

17/01/25

## Lecture 4

### Thermodynamic Potentials (fluid systems)

#### Internal Energy

$$E \Rightarrow E(S, V, N)$$

$S, V, N$  are natural variables of  $E$

$$dE = TdS - PdV + \mu dN$$

$$T = \left( \frac{\partial E}{\partial S} \right)_{P, N}, \quad P = - \left( \frac{\partial E}{\partial V} \right)_{S, N}, \quad \mu = \left( \frac{\partial E}{\partial N} \right)_{S, V}$$

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#### Enthalpy

$$H = E + PV$$

$$E(S, V, N) \longrightarrow H(S, P, N)$$

$$dH = TdS + VdP + \mu dN$$

$$T = \left( \frac{\partial H}{\partial S} \right)_{P, N}, \quad V = \left( \frac{\partial H}{\partial P} \right)_{S, N}, \quad \mu = \left( \frac{\partial H}{\partial N} \right)_{S, P}$$

## Helmholtz free Energy

$$A = E - TS$$

$$E(S, V, N) \rightarrow A(T, V, N)$$

$$dA = -SdT - PdV + \mu dN$$

$$S = -\left(\frac{\partial A}{\partial T}\right)_{V, N} \quad P = -\left(\frac{\partial A}{\partial V}\right)_{T, N} \quad \mu = \left(\frac{\partial A}{\partial N}\right)_{T, V}$$

## Gibbs free Energy

$$G = E + PV - TS$$

$$E(S, V, N) \rightarrow G(T, P, N)$$

$$dG = -SdT + PdV + \mu dN$$

$$S = -\left(\frac{\partial G}{\partial T}\right)_{P, N} \quad P = \left(\frac{\partial G}{\partial V}\right)_{T, N} \quad \mu = \left(\frac{\partial G}{\partial N}\right)_{T, P}$$

## Grand Potential

$$\Lambda = E - TS - \mu N = -PV$$

$$E(S, V, N) \rightarrow \Lambda(T, V, \mu)$$

$$d\Lambda = -SdT - PdV - Nd\mu$$

$$S = -\left(\frac{\partial \Lambda}{\partial T}\right)_{V, \mu} \quad P = -\left(\frac{\partial \Lambda}{\partial V}\right)_{T, \mu} \quad N = -\left(\frac{\partial \Lambda}{\partial \mu}\right)_{T, V}$$

# Thermodynamic Potentials for Spin / Magnetic Systems

Consider the internal energy for such systems,  $dE = TdS + h dM$

We define the free energy (Gibbs),

$$F = E - TS - hM$$

$$\Rightarrow dF = -SdT - Mdh$$

$$\Rightarrow S = -\left(\frac{\partial F}{\partial T}\right)_h \quad \& \quad M = -\left(\frac{\partial F}{\partial h}\right)_T$$

Also since  $dF$  is a total derivative, the order of second derivative doesn't

Matter.  $\Rightarrow \left[ \frac{\partial}{\partial h} \left( \frac{\partial F}{\partial T} \right)_h \right]_T = \left[ \frac{\partial}{\partial T} \left( \frac{\partial F}{\partial h} \right)_T \right]_h$

$$\Rightarrow \left( \frac{\partial S}{\partial h} \right)_T = \left( \frac{\partial M}{\partial T} \right)_h$$

## Relation b/w Magnetic Susceptibility & Correlation function

The interacting Hamiltonian of spin-system is given by  $H_{int} = -\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j$

To understand the magnetization & its fluctuations in the system we add a  $h$ -coupling term of " $-\sum_j h_j S_j$ " to the hamiltonian, where  $h_j$  is the magnetic field present in the  $j^{th}$  cell.

Then finally take the limit  $\{h_j\} \rightarrow 0$

$$\Rightarrow H = -\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j - \sum_j h_j S_j$$

From gibb's canonical ensemble picture the gibb's partition function is,

$$Z = \sum_{\{S_i\}} e^{-\beta H} = \sum_{\{S_i\}} e^{\beta \left[ \frac{1}{2} \sum_{i,j} J_{ij} S_i S_j + \sum_j h_j S_j \right]}$$

$$\Rightarrow Z = \text{Trace}(e^{-\beta \hat{H}})$$

The magnetization at  $j^{\text{th}}$  cell is

$$m_j = \langle S_j \rangle = \frac{\text{Trace}(S_j e^{-\beta \hat{H}})}{Z}$$

$$m_j = \frac{1}{Z\beta} \frac{\partial}{\partial h_j}(Z) = \frac{1}{\beta} \frac{\partial}{\partial h_j} [\ln(Z)]$$

We define  $\chi_{ij} = \frac{\partial m_j}{\partial h_i}$  as the

susceptibility at  $j^{\text{th}}$  cell due to magnetic field at  $i^{\text{th}}$  cell.

$$\chi_{ij} = \frac{\partial}{\partial h_i} \left[ \frac{\text{Trace}(S_j e^{-\beta \hat{H}})}{Z} \right]$$

$$= \frac{\partial}{\partial h_i} \left( \frac{\text{Trace}(S_j e^{-\beta [\hat{H}_{\text{int}} - \sum_k h_k S_k]})}{Z} \right)$$

$$= \frac{\beta}{Z^2} \left[ Z \text{Trace}(S_i S_j e^{-\beta [\hat{H}_{\text{int}} - \sum_k h_k S_k]}) - \text{Trace}(S_j e^{-\beta [\hat{H}_{\text{int}} - \sum_k h_k S_k]}) \frac{\partial Z}{\partial h_i} \right]$$

$$\Rightarrow \frac{\chi_{ij}}{\beta} = \frac{\text{Tr}(S_i S_j e^{-\beta \hat{H}})}{Z} - \frac{\text{Tr}(S_j e^{-\beta \hat{H}}) \text{Tr}(S_i e^{-\beta \hat{H}})}{Z^2}$$

$$\chi_{ij} = \beta [\langle S_i S_j \rangle - \langle S_j \rangle \langle S_i \rangle]$$

The total susceptibility is given by

$$\chi_m = \sum_{i,j} \chi_{ij}$$

$$\Rightarrow \chi_m = \beta \sum_{i,j} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle]$$

If  $j = i + r$  then

$$\chi_m = \beta \sum_{i,r} \underbrace{[\langle S_i S_{i+r} \rangle - \langle S_i \rangle \langle S_{i+r} \rangle]}_{G(r)}$$

(Independent of  $i$  due to Translation Symmetry)

$$\Rightarrow \chi_m = \beta \sum_{i=1}^N \sum_{r=1}^N G(r) = N \beta \sum_r G(r)$$

Susceptibility per spin  $\leftarrow \chi_s = \frac{\chi_m}{N} = \beta \sum_r G(r)$

$$\chi_s = \beta \sum_r G(r)$$

$\Rightarrow$

$$G(r) = \langle S_i S_{i+r} \rangle - \langle S_i \rangle \langle S_{i+r} \rangle$$

Another way to derive relation b/w  $\chi_m$  & correlations is as follows.

We know that,  $M = \sum_i S_i$

$$M^2 = \left( \sum_i S_i \right) \left( \sum_j S_j \right) = \sum_{i,j} S_i S_j$$

$$\Rightarrow \langle M^2 \rangle = \sum_{i,j} \langle S_i S_j \rangle$$

$$\langle M \rangle^2 = \langle \sum_i S_i \rangle \langle \sum_j S_j \rangle$$

$$\langle M \rangle^2 = \sum_{i,j} \langle S_i \rangle \langle S_j \rangle$$

$$\Rightarrow \langle (\Delta M)^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2$$

$$\Rightarrow \langle (\Delta M)^2 \rangle = \sum_{i,j} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle]$$

we had already derived  $\chi_m = \beta \langle (\Delta M)^2 \rangle$

$$\Rightarrow \chi_m = \beta \sum_{i,j} [\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle]$$

$$\chi_m = \beta \sum_i \sum_{\vec{r}} G(\vec{r}) = N \beta \sum_{\vec{r}} G(\vec{r})$$

$$\Rightarrow \chi_s = \beta \sum_{\vec{r}} G(\vec{r})$$

# Heat Capacity at Constant 'h'

Gibbs's partition function is

$$Z(h, T, N) = \text{Trace} (e^{-\beta \hat{H}})$$

$$\hat{H} = -\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j - h \sum_i S_i$$

$$\left( \frac{\partial Z}{\partial \beta} \right)_h = \text{Trace} (-\hat{H} e^{-\beta \hat{H}})$$

$$-\frac{1}{Z} \left( \frac{\partial Z}{\partial \beta} \right)_h = \frac{\text{Trace} (\hat{H} e^{-\beta \hat{H}})}{Z} = \langle \hat{H} \rangle$$

$$\langle \hat{H} \rangle = \langle \mathcal{H}_{\text{int}} - h M \rangle = - \left( \frac{\partial \ln(Z)}{\partial \beta} \right)_h$$

↪ analogous to  
Enthalpy

Now,  $F = -k_B T \ln(Z)$

where  $dF = -S dT - M dh$

$$\Rightarrow \left( \frac{\partial F}{\partial T} \right)_h = -S$$

$$\begin{aligned}
\Rightarrow \langle \hat{H} \rangle &= - \left( \frac{\partial \ln(\Xi)}{\partial \beta} \right)_h = \left[ \frac{\partial}{\partial \beta} (\beta F) \right]_h \\
&= F + \beta \left( \frac{\partial F}{\partial \beta} \right)_h \\
&= F + \frac{1}{k_B T} \left( \frac{\partial T}{\partial \beta} \right) \left( \frac{\partial F}{\partial T} \right)_h \\
&\quad \downarrow \qquad \downarrow \\
&= F + \frac{1}{k_B^2 T} \left( -\frac{1}{\beta^2} \right) (-S)
\end{aligned}$$

$$\Rightarrow \langle \hat{H} \rangle = F + TS$$

$$d\langle \hat{H} \rangle = dF + TdS + SdT$$

$$d\langle \hat{H} \rangle = \underbrace{TdS}_{\delta Q} - Mdh$$

$\delta Q \Rightarrow$  heat exchange.

$$d\langle \hat{H} \rangle = \delta Q - Mdh$$

Heat Capacity

$$C_h = \left( \frac{\delta Q}{dT} \right)_h = \left( \frac{\partial \langle \hat{H} \rangle}{\partial T} \right)_h$$

$$\Rightarrow C_h = \left( \frac{\partial}{\partial T} \left[ - \left( \frac{\partial}{\partial \beta} \ln(Z) \right)_h \right]_h \right)_h$$

$$= - \left( \frac{\partial \beta}{\partial T} \right) \left( \frac{\partial^2}{\partial \beta^2} \ln(Z) \right)_h$$

$$\Rightarrow \boxed{C_h = K_B \beta^2 \left( \frac{\partial^2}{\partial \beta^2} \ln(Z) \right)_h}$$

In terms of temperature,

$$\frac{\partial}{\partial \beta} = \frac{-1}{K_B \beta^2} \frac{\partial}{\partial T} = -K_B T^2 \frac{\partial}{\partial T}$$

$$\& \ln(Z) = \frac{-F}{K_B T}$$

$$\Rightarrow C_h = - \frac{\partial}{\partial T} \left[ -K_B T^2 \frac{\partial}{\partial T} \left( \frac{-F}{K_B T} \right)_h \right]_h$$

$$= -K_B \frac{\partial}{\partial T} \left( T^2 \left[ \frac{1}{K_B T} \left( \frac{\partial F}{\partial T} \right) - \frac{F}{K_B T^2} \right] \right)_h$$

$$= \left[ - \frac{\partial F}{\partial T} - T \frac{\partial^2 F}{\partial T^2} + \frac{\partial F}{\partial T} \right]_h$$

$$\Rightarrow \boxed{C_h = -T \left( \frac{\partial^2 F}{\partial T^2} \right)_h}$$

Thus heat capacity is proportional to 2nd derivative of free energy.

Note: Magnetic Susceptibility is also proportional to 2<sup>nd</sup> derivative of free energy.

$$\chi_m = \frac{1}{\beta} \left[ \frac{\partial^2}{\partial h^2} \ln(Z) \right]_{\beta, N}$$

$$\chi_m = \frac{1}{\beta} \left[ \frac{\partial^2}{\partial h^2} (-\beta F) \right]_{\beta, N}$$

$$\Rightarrow \boxed{\chi_m = - \left( \frac{\partial^2 F}{\partial h^2} \right)_{\beta, N}}$$

Relation b/w  $C_h$  & Fluctuation  $\langle (\Delta H)^2 \rangle$

We know that  $C_h = K_B \beta^2 \frac{\partial^2}{\partial \beta^2} \ln(Z)$

$$\text{where } Z = \text{Tr} (e^{-\beta \hat{H}})$$

$$\Rightarrow \frac{\partial Z}{\partial \beta} = - \text{Tr} (\hat{H} e^{-\beta \hat{H}}) = - Z \langle H \rangle$$

$$\Rightarrow \langle H \rangle = \underbrace{- \frac{1}{Z} \left( \frac{\partial Z}{\partial \beta} \right)}$$

$$\frac{\partial^2 Z}{\partial \beta^2} = \text{Tr} (\hat{H}^2 e^{-\beta \hat{H}}) = Z \langle \hat{H}^2 \rangle$$

$$\Rightarrow \underbrace{\langle \hat{H}^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}}$$

$$C_h = k_B \beta^2 \left[ \frac{\partial^2}{\partial \beta^2} \ln(Z) \right]_h = k_B \beta^2 \frac{\partial}{\partial \beta} \left( \frac{1}{Z} \frac{\partial Z}{\partial \beta} \right)$$

$$= k_B \beta^2 \left[ \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2} - \frac{1}{Z^2} \left( \frac{\partial Z}{\partial \beta} \right)^2 \right]$$

$$= k_B \beta^2 \left[ \frac{\text{Tr}(\hat{H}^2 e^{-\beta \hat{H}})}{Z} - \left( \frac{\text{Tr}(\hat{H} e^{-\beta \hat{H}})}{Z} \right)^2 \right]$$

$$C_h = k_B \beta^2 [\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2]$$

$$\langle (\Delta \hat{H})^2 \rangle = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2$$

$$\Rightarrow \boxed{C_h = k_B \beta^2 \langle (\Delta \hat{H})^2 \rangle}$$

In absence of 'h',  $\hat{H} = -J \sum_{i,j} S_i S_j$

$$\Rightarrow \hat{H}^2 = J^2 \sum_{i,j,k,l} S_i S_j S_k S_l$$

$$\Rightarrow \langle (\Delta \hat{H})^2 \rangle = J^2 \sum_{i,j,k,l} \left[ \langle S_i S_j S_k S_l \rangle - \langle S_i S_j \rangle \langle S_k S_l \rangle \right]$$

# eg. 1 Classical Monoatomic Ideal Gas

$$H = \sum_{i=1}^{3N} \frac{|\vec{p}_i|^2}{2m}$$

$$Q(N, V, T) = \frac{1}{N!} \frac{1}{h^{3N}} \int d^3x d^{3N}p e^{-\beta \sum_{i=1}^{3N} \frac{|\vec{p}_i|^2}{2m}}$$

$$= \frac{1}{N! h^{3N}} V^N \left( \int_0^\infty dp e^{-\beta \frac{p^2}{2m}} \right)^{3N}$$

$$= \frac{V^N}{N! h^{3N}} \left( \int \sqrt{2\pi m k_B T} \right)^{3N}$$

$$= \frac{V^N}{N!} \left[ \frac{\sqrt{2\pi m k_B T}}{h} \right]^{3N}$$

$$= \frac{1}{\lambda^{3N}}$$

$$Q_N = \frac{V^N}{(\lambda^3)^N N!}$$

G.C. ensemble

$$Z = \sum_n e^{\beta \mu N} Q_n$$

$$= \sum_n e^{\beta \mu N} \frac{V^n}{N! \lambda^{3N}}$$

$$= \sum_n \frac{\left( e^{\beta \mu} \frac{V}{\lambda^3} \right)^n}{N!}$$

$$\boxed{Z = \exp \left( \frac{V}{\lambda^3} e^{\beta \mu} \right)}$$

$$A = k_B T \ln(Z) = PV$$

$$PV = k_B T \left( \frac{V}{\lambda^3} e^{\beta \mu} \right)$$

$$\langle n \rangle = \left[ \frac{\partial}{\partial \mu} (\Lambda) \right]_{V, T}$$

$$\langle n \rangle = \frac{1}{\beta} \left( \frac{V}{\lambda^3} e^{\beta \mu} \right)$$

$$\langle n \rangle = \frac{V}{\lambda^3} e^{\beta \mu}$$

$$\Rightarrow \boxed{PV = k_B T \langle n \rangle}$$

Eg. 2 Non-Interacting 1D Ising Spins  
Ising Paramagnet

...  $\downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \uparrow \uparrow \downarrow \uparrow \downarrow \uparrow$  ...

$$(J=0) \Rightarrow H = -h \sum_{i=1}^N S_i$$

$$Z = \sum_{\{S\}} e^{-\beta H_c} = \sum_{\{S\}} \left( e^{\beta h \sum_i S_i} \right)$$

$$Z = \sum_{\{c\}} \left( e^{\beta \hbar S_1} e^{\beta \hbar S_2} \dots e^{\beta \hbar S_N} \right)$$

$$\{c\} = S_i = \pm 1$$

$$Z = \sum_{S_1 = \pm 1} e^{\beta \hbar S_1} \sum_{S_2 = \pm 1} e^{\beta \hbar S_2} \dots \sum_{S_N = \pm 1} e^{\beta \hbar S_N}$$

$$Z = \left( e^{\beta \hbar} + e^{-\beta \hbar} \right)^N$$

$$Z = \left[ 2 \cosh(\beta \hbar) \right]^N$$

Energy of  $i^{\text{th}}$  spin  $\leftarrow E_i = -\vec{\mu}_i \cdot \vec{B}$        $\mu_i = \frac{e}{m} \vec{S}_i$

$$\Rightarrow E_i = -\frac{e}{m} \vec{S}_i \cdot \vec{B}$$

$$\Rightarrow E_i = -\frac{eB}{m} S_i^z \quad \left( S_i^z = \frac{\hbar}{2} S_i \right)$$

$\downarrow$   
 $S = \pm 1$

$$E_i = -\frac{e\hbar}{2m} B S_i = -\underbrace{\mu_B}_{\hbar} B S_i = -\hbar S_i$$

$$\underline{\underline{\hbar = \mu_B B}}$$

(Setting  $\mu_B = 1$ , throughout)

$$\boxed{E_i = -\hbar S_i} \Rightarrow \text{This is what is written in the hamiltonian.}$$

$$M = \langle \sum_i S_i \rangle = \frac{\sum_{\{S\}} \left( \sum_i S_i \right) e^{\beta h \sum_i S_i}}{Z}$$

$$M = \frac{1}{Z} \left[ \frac{\partial}{\partial h} \left( \frac{\sum_{\{S\}} e^{\beta h \sum_i S_i}}{(-h)} \right) \right]_{\beta}$$

$$\Rightarrow M = \frac{1}{\beta Z} \left( \frac{\partial}{\partial h} (Z) \right)_{\beta}$$

$$\Rightarrow M = \frac{1}{\beta} \left( \frac{\partial}{\partial h} \log(Z) \right)_{\beta}$$

$$M = \frac{1}{\beta} \left( \frac{\partial \log(Z)}{\partial h} \right)_{\beta}$$

$$M = \frac{N}{\beta} \left[ \frac{\partial}{\partial h} \log [2 \cosh(\beta h)] \right]_{\beta}$$

$$M = \frac{N}{\beta} [\tanh(\beta h) \times \beta]$$

Eq<sup>n</sup> of  
state

$$\Rightarrow M = N \tanh(\beta h)$$

$$\chi_m = \left( \frac{\partial M}{\partial h} \right)_\beta = N\beta \operatorname{sech}^2(\beta h)$$

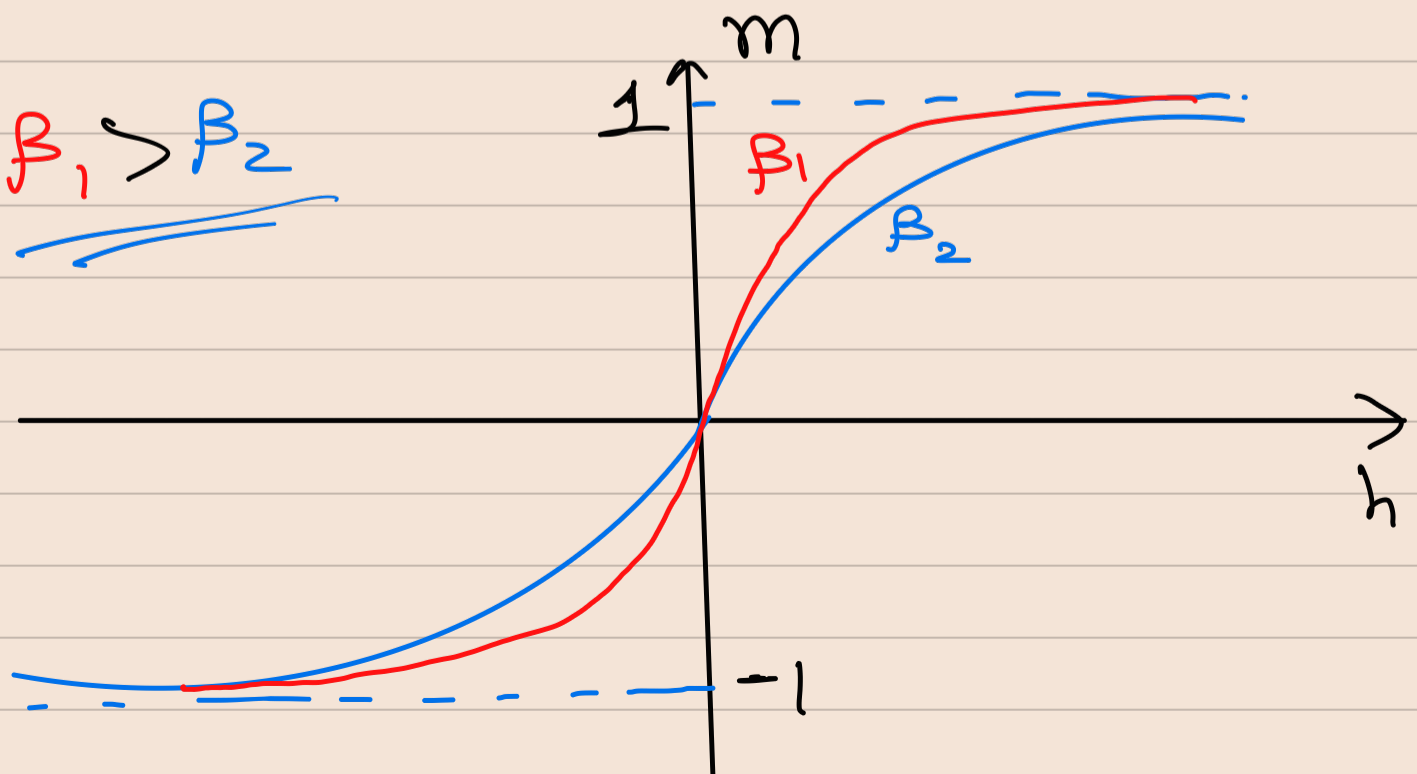
$$\chi_m = N\beta [1 - \tanh^2(\beta h)]$$

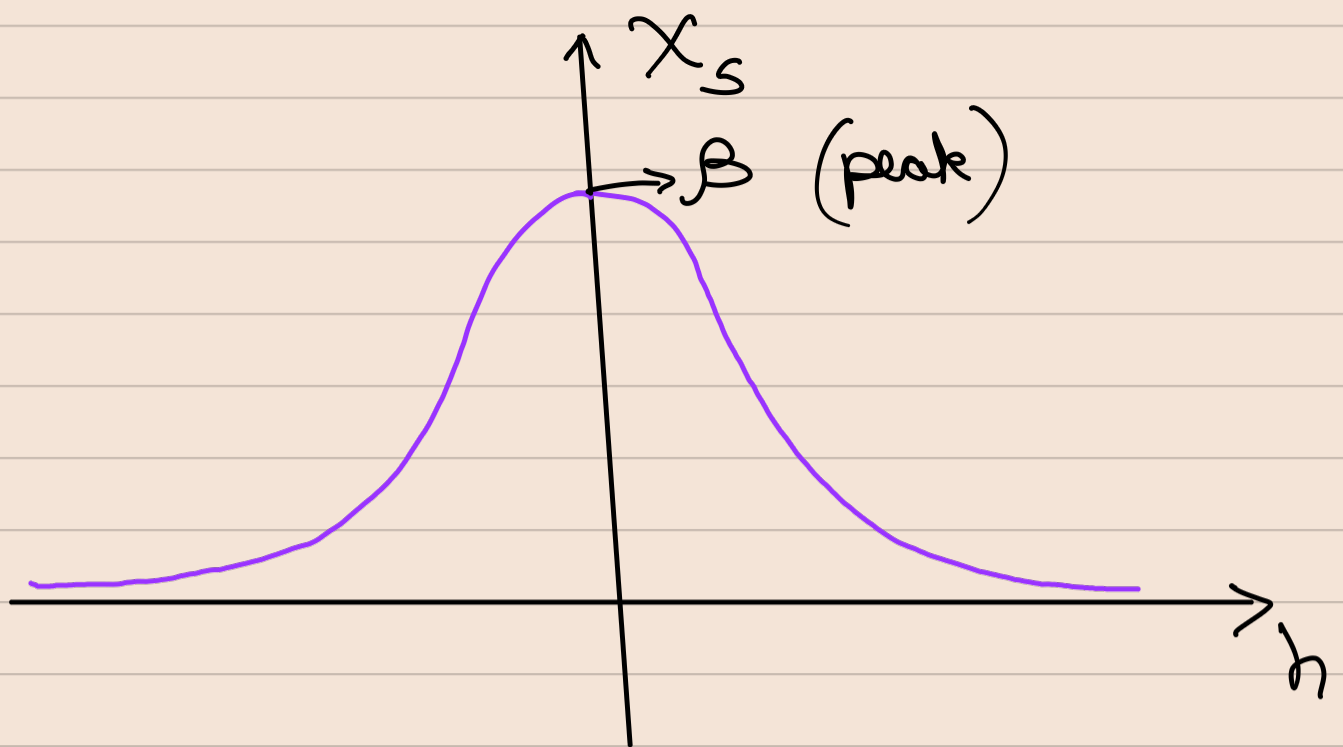
$$\chi_m = N\beta - \beta \frac{M^2}{N}$$

$$\chi_s = \frac{\chi_m}{N} \Rightarrow \text{Susceptibility per Spin}$$

$$m = \frac{M}{N} \Rightarrow \text{Magnetization per Spin}$$

$$\Rightarrow \underline{\underline{\chi_s = \beta (1 - m^2)}}$$





In the limit  $h \rightarrow 0$ ,

$$\chi_s = B = \frac{1}{k_B T}$$

$$\chi_m = \frac{N}{k_B T}$$

But in S.I  $\mu_B$  is present explicitly

$$\underline{M = N \mu_B \tanh(\mu_B B h)}$$

$$\Rightarrow \underline{\chi_m = \frac{N \mu_B^2}{k_B T}} \Rightarrow \text{Curie's law}$$

Eg: 3

## Non-Interacting Classical Heisenberg Spins in a 1D chain

$$\mathcal{H} = -\vec{h} \cdot \sum_i \vec{S}_i$$

$$\vec{h} = h \hat{z} \Rightarrow \mathcal{H} = -h \sum_i S_i^z$$

In 3D polar coordinates,  $S_i^z = \cos(\theta_i)$

$$\Rightarrow \mathcal{H} = -h \sum_i \cos(\theta_i)$$

Partition function for continuous  
Parameter ( $\theta_i$ ) summation is

$$Z = \int \prod_{i=1}^N \int_{\text{Surface}} d^2 S_i e^{\beta h S_i^z} = \int \prod_{i=1}^N \int_{\text{Surface}} d^2 S_i e^{\beta h \cos(\theta_i)}$$

↓  
(On the surface of the sphere in  
the spin space of unit radius)

$$d^2 S_i = \sin(\theta_i) d\theta_i d\varphi_i$$

$$0 < \theta_i < \pi$$

$$0 < \varphi_i < 2\pi$$

For 1 Spin

$$Z = 2\pi \int_0^\pi \sin(\theta) d\theta e^{\beta h \cos(\theta)}$$

$$\mathcal{Z} = \frac{2\pi (e^{\beta h} - e^{-\beta h})}{\beta h}$$

$$\mathcal{Z} = \left( \frac{4\pi}{\beta h} \right) \sinh(\beta h)$$

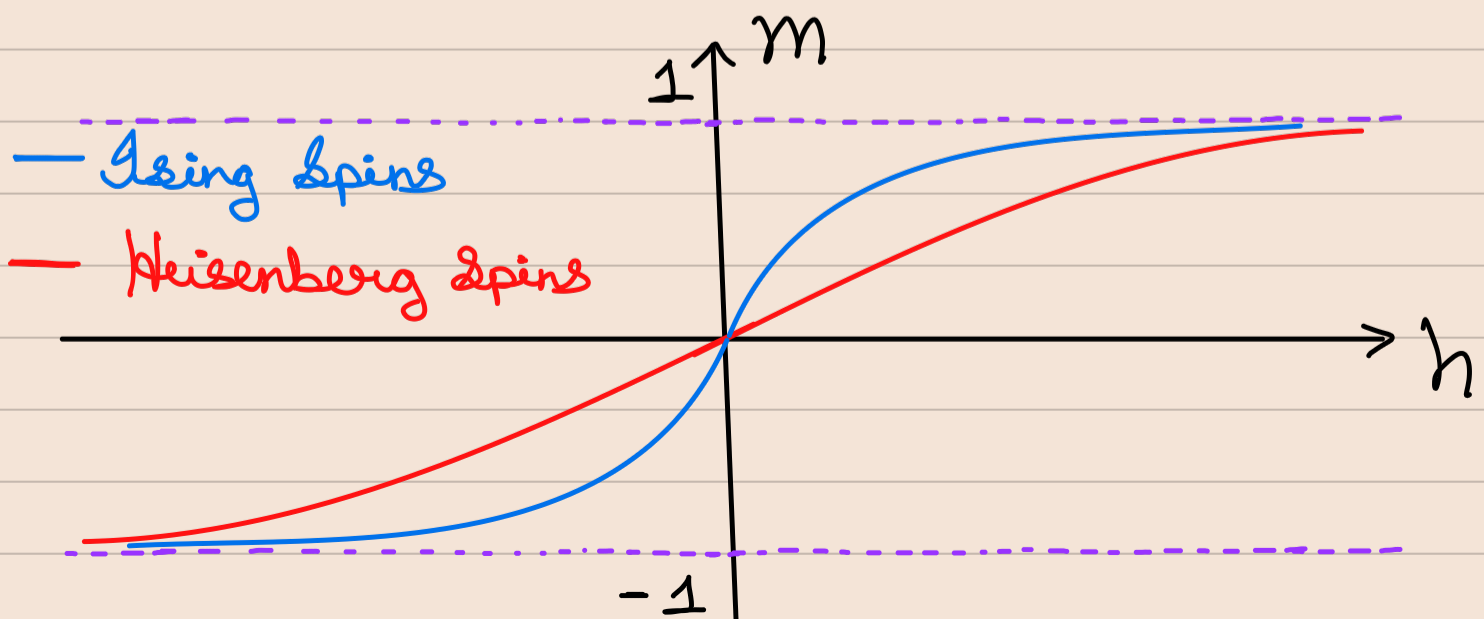
$$m = \frac{1}{\beta} \left( \frac{\partial}{\partial h} \ln(\mathcal{Z}) \right)_T$$

$$= \frac{1}{\beta} \frac{\partial}{\partial h} \left[ \ln(4\pi) + \ln[\sinh(\beta h)] - \ln(\beta h) \right]_T$$

$$= \frac{1}{\beta} \left[ \coth(\beta h) - \frac{1}{\beta h} \right] \times \beta$$

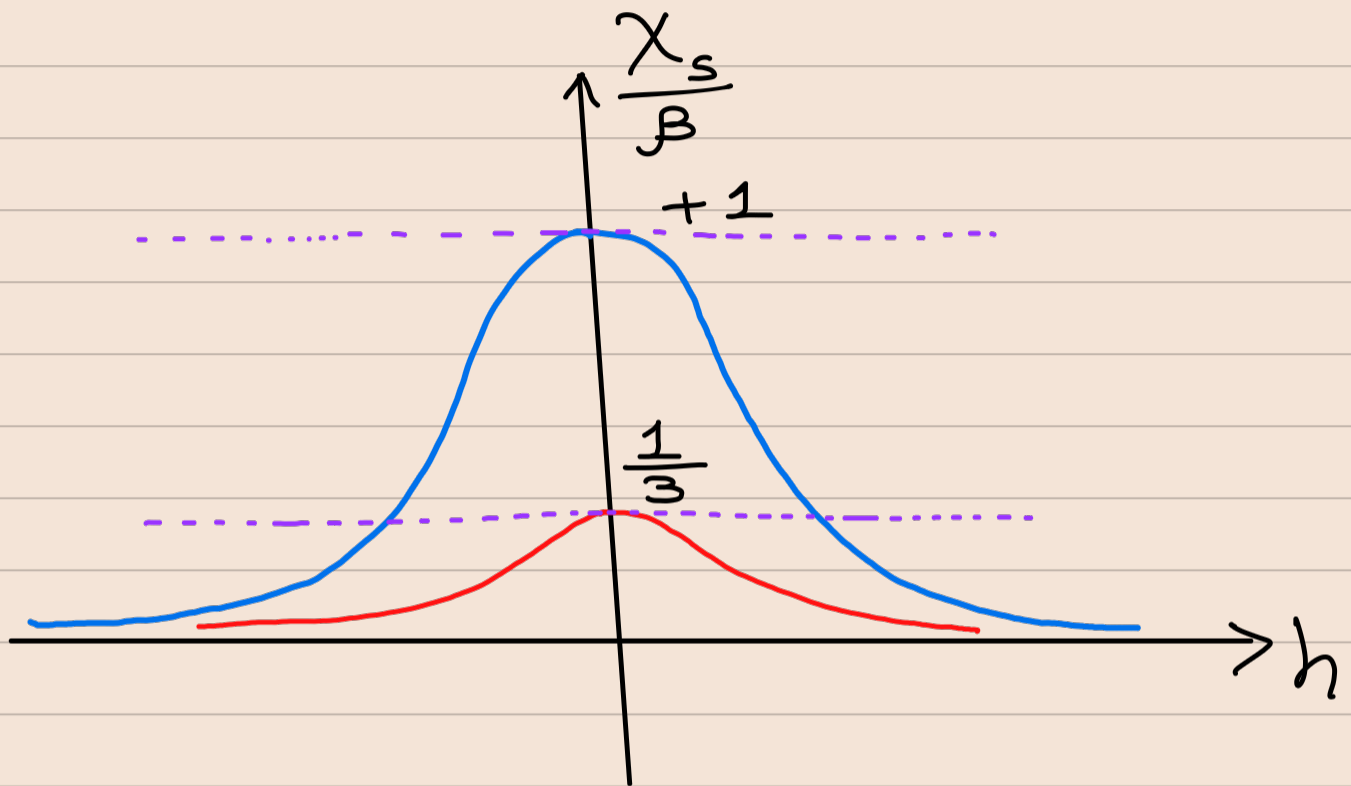
$$\Rightarrow m = \coth(\beta h) - \frac{1}{\beta h} = \mathcal{L}(\beta h) \Rightarrow \text{Langevin Function}$$

$$\mathcal{L}(x) = \frac{x}{3} - \frac{1}{45} x^3 + \dots \quad \text{Series expansion}$$



$$\chi_s = \left( \frac{\partial m}{\partial h} \right)_\beta = -\beta \operatorname{cosech}^2(\beta h) + \frac{1}{\beta h^2}$$

$$\chi_s = \beta \left[ \frac{1}{(\beta h)^2} - \frac{1}{\sinh^2(\beta h)} \right] = \beta \left[ \frac{1}{3} - \frac{(\beta h)^2}{15} + \dots \right]$$



In the limit  $h \rightarrow 0$  we have

Ising spin

$$m = \beta h - \frac{(\beta h)^3}{3} + \dots$$

$$\frac{\chi_s}{\beta} = 1 - (\beta h)^2 + \dots$$

Heisenberg spin

$$m = \frac{\beta h}{3} - \frac{(\beta h)^3}{45} + \dots$$

$$\frac{\chi_s}{\beta} = \frac{1}{3} - \frac{(\beta h)^2}{15} + \dots$$

Eg. 4

## Non-Interacting classical X-Y Spins in a 1D chain

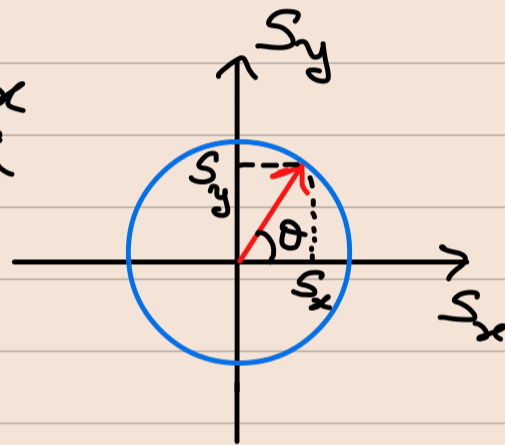
Here  $\vec{S}_i = (S_i^x, S_i^y)$

$$H = -\vec{h} \cdot \sum_i \vec{S}_i$$

Let  $\vec{h} = h \hat{x}$

$$\Rightarrow H = -h \sum_i S_i^x$$

$$S_i^x = \cos(\theta_i)$$



$$Z = \int_0^{2\pi} \prod_{i=1}^N d\theta_i e^{Bh S_i^x} = \int_0^{2\pi} \prod_{i=1}^N d\theta_i e^{Bh \cos(\theta_i)}$$

Single Spin  $\Rightarrow Z = \int_0^{2\pi} d\theta e^{Bh \cos(\theta)} = I_0(Bh)$

$I_0$  is the modified Bessel function of first kind of order Zero.

$$I_0(Bh) = 1 + \frac{1}{4} (Bh)^2 + \frac{1}{64} (Bh)^4 + \dots$$

$$I_0(\beta h) = \sum_{k=0}^{\infty} \frac{(\beta h)^{2k}}{4^k (k!)^2}$$

$$m = \frac{1}{\beta} \left[ \frac{\partial}{\partial h} \log(Z) \right]_{\beta}$$

$$\log(Z) = \log \left( 1 + \frac{1}{4} (\beta h)^2 + O[(\beta h)^4] \right)$$

$$\log(Z) = \frac{1}{4} (\beta h)^2 + O[(\beta h)^4]$$

$$\Rightarrow m = \frac{1}{2} (\beta h) + O[(\beta h)^3]$$

$$\chi_s = \left( \frac{\partial m}{\partial h} \right)_{\beta} = \frac{\beta}{2} + O[(\beta h)^2]$$

$\lim_{h \rightarrow 0} \frac{\chi_s}{\beta} = \frac{1}{2}$

(for  $X-Y$  model)

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\chi_s}{\beta} = \begin{cases} 1 \Rightarrow \text{Ising model} \\ \frac{1}{2} \Rightarrow X-Y \text{ model} \\ \frac{1}{3} \Rightarrow \text{Heisenberg model} \end{cases}$$