hat{\vec{E}}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \sum_{\alpha=\{1,2\}} \left[\vec{\epsilon}_\alpha(\vec{p}) \hat{a}_\alpha(\vec{p}) e^{-ip_\mu x^\mu} - \vec{\epsilon}_\alpha^*(\vec{p}) \hat{a}_\alpha^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]$$

$$\hat{\vec{B}}(x) = \int \frac{d^3q}{\sqrt{(2\pi)^3(2E_q)}} \sum_{\beta=\{1,2\}} i \vec{q} \times \left[\vec{\epsilon}_\beta(\vec{q}) \hat{a}_\beta(\vec{q}) e^{-iq_\mu x^\mu} - \vec{\epsilon}_\beta^*(\vec{q}) \hat{a}_\beta^\dagger(\vec{q}) e^{iq_\mu x^\mu} \right]$$

$$\rightarrow \hat{\vec{P}} = \int \frac{d^3p}{2} \sum_{\alpha,\beta} \left[\begin{aligned} & \vec{\epsilon}_\alpha^*(\vec{p}) \times [\vec{p} \times \vec{\epsilon}_\beta(\vec{p})] \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta(\vec{p}) \\ & + \vec{\epsilon}_\alpha(\vec{p}) \times [\vec{p} \times \vec{\epsilon}_\beta^*(\vec{p})] \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^\dagger(\vec{p}) \\ & + \vec{\epsilon}_\alpha(\vec{p}) \times [\vec{p} \times \vec{\epsilon}_\beta(-\vec{p})] \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(-\vec{p}) e^{-2iE_p t} \\ & + \vec{\epsilon}_\alpha^*(\vec{p}) \times [\vec{p} \times \vec{\epsilon}_\beta^*(-\vec{p})] \hat{a}_\alpha^\dagger(\vec{p}) \hat{a}_\beta^\dagger(-\vec{p}) e^{2iE_p t} \end{aligned} \right]$$

$$\vec{\varepsilon}_\alpha^*(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta(\vec{p})] = [\vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{\varepsilon}_\beta(\vec{p})] \vec{p} - \underbrace{[\vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{p}]}_0 \vec{\varepsilon}_\beta(\vec{p})$$

$$\rightarrow \vec{\varepsilon}_\alpha^*(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta(\vec{p})] = \delta_{\alpha\beta} \vec{p}$$

$$\vec{\varepsilon}_\alpha(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta^*(\vec{p})] = [\vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{\varepsilon}_\beta^*(\vec{p})] \vec{p} - \underbrace{[\vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{p}]}_0 \vec{\varepsilon}_\beta^*(\vec{p})$$

$$\rightarrow \vec{\varepsilon}_\alpha(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta^*(\vec{p})] = \delta_{\alpha\beta} \vec{p}$$

$$\vec{\varepsilon}_\alpha(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta(-\vec{p})] = [\vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{\varepsilon}_\beta(-\vec{p})] \vec{p} - \underbrace{[\vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{p}]}_0 \vec{\varepsilon}_\beta(-\vec{p})$$

$$\rightarrow \vec{\varepsilon}_\alpha(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta(-\vec{p})] = [\vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{\varepsilon}_\beta(-\vec{p})] \vec{p}$$

$$\vec{\varepsilon}_\alpha^*(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta^*(-\vec{p})] = [\vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{\varepsilon}_\beta^*(-\vec{p})] \vec{p} - \underbrace{[\vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{p}]}_0 \vec{\varepsilon}_\beta^*(-\vec{p})$$

$$\rightarrow \vec{\varepsilon}_\alpha^*(\vec{p}) \times [\vec{p} \times \vec{\varepsilon}_\beta^*(-\vec{p})] = [\vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{\varepsilon}_\beta^*(-\vec{p})] \vec{p}$$

$$\rightarrow \hat{\vec{p}} = \int \frac{d^3p}{2} \vec{p} \sum_{\alpha, \beta} \left[\delta_{\alpha\beta} \left[\hat{a}_\alpha^+(\vec{p}) \hat{a}_\beta(\vec{p}) + \hat{a}_\alpha(\vec{p}) \hat{a}_\beta^+(\vec{p}) \right] + [\vec{\varepsilon}_\alpha(\vec{p}) \cdot \vec{\varepsilon}_\beta(-\vec{p})] \hat{a}_\alpha(\vec{p}) \hat{a}_\beta(-\vec{p}) e^{-2iE_p t} + [\vec{\varepsilon}_\alpha^*(\vec{p}) \cdot \vec{\varepsilon}_\beta^*(-\vec{p})] \hat{a}_\alpha^+(\vec{p}) \hat{a}_\beta^+(-\vec{p}) e^{2iE_p t} \right]$$

Consider the term,

$$\begin{aligned} I &= \int \frac{d^3p}{2} \vec{p} \sum_{\alpha, \beta} [\vec{\epsilon}_{\alpha}(\vec{p}) \cdot \vec{\epsilon}_{\beta}(-\vec{p})] \hat{a}_{\alpha}(\vec{p}) \hat{a}_{\beta}(-\vec{p}) e^{-2iE_p t} \\ &\quad \downarrow p \rightarrow -\vec{p} \quad (E_p \rightarrow E_p) \\ &= \int \frac{d^3p}{2} (-\vec{p}) \sum_{\alpha, \beta} [\vec{\epsilon}_{\alpha}(-\vec{p}) \cdot \vec{\epsilon}_{\beta}(\vec{p})] \hat{a}_{\alpha}(-\vec{p}) \hat{a}_{\beta}(\vec{p}) e^{-2iE_p t} \end{aligned}$$

Since α, β are dummy indices
& also as $\hat{a}_{\alpha}, \hat{a}_{\beta}$'s commute,

$$= - \int \frac{d^3p}{2} \vec{p} \sum_{\alpha, \beta} [\vec{\epsilon}_{\alpha}(\vec{p}) \cdot \vec{\epsilon}_{\beta}(-\vec{p})] \hat{a}_{\alpha}(\vec{p}) \hat{a}_{\beta}(-\vec{p}) e^{-2iE_p t}$$

$$\Rightarrow I = -I \Rightarrow \underline{\underline{I = 0}}$$

Similarly we can show that,

$$\int \frac{d^3p}{2} \vec{p} \sum_{\alpha, \beta} [\vec{\epsilon}_{\alpha}^*(\vec{p}) \cdot \vec{\epsilon}_{\beta}^*(-\vec{p})] \hat{a}_{\alpha}^+(\vec{p}) \hat{a}_{\beta}^+(-\vec{p}) e^{2iE_p t} = 0$$

$$\rightarrow \hat{\vec{p}} = \int \frac{d^3p}{2} \vec{p} \sum_{\lambda} [\hat{a}_{\lambda}^+(\vec{p}) \hat{a}_{\lambda}(\vec{p}) + \hat{a}_{\lambda}(\vec{p}) \hat{a}_{\lambda}^+(\vec{p})]$$

$$\text{Or } \mathcal{N}[\hat{\vec{p}}] = \int d^3p \vec{p} \sum_{\lambda} \hat{a}_{\lambda}^+(\vec{p}) \hat{a}_{\lambda}(\vec{p})$$

Thus the Fock space 3-momentum operator is also diagonal & the 3-momentum $\vec{p}$ is the eigen value of any photon created by $\hat{a}_{\lambda}^+(\vec{p})$

Hence, we can show that the complete four momentum operator is diagonal in terms of ladder operators.

$$\hat{p}^{\mu} = \int \frac{d^3p}{2} p^{\mu} \sum_{\lambda} [\hat{a}_{\lambda}^+(\vec{p}) \hat{a}_{\lambda}(\vec{p}) + \hat{a}_{\lambda}(\vec{p}) \hat{a}_{\lambda}^+(\vec{p})]$$

$$\mathcal{N}[\hat{p}^{\mu}] = \int d^3p p^{\mu} \sum_{\lambda} \hat{a}_{\lambda}^+(\vec{p}) \hat{a}_{\lambda}(\vec{p})$$