

Green's Functions

Consider a function $x(t)$ & a linear differential operator $\hat{L}$ with respect to 't' such that, $\hat{L}(t)x(t) = f(t)$

Now, green's function of $\hat{L}$ is given by 'G', which satisfies when $f(t)$ is just a dirac delta function.

$$\rightarrow \underline{\hat{L} G(t, u) = \delta(t - u)}$$

If $G(t, u)$ is known, then, we can find solution for $\hat{L}(t)x(t) = f(t)$ for any $f(t)$. The solution is given by

$$x(t) = x_0(t) + \int du G(t, u) f(u)$$

where $x_0(t)$ solves $\hat{L}(t)x_0(t) = 0$.

$$\begin{aligned} \rightarrow \hat{L}(t) \left[x_0(t) + \int du G(t, u) f(u) \right] \\ = 0 + \int du \delta(t - u) f(u) = \underline{\underline{f(t)}} \end{aligned}$$

Green's function for some known operators

a) $\hat{L} = \nabla^2$ (Laplacian)

$$\nabla^2 = \partial_{x^1}^2 + \partial_{x^2}^2 + \partial_{x^3}^2, \quad \vec{x} = (x^1, x^2, x^3)$$

To find $G(\vec{x}, \vec{y})$ which solves,

$$\nabla^2 G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$$

In Fourier space,

$$G(\vec{x}, \vec{y}) = \int \frac{d^3 k}{(2\pi)^3} G(\vec{k}) e^{i(\vec{x} - \vec{y}) \cdot \vec{k}}$$

$$\nabla^2 G(\vec{x}, \vec{y}) = \int \frac{d^3 k}{(2\pi)^3} [-|\vec{k}|^2] G(\vec{k}) e^{i(\vec{x} - \vec{y}) \cdot \vec{k}}$$

$$\rightarrow \delta(\vec{x} - \vec{y}) = \int \frac{d^3 k [-|\vec{k}|^2] G(\vec{k})}{(2\pi)^3} e^{i(\vec{x} - \vec{y}) \cdot \vec{k}}$$

$$\text{But } \delta(\vec{x} - \vec{y}) = \int \frac{d^3 k}{(2\pi)^3} e^{i(\vec{x} - \vec{y}) \cdot \vec{k}}$$

$$\rightarrow \underline{G(\vec{k}) = -\frac{1}{|\vec{k}|^2}}$$

$$\rightarrow G(\vec{x}, \vec{y}) = - \int \frac{d^3 k}{(2\pi)^3} \frac{1}{|\vec{k}|^2} e^{i(\vec{x}-\vec{y}) \cdot \vec{k}}$$

$d^3 k = |\vec{k}|^2 \sin(\theta_k) d|\vec{k}| d\theta_k d\phi_k$ and
we can choose $(\vec{x}-\vec{y}) \cdot \vec{k} = |\vec{x}-\vec{y}| |\vec{k}| \cos(\theta_k)$

$$\rightarrow G(\vec{x}, \vec{y}) = - \int \frac{d|\vec{k}|}{(2\pi)^3} d\theta_k d\phi_k \sin(\theta_k) \times e^{i|\vec{x}-\vec{y}| |\vec{k}| \cos(\theta_k)}$$

$$\text{Let } k = |\vec{k}|, \quad r = |\vec{x}-\vec{y}|$$

$$\oint d\phi_k = 2\pi, \quad \begin{matrix} 0 < k < \infty, \\ 0 < \theta_k < \pi \end{matrix}$$

$$\rightarrow G(\vec{x}, \vec{y}) = - \int_0^\infty \frac{dk}{(2\pi)^2} \int_0^\pi d\theta_k \sin(\theta_k) \times e^{i r k \cos(\theta_k)}$$

$$\text{Let } u = \cos(\theta)$$

$$du = -\sin(\theta) d\theta$$

$$\rightarrow G(\vec{x}, \vec{y}) = \frac{1}{(2\pi)^2} \int_0^\infty dk \int_{+1}^{-1} du e^{i r k u}$$

$$\begin{aligned}
 \rightarrow G(\vec{x}, \vec{y}) &= -\frac{1}{(2\pi)^2} \int_0^\infty dk \int_{-1}^1 du e^{i\vec{x}k\vec{u}} \\
 &= -\frac{1}{(2\pi)^2} \int_0^\infty dk \left[\frac{e^{i\vec{x}k} - e^{-i\vec{x}k}}{i\vec{x}k} \right] \\
 &= -\frac{2}{(2\pi)^2 \vec{x}} \underbrace{\int_0^\infty d(\vec{x}k) \frac{\sin(\vec{x}k)}{(\vec{x}k)}}_{(\pi/2)}
 \end{aligned}$$

$$\rightarrow G(\vec{x}, \vec{y}) = \frac{-1}{4\pi |\vec{x} - \vec{y}|}$$

$$\rightarrow \nabla^2 \frac{1}{|\vec{x} - \vec{y}|} = -4\pi \delta(\vec{x} - \vec{y})$$

This result is also evident from electrostatics where 'G' represents the electrostatic potential of a unit point charge at $\vec{y}$ as measured in $\vec{x}$.

b) $\hat{L} = \nabla_{\vec{x}}^2 + q^2$ (Wave equation)

Consider $\frac{1}{c^2} \frac{\partial^2 \phi(\vec{x}, t)}{\partial t^2} - \nabla^2 \phi(\vec{x}, t) = 0$,

which is a 3D wave equation.

Now $\phi(\vec{x}, t) = \phi_{\vec{k}}(\vec{x}) e^{-i\omega t}$
represents one particular mode of
the wave in frequency ' ω '.

$\rightarrow \phi_{\vec{k}}(\vec{x})$ satisfies,

$$-\frac{\omega^2}{c^2} \phi_{\vec{k}}(\vec{x}) - \nabla^2 \phi_{\vec{k}}(\vec{x}) = 0$$

$$\text{Let } q^2 = \frac{\omega^2}{c^2}$$

$$\rightarrow \underline{(\nabla_{\vec{x}}^2 + q^2) \phi_{\vec{k}}(\vec{x}) = 0} \quad \left. \vphantom{\frac{1}{c^2}} \right\} \begin{array}{l} \text{This is the} \\ \text{Helmholtz} \\ \text{equation} \end{array}$$

$$\& \quad \underline{\hat{L} \phi_{\vec{k}}(\vec{x}) = 0}$$

The green's function is given by,

$$(\nabla_{\vec{x}}^2 + q^2) G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y})$$

Going to fourier space, we get

$$(-|\vec{k}|^2 + q^2) G(\vec{k}) = 1$$

$$\Rightarrow G(\vec{k}) = \frac{-1}{k^2 - q^2} \quad (k = |\vec{k}|)$$

$$\& G(\vec{x}, \vec{y}) = - \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i(\vec{x} - \vec{y}) \cdot \vec{k}}}{(k^2 - q^2)}$$

$d^3 k = |\vec{k}|^2 \sin(\theta_k) d|\vec{k}| d\theta_k d\phi_k$ and
we can choose $(\vec{x} - \vec{y}) \cdot \vec{k} = |\vec{x} - \vec{y}| |\vec{k}| \cos(\theta_k)$

$$\text{Let } r = |\vec{x} - \vec{y}| \quad \& \int d\phi_k = 2\pi$$

$$0 < k < \infty, \quad 0 < \theta_k < \pi$$

$$\Rightarrow G(\vec{x}, \vec{y}) = - \int \frac{dk k^2 d\theta \sin(\theta) e^{i\mathbf{x} \cdot \mathbf{k} \cos(\theta)}}{(2\pi)^2 (k^2 - q^2)}$$

$$= - \int_0^\infty \frac{dk}{(2\pi)^2} \frac{k^2}{(k^2 - q^2)} \int_{-1}^{+1} du e^{i\mathbf{x} \cdot \mathbf{k} u}$$

$$= - \int_0^\infty \frac{dk}{(2\pi)^2} \frac{k^2}{(k^2 - q^2)} \left[\frac{e^{i\mathbf{x} \cdot \mathbf{k}} - e^{-i\mathbf{x} \cdot \mathbf{k}}}{i\mathbf{x} \cdot \mathbf{k}} \right]$$

$$= - \int_0^\infty \frac{dk}{(2\pi)^2} \frac{k^2}{(k^2 - q^2)} \left[\frac{e^{i\mathbf{x} \cdot \mathbf{k}}}{(i\mathbf{x} \cdot \mathbf{k})} - \underbrace{\frac{e^{-i\mathbf{x} \cdot \mathbf{k}}}{i\mathbf{x} \cdot \mathbf{k}}}_{k \rightarrow -k} \right]$$

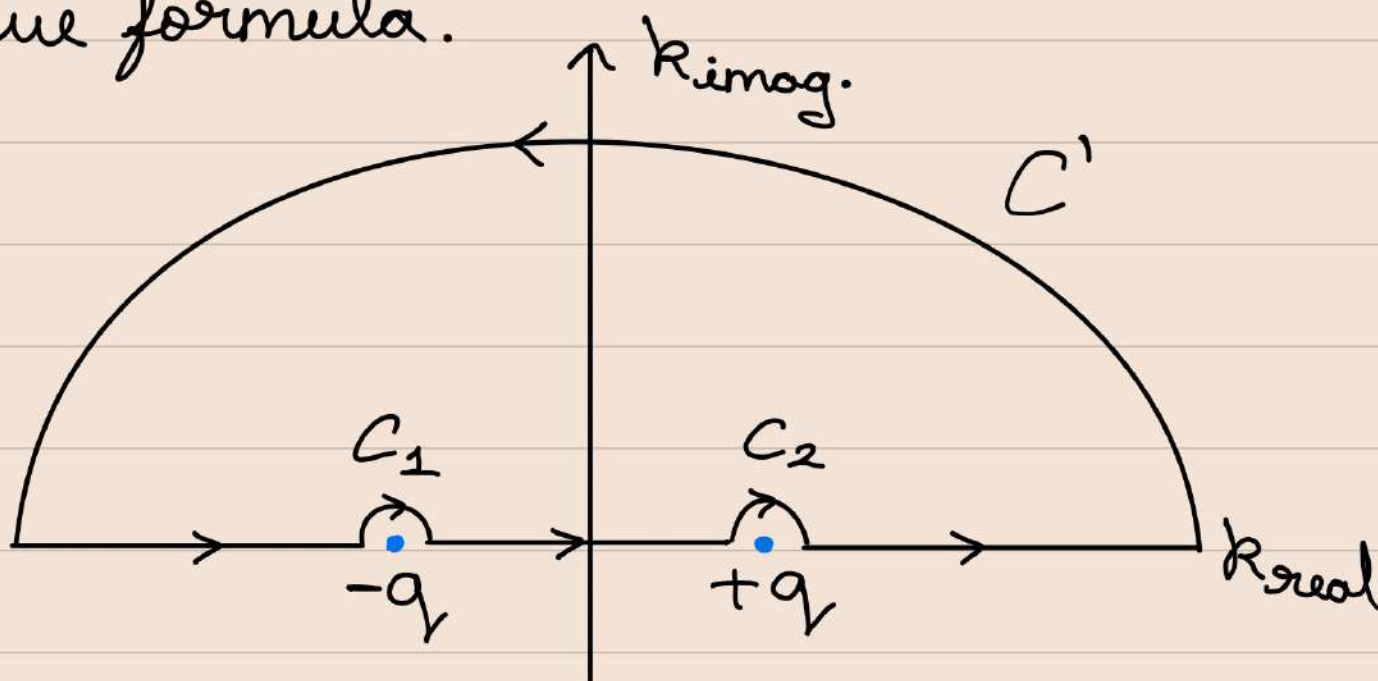
$$= - \int_{-\infty}^\infty \frac{dk}{(2\pi)^2} \frac{k^2}{(k^2 - q^2)} \left[\frac{e^{i\mathbf{x} \cdot \mathbf{k}}}{i\mathbf{x} \cdot \mathbf{k}} \right]$$

$$\Rightarrow G(\vec{x}, \vec{y}) = \frac{i}{(2\pi)^2 \mathcal{H}} \int_{-\infty}^\infty dk \left[\frac{k e^{i\mathbf{x} \cdot \mathbf{k}}}{(k^2 - q^2)} \right]$$

Consider
$$\int_{-\infty}^{\infty} dk \left[\frac{k e^{i\alpha k}}{(k^2 - q^2)} \right]$$

There is a pole at $k = \pm q$.

We can do this integral in complex plane of k and by Cauchy's principle value formula.



$C' \rightarrow$ outer semi circle with infinite radius
 Contribution from $C' = 0$ (Jordan's lemma)

$$\left. \begin{aligned} C_1: z + q &= \varepsilon e^{i\theta} \\ C_2: z - q &= \varepsilon e^{i\theta} \end{aligned} \right\} \begin{aligned} &\theta: \pi \rightarrow 0 \\ &(\varepsilon \rightarrow 0) \\ &(\varepsilon^2 = 0) \end{aligned}$$

$$dz = i\varepsilon e^{i\theta} d\theta$$

$$\underbrace{\oint \frac{z e^{i\pi z}}{(z^2 - q^2)} dz}_{\text{(Cauchy's theorem)}} = \text{PV} \int_{-\infty}^{\infty} \frac{k e^{i\pi k}}{k^2 - q^2} dk + \int_{C_1} \frac{z e^{i\pi z}}{(z^2 - q^2)} dz + \int_{C_2} \frac{z e^{i\pi z}}{(z^2 - q^2)} dz$$

$$\int_{C_1} \frac{z e^{i\pi z}}{(z^2 - q^2)} dz = \lim_{\varepsilon \rightarrow 0} \int_{\pi}^0 \frac{(\varepsilon e^{i\theta} - q) e^{i\pi(\varepsilon e^{i\theta} - q)}}{(\varepsilon e^{i\theta} - 2q)(\cancel{\varepsilon e^{i\theta}})} i \cancel{\varepsilon e^{i\theta}} d\theta$$

$$= \lim_{\varepsilon \rightarrow 0} - \int_0^{\pi} \frac{(\varepsilon e^{i\theta} - q) e^{i\pi \varepsilon e^{i\theta}} e^{-i\pi q}}{(\varepsilon e^{i\theta} - 2q)} i d\theta$$

$$= - \int_0^{\pi} \frac{(-q) e^{-i\pi q}}{(-2q)} i d\theta = \frac{-\pi i e^{-i\pi q}}{2}$$

$$\rightarrow \int_{C_1} \frac{z e^{i\pi z}}{(z^2 - q^2)} dz = \frac{-\pi i e^{-i\pi q}}{2}$$

Similarly $\int_{C_2} \frac{z e^{i\pi z}}{(z^2 - q^2)} dz = \frac{-\pi i e^{+i\pi q}}{2}$

$$\rightarrow \text{PV} \int_{-\infty}^{\infty} \frac{k e^{i\pi k}}{k^2 - q^2} dk = \pi i \left(\frac{e^{i\pi q} + e^{-i\pi q}}{2} \right)$$

$$\rightarrow \boxed{G(\vec{x}, \vec{y}) = \frac{-1}{4\pi\pi} \left[e^{i\pi q} + e^{-i\pi q} \right]}$$

Thus there is a sum of two wave vectors with $k = +q \neq k = -q$.

$$G_q^{(\text{out})}(\vec{x}, \vec{y}) = \frac{-1}{4\pi\pi} e^{i\pi q} \quad \text{is}$$

got by the pole at $k = q$, with the $e^{i\pi q}$ contribution, because of the +ve factor of wave vector (k). This describes the outward going wave.

$$G_q^{(in)}(\vec{x}, \vec{y}) = \frac{-1}{4\pi x} e^{-i x q} \quad \text{is}$$

got by the pole at $k = -q$, with the $e^{-i x q}$ contribution, because of the $-ve$ factor of wave vector (k). This describes the inward going wave.

And the total green's function is sum of outgoing & incoming waves.

Notice that for $q \rightarrow 0$, i.e. $\omega = 0$ or for a time frozen wave, we get the green's function of that of $\hat{L} = \nabla^2$.

Propagators in Quantum Mechanics

In quantum mechanics, the equation which governs the change in the wave function $\Psi(\vec{x}, t)$ is given by,

$$\hat{H}\Psi(\vec{x}, t) = i \frac{\partial \Psi(\vec{x}, t)}{\partial t}$$

where $\hat{H}$ is a function of x & $i\partial_x$.

The propagator is defined as follows,

$$\Psi(\vec{x}, t_x) = \int d^3y K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \Psi(\vec{y}, t_y)$$

$K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \rightarrow$ Propagator of the system.

It takes the system at $\vec{y}, t_y$ and evolves it to $\vec{x}, \vec{y}$. Here the '+' sign means that system evolves forward in time.
i.e. $t_x > t_y$ is a constraint.

$$\text{Now, } \psi(\vec{y}, t_y) = \langle \vec{y}, t_y | \psi \rangle$$

$$\psi(\vec{x}, t_x) = \langle \vec{x}, t_x | \psi \rangle$$

$$\psi(\vec{x}, t_x) = \int d^3y \, K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \psi(\vec{y}, t_y)$$

→

$$\langle \vec{x}, t_x | \psi \rangle = \int d^3y \, K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \langle \vec{y}, t_y | \psi \rangle$$

$$\Rightarrow \langle \vec{x}, t_x | = \int d^3y \, K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \langle \vec{y}, t_y |$$

$$\rightarrow \langle \vec{x}, t_x | \vec{y}', t_y \rangle$$

$$= \int d^3y \, K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \underbrace{\langle \vec{y}, t_y | \vec{y}', t_y \rangle}_{\delta^3(\vec{y} - \vec{y}')}$$

$$= \int d^3y \, K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) \delta^3(\vec{y} - \vec{y}')$$

$$= K^{(+)}(\vec{x}, t_x, \vec{y}', t_y)$$

$$\rightarrow \underline{\langle \vec{x}, t_x | \vec{y}', t_y \rangle = K^{(+)}(\vec{x}, t_x, \vec{y}', t_y)}$$

But the unitary time evolution operator is $\hat{U}(t_2, t_1)$. It takes the state at t_1 & evolves it to t_2 .

$$\rightarrow |\vec{y}, t_y\rangle = \hat{U}(t_y, 0) |\vec{y}\rangle$$

$$\langle \vec{x}, t_x | = \langle \vec{x} | \hat{U}^\dagger(t_x, 0)$$

$$\text{But } \hat{U}^\dagger(t_2, t_1) = \hat{U}^{-1}(t_2, t_1) = \hat{U}(t_1, t_2)$$

$$\rightarrow K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$$

$$= \langle \vec{x} | \hat{U}^\dagger(t_x, 0) \hat{U}(t_y, 0) |\vec{y}\rangle$$

$$= \underbrace{\Theta(t_x - t_y)} \langle \vec{x} | \hat{U}(t_x, t_y) |\vec{y}\rangle$$

To constrain $t_x > t_y$.

$$\underline{K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \Theta(t_x - t_y) \langle \vec{x} | \hat{U}(t_x, t_y) |\vec{y}\rangle}$$

Similarly, we define,

$$\underline{K^{(-)}(\vec{x}, t_x, \vec{y}, t_y) = \Theta(t_x - t_y) \langle \vec{x} | \hat{U}(t_x, t_y) |\vec{y}\rangle}$$

Thus the representation of the time evolution unitary operator in the position basis is the propagator, but with an additional constraint of the order of time (only allow the system to go forward in time).

Relation b/w $\hat{H}$ & $\hat{U}$

We know that the time evolution of a system in Dirac notation is,

$$\rightarrow \hat{H}|\psi(t)\rangle = i \frac{\partial}{\partial t} |\psi(t)\rangle$$

$$\text{But } |\psi(t)\rangle = \hat{U}(t, 0)|\psi(0)\rangle$$

$$\rightarrow \hat{H}\hat{U}(t, 0)|\psi(0)\rangle = i \frac{\partial}{\partial t} \hat{U}(t, 0)|\psi(0)\rangle$$

$$\rightarrow \boxed{\hat{H}\hat{U}(t, 0) = i \frac{\partial \hat{U}(t, 0)}{\partial t}}$$

If the $\hat{H}$ is not time dependent, then,
 $\rightarrow \underline{\hat{U}(t, t_0) = e^{-i(t-t_0)\hat{H}}}$

If the $\hat{H}$ is time dependent, then,
 $\rightarrow \underline{\hat{U}(t, t_0) = \exp \left[-i \int_{t_0}^t dt' \hat{H}(t') \right]}$

Propagator in Position & time domain

Now we shall show that, if the hamiltonian is time independent, then the propagator $K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ becomes the green's function of the operator

$$i \left[\hat{H}_{\vec{x}} - i \frac{\partial}{\partial t_x} \right] \left(\hat{H}_{\vec{x}} \text{ is the rep}^n \text{ of } \hat{H} \text{ in } |\vec{x}\rangle \text{ basis} \right)$$

$\rightarrow$ To prove:

$$i \left[\hat{H}_{\vec{x}} - i \frac{\partial}{\partial t_x} \right] K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \delta(x^u - y^u)$$

Now we have, $K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ when $\hat{H}$ is time independent, as

$$\begin{aligned} K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) &= \Theta(t_x - t_y) \langle \vec{x} | \hat{U}(t_x, t_y) | \vec{y} \rangle \\ &= \Theta(t_x - t_y) \langle \vec{x} | \left[e^{-i(t_x - t_y)\hat{H}} \right] | \vec{y} \rangle \end{aligned}$$

Let $|n\rangle$ be the eigen states of $\hat{H}$, then,

$$\hat{H}|n\rangle = E_n|n\rangle$$

$E_n \rightarrow$ energy eigen values

Then, $\hat{I} = \sum_n |n\rangle \langle n| \rightarrow$ Completeness relation

$$\rightarrow K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) =$$

$$= \sum_n \Theta(t_x - t_y) \langle \vec{x} | \left[e^{-i(t_x - t_y)\hat{H}} \right] | n \rangle \langle n | \vec{y} \rangle$$

$$= \sum_n \Theta(t_x - t_y) \langle \vec{x} | n \rangle e^{-i(t_x - t_y)E_n} \langle n | \vec{y} \rangle$$

using $u_n(\vec{x}) = \langle \vec{x} | n \rangle$, $u_n^*(\vec{y}) = \langle n | \vec{y} \rangle$ as the eigen functions of $\hat{H}$ in the position basis, we get,

$$K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$$

$$= \Theta(t_x - t_y) \sum_n e^{-i(t_x - t_y)E_n} u_n(\vec{x}) u_n^*(\vec{y})$$

Thus we can relate the propagator with the energy eigen wave functions.

Now, $\hat{H}_{\vec{x}} K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ is,

$$= \hat{H}_{\vec{x}} \Theta(t_x - t_y) \sum_n e^{-i(t_x - t_y)E_n} u_n(\vec{x}) u_n^*(\vec{y})$$

$$= \Theta(t_x - t_y) \sum_n e^{-i(t_x - t_y)E_n} \underbrace{\left[\hat{H}_{\vec{x}} u_n(\vec{x}) \right]}_{E_n u_n(\vec{x})} u_n^*(\vec{y})$$

$$= \Theta(t_x - t_y) \sum_n E_n e^{-i(t_x - t_y)E_n} u_n(\vec{x}) u_n^*(\vec{y})$$

Then, $i \frac{\partial}{\partial t_x} K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ is,

$$= \Theta(t_x - t_y) \sum_n E_n e^{-i(t_x - t_y) E_n} u_n(\vec{x}) u_n^*(\vec{y}) \\ + \left[i \frac{\partial}{\partial t_x} \Theta(t_x - t_y) \right] \sum_n e^{-i(t_x - t_y) E_n} u_n(\vec{x}) u_n^*(\vec{y})$$

$$\underline{\frac{\partial}{\partial t_x} \Theta(t_x - t_y) = \delta(t_x - t_y)}$$

$$\rightarrow i \left[\hat{H}_{\vec{x}} - i \frac{\partial}{\partial t_x} \right] K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) =$$

$$= \delta(t_x - t_y) \sum_n e^{-i(t_x - t_y) E_n} u_n(\vec{x}) u_n^*(\vec{y})$$

This expression is non zero only when $t_x = t_y$. Thus we can set

$$\delta(t_x - t_y) e^{-i(t_x - t_y) E_n} = \delta(t_x - t_y)$$

$$\begin{aligned}
\rightarrow i \left[\hat{H}_{\vec{x}} - i \frac{\partial}{\partial t_x} \right] K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) &= \\
&= \delta(t_x - t_y) \sum_n \psi_n(\vec{x}) \psi_n^*(\vec{y}) \\
&= \delta(t_x - t_y) \sum_n \langle \vec{x} | n \rangle \langle n | \vec{y} \rangle \\
&= \delta(t_x - t_y) \underbrace{\langle \vec{x} | \vec{y} \rangle}_{\delta(\vec{x} - \vec{y})} \underbrace{\sum_n |n\rangle \langle n|}_{\hat{I}}
\end{aligned}$$

$$\rightarrow i \left[\hat{H}_{\vec{x}} - i \frac{\partial}{\partial t_x} \right] K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \underbrace{\delta(t_x - t_y) \delta(\vec{x} - \vec{y})}_{\delta(x^\mu - y^\mu)}$$

Thus we showed that,

$$i \left[\hat{H}_{\vec{x}} - i \frac{\partial}{\partial t_x} \right] K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \delta(x^\mu - y^\mu)$$

Hence, the propagator is the green's function of the Schrodinger eqⁿ.

Propagator for Non-relativistic free Particle

Here $\hat{H} = \frac{\hat{\vec{p}}^2}{2m} \rightarrow \hat{H}_{\vec{x}} = -\frac{\hbar^2}{2m} \left[\partial_{x^1}^2 + \partial_{x^2}^2 + \partial_{x^3}^2 \right]$

The eigen function of $\hat{H}$ is the momentum eigen ket $|\vec{p}\rangle$ which follow $\langle \vec{q} | \vec{p} \rangle = \delta(\vec{p} - \vec{q})$

$$\rightarrow \int d^3p |\vec{p}\rangle \langle \vec{p}| = \hat{I}$$

& $\hat{H} |\vec{p}\rangle = E_{\vec{p}} |\vec{p}\rangle$ where $E_{\vec{p}} = \frac{p^2}{2m}$

Thus in position basis, $u_{\vec{p}}(\vec{x}) = \langle \vec{x} | \vec{p} \rangle$

where $u_{\vec{p}}(\vec{x}) = \frac{1}{(2\pi)^{3/2}} e^{i\vec{p} \cdot \vec{x}}$

Now, $K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ is given by,

$$= \Theta(t_x - t_y) \langle \vec{x} | \hat{U}(t_x, t_y) | \vec{y} \rangle$$

$$= \Theta(t_x - t_y) \langle \vec{x} | \left[e^{-i(t_x - t_y)\hat{H}} \right] | \vec{y} \rangle$$

Insert $\hat{I} = \int d^3p |\vec{p}\rangle \langle \vec{p}|$, we get,

$$K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) =$$

$$= \Theta(t_x - t_y) \int d^3p \langle \vec{x} | [e^{-i(t_x - t_y)\hat{H}}] | \vec{p} \rangle \langle \vec{p} | \vec{y} \rangle$$

$$= \Theta(t_x - t_y) \int d^3p e^{-i(t_x - t_y)E_{\vec{p}}} \langle \vec{x} | \vec{p} \rangle \langle \vec{p} | \vec{y} \rangle$$

$$= \Theta(t_x - t_y) \int \frac{d^3p e^{-i(t_x - t_y)E_{\vec{p}}}}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} e^{-i\vec{p} \cdot \vec{y}}$$

$$= \Theta(t_x - t_y) \int \frac{d^3p e^{\frac{-i(t_x - t_y)|\vec{p}|^2}{2m}} e^{i\vec{p} \cdot [\vec{x} - \vec{y}]}}{(2\pi)^3}$$

This is a gaussian integral of three variables $p^1, p^2 \neq p^3$, for a given p^j ,

$$I = \int_{-\infty}^{\infty} \frac{dp^j}{(2\pi)} \exp \left[\frac{-i(t_x - t_y)(p^j)^2}{2m} + ip^j(x^j - y^j) \right]$$

Now, $\int_{-\infty}^{\infty} dx e^{[-ax^2+bx]} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}}$

$$\Rightarrow I = \left(\frac{1}{2\pi}\right) \sqrt{\frac{2m\pi}{i(t_x - t_y)}} e^{\left[\frac{im(\vec{x} - \vec{y})^2}{2(t_x - t_y)}\right]}$$

Thus the propagator for free particle is $K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ given by,

$$\textcircled{H} (t_x - t_y) \sqrt{\frac{m}{2\pi i(t_x - t_y)}} \exp\left[\frac{im(\vec{x} - \vec{y})^2}{2(t_x - t_y)}\right]$$

Thus the propagator depends only on the time difference and also on the distance of propagation.

Propagator in Position & Energy domain

The usual time and space domain integration is given by,

$$K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \\ = \Theta(t_x - t_y) \sum_n e^{-i(t_x - t_y)E_n} u_n(\vec{x}) u_n^*(\vec{y})$$

Without any loss of generality, we shall set $t_y = 0$ and $t_x = t$.

$$K^{(+)}(\vec{x}, t, \vec{y}, 0) = \Theta(t) \sum_n e^{-itE_n} u_n(\vec{x}) u_n^*(\vec{y})$$

We shall define $G^{(+)}(\vec{x}, \vec{y}, E)$ as the propagator in the energy domain which is got by fourier transform of the time domain propagator $K^{(+)}(\vec{x}, t, \vec{y}, 0)$.

$$G^{(+)}(\vec{x}, \vec{y}, E) = \int_{-\infty}^{\infty} K^{(+)}(\vec{x}, t, \vec{y}, 0) e^{iEt} dt$$

$\rightarrow G^{(+)}(\vec{x}, \vec{y}, E)$ is given by,

$$= \int_{-\infty}^{\infty} dt \Theta(t) \sum_n e^{-itE_n} u_n(\vec{x}) u_n^*(\vec{y}) e^{iEt}$$

(Survives only when $t > 0$)

$$= \sum_n u_n(\vec{x}) u_n^*(\vec{y}) \int_0^{\infty} dt e^{i(E-E_n)t}$$

But to ensure convergence, we shall multiply a factor of $e^{-\varepsilon t}$, $\varepsilon > 0$ and then at last take the limit $\varepsilon \rightarrow 0$.

$$\rightarrow G^{(+)}(\vec{x}, \vec{y}, E) =$$

$$= \lim_{\varepsilon \rightarrow 0} \sum_n u_n(\vec{x}) u_n^*(\vec{y}) \int_0^{\infty} dt e^{i(E-E_n+i\varepsilon)t}$$

$$= \lim_{\varepsilon \rightarrow 0} \sum_n u_n(\vec{x}) u_n^*(\vec{y}) \left[\frac{e^{i(E-E_n+i\varepsilon)t}}{i(E-E_n+i\varepsilon)} \right]_{t=0}^{t=\infty}$$

$$= \lim_{\varepsilon \rightarrow 0} \sum_n u_n(\vec{x}) u_n^*(\vec{y}) \left(\frac{-1}{i(E-E_n+i\varepsilon)} \right)$$

Thus

$$G^{(+)}(\vec{x}, \vec{y}, E) = \sum_n \frac{i u_n(\vec{x}) u_n^*(\vec{y})}{E - E_n}$$

Thus we can consider 'E' as a variable complex parameter, where E_n are the poles in the complex E-plane. The residues of these singularities signifies the energy eigen functions.

$$\begin{aligned}
 \text{Now, } G^{(+)}(\vec{x}, \vec{y}, E) &= \sum_n \frac{i u_n(\vec{x}) u_n^*(\vec{y})}{E - E_n} \\
 &= \sum_n \frac{i \langle \vec{x} | n \rangle \langle n | \vec{y} \rangle}{E - E_n} \\
 &= \sum_n i \langle \vec{x} | \left[\frac{1}{E - E_n} \right] | n \rangle \langle n | \vec{y} \rangle \\
 &= \sum_n i \langle \vec{x} | (E - \hat{H})^{-1} | n \rangle \langle n | \vec{y} \rangle \\
 &= i \langle \vec{x} | (E - \hat{H})^{-1} \underbrace{\left[\sum_n | n \rangle \langle n | \right]}_{\hat{I}} | \vec{y} \rangle \\
 &\rightarrow \underline{G^{(+)}(\vec{x}, \vec{y}, E) = i \langle \vec{x} | (E - \hat{H})^{-1} | \vec{y} \rangle}
 \end{aligned}$$

Propagator in Momentum & time domain

Given $K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ as follows,
① $(t_x - t_y) \langle \vec{x} | \hat{U}(t_x, t_y) | \vec{y} \rangle$, we can find momentum space representation by fourier transforming $|\vec{x}\rangle$ & $|\vec{y}\rangle$.

$$|\vec{x}\rangle = \int d^3p |\vec{p}\rangle \langle \vec{p} | \vec{x} \rangle$$

$$\langle \vec{p} | \vec{x} \rangle = \frac{e^{-i\vec{p} \cdot \vec{x}}}{(2\pi)^{3/2}}$$

$$\Rightarrow \langle \vec{x} | = \int \frac{d^3p}{(2\pi)^{3/2}} \langle \vec{p} | e^{i\vec{p} \cdot \vec{x}}$$

$$\& |\vec{y}\rangle = \int \frac{d^3q}{(2\pi)^{3/2}} |\vec{q}\rangle e^{-i\vec{q} \cdot \vec{y}}$$

$$\Rightarrow \langle \vec{x} | \hat{U}(t_x, t_y) | \vec{y} \rangle =$$

$$= \int \frac{d^3q d^3p}{(2\pi)^3} \langle \vec{p} | \hat{U}(t_x, t_y) | \vec{q} \rangle e^{i(\vec{p} \cdot \vec{x} - \vec{q} \cdot \vec{y})}$$

$K^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \Theta(t_x - t_y) \langle \vec{x} | \hat{U}(t_x, t_y) | \vec{y} \rangle$
 can be written as,

$$= \int \frac{d^3 q d^3 p}{(2\pi)^3} \Theta(t_x - t_y) \langle \vec{p} | \hat{U}(t_x, t_y) | \vec{q} \rangle \times e^{i(\vec{q} \cdot \vec{x} - \vec{p} \cdot \vec{y})}$$

Thus we see the position space fourier transform of $K^{(+)}(\vec{x}, t_x, \vec{y}, t_y)$ gives the momentum space propagator,

$$\underline{G^{(+)}(\vec{p}, t_x, \vec{q}, t_y) = \Theta(t_x - t_y) \langle \vec{p} | \hat{U}(t_x, t_y) | \vec{q} \rangle}$$

Thus it takes in a state $|\vec{q}\rangle$ at time ' t_y ' & then results in a state $|\vec{p}\rangle$ at time t_x . If the $\hat{H}$ is time independent, then,

$$\hat{U}(t_x, t_y) = e^{-i(t_x - t_y)\hat{H}}, \text{ and if}$$

$$\hat{H}|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle, \text{ (Free particle), we get,}$$

$$\underline{G^{(+)}(\vec{p}, t_x, \vec{q}, t_y) = \Theta(t_x - t_y) \delta(\vec{p} - \vec{q}) e^{-i(t_x - t_y)E_{\vec{p}}}}$$

Propagator in Momentum & Energy domain

This is got by taking fourier transform of $t \rightarrow$ Energy space as

$$G^{(+)}(\vec{p}, \vec{q}, E) = \int_{-\infty}^{\infty} dt G^{(+)}(\vec{p}, t, \vec{q}, 0) e^{iEt}$$
$$= \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dt \Theta(t) \delta(\vec{p} - \vec{q}) e^{-iE_F t} e^{iEt} e^{-\epsilon t}$$

$$= \lim_{\epsilon \rightarrow 0} \delta(\vec{p} - \vec{q}) \int_0^{\infty} dt e^{i(E - E_F + i\epsilon)t}$$

$$\Rightarrow G^{(+)}(\vec{p}, \vec{q}, E) = \lim_{\epsilon \rightarrow 0} \frac{i \delta(\vec{p} - \vec{q})}{E - E_F + i\epsilon}$$

Thus it gives the amplitude for a particle with energy E , to enter with a momentum $\vec{q}$ & leave with momentum $\vec{p}$.

Thus we can summarize the various forms of propagators in quantum mech.

- In position space & time domain, it is,

$$G^{(+)}(\vec{x}, t_x, \vec{y}, t_y) = \Theta(t_x - t_y) \sum_n e^{-i(t_x - t_y)E_n} \psi_n(\vec{x}) \psi_n^*(\vec{y})$$

- In position space & energy domain, it is,

$$G^{(+)}(\vec{x}, \vec{y}, E) = \lim_{\epsilon \rightarrow 0} \sum_n \frac{i \psi_n(\vec{x}) \psi_n^*(\vec{y})}{E - E_n + i\epsilon}$$

- In momentum space & time domain, it is,

$$G^{(+)}(\vec{p}, t_x, \vec{q}, t_y) = \Theta(t_x - t_y) \delta(\vec{p} - \vec{q}) e^{-i(t_x - t_y)E_{\vec{p}}}$$

- In momentum space & Energy domain, it is,

$$G^{(+)}(\vec{p}, \vec{q}, E) = \lim_{\epsilon \rightarrow 0} \frac{i \delta(\vec{p} - \vec{q})}{E - E_{\vec{p}} + i\epsilon}$$

Propagators for Quantum Fields

In the canonical second quantization picture, we interpreted the quantized excitations of the fields as the Particles and their anti-particles.

But our next task is to keep track of these particles. We can do so using quantities like Propagators, which will have the informations about excitations of the systems and they describe the fate of particles in a field theory.

In most of the field theories, the propagators are directly related to the Green's function of the equation of motion got by the Lagrangian that describes the non-interacting version of the quantum fields (which implies that the theory is diagonalizable using canonical quantization)

From now we shall call the vacuum state for canonically quantizable systems as the state $|0\rangle$. Then the creation operators $\hat{a}^\dagger(\vec{p})$ will create 'a' type non-interacting particle, with the definite momentum of ' $\vec{p}$ ' and energy of $E_{\vec{p}}$.

$$\Rightarrow \hat{a}^\dagger(\vec{p})|0\rangle = |\vec{p}\rangle, \text{ where,}$$

$$\hat{\vec{p}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle \quad \& \quad \hat{H}|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle$$

But in the interacting/non-diagonalizable theory, we cannot simply create particles with definite momenta $\vec{p}$, & it always results in superposition of various momentum states. For such interacting theories we denote the vacuum as $|\Omega\rangle$.

$$\Rightarrow \hat{a}^\dagger(\vec{p})|\Omega\rangle = \begin{cases} |\vec{p}\rangle \rightarrow \text{Non-Interacting Particle} \\ \text{Superposition of states} \rightarrow \text{Interacting Particle} \end{cases}$$

Propagators for Klein-Gordon Fields

The Green's function for the Klein Gordon field which satisfies

$$(\partial_\mu \partial^\mu + m^2) G(x, y) = -i \delta^{(4)}(x - y)$$

becomes the propagator for the particles described by the field operator $\hat{\phi}$, that satisfies the Klein Gordon eqⁿ

$$(\partial_\mu \partial^\mu + m^2) \hat{\phi} = 0$$

To find the green's function, we shall go to fourier space, where,

$$G(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p x} \tilde{G}(p, y)$$

$$\rightarrow \tilde{G}(p, y) = \int d^4 x e^{i p x} G(x, y)$$

$$\neq \delta^{(4)}(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p(x - y)}$$

$$\rightarrow (\partial_\mu \partial^\mu + m^2) G(x, y)$$

$$= \int \frac{d^4 p}{(2\pi)^4} [(i p_\mu)(i p^\mu) + m^2] e^{-i p x} \tilde{G}(p, y)$$

$$= \int \frac{d^4 p}{(2\pi)^4} [-p^\mu p_\mu + m^2] e^{-i p x} \tilde{G}(p, y)$$

$$\text{Since } (\partial_\mu \partial^\mu + m^2) G(x, y) = -i \delta^{(4)}(x - y)$$

$$\begin{aligned} \rightarrow \int \frac{d^4 p}{(2\pi)^4} [-p^\mu p_\mu + m^2] e^{-i p x} \tilde{G}(p, y) \\ = -i \int \frac{d^4 p}{(2\pi)^4} e^{-i p(x-y)} \end{aligned}$$

$$\rightarrow [-p^\mu p_\mu + m^2] \tilde{G}(p, y) = -i e^{i p y}$$

$$\rightarrow \tilde{G}(p, y) = \frac{i e^{i p y}}{(p^\mu p_\mu - m^2)}$$

$$\rightarrow G(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p x} \left[\frac{i e^{i p y}}{(p^\mu p_\mu - m^2)} \right]$$

$$\rightarrow G(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^\mu p_\mu - m^2)} e^{-i p(x-y)}$$

$$\text{But } p^\mu p_\mu - m^2 = (p^0)^2 - |\vec{p}|^2 - m^2 \\ = (p^0)^2 - (E_p)^2$$

$$\text{where } E_p = \sqrt{|\vec{p}|^2 + m^2} \quad (E_p > 0)$$

$$\rightarrow \underline{G(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^0)^2 - E_p^2} e^{-ip(x-y)}}$$

Feynmann Propagator for K.G. fields

The integral above has a pole at $p^0 = E_p$, so we need to go to the complex plane & find its residue.

How we take the contour around the pole, determines the propagator.

$$\text{Let } \Delta_F(p) = \frac{i}{(p^0)^2 - E_p^2 + i\varepsilon}, \quad (\varepsilon > 0)$$

This is the complexified propagator whose inverse fourier transform gives the Feynmann propagator ' $\Delta_F(x, y)$ '

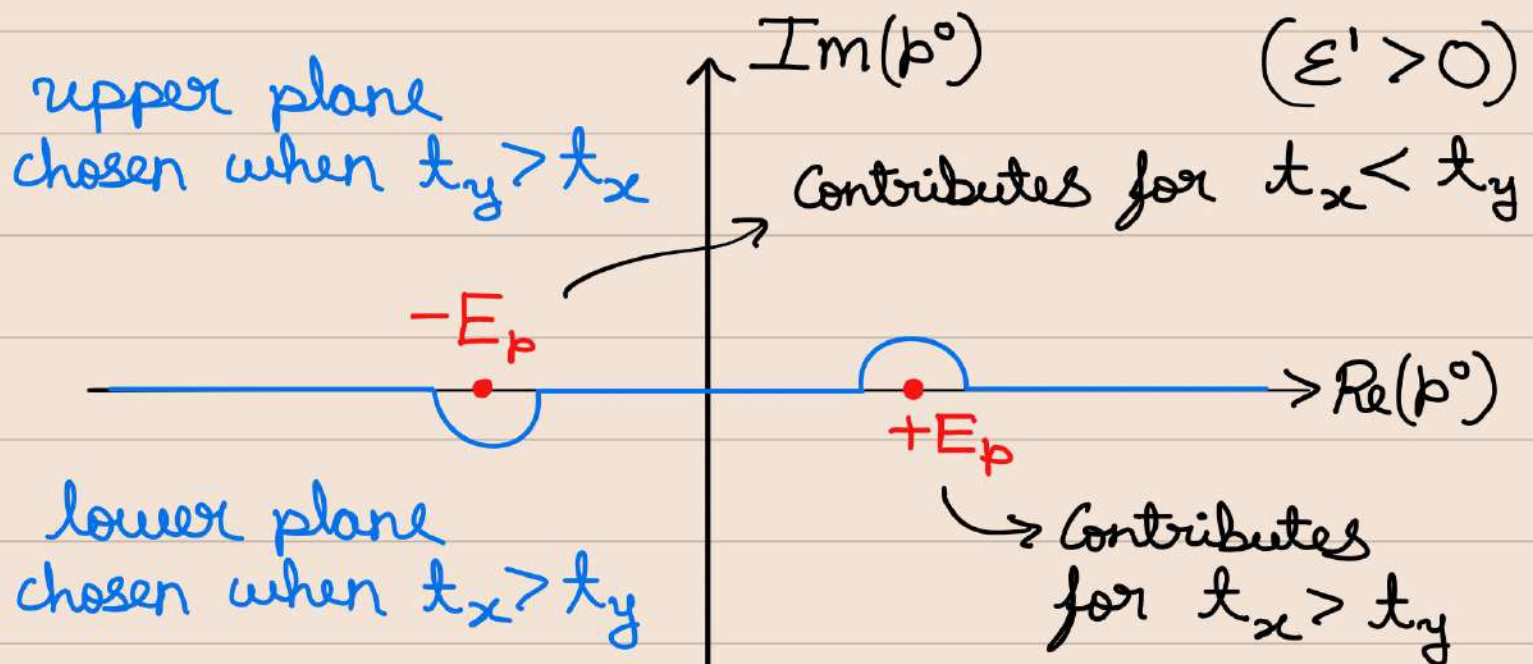
Consider $\tilde{G}(p) = \frac{1}{(p^0)^2 - E_p^2}$ which has two poles at $p^0 = \pm E_p$.

$$\tilde{G}(p) = \frac{1}{(p^0)^2 - E_p^2} = \frac{1}{(p^0 - E_p)(p^0 + E_p)}$$

$$\rightarrow \tilde{G}(p) = \frac{1}{2E_p} \left[\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p} \right]$$

Feynmann's
Prescription

$$\Delta_F(p) = \frac{i}{2E_p} \left[\frac{1}{p^0 - E_p + i\varepsilon'} - \frac{1}{p^0 + E_p - i\varepsilon'} \right]$$



Then we shall call the inverse fourier transform of $\Delta_F(p)$ as $\Delta_F(x, y)$, which still satisfies,

$$(\partial_\mu \partial^\mu + m^2) \Delta_F(x, y) = -i \delta^{(4)}(x - y)$$

This version which captures the poles of $p^0 = E_p - i\varepsilon'$ & $p^0 = -E_p + i\varepsilon'$ is called Feynmann propagator in momentum space.

$$\begin{aligned} \Delta_F(p) &= \frac{i}{(p^0)^2 - E_p^2 + i\varepsilon} \\ &= \frac{i}{2E_p} \left[\frac{1}{p^0 - E_p + i\varepsilon'} - \frac{1}{p^0 + E_p - i\varepsilon'} \right] \end{aligned}$$

ε & ε' are very small parameters that are directly proportional to each other.
($\varepsilon = 2E_p \varepsilon'$)

$$\Rightarrow \Delta_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} \Delta_F(p) e^{-ip(x-y)}$$

$$= i \int \frac{d^4 p}{(2\pi)^4 (2E_p)} e^{-ip(x-y)} \left[\frac{1}{p^0 - E_p + i\varepsilon'} - \frac{1}{p^0 + E_p - i\varepsilon'} \right]$$

Now we first perform p^0 integral.

Consider
$$I(t) = \lim_{\varepsilon \rightarrow 0^+} i \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \frac{e^{-i\zeta t}}{\zeta + i\varepsilon}$$

$$\begin{aligned} \Rightarrow \frac{dI(t)}{dt} &= \lim_{\varepsilon \rightarrow 0^+} \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\zeta (-i\zeta) e^{-i\zeta t}}{\zeta + i\varepsilon} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\zeta e^{-i\zeta t} \end{aligned}$$

$$\Rightarrow \frac{dI(t)}{dt} = \delta(t) \quad \rightarrow I(t) \text{ has to be Heaviside.}$$

$$\Rightarrow \lim_{\varepsilon \rightarrow 0^+} i \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \frac{e^{-i\zeta t}}{\zeta + i\varepsilon} = \Theta(t)$$

$$\Rightarrow \Delta_F(x, y) =$$

$$= i \int \frac{d^4 p}{(2\pi)^4 (2E_p)} e^{-ip(x-y)} \left[\frac{1}{p^0 - E_p + i\varepsilon'} - \frac{1}{p^0 + E_p - i\varepsilon'} \right]$$

$$= \int \frac{d^3 p}{(2\pi)^3 (2E_p)} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \left[i \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \left[\frac{e^{-ip^0(t_x - t_y)}}{p^0 - E_p + i\varepsilon'} - \frac{e^{-ip^0(t_x - t_y)}}{p^0 + E_p - i\varepsilon'} \right] \right]$$

Now by using the above identity

$$i \int_{-\infty}^{\infty} \frac{dp^0 e^{-ip^0(t_x - t_y)}}{2\pi(p^0 - E_p + i\varepsilon')} = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\zeta e^{-i[\zeta + E_p](t_x - t_y)}}{\zeta + i\varepsilon}$$

$(\zeta = p^0 - E_p)$

$$= e^{-iE_p(t_x - t_y)} \underbrace{\left[\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{d\zeta e^{-i\zeta(t_x - t_y)}}{(\zeta + i\varepsilon)} \right]}_{\Theta(t_x - t_y)}$$

$$\rightarrow i \int_{-\infty}^{\infty} \frac{dp^0 e^{-ip^0(t_x - t_y)}}{2\pi(p^0 - E_p + i\varepsilon')} = e^{-iE_p(t_x - t_y)} \Theta(t_x - t_y)$$

Similarly, we can write,

$$-i \int_{-\infty}^{\infty} \frac{dp^0 e^{-ip^0(t_x - t_y)}}{2\pi(p^0 + E_p - i\varepsilon')} = i \int_{-\infty}^{\infty} \frac{d\zeta e^{i(\zeta + E_p)(t_x - t_y)}}{2\pi(\zeta + i\varepsilon)}$$

$(p^0 + E_p = -\zeta)$

$$= e^{iE_p(t_x - t_y)} \underbrace{i \int_{-\infty}^{\infty} \frac{d\zeta e^{-i\zeta(t_y - t_x)}}{2\pi(\zeta + i\varepsilon)}}_{\Theta(t_y - t_x)}$$

$$\rightarrow -i \int_{-\infty}^{\infty} \frac{dp^0 e^{-ip^0(t_x - t_y)}}{2\pi(p^0 + E_p - i\varepsilon')} = e^{iE_p(t_x - t_y)} \Theta(t_y - t_x)$$

$$\Rightarrow \Delta_F(x, y) =$$

$$= \int \frac{d^3p e^{i\vec{p} \cdot (\vec{x} - \vec{y})}}{(2\pi)^3 (2E_p)} \left\{ e^{-iE_p(t_x - t_y)} \Theta(t_x - t_y) + e^{iE_p(t_x - t_y)} \Theta(t_y - t_x) \right\}$$

Now in the 2nd term put $\vec{p} \rightarrow -\vec{p}$

$$\Rightarrow (d^3p) \rightarrow -(d^3p), \quad E_p \rightarrow E_p$$

$$\int_{-\infty}^{\infty} \rightarrow -\int_{-\infty}^{\infty}, \text{ we get}$$

$$\Rightarrow \Delta_F(x, y) =$$

$$= \int \frac{d^3p e^{i\vec{p} \cdot (\vec{x} - \vec{y})}}{(2\pi)^3 (2E_p)} e^{-iE_p(t_x - t_y)} \Theta(t_x - t_y) + \int \frac{d^3p e^{-i\vec{p} \cdot (\vec{x} - \vec{y})}}{(2\pi)^3 (2E_p)} e^{iE_p(t_x - t_y)} \Theta(t_y - t_x)$$

$$\Rightarrow \Delta_F(x, y) =$$

$$= \int \frac{d^3p}{(2\pi)^3 (2E_p)} e^{-ip(x-y)} \Theta(t_x - t_y) + \int \frac{d^3p}{(2\pi)^3 (2E_p)} e^{ip(x-y)} \Theta(t_y - t_x)$$

where $p^0 = E_p$, $p \rightarrow p^\mu = (E_p, \vec{p})$

Thus the final expression for Feynmann propagator in position space for the Klein Gordon eqⁿ is given by,

$$\Rightarrow \Delta_F(x, y) = \int \frac{d^3p}{(2\pi)^3 (2E_p)} \left[\Theta(t_x - t_y) e^{-ip(x-y)} + \Theta(t_y - t_x) e^{ip(x-y)} \right]$$

This expression of $\Delta_F(x, y)$ solves,

$$(\partial_\mu \partial^\mu + m^2) \Delta_F(x, y) = -i \delta^{(4)}(x - y)$$

Physical Interpretation of Feynmann Propagator

Consider a field operator $\hat{\phi}(x)$, that solves Klein-Gordon eqⁿ:

$$\rightarrow (\partial_\mu \partial^\mu + m^2) \hat{\phi}(x) = 0$$

But $\hat{\phi}(x)$ need not be real, we can as well consider $\hat{\phi}(x)$ to be complex. So, in terms of fourier modes, most general expression for $\hat{\phi}(x) \neq \hat{\phi}^\dagger(x)$ is,

$$\hat{\phi}(x) = \frac{\int d^3p [\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{b}^\dagger(\vec{p}) e^{+ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\phi}^\dagger(x) = \frac{\int d^3p [\hat{a}^\dagger(\vec{p}) e^{+ip_\mu x^\mu} + \hat{b}(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

Here $\hat{\phi}(x)$ and $\hat{\phi}^\dagger(x)$ represent free field operators where they create or annihilate two types of excitations. We shall choose to call $\hat{a}^\dagger$ as the scalar particle & $\hat{b}^\dagger$ as anti-particle.

$\rightarrow \hat{\phi}^\dagger(x)$ creates particles & annihilates anti-particles.

$\hat{\phi}(x)$ creates anti-particles and & annihilates particles.

Since, the fields $\hat{\phi}(x)$ and $\hat{\phi}^\dagger(x)$ solve K.G. eqⁿ obtained by free scalar field lagrangian, they represent free fields.

$$\text{Consider } \hat{\phi}^\dagger(y)|0\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} e^{+ik_\mu y^\mu} [\hat{a}^\dagger(\vec{p})|0\rangle]$$

$$\text{and } \langle 0|\hat{\phi}(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} e^{-ik_\mu x^\mu} [\langle 0|\hat{a}(\vec{p})]$$

Then, $\langle 0 | \hat{\phi}(x) \hat{\phi}^\dagger(y) | 0 \rangle$ is given by,

$$= \int \frac{d^3q d^3p e^{-i(q_\mu x^\mu - p_\mu y^\mu)}}{(2\pi)^3 (2) \sqrt{E_q E_p}} \langle 0 | \hat{a}(\vec{q}) \hat{a}^\dagger(\vec{p}) | 0 \rangle$$

$$= \int \frac{d^3q d^3p e^{-i(q_\mu x^\mu - p_\mu y^\mu)}}{(2\pi)^3 (2) \sqrt{E_q E_p}} \underbrace{\langle 0 | \hat{a}(\vec{q}) \hat{a}^\dagger(\vec{p}) | 0 \rangle}_{\delta^3(\vec{q} - \vec{p})}$$

$$= \int \frac{d^3q d^3p e^{-i(q_\mu x^\mu - p_\mu y^\mu)}}{(2\pi)^3 (2) \sqrt{E_q E_p}} \delta^3(\vec{q} - \vec{p})$$

$$\Rightarrow \langle 0 | \hat{\phi}(x) \hat{\phi}^\dagger(y) | 0 \rangle = \int \frac{d^3p e^{-i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 (2 E_p)}$$

Similarly,

$$\hat{\phi}(x) | 0 \rangle = \int \frac{d^3p e^{+i p_\mu x^\mu}}{\sqrt{(2\pi)^3 (2 E_p)}} [\hat{b}^\dagger(\vec{p}) | 0 \rangle]$$

$$\langle 0 | \hat{\phi}^\dagger(y) = \int \frac{d^3p e^{-i p_\mu y^\mu}}{\sqrt{(2\pi)^3 (2 E_p)}} [\langle 0 | \hat{b}(\vec{p})]$$

Then, $\langle 0 | \hat{\phi}^\dagger(y) \hat{\phi}(x) | 0 \rangle$ is given by,

$$= \int \frac{d^3q d^3p e^{-i(q_\mu y^\mu - p_\mu x^\mu)}}{(2\pi)^3 (2) \sqrt{E_q E_p}} \langle 0 | \hat{b}(\vec{p}) \hat{b}^\dagger(\vec{p}) | 0 \rangle$$

$$= \int \frac{d^3q d^3p e^{-i(q_\mu y^\mu - p_\mu x^\mu)}}{(2\pi)^3 (2) \sqrt{E_q E_p}} \underbrace{\langle 0 | \hat{a}(\vec{q}) \hat{a}^\dagger(\vec{p}) | 0 \rangle}_{\delta^3(\vec{q} - \vec{p})}$$

$$= \int \frac{d^3q d^3p e^{-i(q_\mu y^\mu - p_\mu x^\mu)}}{(2\pi)^3 (2) \sqrt{E_q E_p}} \delta^3(\vec{q} - \vec{p})$$

$$\Rightarrow \langle 0 | \hat{\phi}^\dagger(y) \hat{\phi}(x) | 0 \rangle = \int \frac{d^3p e^{i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 (2 E_p)}$$

Then we can write the Feynmann propagator in position space as,

$$\Rightarrow \Delta_F(x, y) = \Theta(t_x - t_y) \langle 0 | \hat{\phi}(x) \hat{\phi}^\dagger(y) | 0 \rangle + \Theta(t_y - t_x) \langle 0 | \hat{\phi}^\dagger(y) \hat{\phi}(x) | 0 \rangle$$

Thus, physically, the free Feynmann propagator gives the vacuum state expectation value of the following,

$$\Delta(x, y) = \begin{cases} \langle 0 | \hat{\phi}(x) \hat{\phi}^+(y) | 0 \rangle, & t_x > t_y \\ \langle 0 | \hat{\phi}^+(y) \hat{\phi}(x) | 0 \rangle, & t_y > t_x \end{cases}$$

$\langle 0 | \hat{\phi}(x) \hat{\phi}^+(y) | 0 \rangle$ represents the probability amplitude of a particle to be created from vacuum at time t_y , position $\vec{y}$ & then to get annihilated at a later time t_x ($t_x > t_y$), position $\vec{x}$.

$\langle 0 | \hat{\phi}^+(y) \hat{\phi}(x) | 0 \rangle$ represents the probability amplitude of anti-particle to be created from vacuum at time t_x , position $\vec{x}$ & then to get annihilated at a later time t_y ($t_y > t_x$), position $\vec{y}$.

We can define the time ordered product of Bosonic operators as follows.

$$T[\hat{A}(x)\hat{B}(y)] = \begin{cases} \hat{A}(x)\hat{B}(y), & x^0 > y^0 \\ \hat{B}(y)\hat{A}(x), & y^0 > x^0 \end{cases}$$

Thus in time ordered product, operators are arranged from left to right in decreasing value of time component.

$$\rightarrow T[\hat{A}(x)\hat{B}(y)] = \Theta(t_x - t_y)\hat{A}(x)\hat{B}(y) + \Theta(t_y - t_x)\hat{B}(y)\hat{A}(x)$$

So, the feynmann propagator for fields that solve K.G eqⁿ are written as:

$$\underline{\Delta_F(x, y) = \langle 0 | T[\hat{\phi}(x)\hat{\phi}^+(y)] | 0 \rangle}$$

As a fourier integral, it can also be written as

$$\underline{\Rightarrow \Delta_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-i p_\mu (x^\mu - y^\mu)}}{(p^0)^2 - E_p^2 + i\epsilon}}$$

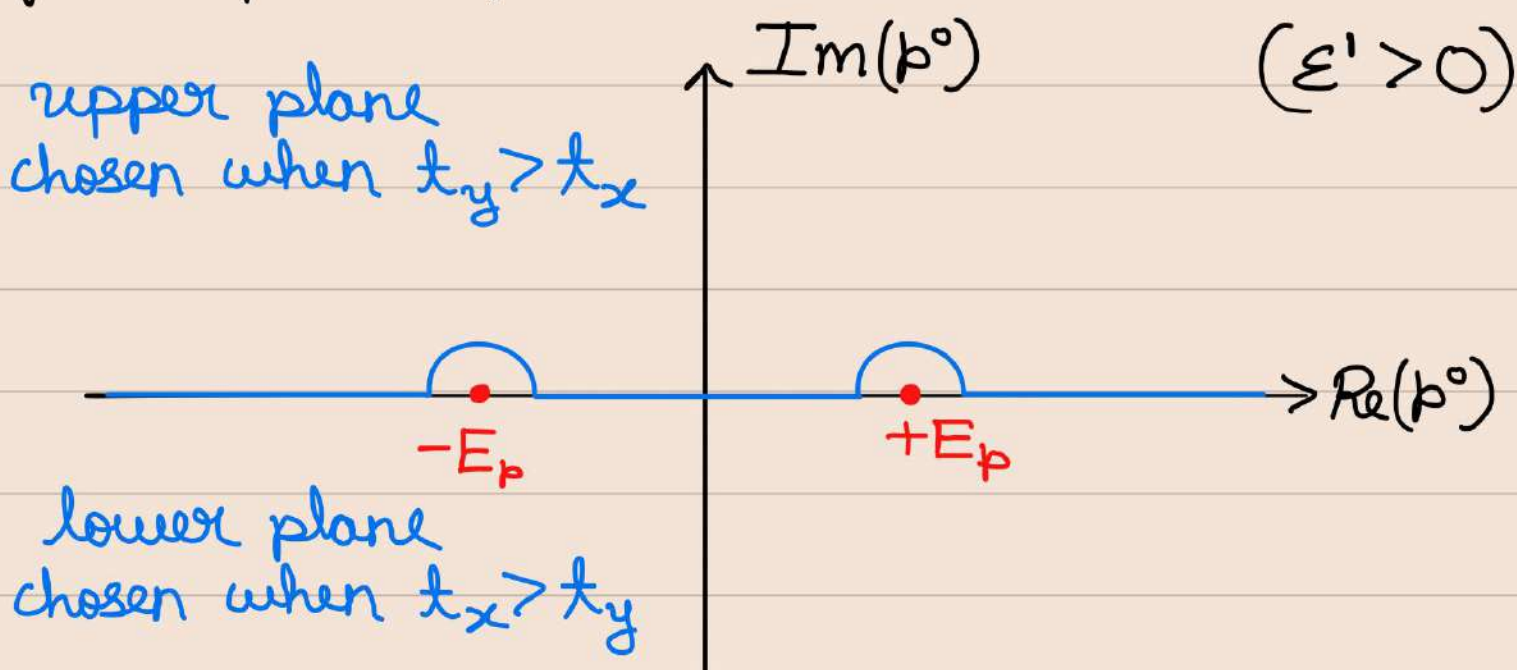
Retarded Green's function for K.G. eqⁿ

Consider the Green's function for the K.G. eqⁿ where,

$$(\partial_\mu \partial^\mu + m^2) G(x, y) = -i \delta^{(4)}(x - y)$$

$$\begin{aligned} \rightarrow G(x, y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p^0)^2 - E_p^2} e^{-ip(x-y)} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(2E_p)} \left[\frac{1}{(p^0 - E_p)} - \frac{1}{(p^0 + E_p)} \right] e^{-ip(x-y)} \end{aligned}$$

If in case we choose the following pole prescription



Then we get Retarded Green's function.

Here both $p^0 = \pm E_p$ contributes only when $t_x > t_y$. Thus it is called retarded Green's function, that describe the retarded propagator for the free scalar field theory denoted as $\Delta_R(x, y)$.

$$\rightarrow \Delta_R(x, y) =$$

$$= i \int \frac{d^4 p}{(2\pi)^4 (2E_p)} e^{-ip(x-y)} \left[\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p + i\epsilon} \right]$$

Now, performing the p^0 integral by using Heaviside identity, we get,

$$i \int_{-\infty}^{\infty} \frac{dp^0}{(2\pi) (p^0 - E_p + i\epsilon)} e^{-ip^0(t_x - t_y)} = e^{-iE_p(t_x - t_y)} \Theta(t_x - t_y)$$

Similarly,

$$i \int_{-\infty}^{\infty} \frac{dp^0}{(2\pi) (p^0 + E_p + i\epsilon)} e^{-ip^0(t_x - t_y)} = e^{iE_p(t_x - t_y)} \Theta(t_x - t_y)$$

$\rightarrow \Delta_R(x, y)$ is given by,

$$\Theta(t_x - t_y) \int \frac{d^3 p e^{i \vec{p} \cdot (\vec{x} - \vec{y})}}{(2\pi)^3 (2E_p)} \left[e^{-iE_p(t_x - t_y)} - e^{iE_p(t_x - t_y)} \right]$$

$$= \Theta(t_x - t_y) \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left[e^{-iE_p(t_x - t_y)} e^{i \vec{p} \cdot (\vec{x} - \vec{y})} - e^{iE_p(t_x - t_y)} e^{i \vec{p} \cdot (\vec{x} - \vec{y})} \right]$$

Put $\vec{p} \rightarrow -\vec{p}$ in 2nd term, and then using $p \equiv p^\mu = (E_p, \vec{p})$, we get,

$$\Delta_R(x, y) = \Theta(t_x - t_y) \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

This can be interpreted as follows, we have already seen,

$$\langle 0 | \hat{\phi}(x) \hat{\phi}^\dagger(y) | 0 \rangle = \int \frac{d^3 p e^{-ip_\mu(x^\mu - y^\mu)}}{(2\pi)^3 (2E_p)}$$

&

$$\langle 0 | \hat{\phi}^\dagger(y) \hat{\phi}(x) | 0 \rangle = \int \frac{d^3 p e^{ip_\mu(x^\mu - y^\mu)}}{(2\pi)^3 (2E_p)}$$

Then $\Theta(t_x - t_y) \langle 0 | \hat{\phi}(x) \hat{\phi}^\dagger(y) | 0 \rangle - \langle 0 | \hat{\phi}^\dagger(y) \hat{\phi}(x) | 0 \rangle$
is equal to the retarded green's func.

$$\begin{aligned}\Delta_R(x, y) &= \Theta(t_x - t_y) \langle 0 | (\hat{\phi}(x) \hat{\phi}^\dagger(y) - \hat{\phi}^\dagger(y) \hat{\phi}(x)) | 0 \rangle \\ &= \Theta(t_x - t_y) \langle 0 | [\hat{\phi}(x), \hat{\phi}^\dagger(y)] | 0 \rangle\end{aligned}$$

Since the commutator of field operators is just some complex no:

$$\underline{\rightarrow \Delta_R(x, y) = \Theta(t_x - t_y) [\hat{\phi}(x), \hat{\phi}^\dagger(y)]}$$

Thus retarded propagator has two contributions for $t_x > t_y$.

$\rightarrow$ A particle created at $(t_y, \vec{y})$ & then annihilated at $(t_x, \vec{x})$.

$\rightarrow$ An anti-particle created at a later time $(t_x, \vec{x})$ & then annihilated at earlier time $(t_y, \vec{y})$.

Scalar Particle creation by classical source

Consider Scalar K.G field equipped with an external classical source.

$$\rightarrow (\partial^2 + m^2)\phi(x) = j(x)$$

where $j(x)$ is some fixed known function of space & time.

Let $j(x)$ be non-zero for some finite interval of time. Then if we originally start with vacuum state, then we would like to know what happens because of $j(x)$ being non-zero.

The Lagrangian that results in the above K.G eqⁿ with a source is

$$\mathcal{L} = \mathcal{L}_{\text{scalar}} + j(x)\phi(x)$$

$$\text{where } \mathcal{L}_{\text{scalar}} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2$$

Let's call the free scalar field operator as,

$$\hat{\phi}_0(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} [\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{a}^\dagger(\vec{p}) e^{ip_\mu x^\mu}]$$

Where $\hat{\phi}_0(x)$ solves $(\partial^2 + m^2)\hat{\phi}_0 = 0$

With $j(x)$ present, the full solution for $(\partial^2 + m^2)\phi(x) = j(x)$ is given using the retarded Green's function that satisfies,

$$(\partial^2 + m^2)\Delta_R(x, y) = -i\delta^{(4)}(x - y)$$

$$\rightarrow \phi(x) = \phi_0(x) + i \int d^4y j(y) \Delta_R(x, y)$$

where,

$$\Delta_R(x, y) = \int \frac{d^3p}{(2\pi)^3(2E_p)} \Theta(t_x - t_y) [e^{-ip(x-y)} - e^{ip(x-y)}]$$

Now if $j(y)$ is turned on only for a finite time & if we are interested in seeing its effect after it is turned off, then when ever $j(y) \neq 0$, we get $t_x > t_y$, so $\Theta(t_x - t_y) = 1$.

Then we get the full solⁿ of $\phi(x)$,

$$\phi(x) = \frac{\int d^3p}{\sqrt{(2\pi)^3(2E_p)}} \times \left\{ \left[\hat{a}(\vec{p}) e^{-ip_\mu x^\mu} + \hat{a}^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right] + i \int \frac{d^4y j(y)}{\sqrt{(2\pi)^3(2E_p)}} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right] \right\}$$

$$= \frac{\int d^3p}{\sqrt{(2\pi)^3(2E_p)}} \times \left\{ \left[\hat{a}(\vec{p}) + i \int \frac{d^4y j(y) e^{ip_\mu y^\mu}}{\sqrt{(2\pi)^3(2E_p)}} \right] e^{-ip_\mu x^\mu} + \left[\hat{a}^\dagger(\vec{p}) - i \int \frac{d^4y j(y) e^{-ip_\mu y^\mu}}{\sqrt{(2\pi)^3(2E_p)}} \right] e^{ip_\mu x^\mu} \right\}$$

Let $\tilde{J}(p) = \int d^4y j(y) e^{ip_\mu y^\mu}$

and $\tilde{J}^*(p) = \int d^4y j(y) e^{-ip_\mu y^\mu} = \tilde{J}(-p)$

$$\rightarrow \phi(x) = \frac{\int d^3p}{\sqrt{(2\pi)^3(2E_p)}} \times \left\{ \left[\hat{a}(\vec{p}) + \frac{i\tilde{J}(\vec{p})}{\sqrt{(2\pi)^3(2E_p)}} \right] e^{-i\vec{p}\cdot\vec{x}} + h.c. \right\}$$

This can be written as,

$$\phi(x) = \phi_0 + \frac{\int d^3p}{\sqrt{(2\pi)^3(2E_p)}} \left[i\tilde{J}(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + h.c. \right]$$

Then the Hamiltonian for this theory is given by replacing $\hat{a}$ of old theory with an extra interaction term

$$\hat{a}(\vec{p}) \rightarrow \hat{a}(\vec{p}) + \frac{i\tilde{J}(\vec{p})}{\sqrt{(2\pi)^3(2E_p)}}$$

Then we get

$$\hat{H} = \int d^3p E_p \left[\hat{a}^\dagger(\vec{p}) - \frac{i\tilde{J}^*(\vec{p})}{\sqrt{(2\pi)^3(2E_p)}} \right] \left[\hat{a}(\vec{p}) + \frac{i\tilde{J}(\vec{p})}{\sqrt{(2\pi)^3(2E_p)}} \right]$$

Thus the Hamiltonian is not diagonal in presence of source term.

$$\hat{H} = \int d^3p E_p \left\{ \hat{a}^\dagger(\vec{p}) \hat{a}(\vec{p}) + i \left[\hat{a}^\dagger(\vec{p}) \tilde{J}(\vec{p}) - \tilde{J}^*(\vec{p}) \hat{a}(\vec{p}) \right] + \frac{\tilde{J}^*(\vec{p}) \tilde{J}(\vec{p})}{(2\pi)^3 (2E_p)} \right\}$$

$$\rightarrow \hat{H} = \hat{H}_0 + \int d^3p E_p i \left[\hat{a}^\dagger(\vec{p}) \tilde{J}(\vec{p}) - \tilde{J}^*(\vec{p}) \hat{a}(\vec{p}) \right] + \int \frac{d^3p}{(2\pi)^3} \frac{|\tilde{J}(\vec{p})|^2}{2}$$

$\hat{H}_0 \rightarrow$ Hamiltonian in absence of source.

Then the vacuum energy after the source is turned off is given by,

$$\langle 0 | \hat{H} | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \left(\frac{|\tilde{J}(\vec{p})|^2}{2} \right)$$

Energy density
of created particles

Thus $\langle 0 | \hat{H} | 0 \rangle$ represents the energy of created particles in the duration for which source was on.

So we can interpret $\frac{1}{2E_p} \left(\frac{|\vec{J}(p)|^2}{(2\pi)^3} \right)$ as probability density to create a particle at mode E_p . Then total no: of particles produced is given by,

$$N = \int \frac{d^3p |\vec{J}(p)|^2}{(2\pi)^3 (2E_p)}$$

But as seen previously,

$$\frac{d^3p}{2E_p} = \frac{d^4p \delta(p_0 - E_p)}{2E_p} = \underbrace{d^4p \delta(p_0^2 - E_p^2) \Theta(p_0)}_{\text{Lorentz invariant}}$$

$$\rightarrow N = \int \frac{d^3p |\vec{J}(p)|^2}{(2\pi)^3 (2E_p)}$$

$$\rightarrow N = \int d^4p \delta(p_0^2 - E_p^2) \Theta(p_0) \left[\frac{|\vec{J}(p)|^2}{(2\pi)^3} \right]$$

This implies that particles of only those fourier modes which resonate with the appropriate mass shell condition which are the K.G. waves are effective in creating particles.

$$\text{Now, } N = \int \frac{d^4 p \delta(p_0 - E_p)}{(2\pi)^3 (2E_p)} |\tilde{J}(p)|^2$$

$$\rightarrow N = \int \frac{d^4 p \delta(p_0 - E_p)}{(2\pi)^3 (2E_p)} |\tilde{J}(p)|^2$$

$$\& \quad |\tilde{J}(p)|^2 = \int d^4 y d^4 x j(x) j(y) e^{i p_\mu (y^\mu - x^\mu)}$$

$$\Rightarrow N = \int d^4 y d^4 x j(x) j(y) \int \frac{d^3 p e^{-i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 (2E_p)}$$

$$\rightarrow N = \int d^4 y d^4 x j(x) j(y) \langle 0 | \hat{\phi}_0(x) \hat{\phi}_0(y) | 0 \rangle$$

$$\text{Since } \langle 0 | \hat{\phi}_0(x) \hat{\phi}_0(y) | 0 \rangle = \int \frac{d^3 p e^{-i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 (2E_p)}$$

gives probability for particles to propagate from $(t_y, \vec{y}) \rightarrow (t_x, \vec{x})$

Propagators for Dirac Spinor field

Dirac eqⁿ is $(i\cancel{\partial} - m)\psi(x) = 0$

where $\cancel{\partial} = \gamma^\mu \partial_\mu$, in chiral basis,
 $\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{bmatrix}$ $\sigma^\mu = (\mathbb{I}_2, \vec{\sigma})$
 $\bar{\sigma}^\mu = (\mathbb{I}_2, -\vec{\sigma})$

$\psi(x)$ is a 4-spinor field,

$$\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}, \quad \psi_A \text{ is } A^{\text{th}} \text{ component of the spinor}$$

The adjoint eqⁿ for Dirac eqⁿ is

$$\bar{\psi}(x)(i\overleftarrow{\cancel{\partial}} - m) = 0, \quad \text{where } \underline{\bar{\psi}} = \psi^\dagger \gamma^0$$

The dispersion relation that the Dirac field satisfies is given by,

$$\underline{E_p^2 = |\vec{p}|^2 + m^2}$$

ψ_A is not Hermitian & is complex.
The mode expansion of $\psi(x)$ & $\bar{\psi}(x)$ is,

$$\hat{\psi}(x) = \int \frac{d^3p \sum_{s=1}^2 [\psi^s(p) \hat{a}_s(\vec{p}) e^{-ip_\mu x^\mu} + \psi^s(p) \hat{b}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

$$\hat{\bar{\psi}}(x) = \int \frac{d^3p \sum_{s=1}^2 [\bar{\psi}^s(p) \hat{a}_s^\dagger(\vec{p}) e^{ip_\mu x^\mu} + \bar{\psi}^s(p) \hat{b}_s(\vec{p}) e^{-ip_\mu x^\mu}]}{\sqrt{(2\pi)^3 (2E_p)}}$$

where $\psi^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \xi_s^s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \xi_s^s \end{bmatrix} \rightarrow \text{Dirac Particle}$

$\psi^s(p) = \begin{bmatrix} \sqrt{p_\mu \sigma^\mu} \eta^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu} \eta^s \end{bmatrix} \rightarrow \text{Dirac anti-particle}$

$\xi_s^1 = \eta^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \text{gives physical spin up state } S_z = +\frac{1}{2} \text{ (+ve helicity)}$

$\xi_s^2 = \eta^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \text{gives physical spin down state } S_z = -\frac{1}{2} \text{ (-ve helicity)}$

The creation and annihilation operators are given as follows,

$\hat{a}_s \& \hat{a}_s^+ \rightarrow$ particles (+ve frequency mode)

$\hat{b}_s \& \hat{b}_s^+ \rightarrow$ Anti-particles (-ve frequency mode)

$\hat{a}_1^+(\vec{p}) \rightarrow$ creates a Dirac particle with 3-momentum $\vec{p}$, energy E_p , with spin up along $\vec{p}$ axis & hence has +ve helicity.

$\hat{a}_2^+(\vec{p}) \rightarrow$ creates a Dirac particle with 3-momentum $\vec{p}$, energy E_p , with spin down along $\vec{p}$ axis & hence has -ve helicity.

$\hat{b}_1^+(\vec{p}) \rightarrow$ creates a Dirac anti-particle with 3-momentum $\vec{p}$, energy E_p , with spin up along $\vec{p}$ axis & hence has +ve helicity.

$\hat{b}_2^+(\vec{p}) \rightarrow$ creates a Dirac anti-particle with 3-momentum $\vec{p}$, energy E_p , with spin down along $\vec{p}$ axis & hence has -ve helicity.

Orthogonal relations of u & v spinors are given by,

- $\bar{u}^r(p) u^s(p) = 2m \delta^{rs}$
- $\bar{v}^r(p) v^s(p) = -2m \delta^{rs}$
- $\bar{u}^s(p) v^r(p) = 0 = \bar{v}^r(p) u^s(p)$
- $u^{r\dagger}(p_0, -\vec{p}) v^s(p_0, \vec{p}) = 0$
- $u^{r\dagger}(p_0, \vec{p}) v^s(p_0, -\vec{p}) = 0$
- $v^{s\dagger}(p_0, \vec{p}) u^r(p_0, -\vec{p}) = 0$
- $v^{s\dagger}(p_0, -\vec{p}) u^r(p_0, \vec{p}) = 0$

• Spin Sums

$$a) \sum_{s=\{1,2\}} u^s(p) \bar{u}^s(p) = \not{p} + m$$

$$b) \sum_{s=\{1,2\}} v^s(p) \bar{v}^s(p) = \not{p} - m$$

All these above results were derived in earlier chapters.

Green's function for Dirac eqⁿ

Here we are looking for the Green's function $\mathcal{S}(x, y)$ for the Dirac operator that satisfies,

$$[i\gamma^\mu \partial_{x^\mu} - m]\mathcal{S}(x, y) = i\delta(x - y)\mathbb{I}_{4 \times 4}$$

Since the Dirac operator ' $i\gamma^\mu \partial_{x^\mu} - m$ ' is a 4×4 object that acts on a 4×1 Spinor, we expect that Green's function to be also a 4×4 object.

This is because it relates to the propagator which in turn relates to the time evolution operator represented in Dirac Spinor representation. Thus Dirac Propagator has to be a 4×4 operator that acts on Dirac Spinors & propagates the Spinor to a new Space-time point.

Consider the fourier integral of

$$\mathcal{S}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \tilde{\mathcal{S}}(p) e^{-i p_\mu (x^\mu - y^\mu)}$$

Acting throughout by $(i\not{\partial}_x - m)$

$$\begin{aligned} (i\not{\partial}_x - m)\mathcal{S}(x, y) &= \int \frac{d^4 p}{(2\pi)^4} \tilde{\mathcal{S}}(p) (i\not{\partial}_x - m) e^{-i p_\mu (x^\mu - y^\mu)} \\ &= \int \frac{d^4 p}{(2\pi)^4} \tilde{\mathcal{S}}(p) (\not{p} - m) e^{-i p_\mu (x^\mu - y^\mu)} \end{aligned}$$

But $(i\not{\partial}_x - m)\mathcal{S}(x, y) = i\mathcal{D}(x - y)$

$$\rightarrow \int \frac{d^4 p}{(2\pi)^4} \tilde{\mathcal{S}}(p) (\not{p} - m) e^{-i p_\mu (x^\mu - y^\mu)} = i\mathcal{D}(x - y)$$

$$\text{Since } \mathcal{D}(x - y) = \int \frac{d^4 p}{(2\pi)^4} e^{-i p_\mu (x^\mu - y^\mu)}$$

$$\rightarrow \underline{\tilde{\mathcal{S}}(p) (\not{p} - m) = i}$$

Now multiply throughout by $(\not{p}+m)$ from left,

$$\rightarrow \tilde{\mathcal{J}}(p)(\not{p}-m)(\not{p}+m) = i(\not{p}+m)$$

$$\text{now } (\not{p}-m)(\not{p}+m) = \not{p}\not{p} - m^2$$

$$\begin{aligned}\not{p}\not{p} &= (\gamma^\mu p_\mu)(\gamma^\nu p_\nu) \\ &= \frac{(\gamma^\mu p_\mu)(\gamma^\nu p_\nu) + (\gamma^\nu p_\nu)(\gamma^\mu p_\mu)}{2}\end{aligned}$$

$$= \frac{(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)(p_\mu p_\nu)}{2}$$

$$= \frac{\{\gamma^\mu, \gamma^\nu\}}{2} p_\mu p_\nu$$

$$= \frac{2\eta^{\mu\nu} p_\mu p_\nu}{2} = p^\nu p_\nu = \underline{\underline{p^2}}$$

$$\rightarrow (\not{p}-m)(\not{p}+m) = p^2 - m^2$$

$$\Rightarrow \tilde{\mathcal{J}}(p)(p^2 - m^2) = (\not{p}+m)$$

Thus Green's function in momentum space is

$$\tilde{\mathcal{G}}(p) = \frac{i(\not{p} + m)}{p^2 - m^2}$$

$$\Rightarrow \mathcal{G}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{p^2 - m^2} \right] e^{-i p_\mu (x^\mu - y^\mu)}$$

Note that we can relate Dirac Green's function to KG Green's function as follows,

$$\mathcal{G}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{p^2 - m^2} \right] e^{-i p_\mu (x^\mu - y^\mu)}$$

$$= (i\not{\partial}_x + m) \underbrace{\int \frac{d^4 p}{(2\pi)^4} \left[\frac{i}{p^2 - m^2} \right] e^{-i p_\mu (x^\mu - y^\mu)}}_{\Delta(x, y)}$$

$$\Rightarrow \underline{\mathcal{G}(x, y) = (i\not{\partial}_x + m) \Delta(x, y)}$$

Thus we can easily obtain the Dirac propagator by using Klein Gordon propagator.

First we can replace $p^2 = (p^0)^2 - |\vec{p}|^2$

$$\Rightarrow p^2 - m^2 = (p^0)^2 - (E_p)^2$$

where $E_p = \sqrt{|\vec{p}|^2 + m^2} \quad (E_p > 0)$

$$\Rightarrow \mathcal{S}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{(p^0)^2 - E_p^2} \right] e^{-i p_\mu (x^\mu - y^\mu)}$$

$$= i \int \frac{d^4 p (\not{p} + m)}{(2\pi)^4 (2E_p)} \left[\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p} \right] e^{-i p_\mu (x^\mu - y^\mu)}$$

$$= (i \not{\partial}_x + m) \left[i \int \frac{d^4 p e^{-i p_\mu (x^\mu - y^\mu)}}{(2\pi)^4 (2E_p)} \left[\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p} \right] \right]$$

Thus there are two poles at $p^0 = \pm E_p$.

Based on how we complexify the integrand to shift the pole from the real p^0 line, we get various types of Green's functions.

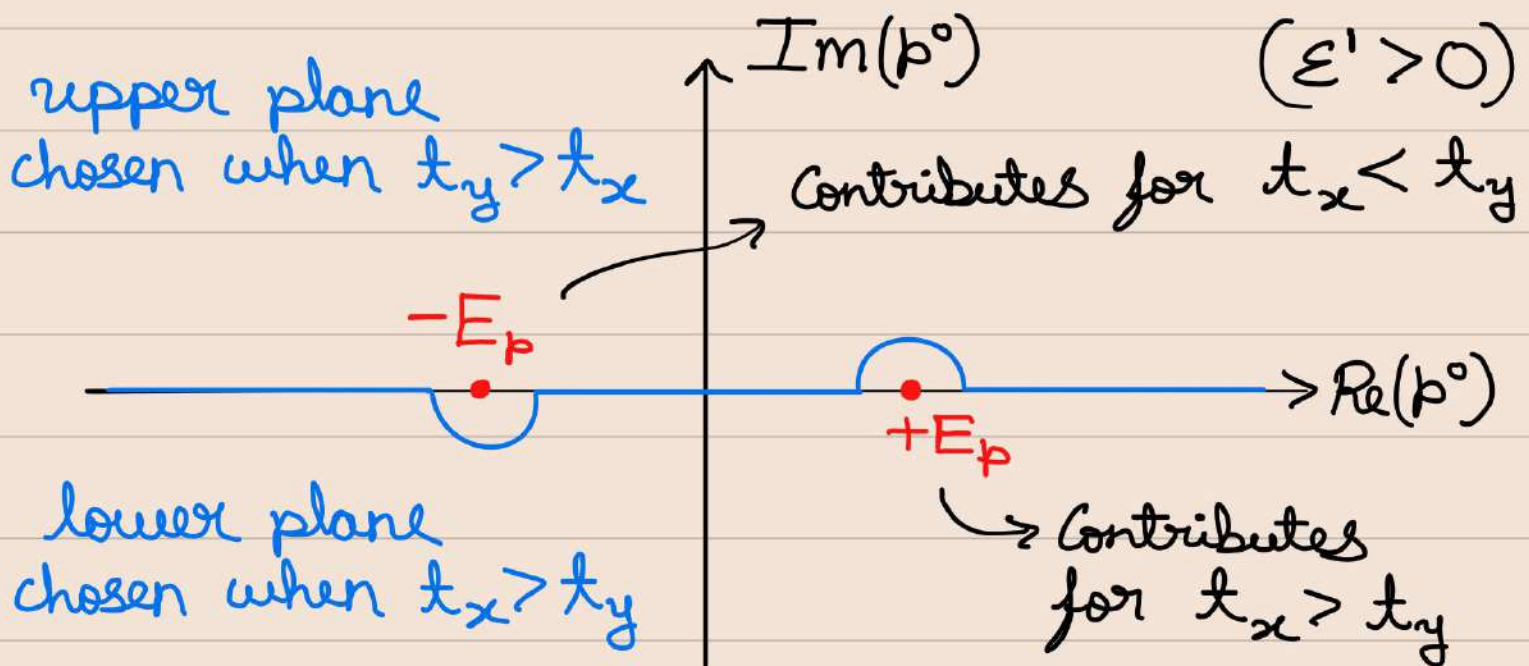
Feynmann Propagator for Dirac fields

To avoid the singularity, we can do the p^0 integral in complex plane. If we use the feynmann's prescription, then we get the feynmann Propagator.

$$\left[\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p} \right]$$

Feynmann's
Prescription

$$\lim_{\varepsilon \rightarrow 0} \left[\frac{1}{p^0 - E_p + i\varepsilon} - \frac{1}{p^0 + E_p - i\varepsilon} \right]$$



Then we get the feynmann propagator

$$\mathcal{S}_F(x, y) = (i\not{\partial}_x + m)\Delta_F(x, y)$$

where,

$$\Delta_F(x, y) = \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[\Theta(t_x - t_y) e^{-ip(x-y)} + \Theta(t_y - t_x) e^{ip(x-y)} \right]_{p^0 = E_p}$$

Now,

$$i\not{\partial}_x \Theta(t_x - t_y) e^{-ip(x-y)} = \left[\Theta(t_x - t_y) \not{p} + i\gamma^0 \delta(t_x - t_y) \right] e^{-ip(x-y)}$$

$$i\not{\partial}_x \Theta(t_y - t_x) e^{ip(x-y)} = - \left[\Theta(t_y - t_x) \not{p} + i\gamma^0 \delta(t_x - t_y) \right] e^{ip(x-y)}$$

$$\Rightarrow \mathcal{S}_F(x, y) = \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[\left(\Theta(t_x - t_y) (\not{p} + m) + i\gamma^0 \delta(t_x - t_y) \right) e^{-ip(x-y)} - \left(\Theta(t_y - t_x) (\not{p} - m) + i\gamma^0 \delta(t_x - t_y) \right) e^{ip(x-y)} \right]$$

$$\mathcal{S}_F(x, y) = \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[\begin{aligned} &\Theta(t_x - t_y)(\not{p} + m)e^{-ip(x-y)} \\ &- \Theta(t_y - t_x)(\not{p} - m)e^{ip(x-y)} \\ &+ i\gamma^0 \delta(t_x - t_y) \\ &\times \left[e^{-ip(x-y)} - e^{ip(x-y)} \right] \end{aligned} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[\begin{aligned} &\Theta(t_x - t_y)(\not{p} + m)e^{-ip(x-y)} \\ &- \Theta(t_y - t_x)(\not{p} - m)e^{ip(x-y)} \\ &+ i\gamma^0 \underbrace{\left[e^{i\vec{p} \cdot (\vec{x} - \vec{y})} - e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} \right]}_{\text{odd function, so vanishes}} \end{aligned} \right]$$

odd function, so vanishes.

$$= \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[\begin{aligned} &\Theta(t_x - t_y)(\not{p} + m)e^{-ip(x-y)} \\ &- \Theta(t_y - t_x)(\not{p} - m)e^{ip(x-y)} \end{aligned} \right]$$

$$\Rightarrow \boxed{\mathcal{S}_F(x, y) = \Theta(t_x - t_y) \int \frac{d^3p}{(2\pi)^3(2E_p)} (\not{p} + m)e^{-ip(x-y)} - \Theta(t_y - t_x) \int \frac{d^3p}{(2\pi)^3(2E_p)} (\not{p} - m)e^{ip(x-y)}}$$

Since $\mathcal{S}_F(x, y)$ is a 4×4 matrix it contains 16 elements which are denoted by the spinor indices $\{A, B, \dots\}$ which takes values $\{1, 2, 3, 4\}$.

$\rightarrow \mathcal{S}_F^{AB}(x, y)$ as element at A^{th} row and B^{th} column. So,

$$\mathcal{S}_F^{AB}(x, y) = \Theta(t_x - t_y) \int \frac{d^3p (\not{p} + m)_{AB}}{(2\pi)^3 (2E_p)} e^{-ip(x-y)} \\ - \Theta(t_y - t_x) \int \frac{d^3p (\not{p} - m)_{AB}}{(2\pi)^3 (2E_p)} e^{ip(x-y)}$$

where $(\not{p} + m)_{AB}$ & $(\not{p} - m)_{AB}$ are elements of respective 4×4 matrices.

Now we shall try to show the relation b/w the above propagator with the particle creation and annihilation operators, same like how we did for K.G. propagator.

Consider,

$$\hat{\psi}(y)|0\rangle = \int \frac{d^3p \sum_{s=1}^2 \bar{u}^s(p) e^{ip_\mu y^\mu} \hat{a}_s^\dagger(\vec{p})}{\sqrt{(2\pi)^3 (2E_p)}} |0\rangle$$

$$\& \langle 0 | \hat{\psi}(x) = \int \frac{d^3q \sum_{s=1}^2 u^s(q) e^{-iq_\mu x^\mu} \langle 0 | \hat{a}_s(\vec{q})}{\sqrt{(2\pi)^3 (2E_q)}}$$

$$\Rightarrow \langle 0 | \hat{\psi}(x) \hat{\psi}(y) | 0 \rangle =$$

$$= \int \frac{d^3q d^3p \sum_{s,s'} u^s(q) \bar{u}^{s'}(p) e^{i(p_\mu y^\mu - q_\mu x^\mu)}}{(2\pi)^3 2\sqrt{E_q E_p}} \times \underbrace{\langle 0 | \hat{a}_s(\vec{q}) \hat{a}_{s'}^\dagger(\vec{p}) | 0 \rangle}_{\delta_{ss'} \delta(\vec{q} - \vec{p})}$$

$$= \int \frac{d^3p \sum_{s=1}^2 u^s(p) \bar{u}^s(p) e^{-ip(x-y)}}{(2\pi)^3 2E_p}$$

$$\left(\begin{array}{l} \text{Spin Sum} \\ \text{relation} \end{array} \rightarrow \sum_{s=1}^2 u^s(p) \bar{u}^s(p) = \not{p} + m \right)$$

$$\Rightarrow \langle 0 | \hat{\psi}(x) \hat{\psi}(y) | 0 \rangle = \int \frac{d^3p (\not{p} + m) e^{-ip(x-y)}}{(2\pi)^3 (2E_p)}$$

Consider, $\hat{\psi}(x)|0\rangle = \int \frac{d^3p \sum_{s=1}^2 \psi^s(p) e^{ip_\mu x^\mu} \hat{b}_s^\dagger(\vec{p})}{\sqrt{(2\pi)^3 (2E_p)}} |0\rangle$

& $\langle 0|\hat{\bar{\psi}}(y) = \int \frac{d^3q \sum_{\bar{s}=1}^2 \bar{\psi}^{\bar{s}}(q) e^{-iq_\mu y^\mu} \langle 0|\hat{b}_{\bar{s}}(\vec{q})}{\sqrt{(2\pi)^3 (2E_q)}}$

Observe that the object ' $\hat{\bar{\psi}}(y)\hat{\psi}(x)$ ' is a 1×1 object. As $\hat{\bar{\psi}}(y)$ is (1×4) row-spinor & $\hat{\psi}(x)$ is 4×1 column spinor. But we need to obtain a 4×4 matrix operator that acts on a 4-spinor. Thus we shall define

$$\begin{aligned} \rightarrow \langle 0|\hat{\bar{\psi}}(y)\hat{\psi}(x)|0\rangle &\equiv [\langle 0|\hat{\bar{\psi}}(y)]^T [\hat{\psi}(x)|0\rangle]^T \\ &= \int \frac{d^3q d^3p \sum_{\bar{s}, s} [\bar{\psi}^{\bar{s}}(q)]^T [\psi^s(p)]^T e^{i(p_\mu x^\mu - q_\mu y^\mu)}}{(2\pi)^3 2\sqrt{E_q E_p}} \times \underbrace{\langle 0|\hat{b}_{\bar{s}}(\vec{q}) \hat{b}_s^\dagger(\vec{p})|0\rangle}_{\delta_{\bar{s}s} \delta(\vec{q}-\vec{p})} \end{aligned}$$

[Note: Transpose acts only on spinor & 4×4 operators that act on spinors]

$$\rightarrow \langle 0 | \hat{\bar{\psi}}(y) \hat{\psi}(x) | 0 \rangle \equiv$$

$$= \int \frac{d^3 p \sum_s [\psi^s(p) \bar{\psi}^s(p)]^T e^{i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 2E_p}$$

$$= \int \frac{d^3 p \left[\sum_s \psi^s(p) \bar{\psi}^s(p) \right]^T e^{i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 2E_p}$$

$$\left(\begin{array}{l} \text{spin sum} \\ \text{relation} \end{array} \rightarrow \sum_{s=1}^2 \psi^s(p) \bar{\psi}^s(p) = \not{p} - m \right)$$

$$\rightarrow \langle 0 | \hat{\bar{\psi}}(y) \hat{\psi}(x) | 0 \rangle \equiv$$

$$= \int \frac{d^3 p [\not{p} - m]^T e^{i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 2E_p}$$

$$\rightarrow \langle 0 | \hat{\bar{\psi}}_B(y) \hat{\psi}_A(x) | 0 \rangle \equiv$$

$$= \int \frac{d^3 p (\not{p} - m)_{AB} e^{i p_\mu (x^\mu - y^\mu)}}{(2\pi)^3 2E_p}$$

Thus we can write Feynmann propagator for Dirac eqⁿ as,

$$\mathcal{S}_F^{AB}(x, y) = \Theta(t_x - t_y) \langle 0 | \hat{\Psi}_A(x) \hat{\bar{\Psi}}_B(y) | 0 \rangle \\ - \Theta(t_y - t_x) \langle 0 | \hat{\bar{\Psi}}_B(y) \hat{\Psi}_A(x) | 0 \rangle$$

Thus, physically, the free feynmann propagator for Dirac field gives the vacuum state expectation value of

$$\mathcal{S}_F^{AB}(x, y) = \begin{cases} \langle 0 | \hat{\Psi}_A(x) \hat{\bar{\Psi}}_B(y) | 0 \rangle, & t_x > t_y \\ -\langle 0 | \hat{\bar{\Psi}}_B(y) \hat{\Psi}_A(x) | 0 \rangle, & t_y > t_x \end{cases}$$

Note that $\{\hat{\Psi}_A(x)\} \neq \{\hat{\bar{\Psi}}_B(y)\}$ are set of fermionic operators that follow the anti-commutation relations, thus whenever we exchange two fermionic operators, we get a -ve sign.

(Odd no: of exchanges result in -ve sign).

Thus we can derive time ordered product for fermion operators as

$$T\{\hat{\Psi}_A(x)\hat{\Psi}_B(y)\} = \begin{cases} \hat{\Psi}_A(x)\hat{\Psi}_B(y), & x^0 > y^0 \\ -\hat{\Psi}_B(y)\hat{\Psi}_A(x), & y^0 > x^0 \end{cases}$$

↓
Curly braces for operators
with anti-commuting relations

Thus in time ordered product of fermionic operators, they are arranged from left to right in decreasing value, of time component multiplied with (-1) powered number of exchanges done.

So, the feynmann propagator for fields that solve dirac eqⁿ are written as:

$$\boxed{\mathcal{S}_F^{AB}(x, y) = \langle 0 | T\{\hat{\Psi}_A(x)\hat{\Psi}_B(y)\} | 0 \rangle}$$

And as a fourier integral,

$$\mathcal{S}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{(p^0)^2 - E_p^2 + i\epsilon} \right] e^{-i p_\mu (x^\mu - y^\mu)}$$

Consider $\hat{\psi}(x) = \begin{bmatrix} \hat{\psi}_L(x) \\ \hat{\psi}_R(x) \end{bmatrix} \quad \&$

$$\hat{\bar{\psi}}(y) = \begin{bmatrix} \hat{\psi}_R^\dagger(y) & \hat{\psi}_L^\dagger(y) \end{bmatrix}$$

$$\rightarrow \hat{\psi}(x) \hat{\bar{\psi}}(y) = \begin{bmatrix} \hat{\psi}_L(x) \hat{\psi}_R^\dagger(y) & \hat{\psi}_L(x) \hat{\psi}_L^\dagger(y) \\ \hat{\psi}_R(x) \hat{\psi}_R^\dagger(y) & \hat{\psi}_R(x) \hat{\psi}_L^\dagger(y) \end{bmatrix}$$

Thus,

$$\mathcal{S}_F(x, y) = \begin{bmatrix} \langle 0 | T \{ \hat{\psi}_L(x) \hat{\psi}_R^\dagger(y) \} | 0 \rangle & \langle 0 | T \{ \hat{\psi}_L(x) \hat{\psi}_L^\dagger(y) \} | 0 \rangle \\ \langle 0 | T \{ \hat{\psi}_R(x) \hat{\psi}_R^\dagger(y) \} | 0 \rangle & \langle 0 | T \{ \hat{\psi}_R(x) \hat{\psi}_L^\dagger(y) \} | 0 \rangle \end{bmatrix}$$

Thus we can represent the probability amplitude of a Dirac particle or a Dirac anti-particle of certain chirality to be created from vacuum, propagate to another space-time point with possible oscillations to a different chirality & then annihilate.

Retarded Green's function for Dirac eqⁿ

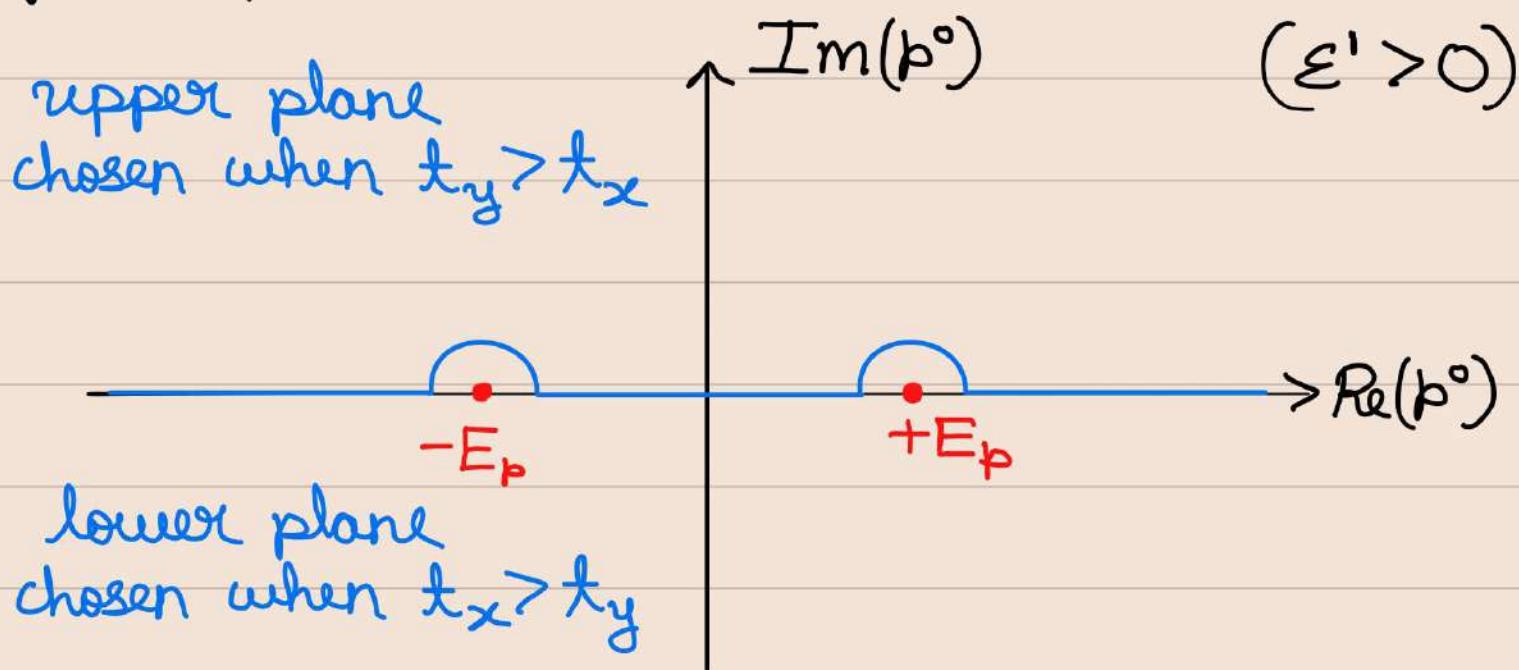
Consider the Green's function for the Dirac eqⁿ where,

$$(i\partial - m)\mathcal{S}(x, y) = i\delta(x - y)$$

$$\rightarrow \mathcal{S}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \left[\frac{i(\not{p} + m)}{(p^0)^2 - E_p^2} \right] e^{-i p_\mu (x^\mu - y^\mu)}$$

$$= (i\not{\partial}_x + m) \left[i \int \frac{d^4 p e^{-i p_\mu (x^\mu - y^\mu)}}{(2\pi)^4 (2E_p)} \left[\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p} \right] \right]$$

If in case we choose the following pole prescription



Then we get Retarded Green's function.

Here both $p^0 = \pm E_p$ contributes only when $t_x > t_y$. Thus it is called retarded Green's function, that describe the retarded propagator for the free Dirac field theory denoted as $\mathcal{S}_R(x, y)$.

$$\rightarrow \mathcal{S}_R(x, y) = (i\cancel{\partial}_x + m)\Delta_R(x, y)$$

$$\Delta_R(x, y) = \Theta(t_x - t_y) \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

Now, $(i\cancel{\partial}_x + m)\Delta_R(x, y)$ is

$$(i\cancel{\partial}_x + m) \left[\Theta(t_x - t_y) \int \frac{d^3p}{(2\pi)^3(2E_p)} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right] \right]$$

$$= \int \frac{d^3p}{(2\pi)^3(2E_p)} \left((i\cancel{\partial}_x + m) \left[\Theta(t_x - t_y) e^{-ip(x-y)} \right] - (i\cancel{\partial}_x + m) \left[\Theta(t_x - t_y) e^{ip(x-y)} \right] \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left(\begin{aligned} & \Theta(t_x - t_y) (\not{p} + m) e^{-ip(x-y)} \\ & + \Theta(t_x - t_y) (\not{p} - m) e^{ip(x-y)} \\ & + i\gamma^0 \delta(t_x - t_y) \left[e^{-ip(x-y)} - e^{ip(x-y)} \right] \end{aligned} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left(\begin{aligned} & \Theta(t_x - t_y) (\not{p} + m) e^{-ip(x-y)} \\ & + \Theta(t_x - t_y) (\not{p} - m) e^{ip(x-y)} \\ & + i\gamma^0 \underbrace{\left[e^{i\vec{p} \cdot (\vec{x} - \vec{y})} - e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} \right]}_{\text{odd function, so vanishes.}} \end{aligned} \right)$$

odd function, so vanishes.

$$\Rightarrow \mathcal{S}_R(x, y) =$$

$$= \Theta(t_x - t_y) \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left[(\not{p} + m) e^{-ip(x-y)} + (\not{p} - m) e^{ip(x-y)} \right]$$

In terms of matrix elements of $\mathcal{S}_R^{AB}(x, y)$

$$\Rightarrow \mathcal{S}_R^{AB}(x, y) =$$

$$= \Theta(t_x - t_y) \int \frac{d^3 p}{(2\pi)^3 (2E_p)} \left[(\not{p} + m)_{AB} e^{-ip(x-y)} + (\not{p} - m)_{AB} e^{ip(x-y)} \right]$$

Recall that,

$$\langle 0 | \hat{\psi}_A(x) \hat{\psi}_B(y) | 0 \rangle = \int \frac{d^3 p (\not{p} + m)_{AB} e^{-i p(x-y)}}{(2\pi)^3 2E_p}$$

$$\langle 0 | \hat{\psi}_B(y) \hat{\psi}_A(x) | 0 \rangle = \int \frac{d^3 p (\not{p} - m)_{AB} e^{i p(x-y)}}{(2\pi)^3 2E_p}$$

Thus in terms of Dirac Spinor components,

$$\rightarrow \mathcal{S}_R^{AB}(x, y) = \Theta(t_x - t_y) \left[\langle 0 | \hat{\psi}_A(x) \hat{\psi}_B(y) | 0 \rangle + \langle 0 | \hat{\psi}_B(y) \hat{\psi}_A(x) | 0 \rangle \right]$$

$$= \Theta(t_x - t_y) \left[\langle 0 | (\hat{\psi}_A(x) \hat{\psi}_B(y) + \hat{\psi}_B(y) \hat{\psi}_A(x)) | 0 \rangle \right]$$

$$\rightarrow \boxed{\mathcal{S}_R^{AB}(x, y) = \Theta(t_x - t_y) \langle 0 | \{ \hat{\psi}_A(x), \hat{\psi}_B(y) \} | 0 \rangle}$$

Notice that the results are very much similar to that of Klein Gordon case, except for anti-commutator brackets in place of commutators.

Propagator for Electromagnetic field

We have seen earlier that the EM fields are encoded in the Faraday tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}$$

Note: Greek indices $\mu, \nu, \dots = \{0, 1, 2, 3\}$ are space-time Minkowski indices.

The non-homogeneous Maxwell's eqⁿ are given by

$$\partial_\mu F^{\mu\nu} = J^\nu$$

where $J^\nu = (\rho, \vec{J}) \rightarrow 4\text{-Current density}$

But in terms of Gauge fields ' A^μ ', we can define

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\rightarrow \partial_\mu [\partial^\mu A^\nu - \partial^\nu A^\mu] = J^\nu$$

$$\rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\mu \partial^\nu A_\mu = J^\nu$$

$$\Rightarrow \square^2 \eta^{\nu\sigma} A_\sigma - \partial^\sigma \partial^\nu A_\sigma = J^\nu$$

$$\Rightarrow (\square^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\mu = \partial_\mu J^\mu$$

Thus the above equation becomes the differential equations solved by gauge fields in the presence of a source.

The above equation can be solved, provided we know the Green's function denoted by $D_{\mu\nu}(x, y)$, which satisfies

$$\underline{(\square^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) D_{\mu\sigma}(x, y) = i \partial_\sigma^\nu \delta(x - y)}$$

Notice that we have not yet made any explicit gauge choice until now.

To get $D_{\mu\sigma}(x,y)$, we go to fourier space,

$$D_{\mu\sigma}(x,y) = \int \frac{d^4 p}{(2\pi)^4} \tilde{D}_{\mu\sigma}(p) e^{-ip(x-y)}$$

$$\Rightarrow (p^2 \eta^{\mu\nu} - p^\mu p^\nu) \tilde{D}_{\mu\sigma}(p) = -i \delta_\sigma^\nu$$

$$\Rightarrow i(p^2 \eta^{\mu\nu} - p^\mu p^\nu) \tilde{D}_{\mu\sigma}(p) = \delta_\sigma^\nu$$

Thus $\tilde{D}_{\mu\sigma}(p)$ has to be inverse of tensor given by $i(p^2 \eta^{\mu\nu} - p^\mu p^\nu)$. Now inverse of this has to be some functions of $p = \sqrt{p^\mu p_\mu}$, $\eta_{\mu\sigma}$ & $p_\mu p_\sigma$. But such that the indices are matched. Hence only possible & most general inverse has to be of the form,

$$C(p) \eta_{\mu\sigma} + H(p) p_\mu p_\sigma$$

$$\begin{aligned} &\rightarrow i(p^2 \eta^{\mu\nu} - p^\mu p^\nu) [C(p) \eta_{\mu\sigma} + H(p) p_\mu p_\sigma] \\ &= i \left[p^2 \delta_\sigma^\nu C(p) + p^2 H(p) p^\nu p_\sigma - p^\nu p_\sigma C(p) - p^2 p^\nu p_\sigma H(p) \right] \end{aligned}$$

$$\rightarrow [p^2 \partial_\sigma^2 - p^\nu p_\sigma] C(p) = -i \partial_\sigma^2$$

$$\rightarrow [p^2 C(p) + i] \partial_\sigma^2 = p^\nu p_\sigma C(p)$$

But $p^\nu p_\sigma$ is never diagonal, i.e. $p^\nu p_\sigma \propto \partial_\sigma^2$ is meaningless. Thus we can never write inverse of the tensor $i(p^2 \eta^{\mu\nu} - p^\mu p^\nu)$.

Moreover it is not invertible because it has eigen values of zero. i.e.

$$i[p^2 \eta^{\mu\nu} - p^\mu p^\nu] p_\mu = p^2 p^\nu - p^2 p^\nu = 0$$

$$\rightarrow i[p^2 \eta^{\mu\nu} - p^\mu p^\nu] p_\mu = 0$$

Thus there is a null space for ' $p^2 \eta^{\mu\nu} - p^\mu p^\nu$ ' which give eigen value zero. This problem arises due to the gauge degree of freedom in EM theory.

Gauge Fixing term in EM Lagrangian

Consider the lagrangian for EM field but with an extra parameter ξ_3 -term which can be used to fix the Gauge. So, consider

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu - \frac{1}{2\xi_3} (\partial_\mu A^\mu)^2$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} - \frac{\eta^{\mu\nu}}{\xi_3} (\partial_\sigma A^\sigma)$$

Eqⁿ of motion becomes,

$$\partial_\mu F^{\mu\nu} + \partial_\mu \left[\frac{\eta^{\mu\nu} (\partial_\sigma A^\sigma)}{\xi_3} \right] = J^\nu$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + \frac{\partial^\nu \partial^\sigma A_\sigma}{\xi_3} = J^\nu$$

$$\Rightarrow \partial_\mu [\partial^\mu A^\nu - \partial^\nu A^\mu] + \frac{\partial^\nu \partial^\sigma A_\sigma}{\xi} = J^\nu$$

$$\Rightarrow \partial_\mu \partial^\mu A^\nu - \partial^\mu \partial^\nu A_\mu + \frac{\partial^\nu \partial^\sigma A_\sigma}{\xi} = J^\nu$$

$$\Rightarrow \square^2 A^\nu + \left(\frac{1}{\xi} - 1 \right) \partial^\nu \partial^\sigma A_\sigma = J^\nu$$

$$\Rightarrow \square^2 \eta^{\mu\nu} A_\mu + \left(\frac{1}{\xi} - 1 \right) \partial^\nu \partial^\mu A_\mu = J^\nu$$

$$\Rightarrow \left[\square^2 \eta^{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \partial^\mu \partial^\nu \right] A_\mu = J^\nu$$

Now we see that the gauge fixing term will force the differential operator to be well behaved that has inverse.

But also we can show that setting ' ξ ' value appropriately will be equivalent to working in a required gauge.

Green's function for modified Maxwell eqⁿ

So now we shall try to find the green's function $D_{\mu\nu}(x, y)$ that satisfies

$$\left[\square^2 \eta^{\mu\nu} + \left(\frac{1}{\epsilon_s} - 1 \right) \partial^\mu \partial^\nu \right] D_{\mu\sigma}(x, y) = i \delta_\sigma^\nu \delta(x - y)$$

This eqⁿ in fourier space becomes,

$$\left[-p^2 \eta^{\mu\nu} - \left(\frac{1}{\epsilon_s} - 1 \right) p^\mu p^\nu \right] \tilde{D}_{\mu\sigma}(p) = i \delta_\sigma^\nu$$

$$-\left[p^2 \eta^{\mu\nu} + \left(\frac{1}{\epsilon_s} - 1 \right) p^\mu p^\nu \right] \tilde{D}_{\mu\sigma}(p) = i \delta_\sigma^\nu$$

$$\text{where } D_{\mu\sigma}(x, y) = \int \frac{d^4 p}{(2\pi)^4} \tilde{D}_{\mu\sigma}(p) e^{-ip(x-y)}$$

$$\text{Let } M^{\mu\nu} = -\left[p^2 \eta^{\mu\nu} + \left(\frac{1}{\epsilon_s} - 1 \right) p^\mu p^\nu \right] = M^{\nu\mu}$$

$$\Rightarrow M^{\mu\nu} \tilde{D}_{\mu\sigma} = i \delta_\sigma^\nu$$

$$\Rightarrow \boxed{\tilde{D}_{\mu\nu} = i (M^{-1})_{\mu\nu}}$$

We can show that the inverse tensor components for M are,

$$(M^{-1})_{\mu\nu} = \frac{-\left[\eta_{\mu\nu} + (\epsilon_s - 1) \frac{p_\mu p_\nu}{p^2}\right]}{p^2}$$

Because, $M^{\mu\nu}(M^{-1})_{\sigma\nu}$ is given by

$$\left[p^2 \eta^{\mu\nu} + \left(\frac{1}{\epsilon_s} - 1\right) p^\mu p^\nu\right] \frac{\left[\eta_{\sigma\nu} + (\epsilon_s - 1) \frac{p_\sigma p_\nu}{p^2}\right]}{p^2}$$

$$= \eta^{\mu\nu} \eta_{\sigma\nu} + \eta^{\mu\nu} (\epsilon_s - 1) \frac{p_\sigma p_\nu}{p^2} + \left(\frac{1}{\epsilon_s} - 1\right) \frac{p^\mu p^\nu}{p^2} \eta_{\sigma\nu} + \left(\frac{1}{\epsilon_s} - 1\right) (\epsilon_s - 1) \frac{p^\mu p^\nu p_\sigma p_\nu}{p^4}$$

$$= \delta_\sigma^\mu + (\epsilon_s - 1) \frac{p^\mu p_\sigma}{p^2} + \left(\frac{1}{\epsilon_s} - 1\right) \frac{p^\mu p_\sigma}{p^2} + \left(2 - \epsilon_s - \frac{1}{\epsilon_s}\right) \frac{p^\mu p_\sigma}{p^2}$$

$$= \delta_\sigma^\mu + \left[(\epsilon_s - 1) + \left(\frac{1}{\epsilon_s} - 1\right) + \left(2 - \epsilon_s - \frac{1}{\epsilon_s}\right) \right] \frac{p^\mu p_\sigma}{p^2}$$

$$\Rightarrow M^{\mu\nu}(M^{-1})_{\sigma\nu} = \delta_\sigma^\mu$$

Thus the Green's function of Gauge fixed maxwell's eqⁿ in the momentum space is given by,

$$\tilde{D}_{\mu\nu} = -i \frac{\left[\eta_{\mu\nu} + (\epsilon_s - 1) \frac{p_\mu p_\nu}{p^2} \right]}{p^2}$$

& in position space as a fourier integral

$$D_{\mu\nu}(x, y) = -i \int \frac{d^4 p}{(2\pi)^4} \frac{\left[\eta_{\mu\nu} + (\epsilon_s - 1) \frac{p_\mu p_\nu}{p^2} \right]}{p^2} e^{-ip(x-y)}$$

Now poles arise when $p^2 = 0$, i.e. when $(p^0)^2 = (E_p)^2$. But $\lim_{p \rightarrow 0} \frac{p_\mu p_\nu}{p^2}$

doesn't cause any additional singularities. Thus it is sufficient to complexify the expression as follows,

$$\tilde{D}_{\mu\nu} = -i \frac{\left[\eta_{\mu\nu} + (\epsilon_s - 1) \frac{p_\mu p_\nu}{p^2} \right]}{p^2 + i\epsilon}$$

Types of Lorentz invariant gauges

- $\xi = 0$: This results in the Landau gauge condition, where we impose the hard condition of $\langle 0 | \partial^\mu \hat{A}_\mu | 0 \rangle = 0$.

So we set,

$$\lim_{\xi \rightarrow 0} \frac{1}{\xi} (\partial_\mu A^\mu)^2 = 0$$

(Gauge fixing term won't blow up)

$$\rightarrow \tilde{D}_{\mu\nu} = \frac{-i \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right]}{p^2 + i\epsilon} \quad \left. \vphantom{\frac{-i \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right]}{p^2 + i\epsilon}} \right\} \begin{array}{l} \text{Thus the} \\ \text{longitudinal} \\ \text{modes are} \\ \text{removed.} \end{array}$$

This projects into transverse subspace.

- $\xi = 1$: This results in Feynmann Propagator for Photons, And hence it is called as Feynman Gauge Condition.

$$\tilde{D}_{\mu\nu} = \frac{-i \eta_{\mu\nu}}{p^2 + i\epsilon}$$

Here even though all the components of A^μ are allowed, even unphysical polarizations are allowed. But when we calculate the physical observables, the unphysical modes don't contribute which will be shown later by Ward identities.

Faddeev-Popov procedure guarantees that the value of any correlation function of any Gauge invariant operators when computed from Feynmann diagrams, they will be independent of the value of ξ , as long as a particular value of ξ is used consistently throughout.

All these will be shown later in QED.

Note that $\xi=1$ in differential eqⁿ for $\{A_\mu\}$ results in $\square^2 \eta^{\mu\nu} A_\mu = J^\nu$

Equivalent to the result of Lorenz gauge.

But even Landau gauge treats $\partial_\mu A^\mu = 0$ as a hard condition. The difference being the propagator has no longitudinal modes where as propagator in Feynmann Gauge has even unphysical modes in it.

Since $\square^2 A^\mu = 0$, when $J^\mu = 0$, then all component of A^μ satisfy massless K.G. eqⁿ ($m=0$), so mode expansion is given by,

$$\hat{A}^\mu(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3(2E_p)}} \left[\sum_{\lambda=0}^3 \left[\varepsilon_\lambda^\mu(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} + \varepsilon_\lambda^{\mu*}(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right] \right]$$

Here $\{\varepsilon_0^\mu(\vec{p}), \varepsilon_1^\mu(\vec{p}), \varepsilon_2^\mu(\vec{p}), \varepsilon_3^\mu(\vec{p})\}$ form a basis of polarization vectors. So they follow minkowski orthogonality given by,

$$\underline{\eta_{\mu\nu} \varepsilon_\lambda^{\mu*} \varepsilon_{\lambda'}^\nu = \eta_{\lambda\lambda'}}$$

And in lorentz gauge $\partial_\mu A^\mu = 0$, so we need

$$\underline{p_\mu \varepsilon_\gamma^\mu(p) = p_\mu \varepsilon_\gamma^{\mu*}(p) = 0}$$

Feynmann Propagator for Photons

We use the Feynmann Gauge where

$$\tilde{D}_{\mu\nu} = \frac{-i\eta_{\mu\nu}}{p^2 + i\epsilon}$$

& in position space as a fourier integral

$$D_{\mu\nu}(x, y) = -i\eta_{\mu\nu} \int \frac{d^4p e^{-ip(x-y)}}{(2\pi)^4 [(p^0)^2 - E_p^2]}$$

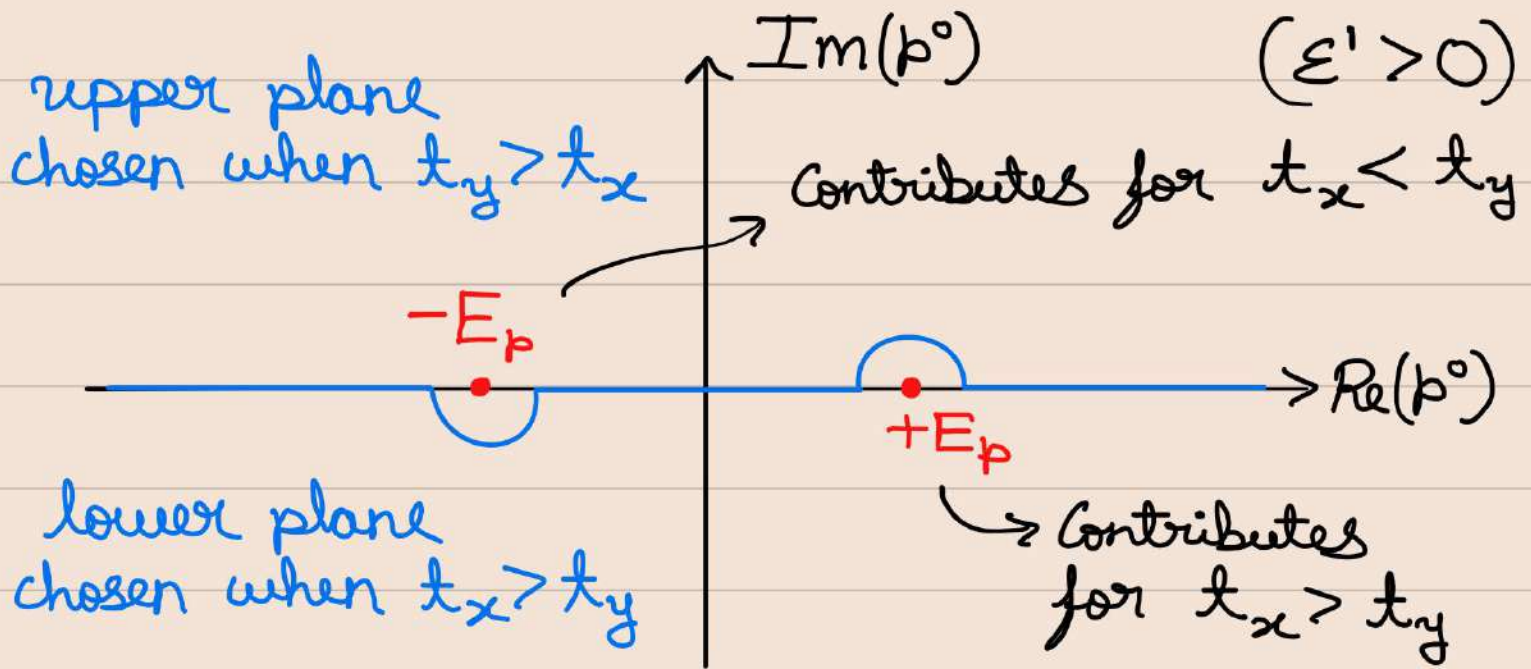
$$(E_p = |\vec{p}|)$$

$$= -i\eta_{\mu\nu} \int \frac{d^4p e^{-ip(x-y)}}{(2\pi)^4 (2E_p)} \left[\frac{1}{p^0 - E_p} - \frac{1}{p^0 + E_p} \right]$$

$$= -\eta_{\mu\nu} \int \frac{d^3p e^{i\vec{p} \cdot (\vec{x} - \vec{y})}}{(2\pi)^3 (2E_p)}$$

↓ Feynmann's Prescription

$$\times \lim_{\epsilon \rightarrow 0} i \int_{-\infty}^{\infty} \frac{dp^0 e^{-ip^0(t_x - t_y)}}{(2\pi)} \left[\frac{1}{p^0 - E_p + i\epsilon} - \frac{1}{p^0 + E_p - i\epsilon} \right]$$



$$\rightarrow i \int_{-\infty}^{\infty} dp^0 \frac{e^{-ip^0(t_x - t_y)}}{2\pi(p^0 - E_p + i\varepsilon')} = e^{-iE_p(t_x - t_y)} \Theta(t_x - t_y)$$

$$\& -i \int_{-\infty}^{\infty} dp^0 \frac{e^{-ip^0(t_x - t_y)}}{2\pi(p^0 + E_p - i\varepsilon')} = e^{iE_p(t_x - t_y)} \Theta(t_y - t_x)$$

Then, $D_{\mu\nu}^F(x, y)$ is given by

$$-\eta_{\mu\nu} \left[\int \frac{d^3p}{(2\pi)^3(2E_p)} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \Theta(t_x - t_y) e^{-iE_p(t_x - t_y)} + \int \frac{d^3p}{(2\pi)^3(2E_p)} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \Theta(t_y - t_x) e^{iE_p(t_x - t_y)} \right]$$

In the 2nd term we can put $\vec{p} \rightarrow -\vec{p}$
 & then setting $p^0 = E_p$, $p^\mu = (E_p, \vec{p})$,
 we get,

$$D_{\mu\nu}^F(x, y) = -\eta_{\mu\nu} \int \frac{d^3p}{(2\pi)^3 (2E_p)} \left[\Theta(t_x - t_y) e^{-ip(x-y)} + \Theta(t_y - t_x) e^{ip(x-y)} \right]$$

Thus this is similar to the Feynmann propagator for K.G. eqⁿ $\Delta_F(x, y)$ except that $m=0$, $p^0 = E_p = |\vec{p}|$ & $p^2 = 0$

$$\Delta_F(x, y) = \int \frac{d^3p}{(2\pi)^3 (2E_p)} \left[\Theta(t_x - t_y) e^{-ip(x-y)} + \Theta(t_y - t_x) e^{ip(x-y)} \right]$$

$$\rightarrow D_{\mu\nu}^F(x, y) = -\eta_{\mu\nu} \Delta_F(x, y)$$

Similarly the retarded Propagator can also be written as,

$$\rightarrow D_{\mu\nu}^R(x, y) = -\eta_{\mu\nu} \Delta_R(x, y)$$

Feynmann Photon Propagator in terms of field operators

Consider,

$$\hat{A}^\mu(x) = \int \frac{d^3p}{(2\pi)^3(2E_p)} \sum_{\lambda=0}^3 \left[\varepsilon_\lambda^\mu(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{-ip_\mu x^\mu} + \varepsilon_\lambda^{\mu*}(\vec{p}) \hat{a}_\lambda^\dagger(\vec{p}) e^{ip_\mu x^\mu} \right]$$

$$\hat{A}^\nu(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3(2E_p)} \sum_{\lambda=0}^3 \left[\varepsilon_\lambda^\nu(\vec{p}) \hat{a}_\lambda(\vec{p}) e^{ip_\mu y^\mu} \right]$$

$$\langle 0|\hat{A}^\mu(x) = \int \frac{d^3q}{(2\pi)^3(2E_q)} \sum_{\lambda'=0}^3 \left[\varepsilon_{\lambda'}^\mu(\vec{q}) \hat{a}_{\lambda'}(\vec{q}) e^{-iq_\mu x^\mu} \right]$$

$\rightarrow \langle 0|\hat{A}^\mu(x)\hat{A}^\nu(y)|0\rangle$ becomes,

$$= \int \frac{d^3p d^3q}{(2\pi)^3 2\sqrt{E_p E_q}} \sum_{\lambda, \lambda'} \varepsilon_{\lambda'}^\mu(\vec{q}) \varepsilon_\lambda^{\nu*}(\vec{p}) \langle 0|\hat{a}_{\lambda'}(\vec{q}) \hat{a}_\lambda^\dagger(\vec{p})|0\rangle \times e^{i(p_\mu y^\mu - q_\mu x^\mu)}$$

$$\langle 0|\hat{a}_{\lambda'}(\vec{q}) \hat{a}_\lambda^\dagger(\vec{p})|0\rangle = \eta_{\lambda\lambda'} \delta^3(\vec{q} - \vec{p})$$

Since λ, λ' are minkowski indices

we can perform d^3p integral.

$$\rightarrow \langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\sum_{\lambda, \lambda'} \varepsilon_\lambda^\mu(\vec{p}) \varepsilon_{\lambda'}^\nu{}^*(\vec{p}) \eta_{\lambda\lambda'} \right] e^{-ip(x-y)}$$

Choosing Polarization states such that,

$$\sum_{\lambda, \lambda'} \varepsilon_\lambda^\mu(\vec{p}) \varepsilon_{\lambda'}^\nu{}^*(\vec{p}) \eta_{\lambda\lambda'} = -\eta^{\mu\nu} \quad \left. \vphantom{\sum_{\lambda, \lambda'}} \right\} \begin{array}{l} \text{Choice of} \\ \text{Polarization} \\ \text{sums} \end{array}$$

$$\rightarrow \langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(y) | 0 \rangle = -\eta^{\mu\nu} \int \frac{d^3p}{(2\pi)^3 2E_p} e^{-ip(x-y)}$$

Similarly,

$$\rightarrow \langle 0 | \hat{A}^\nu(y) \hat{A}^\mu(x) | 0 \rangle = -\eta^{\mu\nu} \int \frac{d^3p}{(2\pi)^3 2E_p} e^{ip(x-y)}$$

Thus we get,

$$\begin{aligned} \mathcal{D}_{\mu\nu}^F(x, y) &= \Theta(t_x - t_y) \langle 0 | \hat{A}_\mu(x) \hat{A}_\nu(y) | 0 \rangle \\ &\quad + \Theta(t_y - t_x) \langle 0 | \hat{A}_\nu(y) \hat{A}_\mu(x) | 0 \rangle \end{aligned}$$

Since ' $\hat{A}_\mu(x)$ ' are bosonic, we can write,

$$\mathcal{D}_{\mu\nu}^F(x, y) = \langle 0 | T [\hat{A}_\mu(x) \hat{A}_\nu(y)] | 0 \rangle$$

The retarded propagator is,

$$D_{\mu\nu}^R(x, y) = -\eta_{\mu\nu} \Delta_R(x, y)$$

$$\Delta_R(x, y) = \Theta(t_x - t_y) \int \frac{d^3p}{(2\pi)^3 (2E_p)} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

$$\begin{aligned} \rightarrow D_{\mu\nu}^R(x, y) &= \Theta(t_x - t_y) [-\eta_{\mu\nu}] \int \frac{d^3p}{(2\pi)^3 (2E_p)} e^{-ip(x-y)} \\ &\quad - \Theta(t_x - t_y) [-\eta_{\mu\nu}] \int \frac{d^3p}{(2\pi)^3 (2E_p)} e^{ip(x-y)} \end{aligned}$$

$\rightarrow$

$$D_{\mu\nu}^R(x, y) = \Theta(t_x - t_y) \left[\langle 0 | \hat{A}_\mu(x) \hat{A}_\nu(y) | 0 \rangle - \langle 0 | \hat{A}_\nu(y) \hat{A}_\mu(x) | 0 \rangle \right]$$

$$= \Theta(t_x - t_y) \langle 0 | [\hat{A}_\mu(x), \hat{A}_\nu(y)] | 0 \rangle$$

$$\rightarrow \boxed{D_{\mu\nu}^R(x, y) = \Theta(t_x - t_y) \langle 0 | [\hat{A}_\mu(x), \hat{A}_\nu(y)] | 0 \rangle}$$

Consider $\lim_{\substack{x \rightarrow y \\ t_x \rightarrow t_y^+}} \langle 0 | T [\hat{A}_\mu(x) \hat{A}_\nu(y)] | 0 \rangle$

$$= \langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(x) | 0 \rangle$$

$$= -\eta^{\mu\nu} \underbrace{\int \frac{d^3p}{(2\pi)^3 2E_p}}_{\text{Total no. of allowed modes}}, E_p = |\vec{p}|$$

$$\Rightarrow \langle 0 | \hat{A}^\mu(x) \hat{A}^\nu(x) | 0 \rangle \propto -\eta^{\mu\nu}$$

$$\begin{aligned} \Rightarrow \langle 0 | \hat{A}^i(x) \hat{A}^j(x) | 0 \rangle &\propto \delta^{ij} \\ \Rightarrow \text{norm}(\hat{A}^i(x) | 0 \rangle) &\text{ is +ve} \end{aligned}$$

But $\langle 0 | \hat{A}^0(x) \hat{A}^0(x) | 0 \rangle$ is -ve.

$\Rightarrow \text{norm}(\hat{A}^0(x) | 0 \rangle)$ is -ve.

So photons created by $\hat{A}^i(x)$ are real & observable. But in QED, it can be shown that -ve norm state created by $\hat{A}^0(x)$ is never produced in any physical process.