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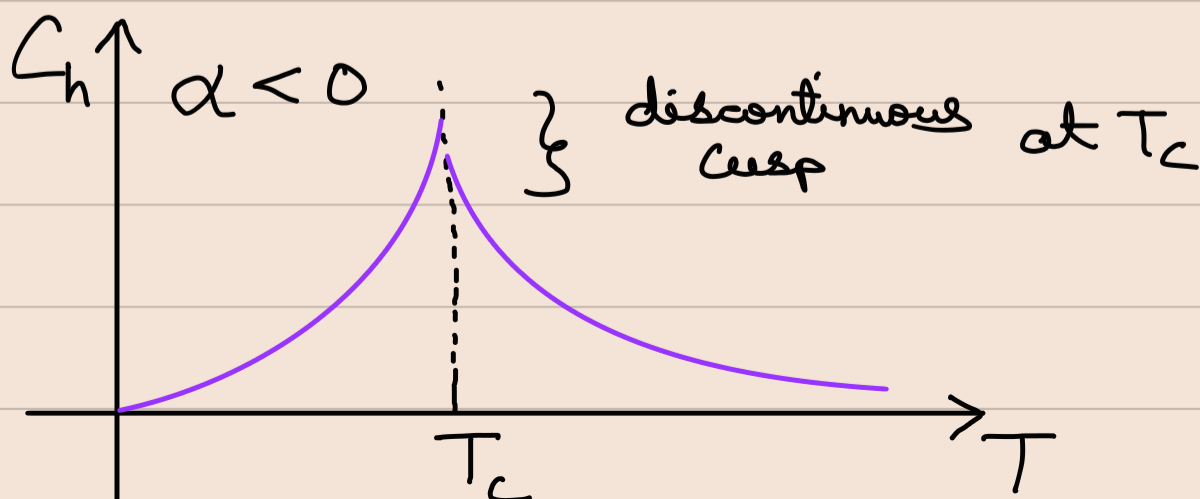
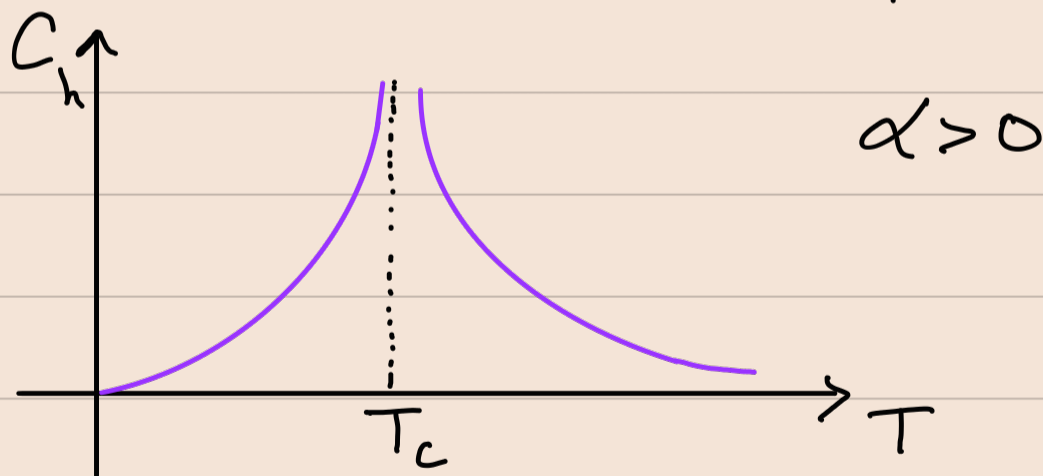
Lecture 11

Critical Exponents

1) Specific heat exponent (α) (at $h=0$)

' α ' is defined as $C_h \sim |t|^{-\alpha}$

In some theories ' α ' can be +ve which implies C_h diverges
But for -ve ' α ' we get C_h to be a discontinuous cusp.



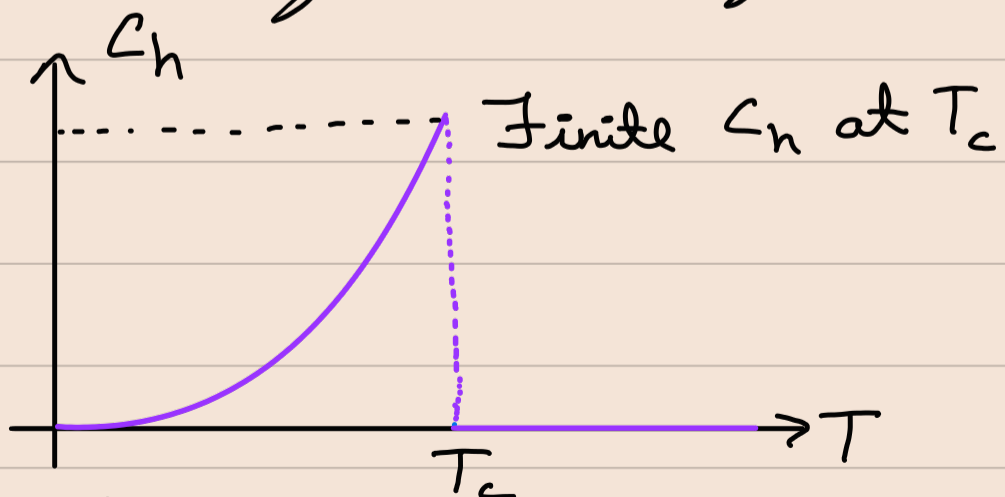
Even though $C_h \sim |t|^{-\alpha}$ but a more proper way of defining " α " is, $\alpha = - \lim_{t \rightarrow 0} \left[\frac{C_h}{\ln|t|} \right]$

If $\alpha=0$, it means a jump at T_c or log. singularity.

For mean field theory, we have C_h to be finite for all values of " t " it doesn't diverge. But as $t \rightarrow 0$ $\ln|t| \rightarrow -\infty$.

$$\Rightarrow \underline{\underline{\alpha \rightarrow 0 \text{ as } t \rightarrow 0.}}$$

Thus $\alpha=0$ for Mean field theory.



Here $\alpha=0$ implies, C_h has a jump discontinuity at T_c .

2) Spontaneous Magnetization exponent (β) (m_s) (at $h=0$)

$$m_s \propto |t|^\beta \quad (\text{as } t \rightarrow 0)$$

For mean field theory we have already seen $m \propto \sqrt{t}$
 $\Rightarrow \underline{\beta = 1/2}$

3) Susceptibility exponent (γ) (at $h=0$)

$$\chi \propto |t|^{-\gamma} \quad (\text{as } t \rightarrow 0)$$

For mean field theory

$$\chi \propto \frac{1}{|t|} \Rightarrow \underline{\underline{\gamma = 1}}$$

χ diverges at critical point.

4) Magnetization exponent (β)
(at $t=0$)

$$\underline{m \propto |h|^{1/3}}$$

For mean field theory we got

$$m \propto h^{1/3}$$

$$\Rightarrow \underline{\underline{\beta = 3}}$$

5) Correlation length exponent (ν)
(at $h=0$)

$$\underline{\xi \sim |t|^{-\nu}}$$

For mean field theory we had gotten previously,

$$\xi \propto |T - T_c|^{-\frac{1}{2}}$$

$$\Rightarrow \underline{\underline{\nu = \frac{1}{2}}}$$

6) Correlation function exponent (η) (at $t=0, h=0$)

$$G(r) \propto \frac{e^{-r/\xi_g}}{r^{d-2+\eta}}$$

$d \Rightarrow$ spatial dimension of system

For mean field theory

$$G(r) \propto \frac{e^{-r/\xi_g}}{r} \quad (\text{for } d=3)$$

But $\xi_g \rightarrow \infty$ at $t=0$.

$$\Rightarrow G(r) \propto \frac{1}{r} \quad (\text{for } d=3)$$

$$\Rightarrow \underline{\underline{\eta=0}} \quad (\text{for } d=3)$$

In 1D exact solⁿ for ising model ($d=1$)

$$G(r) \propto e^{-r/\xi_g} \Rightarrow d-2+\eta=0 \Rightarrow \underline{\underline{\eta=1}}$$

Mean field Exponents

$$\alpha = 0 \quad \beta = 1/2 \quad \gamma = 1$$

$$\delta = 3 \quad \nu = 1/2 \quad \eta = 0 \quad (d=3)$$

Scaling Hypothesis

According to Scaling Hypothesis, "Near the critical point, the correlation length ' ξ_c ' is the only characteristic length scale of the system in terms of which other lengths must be measured." — Huang.

Dimensional Analysis

Here we assume that any thermodynamic variable of dimension $(\text{length})^k$ will be proportional to ξ_c^k .

- Consider the free energy ' F ' in the units of $k_B T_c$ as they both have energy units.

$\Rightarrow \frac{F}{k_B T_c}$ is dimensionless quantity

Then free energy per unit volume has dimensions of $(\text{length})^{-d}$ where $d \Rightarrow$ Spatial dimension of the system.

$$f = \frac{(F/k_B T_c)}{\text{Volume}} \Rightarrow \text{dimensionless free energy per unit volume.}$$

$$\text{Volume} \propto (\text{length})^d.$$

$$\Rightarrow \text{dim. of } f = [f] = \underline{[\text{length}]^{-d}}$$

- We know that the correlation function is given by,

$$\Gamma(r) \propto \frac{e^{-r/\xi}}{r^{d-2+\eta}} \Rightarrow \left\{ \begin{array}{l} \text{Volume} \\ \text{normalized} \\ \text{correlation} \\ \text{function} \end{array} \right.$$

$$\Rightarrow \text{dim. of } \Gamma(r) \Rightarrow [\Gamma(r)] = (\text{length})^{2-d-\eta}$$

$$\text{We know } G(r) = \langle S_i S_{i+r} \rangle - \langle S_i \rangle \langle S_{i+r} \rangle$$

$$\Rightarrow [G(r)] = [m]^2 \quad m = \frac{M_{\text{net}}}{N}$$

But in volume normalized variables,

$$[\Gamma(r)] = \left[\frac{M_{\text{net}}}{V} \right]^2 \Rightarrow \underline{\left[\frac{M_{\text{net}}}{V} \right] = (\text{length})^{(2-d-\eta)/2}}$$

- By the definition of $M_{\text{net}} = \frac{\partial F}{\partial H}$
 $H \Rightarrow$ Magnetic field in SI units, where
as $h = \mu_B H \Rightarrow$ Magnetic field in the
energy units ($\mu_B = \text{Bohr Magnetron}$). Here M_{net}
is in the SI units. To get it into our
usual units do $M \rightarrow \text{Energy} \times M$ & $H \rightarrow h$

$$\Rightarrow [h] = \frac{[F]}{[M_{\text{net}} \times \text{Energy}]} = \frac{\left[\frac{F}{\text{Energy} \times V} \right]}{\left[M_{\text{net}} / V \right]}$$

$$\Rightarrow [h] = \frac{(\text{length})^{-d}}{(\text{length})^{(2-d-\eta)/2}} = (\text{length})^{\frac{\eta-d-2}{2}}$$

in energy units

$$\Rightarrow [h] = (\text{length})^{\frac{\eta-d-2}{2}}$$

- $\chi_{\text{net}} = \frac{\partial M_{\text{net}}}{\partial H} (\Rightarrow \text{in SI})$

$$\Rightarrow [\text{Energy} \times \chi_{\text{net}}] = \frac{[M_{\text{net}} \times \text{energy}]}{[H]}$$

$$\Rightarrow [\text{Energy} \times \chi_{\text{net}}] = \frac{[M_{\text{net}}]}{[H/\text{Energy}]} = \left[\frac{M_{\text{net}}}{h} \right]$$

$$\Rightarrow \left[\frac{\text{Energy} \times \chi_{\text{net}}}{\text{Volume}} \right] = \frac{[M_{\text{net}}/\text{Volume}]}{[h]}$$

$\chi \Rightarrow$ in our usual units

$$\Rightarrow [\chi] = \frac{(\text{length})^{(2-d-\eta)/2}}{(\text{length})^{\frac{\eta-d-2}{2}}} = [\text{length}]^{2-\eta}$$

$$\Rightarrow \text{dim. of } \chi = [\chi] = (\text{length})^{2-\eta}$$

Thus we got the length dimensions of

- $[f] = [\text{length}]^{-d}$
- $\left[\frac{M_{\text{net}}}{V} \right] = (\text{length})^{(2-d-\eta)/2}$
- $[k_B T \chi] = (\text{length})^{2-\eta}$
- $[h] = (\text{length})^{\frac{\eta-d-2}{2}}$
- $[\tau \phi] = (\text{length})^{2-d-\eta}$

Now near critical point all length scales should be measured in terms of ' ξ ' $\Rightarrow$ This is Scaling hypothesis.
 $\Rightarrow$ replace, 'length' $\rightarrow \xi \rightarrow |t|^{-\nu}$

- $\left[\frac{M_{\text{net}}}{V} \right] \sim |t|^\beta = \xi^{(2-d-\eta)/2} = |t|^{-\nu \left(\frac{2-d-\eta}{2} \right)}$

$$\Rightarrow \boxed{\beta = -\nu \left(\frac{2-d-\eta}{2} \right)} \rightarrow (a)$$

- $[k_B T \chi] \sim |t|^{-\gamma} = (\text{length})^{2-\eta} = |t|^{-\nu(2-\eta)}$

$$\Rightarrow \boxed{\gamma = \nu(2-\eta)} \rightarrow (b)$$

$\hookrightarrow$ Fisher's Scaling law

- $\left[\frac{h}{k_B T} \right] \sim m^2 \sim |t|^{\beta\beta} \sim (\text{length})^{\frac{\eta-d-2}{2}} \sim \xi^{\frac{\eta-d-2}{2}}$

$$\Rightarrow \beta\beta = -\nu \frac{(\eta-d-2)}{2}$$

$$\Rightarrow \boxed{\beta\beta = \nu \frac{(2+d-\eta)}{2}} \rightarrow (c)$$

- Specific heat

$$C_{\text{net}} = -T \frac{\partial^2 F}{\partial T^2} = -\frac{T_c(t+1)}{\frac{T_c^2}{T_c}} \frac{\partial^2 F}{\partial t^2}$$

$$\left(t = \frac{T-T_c}{T_c}\right) \quad C_{\text{net}} = K_B \left[-\frac{(t+1)}{K_B T_c} \frac{\partial^2 F}{\partial t^2} \right]$$

$$\Rightarrow C = \frac{C_{\text{net}}}{\text{Volume}} = K_B \left[-\frac{(t+1) \partial^2 \left(\frac{F}{K_B T_c \text{Volume}} \right)}{\partial t^2} \right]$$

$$\Rightarrow C = K_B \left[-\frac{(t+1) \partial^2 f}{\partial t^2} \right]$$

$\Rightarrow$ 'f' related to double integral of 'C'

$$\rightarrow [f] \sim \mathcal{E}^{-d} = |t|^{-\nu d} = |t|^{-\alpha+2} \quad \uparrow$$

+2 because of double integral

$$\Rightarrow -\alpha+2 = -\nu d$$

$$\Rightarrow \boxed{2-\alpha = \nu d} \rightarrow (d)$$

$\hookrightarrow$ Josephson's
Scaling law

- If we perform (c) - (a) then

$$\beta(\beta-1) = \frac{\nu}{2}[(2+d-\eta) + (2-d-\eta)]$$

$$= \frac{\nu}{2}(4-2\eta) = \nu(2-\eta) = \gamma$$

$$\Rightarrow \boxed{\beta(\beta-1) = \gamma} \Rightarrow \text{Widom's Scaling law}$$

- If we find $\alpha + 2\beta + \gamma$ by using (a), (b) & (d) we get

$$\Rightarrow \boxed{\alpha + 2\beta + \gamma = 2} \Rightarrow \text{Rushbrooke's Scaling law}$$

Scaling laws

- $\gamma = \nu(2-\eta) \Rightarrow \text{Fisher's law}$
- $\alpha + 2\beta + \gamma = 2 \Rightarrow \text{Rushbrooke's law}$
- $\gamma = \beta(\beta-1) \Rightarrow \text{Widom's law}$
- $\alpha = 2 - \nu d \Rightarrow \text{Josephson's law}$

In any d-dimensional system out of six critical exponents only two of them are independent as there are four relating equations.

Scaling forms

Scaling form refers to a mathematical expression that describes how a particular thermodynamic variable near a critical point scales with the dimensionless relative temperature (t). Here we try to relate two variables by scaling them w.r.t $|t|$.

For example consider London's field theory. The eqⁿ of state is

$$\text{where } h = 2am + 4bm^3 \quad \left| \begin{array}{l} c_1 = \pm \frac{2a}{|t|} \\ c_2 = 4b \end{array} \right.$$
$$a \propto t = \frac{T - T_c}{T_c}$$

$$\Rightarrow h = \pm c_1 |t| m + c_2 m^3$$

$$\left. \begin{array}{l} \hookrightarrow '+' \text{ if } a > 0 \\ \quad \quad \quad '-' \text{ if } a < 0 \end{array} \right\} \begin{array}{l} \text{This fixes} \\ c_1 > 0. \end{array}$$

$$\Rightarrow \frac{h}{|t|^{3/2}} = \pm \frac{c_1 m}{|t|^{1/2}} + \frac{c_2 m^3}{|t|^{3/2}}$$

$$\Rightarrow \frac{h}{|z|^{3/2}} = \frac{m}{|z|^{1/2}} \left[\pm C_1 + \frac{C_2 m^2}{|z|} \right]$$

$$\Rightarrow \frac{h}{|z|^{3/2}} = \frac{m}{|z|^{1/2}} \left[\pm C_1 + C_2 \left(\frac{m}{|z|^{1/2}} \right)^2 \right]$$

Let $C_1 = \frac{1}{BD}$ & $C_2 = \frac{1}{B^3 D}$

$$\Rightarrow \frac{h}{|z|^{3/2}} = \frac{m}{|z|^{1/2}} \left[\pm \frac{1}{BD} + \frac{1}{B^3 D} \left(\frac{m}{|z|^{1/2}} \right)^2 \right]$$

$$\Rightarrow \frac{h}{|z|^{3/2}} = \frac{m}{|z|^{1/2} BD} \left[\pm 1 + \frac{1}{B^2} \left(\frac{m}{|z|^{1/2}} \right)^2 \right]$$

$$\Rightarrow D \frac{h}{|z|^{3/2}} = \frac{m}{B |z|^{1/2}} \left[\pm 1 + \left(\frac{m}{B |z|^{1/2}} \right)^2 \right]$$

$$\Rightarrow \frac{D h}{|z|^{3/2}} = F_{\pm}^{(M.F.)} \left(\frac{m}{B |z|^{1/2}} \right)$$

$F_{\pm}^{(M.F.)}$ "Some function"

$$\Rightarrow \frac{m}{B |z|^{1/2}} = W_{\pm}^{(M.F.)} \left(\frac{D h}{|z|^{3/2}} \right)$$

$W_{\pm}^{(M.F.)}$ is inverse function of $F_{\pm}^{(M.F.)}$.

where '+' $\Rightarrow$ for $t > 0$
'-' $\Rightarrow$ for $t < 0$

For any other system in the vicinity of critical phase transition we can always write it in the form,

$$\frac{m}{B|t|^\beta} = W_\pm \left(\frac{Dh}{|t|^\Delta} \right)$$

This is an eqⁿ of state with two scaled variables. Here ' Δ ' is called the gap exponent which depends on other critical exponents.

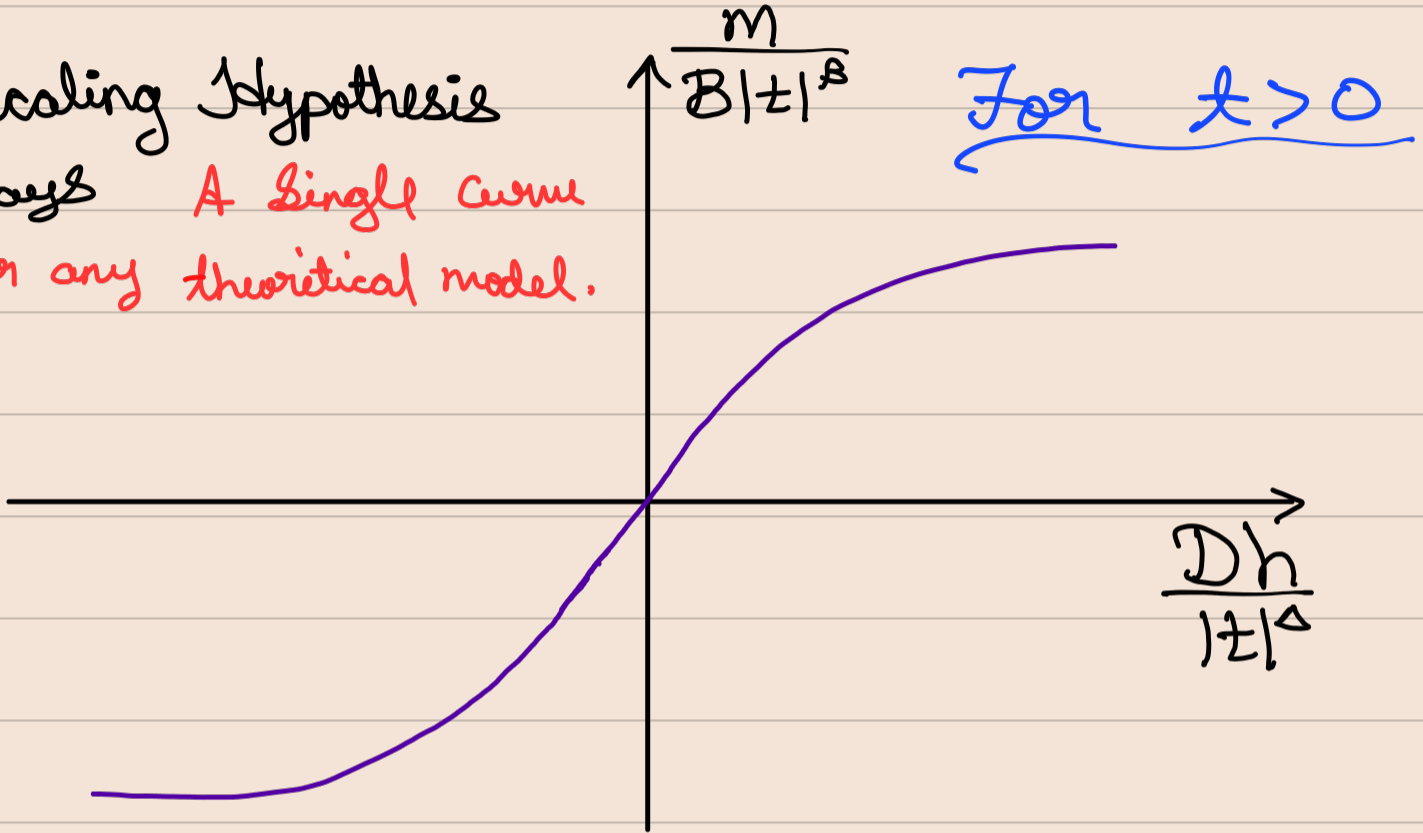
According to Scaling Hypothesis even for completely different systems with very different T_c values, still they behave exactly the same way near the critical phase transition.

i.e. the form of $W_\pm$ will be similar various kinds of system, but only change in the values of B or Δ .

Hence if we plot the graph of $\frac{m}{B|\pm|^B}$ vs $\frac{Dh}{|\pm|^A}$, for all kinds of system then they all will lie in the same master curve.
($B \neq D \rightarrow$ depend on Material Properties.)

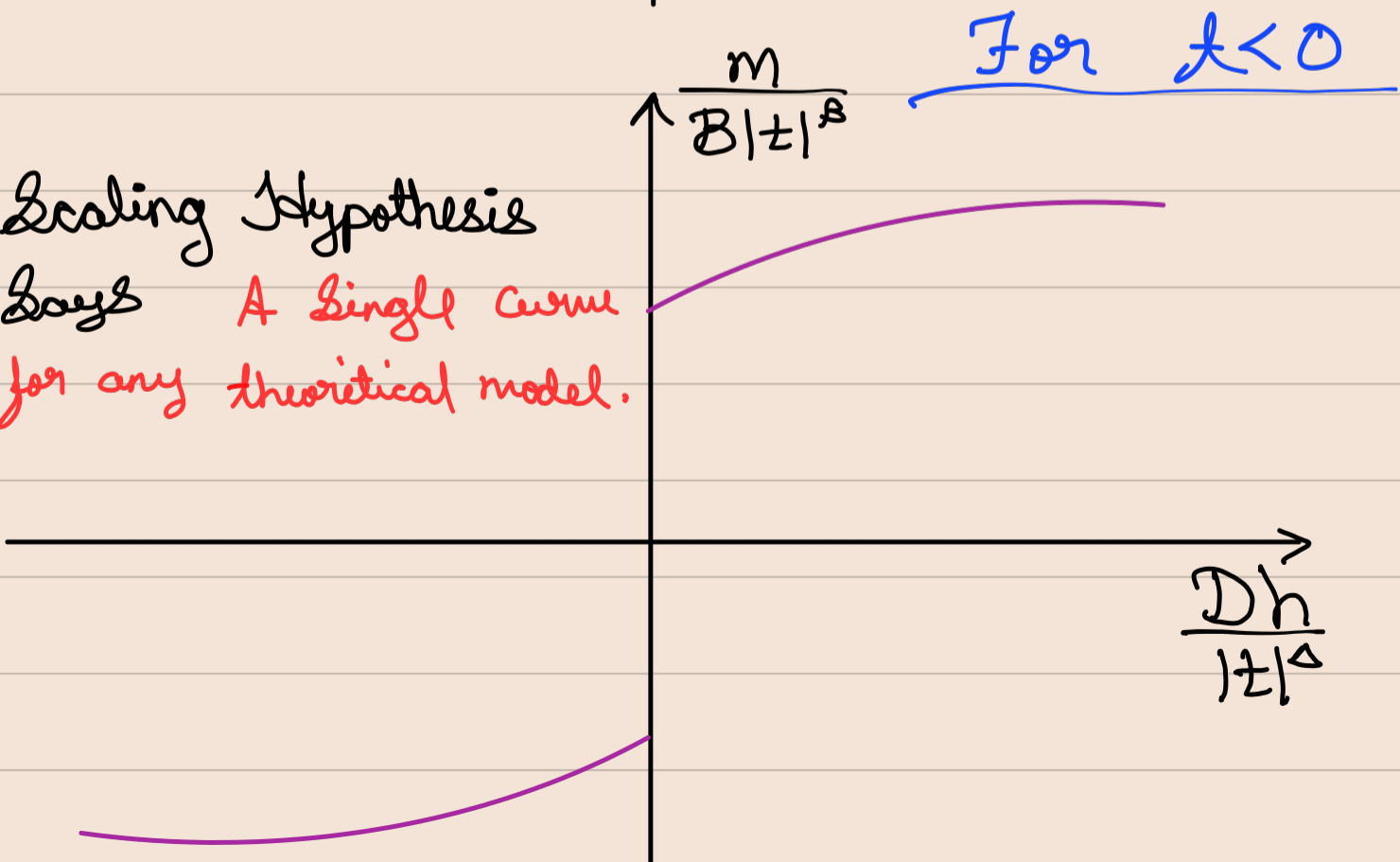
Scaling Hypothesis

Says A single curve for any theoretical model.



Scaling Hypothesis

Says A single curve for any theoretical model.



Consider the Scaled eqⁿ of state.

$$\frac{m}{|t|^\beta} = B W_\pm \left(\frac{Dh}{|t|^\Delta} \right)$$

$$\Rightarrow m = |t|^\beta B W_\pm \left(\frac{Dh}{|t|^\Delta} \right)$$

Since $m \propto h^{1/3}$ at $t=0$.

$\Rightarrow W_\pm$ should be such that as $|t| \rightarrow 0$, $W_\pm$ should vary as a power law w.r.t its variables such that it can cancel the remanent 't' dependencies.

$$\Rightarrow W_\pm \left(\frac{Dh}{|t|^\Delta} \right) \sim \left(\frac{Dh}{|t|^\Delta} \right)^{\beta/\Delta} \quad (\text{as } |t| \rightarrow 0)$$

$$\Rightarrow m = |t|^\beta B \frac{(Dh)^{\beta/\Delta}}{|t|^\beta}$$

$$\Rightarrow m \propto h^{\beta/\Delta} \quad \underline{\text{as } |t| \rightarrow 0}$$

But we know $m \propto h^{1/2}$ as $|t| \rightarrow 0$
 $\Rightarrow \beta = \frac{\Delta}{B}$ or $\Delta = \beta B$

Now we can relate other exponent using the scaling form eqⁿ of state given by,
$$m = |t|^\beta B W_\pm \left(\frac{Dh}{|t|^\Delta} \right)$$

$$\Rightarrow \frac{\partial m}{\partial h} = B |t|^\beta \frac{D}{|t|^\Delta} W'_\pm \left(\frac{Dh}{|t|^\Delta} \right)$$

$$\Rightarrow \chi = |t|^{\beta-\Delta} B D W'_\pm \left(\frac{Dh}{|t|^\Delta} \right)$$

Now at $h=0$ and as $t \rightarrow 0$ we get
$$\chi_{h=0} \propto |t|^{\beta-\Delta} = |t|^{B(1-\beta)}$$

(as $\Delta = \beta B$)

But $\chi_{h=0} \propto |t|^{-\gamma}$ by the definition of susceptibility exponent

$$\Rightarrow \boxed{\gamma = B(\beta - 1)} \Rightarrow \text{Widom's law}$$

Scale Invariance

According to scaling hypothesis, ' ξ ' is the only characteristic length when $|t| \rightarrow 0$. Since, ' ξ ' will diverge at the critical temperature, then the system's characteristic length is undefined. Thus the system remains invariant under the change of length scale.

Scale invariance means, if a part of the system is magnified until it looks as large as original system, then one cannot tell the difference b/w the original & the magnified part of the system. This is valid even when another small part of initial small part is again magnified to original size, so on...

In a Scale invariant system all the thermodynamic functions are homogenous.

Example: Consider the Correlation function $\Gamma(r) \propto \frac{e^{-r/\xi_s}}{r^{d-2+\eta}}$

at critical point $\xi_s \rightarrow \infty$

$$\Rightarrow \Gamma(r) \propto r^{-(d-2-\eta)}$$

If $r \rightarrow r' = \frac{r}{b}$, then, $r = br'$

$$\Rightarrow \Gamma(r') = b^{-(d-2-\eta)} (r')^{-(d-2-\eta)}$$

$\Rightarrow \Gamma(r)$ is homogenous function of order $-(d-2-\eta)$.

Thus the correlation function is homogenous only at critical point where $\xi_s \rightarrow \infty$.

Scaling form for free energy

The dimensionless free energy per unit volume is

$$f = \frac{F}{k_B T_c V}$$

$[f] = (\text{length})^{-d}$, Thus in a scale invariant system,

$$\Rightarrow \underline{f(h, t) = b^{-d} f(b^{D_h} h, b^{D_t} t)}$$

D_t & D_h are scaling exponents needed to make 'f' homogenous.

now $\xi_s \sim t^{-\nu} \Rightarrow t \sim \xi_s^{-1/\nu}$

$$\Rightarrow f(h, t) = b^{-d} f[b^{D_h} h, b^{D_t} \xi_s^{-1/\nu}]$$

But $\xi_s \Rightarrow$ length scale. So if $\xi_s^{-1/\nu} \rightarrow b^{D_t} \xi_s^{-1/\nu}$ then 'f' has to get another additional factor to compensate the change.

Thus if $D_t = \frac{1}{2}$ then it no change occurs & hence this makes $f(h, t)$ homogenous.

If we choose $b^{D_t} t = 1$
 (choosing the scale) $\Leftrightarrow b = t^{-1/D_t}$

$$\begin{aligned} \Rightarrow f(h, t) &= b^{-d} f[b^{D_h} h, 1] \\ &= b^{-d} \mathcal{F}_{\pm}(b^{D_h} h) \\ &= t^{d/D_t} \mathcal{F}_{\pm}(t^{-D_h/D_t} h) \end{aligned}$$

$$\Rightarrow f(h, t) = t^{d/D_t} \mathcal{F}_{\pm}\left[\frac{h}{t^{D_h/D_t}}\right]$$

This is Widom's Scaling form.

where $\mathcal{F}_{\pm}(x) = f(h, \pm 1)$. We know

that $f \sim t^{2-\alpha} \Rightarrow t^{2-\alpha} \sim t^{d/D_t}$

$$\Rightarrow \boxed{2-\alpha = \frac{d}{D_t}}$$

$$\neq m = \frac{\partial f}{\partial h} = t^{d/D_t} t^{-D_h/D_t} F_{\pm}'\left(\frac{h}{t^{D_h/D_t}}\right)$$

$$\Rightarrow \text{as } h \rightarrow 0 \quad m \sim t^{\beta}$$

$$\Rightarrow t^{\beta} = t^{\left(\frac{d-D_h}{D_t}\right)}$$

$$\Rightarrow \boxed{\beta = \frac{d-D_h}{D_t}}$$

And we previously chose $\boxed{\gamma = \frac{1}{D_t}}$

$$\text{Now } m = [t]^{\beta} F_{\pm}'\left[\frac{h}{t^{\Delta}}\right], \quad \Delta = \frac{D_h}{D_t}$$

By same analogy as before we get

$$\beta\gamma = \Delta = \frac{D_h}{D_t}$$

$$\Rightarrow \boxed{\gamma = \frac{D_h}{d-D_h}}$$

$$\chi = \frac{\partial m}{\partial h} = |t|^{B-\Delta} F_{\pm}''\left(\frac{h}{t^{\Delta}}\right)$$

$$\Rightarrow -\gamma = B - \Delta = B - \frac{D_h}{D_t}$$

$$\Rightarrow \gamma = \frac{D_h}{D_t} - \left(\frac{d - D_h}{D_t}\right)$$

$$\Rightarrow \boxed{\gamma = \frac{2D_h - d}{D_t}}$$

By Fisher's scaling law,

$$\eta = 2 - \frac{\gamma}{\nu} = 2 - (2D_h - d)$$

$$\Rightarrow \boxed{\eta = 2 + d - 2D_h}$$

Thus just by knowing how the free energy changes with the scale (i.e. by knowing D_t & D_h) we can define all other critical exponents. We can also derive all the scaling laws using these relations.

The scaling form for the singular part of free energy

$$f(h, t) = t^{2-\alpha} \mathcal{F}_{\pm} \left[\frac{h}{t^{\beta\gamma}} \right]$$

This is Widom's scaling form.

$$\frac{\partial f}{\partial h} \underset{m}{=} t^{2-\alpha} \left(\frac{1}{t^{\beta\gamma}} \right) \mathcal{F}'_{\pm} \left(\frac{h}{t^{\beta\gamma}} \right)$$

$$\Rightarrow m = t^{(2-\alpha-\beta\gamma)} \mathcal{F}'_{\pm} \left(\frac{h}{t^{\beta\gamma}} \right)$$

at $h=0$ we know $m \sim |t|^{\beta}$

$$\Rightarrow \beta = 2 - \alpha - \beta\gamma$$

$$\text{But } \gamma = \beta(\gamma - 1)$$

$$\Rightarrow \beta\gamma = \gamma + \beta$$

$$\Rightarrow \beta = 2 - \alpha - \gamma - \beta$$

$$\Rightarrow \underline{\alpha + 2\beta + \gamma = 2}$$

$\hookrightarrow$ We recovered Rushbrooke law.

Scaling form for Correlation Function

When $h=0$, $G(r)$ is given by the Fisher's scaling form

$$G(r) \sim \frac{Y_{\pm}(r/\xi_s)}{r^{d-2+\eta}}$$

Now $\chi = \beta \sum_r G(r)$
(per spin) $\leftarrow$

$$= \beta \int d^d r \left[\frac{Y_{\pm}(r/\xi_s)}{r^{d-2+\eta}} \right]$$

Put $x = r/\xi_s \Rightarrow d^d r = \xi_s^d d^d x$

$$\chi = \beta \frac{\xi_s^d}{\xi_s^{d-2+\eta}} \left[\int d^d x \frac{Y_{\pm}(x)}{x^{d-2+\eta}} \right]$$

$$\Rightarrow \chi \propto \frac{\xi_s^{2-\eta}}{T} = \frac{\xi_s^{2-\eta}}{T_c(1 \pm t)}$$

when $T \rightarrow T_c$ $t \rightarrow 0$, $\frac{1}{1 \pm t} \rightarrow 1$.

$$\chi \sim |t|^{-\gamma} \text{ \& } \xi_s \sim |t|^{-\nu}$$

$$\Rightarrow |t|^{-\gamma} \sim |t|^{-\nu(2-\eta)}$$

$$\Rightarrow \boxed{\gamma = \nu(2-\eta)} \rightarrow \text{Fisher's Scaling law}$$