

17/02/25

Lecture 12

Fluid - Magnet Analogy

The classical canonical partition of an N -particle system whose hamiltonian is $\mathcal{H}(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N, \vec{p}_1, \vec{p}_2, \dots, \vec{p}_N)$ is given by,

$$Q(N, V, T) = \frac{\int d^3N \vec{r} d^3N \vec{p} e^{-\beta \mathcal{H}}}{N! h^{3N}}$$

$h \rightarrow$ Plank's constant

$N! \Rightarrow$ As the particles are indistinguishable

The classical Hamiltonian for N -particle system can be written as

$$\mathcal{H} = \frac{1}{2m} \left[\sum_{i=1}^N (p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2) \right] + U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$$

where $\vec{p}_i = (p_{x_i}, p_{y_i}, p_{z_i}) \Rightarrow$ Momentum vector of i^{th} particle

& $\vec{r}_i = (x_i, y_i, z_i) \Rightarrow$ Position vector of i^{th} particle

Here $U(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)$ is the Potential energy of the interacting particle system.

The momentum integration is simple as we get $3N$ -gaussian integrals to integrate.

$$Q(N, V, T) = \frac{1}{N! h^{3N}} \underbrace{\int d^{3N} p e^{-\beta(K.E)}}_{I_K} \int d^{3N} r e^{-\beta U}$$

$$I_K = \int d^{3N} p e^{-\frac{\beta}{2m} \sum_{i=1}^N (p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2)}$$

$$= \prod_{i=1}^N \left[\int d^3 p_i e^{-\frac{\beta}{2m} (p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2)} \right]$$

$$= \prod_{i=1}^N \left[\int_{-\infty}^{\infty} dp_{x_i} e^{-\frac{\beta}{2m} p_{x_i}^2} \int_{-\infty}^{\infty} dp_{y_i} e^{-\frac{\beta}{2m} p_{y_i}^2} \int_{-\infty}^{\infty} dp_{z_i} e^{-\frac{\beta}{2m} p_{z_i}^2} \right]$$

$$= \prod_{i=1}^N \left(\sqrt{\frac{2m}{\beta}} \pi \right)^3 = \left[\sqrt{2m\pi K_B T} \right]^{3N}$$

$$\Rightarrow Q(N, V, T) = \frac{\left(\sqrt{2m\pi k_B T} \right)^{3N}}{N! h^{3N}} \int d^{3N} \mathbf{r} e^{-\beta U}$$

$$= \frac{1}{N!} \left[\frac{\sqrt{2m\pi k_B T}}{h} \right]^{3N} \int d^{3N} \mathbf{r} e^{-\beta U}$$

we define thermal de Broglie wavelength as

$$\lambda = \frac{h}{\sqrt{2m\pi k_B T}}$$

The integral $Z_N^{(c)}(V, T) = \int d^{3N} \mathbf{r} e^{-\beta U}$ is

the configurational integral which is System Specific.

$$\Rightarrow Q(N, V, T) = \frac{Z_N^{(c)}(V, T)}{\lambda^{3N} N!}$$

The potential energy is a function of all the particle's position & can be approximated to pair-interactions.

Let $V_{ij}(\vec{r}_i, \vec{r}_j)$ be the interaction between the particles 'i' & 'j' whose position vectors are $\vec{r}_i$ & $\vec{r}_j$ respectively.

Then to a good approximation we can write $U(\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots, \vec{r}_N)$ as a sum of all possible pair interactions.

$$\Rightarrow U = \begin{aligned} &V_{12} + V_{13} + V_{14} + \dots + V_{1N} \\ &\quad + V_{23} + V_{24} + \dots + V_{2N} \\ &\quad \quad + V_{34} + \dots + V_{3N} \\ &\quad \quad \quad + \dots + V_{N-1,N} \end{aligned} \left. \begin{array}{l} \frac{N(N-1)}{2} \\ \text{terms} \\ \text{are} \\ \text{present} \end{array} \right\}$$

$$\Rightarrow U = \sum_{i < j} V_{ij} = \frac{1}{2} \sum_{i \neq j} V_{ij}$$

The configurational can be written as,

$$Z_N = \int d^{3N} \mathcal{r} e^{-\beta \sum_{i < j} V_{ij}}$$

$$\Rightarrow Z_N = \int d^{3N} \mathcal{r} \prod_{i < j} e^{-\beta V_{ij}}$$

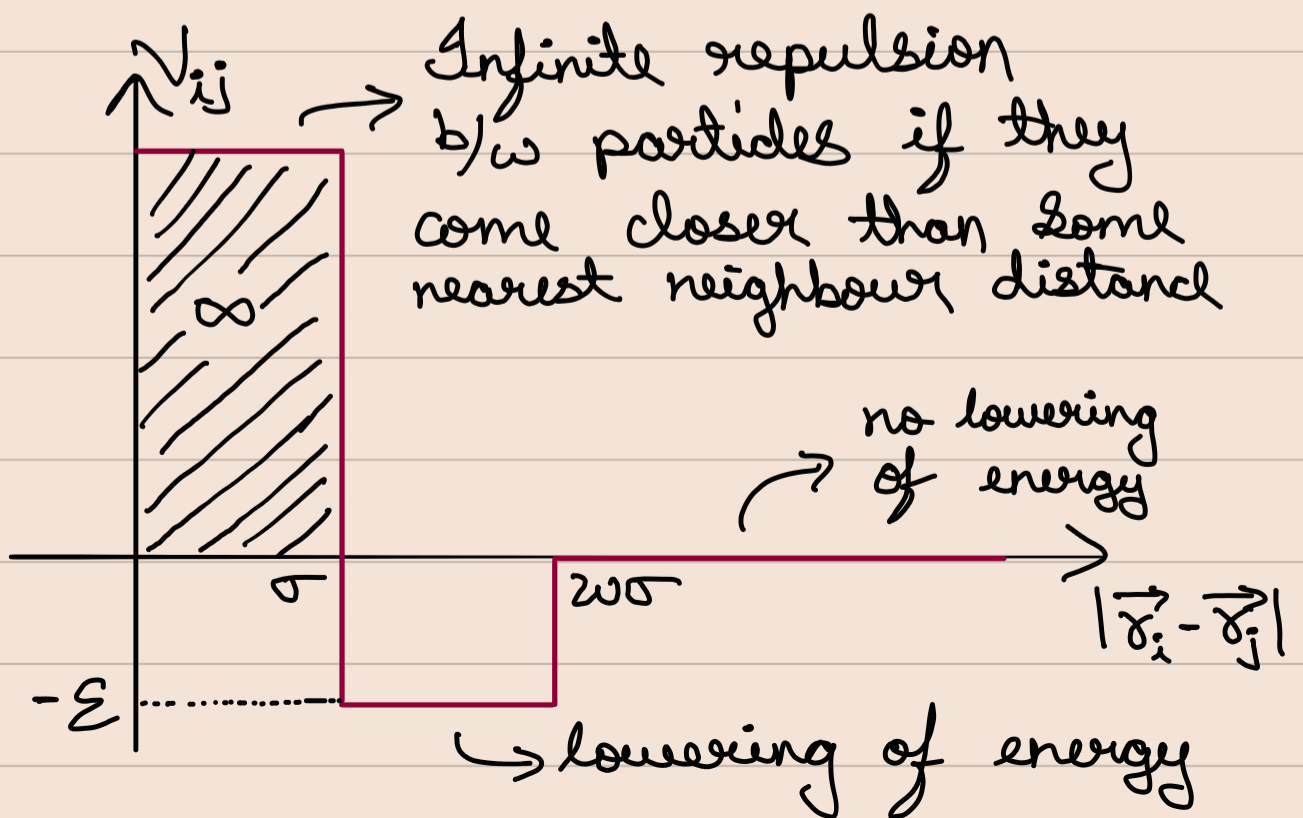
Lattice Gas Model

A lattice gas is a collection of atoms whose positions can take only discrete values which form a lattice.

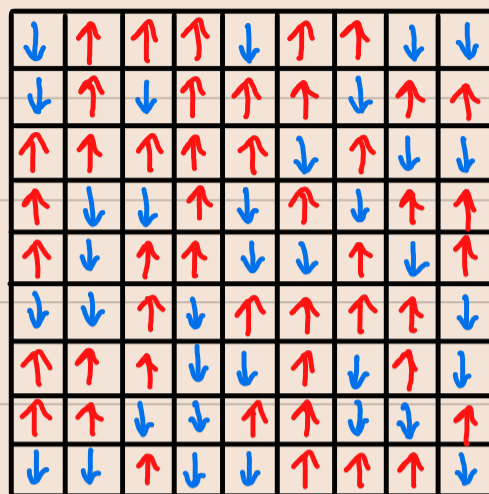
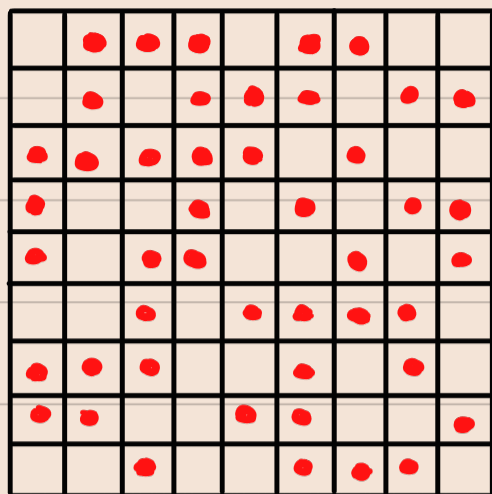
Each lattice site can be occupied by a maximum of one atom per site.

Here we neglect the kinetic energy calculations & focus only on interaction energy between the particles.

We can model the interaction energy b/w particles i & j using V_{ij} which is a function of $|\vec{r}_i - \vec{r}_j|$.



The 2-D lattice gas system with a random configuration is shown below.



We can show a very good analogy between lattice gas & Ising ferromagnet.

There is lowering of energy when two nearby spins are along same direction, similarly in lattice gas, we get lowering of energy when particles are present in nearby lattice positions.

The configuration integral for lattice gas is

$$Z_N^{(c)} = N! (a_0)^{3N} \sum_{\{n_k\}} \delta(N - \sum_k n_k) e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} V_{ij} n_i n_j}$$

↪ To remove double counting

$i, j = \{1, 2, 3, 4, \dots, N_s\}$ $N_s \Rightarrow$ No. of lattice sites

$V_{ij} \Rightarrow$ Lattice gas potential, $V_{ij} = \begin{cases} \infty, & j = i \\ -\epsilon, & j \in nn(i) \\ 0, & j \notin nn(i) \end{cases}$
 $nn(i) \Rightarrow$ nearest neighbour of 'i'

where $n_k \Rightarrow$ occupation of k^{th} site
 $n_k = 0$ for no particle & $n_k = 1$
 if there is a particle at k^{th} site.

' a_0 ' $\Rightarrow$ lattice spacing. The total
 number of particles is given by

$$N = \sum_k n_k \quad k = \{1, 2, 3, \dots, N_s\}$$

$N = \sum_k n_k$ is a constraint on the configuration
 $\{n_k\}$ if N is fixed in the canonical
 ensemble. Hence the $\delta(N - \sum_k n_k)$ term.

In a system of ' N ' particles, there are
 $N!$ ways of having the same number of
 total occupancy of N . Thus there is a
 factor of $N!$. Now we can associate spins
 for each lattice site where it is up spin
 if particle is present & down spin if
 particle is absent.

$$\Rightarrow S_k = 2n_k - 1 \quad \text{or} \quad n_k = \frac{S_k + 1}{2}$$

$$\left. \begin{array}{l} n_k = 1 \Rightarrow S_k = +1 \\ n_k = 0 \Rightarrow S_k = -1 \end{array} \right\} \text{Ising spins}$$

$$\Rightarrow n_i n_j = \frac{1}{4} (S_i + 1)(1 + S_j) = \frac{(1 + S_i + S_j + S_i S_j)}{4}$$

$$\Rightarrow \sum_{\langle i,j \rangle} V_{ij} n_i n_j = \sum_{\langle i,j \rangle} \frac{V_{ij}}{4} (1 + S_i + S_j + S_i S_j)$$

$$= \frac{1}{4} \left[\sum_{\langle i,j \rangle} V_{ij} + \sum_{\langle i,j \rangle} V_{ij} S_i + \sum_{\langle i,j \rangle} V_{ij} S_j + \sum_{\langle i,j \rangle} V_{ij} S_i S_j \right]$$

$$\sum_{\langle i,j \rangle} = \sum_i \sum_{j \in n(i)} = \sum_j \sum_{i \in n(j)}$$

$n(i) \Rightarrow$ nearest neighbours of 'i'

And $V_{ij} = V_{ji}$ as $V_{ij} \propto \underbrace{|\vec{r}_i - \vec{r}_j|}_{\text{Symmetric}}$

$$\sum_{\langle i,j \rangle} V_{ij} S_i = \sum_i \sum_{j \in n(i)} V_{ij} S_i = \sum_i S_i \sum_{j \in n(i)} V_{ij}$$

defining $\sum_{j \in n(i)} V_{ij} = \mathcal{U} \Rightarrow$ (independent of site)

$$\text{we get } \sum_{\langle i,j \rangle} V_{ij} S_i = \sum_{\langle i,j \rangle} V_{ij} S_j = \mathcal{U} \sum_i S_i$$

$$\& \sum_{\langle i,j \rangle} V_{ij} = \sum_i \mathcal{U} = \mathcal{U} \sum_i (1) = N_s \mathcal{U}$$

$N_s \Rightarrow$ No: of sites in the lattice

$$\Rightarrow \sum_{\langle ij \rangle} V_{ij} n_i n_j = \sum_{\langle ij \rangle} \frac{V_{ij}}{4} (1 + S_i + S_j + S_i S_j)$$

$$= \frac{1}{4} \left[\sum_{\langle ij \rangle} V_{ij} + \sum_{\langle ij \rangle} V_{ij} S_i + \sum_{\langle ij \rangle} V_{ij} S_j + \sum_{\langle ij \rangle} V_{ij} S_i S_j \right]$$

$$\Rightarrow \sum_{\langle ij \rangle} V_{ij} n_i n_j = \frac{1}{4} \left[N_s u + 2u \sum_i S_i + \sum_{\langle ij \rangle} V_{ij} S_i S_j \right]$$

We know $N_{\uparrow} + N_{\downarrow} = N_s$ & $N_{\uparrow} = N$

we define $M = N_{\uparrow} - N_{\downarrow}$

$$\Rightarrow N_{\downarrow} = N_{\uparrow} - M$$

$$\Rightarrow N_{\uparrow} + N_{\uparrow} - M = N_s$$

$$\Rightarrow N_{\uparrow} = N = \frac{(N_s + M)}{2}$$

$$\Rightarrow M = 2N - N_s$$

$$N = \sum_{\mathbf{k}} n_{\mathbf{k}}$$

$$\Rightarrow M = 2 \left(\sum_{\mathbf{k}} n_{\mathbf{k}} \right) - N_s$$

$$\Rightarrow M = \sum_{\mathbf{k}} (1 + S_{\mathbf{k}}) - N_s = \sum_{\mathbf{k}} S_{\mathbf{k}}$$

($\sum_{\mathbf{k}} (1) = N_s$)

$\Rightarrow$ The constraint is now $\sum_{\mathbf{k}} S_{\mathbf{k}} = M$.

$\Rightarrow$ M is analogous to net magnetization & has usual defⁿ of $\sum_{\mathbf{k}} S_{\mathbf{k}}$.

$$\Rightarrow Z_N^{(c)} = N! (a_0)^{3N} \sum_{\{n_k\}} \delta(N - \sum_k n_k) e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} V_{ij} n_i n_j}$$

$$= N! (a_0)^{3N} \sum_{\{s_i\}} \delta(M - \sum_k s_k) e^{-\beta/2 \left(\frac{1}{4} \left[N_3 u + 2u \sum_i s_i + \sum_{\langle ij \rangle} V_{ij} s_i s_j \right] \right)}$$

$$= N! (a_0)^{3N} \sum_{\{s_i\}} \delta(M - \sum_k s_k) e^{-\beta \left[\frac{N_3 u}{8} - \frac{uM}{4} - \frac{1}{2} \sum_{\langle ij \rangle} \left(\frac{1}{4} V_{ij} \right) s_i s_j \right]}$$

let $J_{ij} = \frac{V_{ij}}{4}$

$$\Rightarrow Z_N^{(c)} = N! (a_0)^{3N} e^{-\left[\frac{\beta N_3 u}{8} + \frac{\beta u M}{4} \right]} \sum_{\{s_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} s_i s_j} \times \delta(M - \sum_k s_k)$$

Now the Canonical partition function is given by, $Q_N = \frac{Z_N^{(c)}(V, T)}{\lambda^{3N} N!}$
 $\lambda \rightarrow$ due to K.E

$$\Rightarrow Q_N = \left(\frac{a_0}{\lambda} \right)^{3N} e^{-\left[\frac{\beta N_3 u}{8} + \frac{\beta u M}{4} \right]} \sum_{\{s_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} s_i s_j} \times \delta(M - \sum_k s_k)$$

The grand canonical ensemble is given by,

$$Z_{g.c.}^{(fluid)} = \sum_{N=0}^{\infty} e^{\beta \mu N} Q_N$$

This removes the constraint.
 $\mu \Rightarrow$ chemical potential

Use $N = \frac{1}{2}(N_s + M)$

$N = 0 \Rightarrow M = -N_s \quad \& \quad N = N_s \Rightarrow M = N_s$

$\Rightarrow$ 'M' varies b/w $-N_s$ to N_s

$$Z_{g.c}^{(fluid)} = \sum_{M=-N_s}^{N_s} e^{\frac{\beta \mu}{2}(N_s + M)} Q_M$$

$$Q_M = \left(\frac{a_0}{\lambda}\right)^{\frac{3}{2}(N_s + M)} e^{-\left[\frac{\beta N_s u}{8} + \frac{\beta 2\mu M}{4}\right]} \sum_{\{S_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j} \delta\left(M - \sum_k S_k\right)$$

$$Z_{g.c}^{(fluid)} = \sum_{M=-N_s}^{N_s} \sum_{\{S_i\}} \delta\left(M - \sum_k S_k\right) e^{\frac{\beta \mu}{2}(N_s + M)} \left(\frac{a_0}{\lambda}\right)^{\frac{3}{2}(N_s + M)} \times e^{-\left[\frac{\beta N_s u}{8} + \frac{\beta 2\mu M}{4}\right]} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j}$$

Exchange

$$= \sum_{\{S_i\}} e^{\frac{\beta \mu}{2}(N_s + M)} \left(\frac{a_0}{\lambda}\right)^{\frac{3}{2}(N_s + M)} e^{-\left[\frac{\beta N_s u}{8} + \frac{\beta 2\mu M}{4}\right]} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j}$$

$S_i = \{\pm 1\}$ (where $M = \sum_k S_k$, by using δ -function)

$$\Rightarrow Z_{g.c}^{(fluid)} = e^{\frac{\beta \mu}{2}(N_s + M)} \left(\frac{a_0}{\lambda}\right)^{\frac{3}{2}(N_s + M)} e^{-\left[\frac{\beta N_s u}{8} + \frac{\beta 2\mu M}{4}\right]} \times \sum_{\{S_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j}$$

$$\Rightarrow Z_{g.c}^{(fluid)} = \left[\frac{e^{\beta \mu} a_0^3}{\lambda^3} \right]^{\frac{N_s + M}{2}} e^{-\left[\frac{\beta N_s u}{8} + \frac{\beta 2\mu M}{4}\right]} \times \sum_{\{S_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j}$$

$$\Rightarrow Z_{g.c.}^{(fluid)} = \left[\frac{e^{\beta \mu} a_0^3}{\lambda^3} \right]^{\frac{N_s}{2}} e^{-\frac{\beta N_s u}{8}} \left[\frac{e^{\beta \mu} a_0^3}{\lambda^3} \right]^{\frac{M}{2}} e^{-\left[\frac{\beta u M}{4} \right]}$$

$$\times \sum_{\{S_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j}$$

$$\text{use } \left(\frac{a_0}{\lambda} \right)^3 = e^{3 \ln \left(\frac{a_0}{\lambda} \right)} = e^{\beta [-3 k_B T \ln \left(\frac{\lambda}{a_0} \right)]}$$

$$\Rightarrow Z_{g.c.}^{(fluid)} = e^{\beta \left[\frac{\mu}{2} - \frac{3}{2} k_B T \ln \left(\frac{\lambda}{a_0} \right) - \frac{u}{8} \right] N_s}$$

$$\times e^{\beta \left[\frac{\mu}{2} - \frac{3}{2} k_B T \ln \left(\frac{\lambda}{a_0} \right) - \frac{u}{4} \right] M}$$

$$\times \sum_{\{S_i\}} e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j}$$

$$\text{Let } h = \frac{\mu}{2} - \frac{3}{2} k_B T \ln \left(\frac{\lambda}{a_0} \right) - \frac{u}{4}$$

$$\& \mathcal{H}_{mag.}^{int} = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j$$

$$\Rightarrow Z_{g.c.}^{(fluid)} = e^{\beta \left(h + \frac{u}{8} \right) N_s} \sum_{\{S_i\}} e^{\beta h M} e^{-\beta \mathcal{H}_{mag.}^{int.}}$$

$$\Rightarrow Z_{g.c.}^{(fluid)} = e^{\beta \left(h + \frac{u}{8} \right) N_s} \sum_{\{S_i\}} e^{-\beta \left(\mathcal{H}_{mag.}^{int.} - h M \right)}$$

$$(M = \sum_i S_i)$$

Now $H = H_{\text{mag.}}^{\text{int.}} - hM$ is analogous to ising hamiltonian in an external field.

Thus we get the gibbs partition function analogue here as

$$Z_{\text{mag.}}(h, B) = \sum_{\{S_i\}} e^{-B(H_{\text{mag.}}^{\text{int.}} - hM)}$$

$$\Rightarrow Z_{g-c}^{(\text{fluid.})} = e^{B(h + \frac{u}{8})N_s} Z_{\text{mag.}}$$

Thus if we are given the canonical partition function of lattice gas as,

$$Q_N = \frac{(a_0)^{3N}}{\lambda^{3N}} \sum_{\{n_k\}} \delta(N - \sum_k n_k) e^{-\frac{B}{2} \sum_{\langle ij \rangle} V_{ij} n_i n_j}$$

$n_i \Rightarrow$ occupation number of i^{th} lattice site

$(a_0)^3 \Rightarrow$ Volume of a unit cell

$\lambda \Rightarrow$ Thermal de Broglie wavelength

$N \Rightarrow$ Total no. of particles in lattice

$V_{ij} \Rightarrow$ Lattice gas potential

$$V_{ij} = \begin{cases} \infty, & j=i \\ -\epsilon, & j \in n(i) \\ 0, & j \notin n(i) \end{cases}$$

Then we can write an equivalent using ferromagnetic gibbs partition function of the form

$$Z_{\text{mag.}}(h, \beta) = \sum_{\{S_i\}} e^{-\beta(H_{\text{mag.}}^{\text{int.}} - hM)}$$

which relates to the grand partition function of lattice gas by,

$$Z_{\text{g.c.}}^{(\text{fluid.})} = e^{\beta(h + \frac{\mathcal{U}}{8})N_s} Z_{\text{mag.}}$$

where,

$$h = \frac{\mu}{2} - \frac{3}{2} k_B T \ln\left(\frac{\lambda}{a_0}\right) - \frac{\mathcal{U}}{4}$$

$$\mathcal{U} = \sum_{j \in n(i)} V_{ij} \quad i, j = \{1, 2, 3, \dots, N_s\}$$

$$H_{\text{mag.}}^{\text{int.}} = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} S_i S_j$$

$$J_{ij} = \frac{V_{ij}}{4} \Rightarrow J = \frac{\mathcal{U}}{4}$$

$$S_i = 2n_i - 1$$

$$M = \sum_i S_i = N_{\uparrow} - N_{\downarrow} = 2N - N_s$$

$$Z_{\text{g.c.}}^{\text{fluid}} = e^{-\beta \Lambda}$$

$\Lambda \Rightarrow$ grand Potential = $-PV$

$P \Rightarrow$ Pressure

$V \Rightarrow$ Volume of Container

Volume of lattice gas = $V = N_s (a_0)^3$
 $a_0^3 \Rightarrow$ Volume per site

$$\Rightarrow Z_{g.c.}^{(fluid)} = e^{\beta P N_s a_0^3}$$

We can get the eqⁿ of state of lattice gas easily as follows.

$$\text{Since, } Z_{g.c.}^{(fluid)} = e^{\beta(h + \frac{u}{8})N_s} Z_{mag.}$$

$$\text{And } Z_{mag.} = e^{-\beta F_{mag.}} = e^{-\beta N_s f_{mag.}}$$

$F_{mag.} \Rightarrow$ Ising magnetic free energy

$f_{mag.} \Rightarrow$ Ising magnetic free energy per site.

$$\Rightarrow e^{\beta P N_s a_0^3} = e^{\beta(h + \frac{u}{8})N_s} e^{-\beta N_s f_{mag.}}$$

$$\Rightarrow P a_0^3 = h + \frac{u}{8} - f_{mag.}$$

We can also get a relation between magnetization per site & volume fraction occupied by the gas particles.

$$M = 2N - N_s$$

$$\Rightarrow \frac{M}{N_s} = \frac{2N}{N_s} - 1$$

$$\Rightarrow \frac{N}{N_s} = \frac{1}{2} \left(\frac{M}{N_s} + 1 \right)$$

Put $N = \rho V$, $\frac{M}{N_s} = m$ & $N_s a_0^3 = V$

$\rho \Rightarrow$ number density

$$\Rightarrow \underbrace{\rho a_0^3}_{\text{Volume fraction occupied by gas particles}} = \frac{1}{2}(m+1)$$

Note: In Ising model the order parameter is magnetization per site, $m = \frac{M}{N_s}$. And here the external magnetic field couples to the order parameter. But in lattice gas, the order parameter is number density (ρ) where the chemical potential couples to order parameter. ' μ ' controls ' ρ ' value.

Phase transition in Lattice gas

Using the knowledge of Ising model we can actually get the phase transition behaviour of lattice gas.

Here the order parameter is the number density (ρ), which is related to the fraction of volume of gas particles occupying the container (lattice).

$$\rho \sim \frac{1}{2}(m+1) \quad (a_0^3 \sim 1)$$

The critical temperature is $T_c = \frac{\mathcal{U}}{4K_B}$

When $T < T_c$ & if $h=0$, then there is a phase transition. i.e. we need $\mu = \mu_c$

Such that
$$\frac{\mu_c}{2} - \frac{3}{2} K_B T \ln\left(\frac{\lambda}{a_0}\right) - \frac{\mathcal{U}}{4} = 0$$

$\mu_c \rightarrow$ critical chemical potential given by

$$\mu_c = 2K_B T_c + 3K_B T \ln\left(\frac{\lambda}{a_0}\right)$$

Analogous to $h=0$ case of Ising model.

The chemical potential needed for a phase transition is a function of temperature.

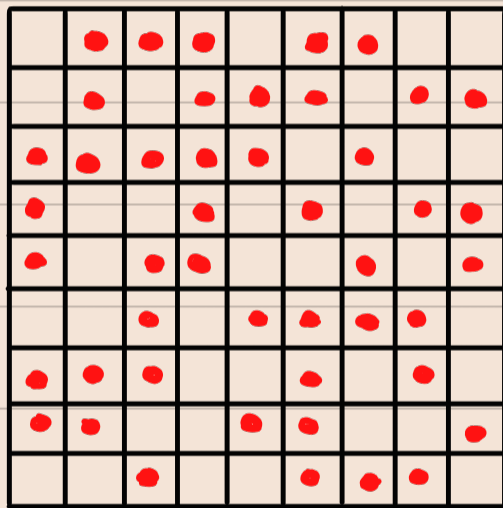
→ Above critical temperature $m=0$,
so, $P \leq \frac{1}{2} \Rightarrow$ low density phase.

→ Below critical temperature $m>0$,
so, $P \geq \frac{1}{2} \Rightarrow$ High density phase.



Super Fluid Model

Here we characterize the Super-fluid using bosonic particles by using the Second-quantization approach. The lattice has various sites where only one boson can occupy it. This is also called the Hard core boson model.



• represents a single boson.

Here Boson refers to the fact that the creation & annihilation operators are bosonic. i.e. they have commutation relations & under particle exchange they are symmetric.

Consider a lattice of N_s number of sites in a d -dimensional space.

Any j^{th} site can have only one boson or no boson. This is like having either an up-spin or a down spin. The occupancy number of the j^{th} site is ' n_j ', which can be either '0' or '1'.

Mapping this to the spin can be done by representing the j^{th} spin state by a spin- $1/2$ kets & corresponding spin operators given by,

$$\hat{S}_j^{(x)} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\hat{S}_j^{(y)} = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\hat{S}_j^{(z)} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{S}_j^{(\pm)} = \hat{S}_j^{(x)} \pm i \hat{S}_j^{(y)}$$

$$\Rightarrow \hat{S}_j^{(+)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{S}_j^{(-)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The up-spin state is $|\uparrow\rangle_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & the down-spin state is $|\downarrow\rangle_j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If we observe, $\hat{S}_j^{(+)}|\uparrow\rangle_j = 0$
 $\hat{S}_j^{(-)}|\downarrow\rangle_j = 0$

But $\hat{S}_j^{(+)}|\downarrow\rangle_j = |\uparrow\rangle_j$ & $\hat{S}_j^{(-)}|\uparrow\rangle_j = |\downarrow\rangle_j$

This gives us $[\hat{S}_j^{(+)}]^2 = [\hat{S}_j^{(-)}]^2 = 0$.

Thus if we consider $|\uparrow\rangle_j$ & $|\downarrow\rangle_j$ as particle number ket by relabelling them as $|1\rangle_j$ & $|0\rangle_j$ respectively & calling them as the particle occupancy kets. $|1\rangle_j \Rightarrow$ one boson present at j^{th} site & $|0\rangle_j \Rightarrow$ no boson present at j^{th} site.

Then we can also consider the ladder operators $\hat{S}_j^{(+)}$ as the creation operator & $\hat{S}_j^{(-)}$ as the annihilation operator also ensuring that there can be only one particle or no particle at any j^{th} site.

Each j^{th} lattice site is associated with its own two dimensional Hilbert Space $\mathbb{H}_j$ spanned by the kets $|1\rangle$ & $|0\rangle$. Then the Hilbert Space associated with the entire lattice is then the direct product given by,

$$\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \mathbb{H}_3 \dots \otimes \mathbb{H}_j \dots \otimes \mathbb{H}_{N_s}$$

Then the basis kets for $\mathbb{H}$ are represented by $|n_1, n_2, n_3, \dots, n_j, \dots, n_{N_s}\rangle$. This basis is also called as the 'Fock-basis' which can be written as the direct product $|n_j\rangle$ kets.

$$|n_1, n_2, \dots, n_j, \dots, n_{N_s}\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \dots \otimes |n_j\rangle \dots \otimes |n_{N_s}\rangle$$

The crucial & the deciding aspect of why these particles are bosons is because of the commutation properties of $\hat{S}_j^{(+)}$ & $\hat{S}_j^{(-)}$ operators.

$$[\hat{S}_j^{(-)}, \hat{S}_k^{(+)}] = [S_j^{(x)} - iS_j^{(y)}, S_k^{(x)} + iS_k^{(y)}]$$

The commutation relation for the angular momentum components is given by $[\hat{S}_j^{(\alpha)}, \hat{S}_k^{(\beta)}] = i\epsilon_{\alpha\beta\gamma} \hat{S}_j^{(\gamma)} \delta_{jk}$ where $\alpha, \beta, \gamma, \dots$ represents the x, y, z components. The above relation implies that the angular momentum operators of different sites will commute and within the same lattice site they have usual commutation relation.

The state where no particles exist in any lattice is $|0\rangle = |0, 0, 0, \dots, 0, \dots, 0\rangle$ when $\hat{S}_j^{(+)}$ acts on the $|0\rangle$ it results in $|0, 0, 0, \dots, 1, \dots, 0\rangle$ '1' at j^{th} site.

$$\rightarrow \hat{S}_j^{(+)} |0, 0, 0, \dots, 0, \dots, 0\rangle = |0, 0, 0, \dots, 1, \dots, 0\rangle$$

$$\& \hat{S}_j^{(-)} |0, 0, 0, \dots, 1, \dots, 0\rangle = |0, 0, 0, \dots, 0, \dots, 0\rangle$$

$$\Rightarrow \langle 0 | \hat{S}_j^{(-)} \hat{S}_j^{(+)} | 0 \rangle = 1 \quad \& \quad \langle 0 | \hat{S}_j^{(+)} \hat{S}_j^{(-)} | 0 \rangle = 0$$

Let $|1_j\rangle = |0, 0, \dots, 1, \dots, 0\rangle$ ('1' at j^{th} site)

$$\langle 1_j | \hat{S}_j^{(+)} \hat{S}_j^{(-)} | 1_j \rangle = 1 \quad \& \quad \langle 1_j | \hat{S}_j^{(-)} \hat{S}_j^{(+)} | 1_j \rangle = 0$$

Thus $\hat{S}_j^{(+)} \hat{S}_j^{(-)}$ acts as the number operator which gives the number of particles present at j^{th} site.

$$\Rightarrow \underline{\hat{n}_j = \hat{S}_j^{(+)} \hat{S}_j^{(-)}} \Rightarrow j^{\text{th}} - \text{Number operator} \quad (\text{it is hermitian})$$

The total boson number is given by $\sum_j \hat{n}_j = \hat{N}$ which gives the number of bosons present in the full lattice.

Observe the relations given by,

$$\langle 0 | \hat{S}_j^{(-)} \hat{S}_j^{(+)} | 0 \rangle = 1 \quad \& \quad \langle 1_j | \hat{S}_j^{(-)} \hat{S}_j^{(+)} | 1_j \rangle = 0$$

$$\Rightarrow \hat{S}_j^{(-)} \hat{S}_j^{(+)} \text{ acts as } (\hat{n}_j + 1)_{\text{mod } 2}$$

$$\text{where } \langle 0 | \hat{S}_j^{(-)} \hat{S}_j^{(+)} | 0 \rangle = (0+1)_{\text{mod } 2} = 1$$

$$\langle 1_j | \hat{S}_j^{(-)} \hat{S}_j^{(+)} | 1_j \rangle = (1+1)_{\text{mod } 2} = 0$$

$$\Rightarrow \hat{S}_j^{(-)} \hat{S}_j^{(+)} - \hat{S}_j^{(+)} \hat{S}_j^{(-)} = \left[(\hat{n}_j + 1)_{\text{mod } 2} - \hat{n}_j \right]_{\text{mod}(2)} = 1$$

$$\Rightarrow \text{Thus } \left[\hat{S}_j^{(-)}, \hat{S}_j^{(+)} \right]_{\text{mod}(2)} = 1 \quad \text{which is}$$

is analogous to $[\hat{a}, \hat{a}^\dagger] = 1$. But not exactly the same.

The above formulation is common for any hard-core Boson system which is used to formulate Super-fluids.

The Hamiltonian for Super-fluid is,

$$\hat{H} = \sum_{j,k} T_{jk} \left[\hat{S}_j^{(+)} \hat{S}_k^{(-)} + \hat{S}_j^{(-)} \hat{S}_k^{(+)} \right] + \sum_j \frac{U_j}{2} \hat{n}_j (\hat{n}_j - 1) - \mu \sum_j \hat{n}_j$$

The interaction term is given by

$$\hat{H}_{\text{int.}}^{(S-f)} = \sum_{j,k} T_{jk} \left[\hat{S}_j^{(+)} \hat{S}_k^{(-)} + \hat{S}_j^{(-)} \hat{S}_k^{(+)} \right]$$

This actually resembles the X-γ model.

Since $\hat{S}_j^{(\pm)} = \hat{S}_j^{(x)} \pm i\hat{S}_j^{(y)}$ where

$$\hat{S}_j^{(+)} \hat{S}_k^{(-)} = \hat{S}_j^{(x)} \hat{S}_k^{(x)} + \hat{S}_j^{(y)} \hat{S}_k^{(y)} + i \left[\hat{S}_j^{(y)} \hat{S}_k^{(x)} - \hat{S}_j^{(x)} \hat{S}_k^{(y)} \right]$$

$$\hat{S}_j^{(-)} \hat{S}_k^{(+)} = \hat{S}_j^{(x)} \hat{S}_k^{(x)} + \hat{S}_j^{(y)} \hat{S}_k^{(y)} + i \left[\hat{S}_j^{(x)} \hat{S}_k^{(y)} - \hat{S}_j^{(y)} \hat{S}_k^{(x)} \right]$$

$$\Rightarrow \hat{S}_j^{(+)} \hat{S}_k^{(-)} + \hat{S}_j^{(-)} \hat{S}_k^{(+)} = 2 \left[\hat{S}_j^{(x)} \hat{S}_k^{(x)} + \hat{S}_j^{(y)} \hat{S}_k^{(y)} \right]$$

$$\Rightarrow \underline{\hat{H}_{int}^{(S.S.)} = \sum_{j,k} 2T_{jk} \left[\hat{S}_j^{(x)} \hat{S}_k^{(x)} + \hat{S}_j^{(y)} \hat{S}_k^{(y)} \right]}$$

This interaction term resembles the X-Y model interaction. This gives rise to the Super-fluid phase transition. The order parameter for this transition is the quantum phase of the system. Phases as we know are angles which trace out a circle on a plane. This plane can be chosen to be some X-Y plane. Thus the interaction can be modelled using the X-Y Spin model.

For the same bosonic system there can also be a liq-vapour phase transition which is actually modelled by the Ising interaction which was modelled previously in lattice-gas model. This interaction is given by,

$$\hat{H}_{int.}^{(l.v.)} = \sum_{j,k} J_{jk}^{(z)} \hat{S}_j^{(z)} \hat{S}_k^{(z)}$$

Then the net interaction can be written as,

$$\hat{H}_{int.} = \hat{H}_{int.}^{(s.s.)} + \hat{H}_{int.}^{(l.v.)}$$

$$\hat{H}_{int.} = \sum_{j,k} \left[J_{jk}^{(xy)} \left[\hat{S}_j^{(x)} \hat{S}_k^{(x)} + \hat{S}_j^{(y)} \hat{S}_k^{(y)} \right] + J_{jk}^{(z)} \hat{S}_j^{(z)} \hat{S}_k^{(z)} \right]$$

This resembles the anisotropic Heisenberg model interaction where x - y term gives rise to the Super-fluid phase transition & z -term gives rise to the liq-vapour phase transition.