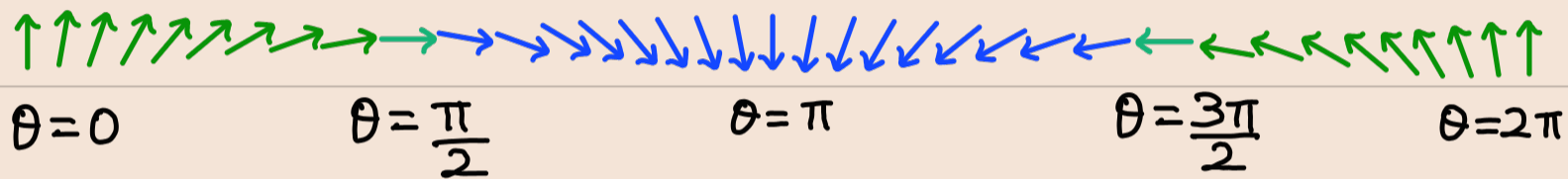


05/03/25

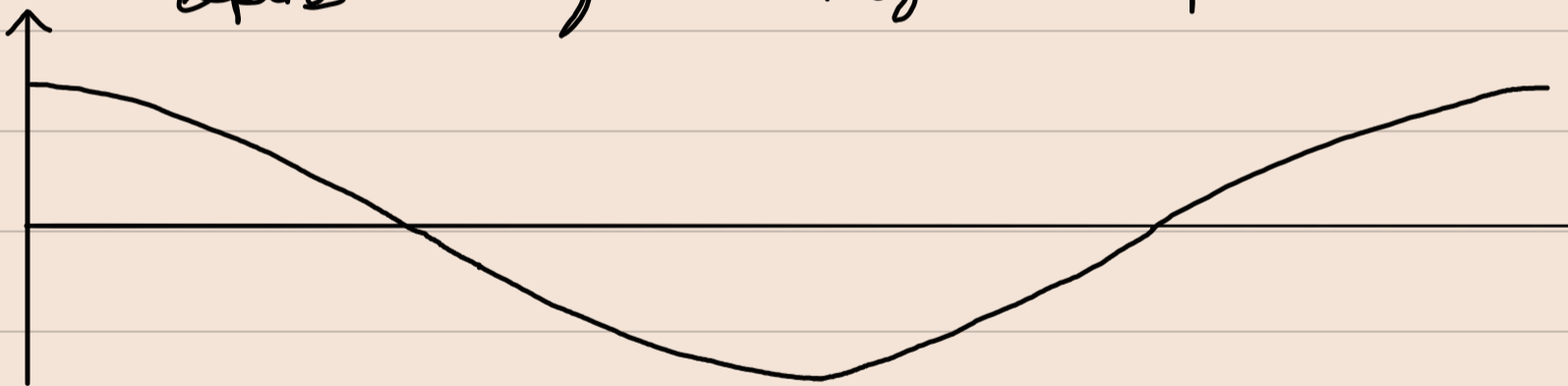
Lecture 17

Spin Waves & Magnons

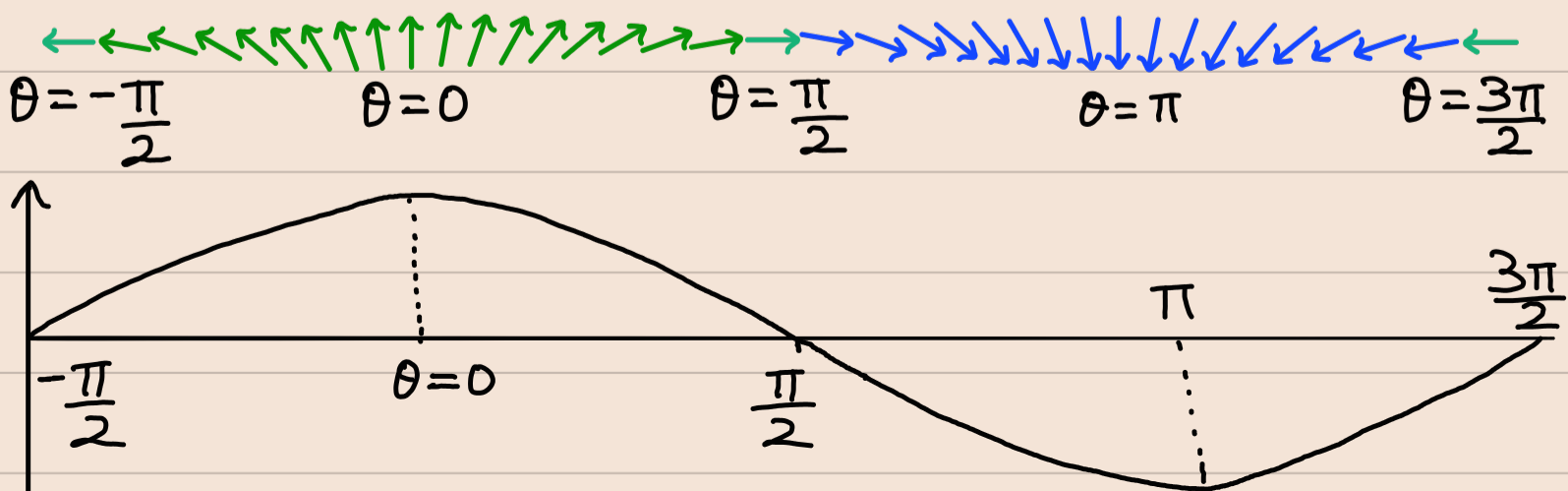
In 1D heisenberg model, lowest excited state can be achieved by allowing all the spins to share the spin reversal.



Spins as a function of lattice position



Complete wave like structure shown below



These are the elementary excitations of Spin System having a wave like form which are called Magnons.

Spin waves are oscillations in the relative orientations of spins in a lattice, & the quantized oscillations are called Magnons.

This is analogous to phonons which are the quantized oscillations of relative positions of atoms in lattice.

Spin-half Heisenberg Ferromagnet

$$\hat{H} = -J \sum_{\langle i,j \rangle} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j \quad \left(\begin{array}{l} d\text{-dimensional} \\ \text{Cubic lattice.} \end{array} \right)$$

$$\vec{S}_j = \frac{\vec{\sigma}_j}{2} \quad \left(\begin{array}{l} \text{for Spin-half} \\ \text{particles} \end{array} \right)$$

(eigen vals of $\hat{S}^{(z)}$
are $\pm 1/2$)

$\vec{\sigma}_j \rightarrow$ Pauli matrices
for j^{th} spin

$$\hat{S}_k^{(\pm)} = \hat{S}_k^{(x)} \pm i \hat{S}_k^{(y)}, \quad [\hat{S}_j^{(\alpha)}, \hat{S}_k^{(\beta)}] = \delta_{jk} i \hbar \epsilon_{\alpha\beta\gamma} \hat{S}_j^{(\gamma)}$$

$$\begin{aligned}
\vec{\hat{S}}_i \cdot \vec{\hat{S}}_j &= \hat{S}_i^{(x)} \hat{S}_j^{(x)} + \hat{S}_i^{(y)} \hat{S}_j^{(y)} + \hat{S}_i^{(z)} \hat{S}_j^{(z)} \\
&= \frac{[\hat{S}_i^{(+)} + \hat{S}_i^{(-)}]}{2} \frac{[\hat{S}_j^{(+)} + \hat{S}_j^{(-)}]}{2} + \frac{[\hat{S}_i^{(+)} - \hat{S}_i^{(-)}]}{2i} \frac{[\hat{S}_j^{(+)} - \hat{S}_j^{(-)}]}{2i} \\
&\quad + \hat{S}_i^{(z)} \hat{S}_j^{(z)} \\
&= \hat{S}_i^{(z)} \hat{S}_j^{(z)} + \frac{\hat{S}_i^{(+)} \hat{S}_j^{(+)} + \hat{S}_i^{(-)} \hat{S}_j^{(-)} + \hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}}{4} \\
&\quad - \left[\frac{\hat{S}_i^{(+)} \hat{S}_j^{(+)} + \hat{S}_i^{(-)} \hat{S}_j^{(-)} - \hat{S}_i^{(+)} \hat{S}_j^{(-)} - \hat{S}_i^{(-)} \hat{S}_j^{(+)}}{4} \right]
\end{aligned}$$

$$\Rightarrow \vec{\hat{S}}_i \cdot \vec{\hat{S}}_j = \hat{S}_i^{(z)} \hat{S}_j^{(z)} + \frac{1}{2} \left(\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)} \right)$$

$$\xrightarrow{\text{(Heisenberg)}} \hat{H} = -J \sum_{\langle i, j \rangle} \left(\hat{S}_i^{(z)} \hat{S}_j^{(z)} + \frac{1}{2} \left[\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)} \right] \right)$$

Let $|G\rangle$ be a state where all spins are up w.r.t z-direction. This actually an eigen state of the Heisenberg model, which is an ordered ground state. The ground state energy is $E_g = -\frac{NJ}{4}$ which will be proved later.

Now consider a state where all the spins are up but one spin is down w.r.t Z -axis, even though this is eigen state to the ising model, for Heisenberg model this state is not an eigen state.

We define $|\vec{r}\rangle = \hat{S}_{\vec{r}}^{(-)} |G\rangle$. This is a state where all the spins are up except the spin at the $\vec{r}$ location, which is in down state due to the action of $\hat{S}_{\vec{r}}^{(-)}$.

The number of nearest neighbours in a d -dimensional cubic lattice is $z = 2d$. The action of the $\hat{H}$ on this state $|\vec{r}\rangle$ is,

$$\begin{aligned} \hat{H}|\vec{r}\rangle &= -J \sum_{\langle i,j \rangle} \left[\hat{S}_i^{(z)} \hat{S}_j^{(z)} + \frac{1}{2} (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) \right] |\vec{r}\rangle \\ &= \underbrace{-J \sum_{\langle i,j \rangle} \hat{S}_i^{(z)} \hat{S}_j^{(z)} (\hat{S}_{\vec{r}}^{(-)} |G\rangle)}_{I_1} - \underbrace{\frac{J}{2} \sum_{\langle i,j \rangle} (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) (\hat{S}_{\vec{r}}^{(-)} |G\rangle)}_{I_2} \end{aligned}$$

we use $[\hat{S}_j^{(z)}, \hat{S}_i^{(-)}] = 2_{j \neq i} (-\hat{S}_i^{(-)})$

$$\Rightarrow \hat{S}_j^{(z)} \hat{S}_i^{(-)} = -2_{j \neq i} \hat{S}_i^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(z)}$$

$$\Rightarrow \hat{S}_i^{(z)} \hat{S}_j^{(z)} \hat{S}_i^{(-)} = \hat{S}_i^{(z)} [-2_{j \neq i} \hat{S}_i^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(z)}]$$

$$= (-2_{j \neq i}) [-2_{i \neq j} \hat{S}_i^{(-)} + \hat{S}_i^{(-)} \hat{S}_i^{(z)}] + [-2_{i \neq j} \hat{S}_i^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(z)}] \hat{S}_j^{(z)}$$

$$= 2_{ij} \hat{S}_i^{(-)} - 2_{j \neq i} \hat{S}_i^{(-)} \hat{S}_i^{(z)} - 2_{i \neq j} \hat{S}_i^{(-)} \hat{S}_j^{(z)} + \hat{S}_i^{(-)} \hat{S}_i^{(z)} \hat{S}_j^{(z)}$$

$$I_1 = -J \sum_{\langle i, j \rangle} \hat{S}_i^{(z)} \hat{S}_j^{(z)} \hat{S}_i^{(-)} |G\rangle$$

$$= -J \sum_{\langle i, j \rangle} [2_{ij} \hat{S}_i^{(-)} - 2_{j \neq i} \hat{S}_i^{(-)} \hat{S}_i^{(z)} - 2_{i \neq j} \hat{S}_i^{(-)} \hat{S}_j^{(z)} + \hat{S}_i^{(-)} \hat{S}_i^{(z)} \hat{S}_j^{(z)}] |G\rangle$$

$$\hat{S}_i^{(z)} |G\rangle = \frac{1}{2} |G\rangle \quad \left(\begin{array}{l} \text{as all spins are up} \\ \text{w.r.t } z. \end{array} \right)$$

$$\Rightarrow I_1 = -J \sum_{\langle i, j \rangle} \left[2_{ij} \hat{S}_i^{(-)} |G\rangle - \frac{1}{2} 2_{j \neq i} \hat{S}_i^{(-)} |G\rangle - \frac{1}{2} 2_{i \neq j} \hat{S}_i^{(-)} |G\rangle + \frac{1}{4} \hat{S}_i^{(-)} |G\rangle \right]$$

now $\sum_{\langle i, j \rangle} = \frac{1}{2} \sum_i \sum_{j \in n \cdot n(i)} = \frac{1}{2} \sum_j \sum_{i \in n \cdot n(j)}$

$$\Rightarrow \sum_{\langle i,j \rangle} \delta_{ij} \hat{S}_{\vec{r}}^- |G\rangle = 0 \quad (i, j \text{ are never equal})$$

$$\sum_{\langle i,j \rangle} \delta_{j\vec{r}} \frac{\hat{S}_{\vec{r}}^{(-)}}{2} |G\rangle = \frac{1}{2} \sum_j \sum_{i \in \text{nn}(\vec{r})} \delta_{j\vec{r}} \frac{\hat{S}_{\vec{r}}^{(-)}}{2} |G\rangle$$

$$= \frac{1}{4} \sum_j \delta_{j\vec{r}} \hat{S}_{\vec{r}}^{(-)} |G\rangle \sum_{i \in \text{nn}(\vec{r})} (1)$$

$3 \Rightarrow$ number of nearest neighbours

$$\Rightarrow \sum_{\langle i,j \rangle} \delta_{j\vec{r}} \frac{\hat{S}_{\vec{r}}^{(-)}}{2} |G\rangle = \frac{3}{4} \hat{S}_{\vec{r}}^{(-)} |G\rangle = \frac{3}{4} |\vec{r}\rangle$$

& $\sum_{\langle i,j \rangle} \delta_{i\vec{r}} \frac{\hat{S}_{\vec{r}}^{(-)}}{2} |G\rangle$ is also same as above

Since $\sum_{\langle i,j \rangle}$ is symmetric over i & j

$$\& \sum_{\langle i,j \rangle} \frac{\hat{S}_{\vec{r}}^{(-)}}{4} |G\rangle = \frac{\hat{S}_{\vec{r}}^{(-)}}{4} |G\rangle \sum_{\langle i,j \rangle} (1) = \frac{3N}{8} |\vec{r}\rangle$$

$(\frac{3N}{2})$

$$\Rightarrow I_1 = -\frac{3NJ}{8} |\vec{r}\rangle + \frac{3J}{2} |\vec{r}\rangle$$

$$\Rightarrow I_1 = \left(E_g + \frac{3J}{2} \right) |\vec{r}\rangle \quad \left(E_g = -\frac{3NJ}{8} \right)$$

$$I_2 = -\frac{J}{2} \sum_{\langle i, j \rangle} (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) (\hat{S}_{\vec{r}}^{(-)} |G\rangle)$$

$$[\hat{S}_j^{(+)}, \hat{S}_r^{(-)}] = 2\delta_{jr} \hat{S}_r^{(z)}$$

$$\Rightarrow \hat{S}_j^{(+)} \hat{S}_r^{(-)} = 2\delta_{jr} \hat{S}_r^{(z)} + \hat{S}_r^{(-)} \hat{S}_j^{(+)}$$

$$\begin{aligned} (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) \hat{S}_{\vec{r}}^{(-)} &= \left[2\delta_{ir} \hat{S}_r^{(z)} + \hat{S}_r^{(-)} \hat{S}_i^{(+)} \right] \hat{S}_j^{(-)} \\ &\quad + \hat{S}_i^{(-)} \left[2\delta_{jr} \hat{S}_r^{(z)} + \hat{S}_r^{(-)} \hat{S}_j^{(+)} \right] \end{aligned}$$

$$\begin{aligned} &= \left(2\delta_{ir} \hat{S}_r^{(z)} \hat{S}_j^{(-)} + \hat{S}_r^{(-)} \left[2\delta_{ij} \hat{S}_i^{(z)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)} \right] \right) \\ &\quad \text{(as } i \neq j) + \hat{S}_i^{(-)} \left[2\delta_{jr} \hat{S}_r^{(z)} + \hat{S}_r^{(-)} \hat{S}_j^{(+)} \right] \end{aligned}$$

$$\hat{S}_j^{(+)} |G\rangle = 0 \quad \text{as it is spin half system \& cannot increase spin after } +1/2$$

$$\begin{aligned} &\Rightarrow (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) \hat{S}_{\vec{r}}^{(-)} |G\rangle \\ &= 2\delta_{ir} \hat{S}_r^{(z)} \hat{S}_j^{(-)} |G\rangle + 2\delta_{jr} \hat{S}_i^{(-)} \hat{S}_r^{(z)} |G\rangle \end{aligned}$$

$$I_2 = -\frac{J}{2} \sum_{\langle i,j \rangle} \left(\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)} \right) \left(\hat{S}_{\vec{r}}^{(-)} |G\rangle \right)$$

$$= -J \sum_{\langle i,j \rangle} \left[\underbrace{\delta_{i\vec{r}} \hat{S}_{\vec{r}}^{(z)} \hat{S}_j^{(-)} |G\rangle}_{I_2'} + \underbrace{\delta_{j\vec{r}} \hat{S}_i^{(-)} \hat{S}_{\vec{r}}^{(z)} |G\rangle}_{I_2''} \right]$$

$$I_2' = -\frac{J}{2} \sum_i \sum_{j \in \text{nn}(\vec{r})} \delta_{i\vec{r}} \hat{S}_{\vec{r}}^{(z)} \hat{S}_j^{(-)} |G\rangle$$

$$= -\frac{J}{2} \sum_i \delta_{i\vec{r}} \hat{S}_{\vec{r}}^{(z)} \sum_{j \in \text{nn}(\vec{r})} \hat{S}_j^{(-)} |G\rangle$$

$$= -\frac{J}{2} \hat{S}_{\vec{r}}^{(z)} \sum_{j \in \text{nn}(\vec{r})} \hat{S}_j^{(-)} |G\rangle$$

$$= -\frac{J}{2} \hat{S}_{\vec{r}}^{(z)} \left[|\vec{r} + \hat{x}_1\rangle + |\vec{r} + \hat{x}_2\rangle + \dots + |\vec{r} + \hat{x}_d\rangle \right. \\ \left. + |\vec{r} - \hat{x}_1\rangle + |\vec{r} - \hat{x}_2\rangle + \dots + |\vec{r} - \hat{x}_d\rangle \right]$$

here all neighbours of $\vec{r}$ are down but $\vec{r}$ spin is up.

$$\rightarrow I_2' = -\frac{J}{4} \left[|\vec{r} + \hat{x}_1\rangle + |\vec{r} + \hat{x}_2\rangle + \dots + |\vec{r} + \hat{x}_d\rangle \right. \\ \left. + |\vec{r} - \hat{x}_1\rangle + |\vec{r} - \hat{x}_2\rangle + \dots + |\vec{r} - \hat{x}_d\rangle \right]$$

$$I_2'' = -J \sum_{\langle i, j \rangle} \hat{S}_i^{(-)} \hat{S}_j^{(+)} |G\rangle$$

$$= -\frac{J}{2} \sum_j \hat{S}_j \sum_{i \in \text{nn}(j)} \frac{\hat{S}_i^{(-)} |G\rangle}{2} = -\frac{J}{4} \sum_{i \in \text{n.n.}(\vec{r})} \hat{S}_i^{(-)} |G\rangle$$

$$= -\frac{J}{4} \left[|\vec{r} + \hat{x}_1\rangle + |\vec{r} + \hat{x}_2\rangle + \dots + |\vec{r} + \hat{x}_d\rangle + |\vec{r} - \hat{x}_1\rangle + |\vec{r} - \hat{x}_2\rangle + \dots + |\vec{r} - \hat{x}_d\rangle \right]$$

$$\Rightarrow I_2 = -\frac{J}{4} \left[|\vec{r} + \hat{x}_1\rangle + |\vec{r} + \hat{x}_2\rangle + \dots + |\vec{r} + \hat{x}_d\rangle + |\vec{r} - \hat{x}_1\rangle + |\vec{r} - \hat{x}_2\rangle + \dots + |\vec{r} - \hat{x}_d\rangle \right]$$

$$\Rightarrow I_2' + I_2'' = -\frac{J}{2} \left[|\vec{r} + \hat{x}_1\rangle + |\vec{r} + \hat{x}_2\rangle + \dots + |\vec{r} + \hat{x}_d\rangle + |\vec{r} - \hat{x}_1\rangle + |\vec{r} - \hat{x}_2\rangle + \dots + |\vec{r} - \hat{x}_d\rangle \right]$$

$$\Rightarrow \hat{H} |\vec{r}\rangle = I_1 + I_2 \quad \text{which gives}$$

$$\hat{H} |\vec{r}\rangle = \left(E_g + \frac{3J}{2} \right) |\vec{r}\rangle - \frac{J}{2} \left[|\vec{r} + \hat{x}_1\rangle + |\vec{r} + \hat{x}_2\rangle + \dots + |\vec{r} + \hat{x}_d\rangle + |\vec{r} - \hat{x}_1\rangle + |\vec{r} - \hat{x}_2\rangle + \dots + |\vec{r} - \hat{x}_d\rangle \right]$$

$$\Rightarrow \hat{H} |\vec{r}\rangle = \left(E_g + \frac{3J}{2} \right) |\vec{r}\rangle - \frac{J}{2} \sum_{\vec{r}' \in \text{n.n.}(\vec{r})} |\vec{r}'\rangle$$

Thus $|\vec{r}\rangle$ is not an eigen state of the Heisenberg Hamiltonian.

We can show that $|G\rangle$ is an eigen state (also the ground state) for $\hat{H}$.

$$\begin{aligned}\hat{H}|G\rangle &= -J \sum_{\langle i,j \rangle} \left[\hat{S}_i^z \hat{S}_j^z + \frac{1}{2} (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) \right] |G\rangle \\ &= -J \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z |G\rangle - \frac{J}{2} \sum_{\langle i,j \rangle} (\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)}) |G\rangle\end{aligned}$$

$$\hat{S}_i^z \hat{S}_j^z |G\rangle = \frac{1}{4} |G\rangle$$

$$\hat{S}_i^{(+)} \hat{S}_j^{(-)} = 2\delta_{ij} \hat{S}_i^{(z)} + \hat{S}_j^{(-)} \hat{S}_i^{(+)}$$

& $\hat{S}_j^{(+)} |G\rangle = 0$, since it is spin half system
& cannot increase spin after $1/2$

$$\begin{aligned}\Rightarrow \hat{H}|G\rangle &= -\frac{N}{4} |G\rangle \sum_{\langle i,j \rangle} (1) - \frac{J}{2} \underbrace{\sum_{\langle i,j \rangle} \delta_{ij}}_0 |G\rangle \\ &= -\frac{3NJ}{8} |G\rangle\end{aligned}$$

as $i \neq j$
in any term

$$\Rightarrow \underline{\hat{H}|G\rangle = E_g |G\rangle} \quad \left(E_g = -\frac{3NJ}{8} \right)$$

$\Rightarrow |G\rangle$ is an eigen state with an eigen value $-\frac{3NJ}{8}$

So we got $|\vec{r}\rangle$ to not be an eigen state of $\hat{H}_{\text{Heisenberg}}$, thus a localized opposite spin cannot exist in the system.

Consider a state $|\vec{q}\rangle$ given by,

$$|\vec{q}\rangle = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{q} \cdot \vec{r}} |\vec{r}\rangle$$

This is a state where the opposite spins are smeared all over the sites of lattice with a weight $e^{i\vec{q} \cdot \vec{r}}$ which can also be understood as the discrete fourier transformed state of the $|\vec{r}\rangle$. We can show that this is an eigen state of $\hat{H}$.

$$\begin{aligned} \hat{H}|\vec{q}\rangle &= \hat{H} \left[\sum_{\vec{r}} \frac{e^{i\vec{q} \cdot \vec{r}}}{\sqrt{N}} |\vec{r}\rangle \right] \\ &= \sum_{\vec{r}} \frac{e^{i\vec{q} \cdot \vec{r}}}{\sqrt{N}} \hat{H}|\vec{r}\rangle \end{aligned}$$

$$\Rightarrow \hat{H}|\vec{r}\rangle = \sum_{\vec{r}} \frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} \left[\left(E_g + \frac{3J}{2}\right)|\vec{r}\rangle - \frac{J}{2} \sum_{\vec{r}' \in n.n.(\vec{r})} |\vec{r}'\rangle \right]$$

$$= \left[E_g + \frac{3J}{2}\right] \sum_{\vec{r}} \frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} |\vec{r}\rangle - \frac{J}{2} \sum_{\vec{r}} \frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} \sum_{\vec{r}' \in n.n.(\vec{r})} |\vec{r}'\rangle$$

$$= \left[E_g + \frac{3J}{2}\right] |\vec{r}\rangle - \frac{J}{2} \sum_{\vec{r}} \sum_{\vec{r}' \in n.n.(\vec{r})} \frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} |\vec{r}'\rangle$$

Consider $\sum_{\vec{r}} \sum_{\vec{r}' \in n.n.(\vec{r})} e^{i\vec{q}\cdot\vec{r}} |\vec{r}'\rangle$

$$= \sum_{\vec{r}} \sum_{n=1}^d \frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} \left[|\vec{r} + \hat{x}_n\rangle + |\vec{r} - \hat{x}_n\rangle \right]$$

[where $\hat{x}_n \rightarrow$ unit vector
along n^{th} coordinate axis]

$$= \sum_{\vec{r}} \sum_{n=1}^d \left[\underbrace{\frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} |\vec{r} + \hat{x}_n\rangle}_{\vec{r} \rightarrow \vec{r} - \hat{x}_n} + \underbrace{\frac{e^{i\vec{q}\cdot\vec{r}}}{\sqrt{N}} |\vec{r} - \hat{x}_n\rangle}_{\vec{r} \rightarrow \vec{r} + \hat{x}_n} \right]$$

$$\Rightarrow \sum_{\vec{r}} \sum_{\vec{r}' \in n \cdot n(\vec{r})} \frac{e^{i\vec{q} \cdot \vec{r}'} |\vec{r}'\rangle}{\sqrt{N}}$$

$$= \sum_{\vec{r}} \sum_{n=1}^d \left[\frac{e^{i\vec{q} \cdot (\vec{r} - \hat{x}_n)}}{\sqrt{N}} |\vec{r}\rangle + \frac{e^{i\vec{q} \cdot (\vec{r} + \hat{x}_n)}}{\sqrt{N}} |\vec{r}\rangle \right]$$

$$= \sum_{\vec{r}} \sum_{n=1}^d \frac{(e^{-i q_n} + e^{i q_n})}{\sqrt{N}} e^{i\vec{q} \cdot \vec{r}} |\vec{r}\rangle$$

$(q_n = \vec{q} \cdot \hat{x}_n)$

$$= \sum_{n=1}^d 2 \cos(q_n) \underbrace{\sum_{\vec{r}} \frac{e^{i\vec{q} \cdot \vec{r}}}{\sqrt{N}}}_{|\vec{q}\rangle}$$

$$\Rightarrow \sum_{\vec{r}} \sum_{\vec{r}' \in n \cdot n(\vec{r})} \frac{e^{i\vec{q} \cdot \vec{r}'} |\vec{r}'\rangle}{\sqrt{N}} = \sum_{n=1}^d 2 \cos(q_n) |\vec{q}\rangle$$

$$\Rightarrow \hat{H} |\vec{q}\rangle = \left[E_g + \frac{3J}{2} \right] |\vec{q}\rangle - \frac{J}{2} \sum_{n=1}^d 2 \cos(q_n) |\vec{q}\rangle$$

$$\Rightarrow \hat{H} |\vec{q}\rangle = \left[E_g + \frac{3J}{2} - J \sum_{n=1}^d \cos(q_n) \right] |\vec{q}\rangle$$

Now $z = 2d$ for Cubic lattice in d -dimensions. Thus $\frac{z}{2} = \sum_{n=1}^d (1)$

$$\Rightarrow \hat{H}|\vec{q}\rangle = \left(E_g + J \sum_{n=1}^d [1 - \cos(q_n)]\right) |\vec{q}\rangle$$

This is called the dispersion relation of the spin wave as it predicts that one opposite spin is distributed throughout the spin lattice. A spin wave is a state with superposition of all possible one opposite spin. The penalty for having long wave length spin waves is very negligible. As it minimizes the misalignment of any two neighbouring spins.

$|\vec{q}=0\rangle = \sum_{\vec{r}} \frac{|\vec{r}\rangle}{\sqrt{N}}$ is a single down spin superposed state with equal weight in all the lattice site. This state has energy equal to the ground state & it generates a spin wave of zero frequency & infinite wavelength

For any $\vec{q} \neq 0$, $|\vec{q}\rangle$ describes a spin wave of well defined crystal momentum ' $\vec{q}$ '.

$$E(q) = E_g + J \sum_{n=1}^d [1 - \cos(qr_n)]$$

$$\underbrace{\Delta E_{\vec{q}}}_{\hbar \omega_{\vec{q}}} = E_{\vec{q}} - E_g \Rightarrow \text{Excess energy because of spin wave of some momentum } \vec{q}.$$

$(\hbar = 1)$

$$\Rightarrow \underline{\omega_{\vec{q}} = J \sum_{n=1}^d [1 - \cos(qr_n)]}$$

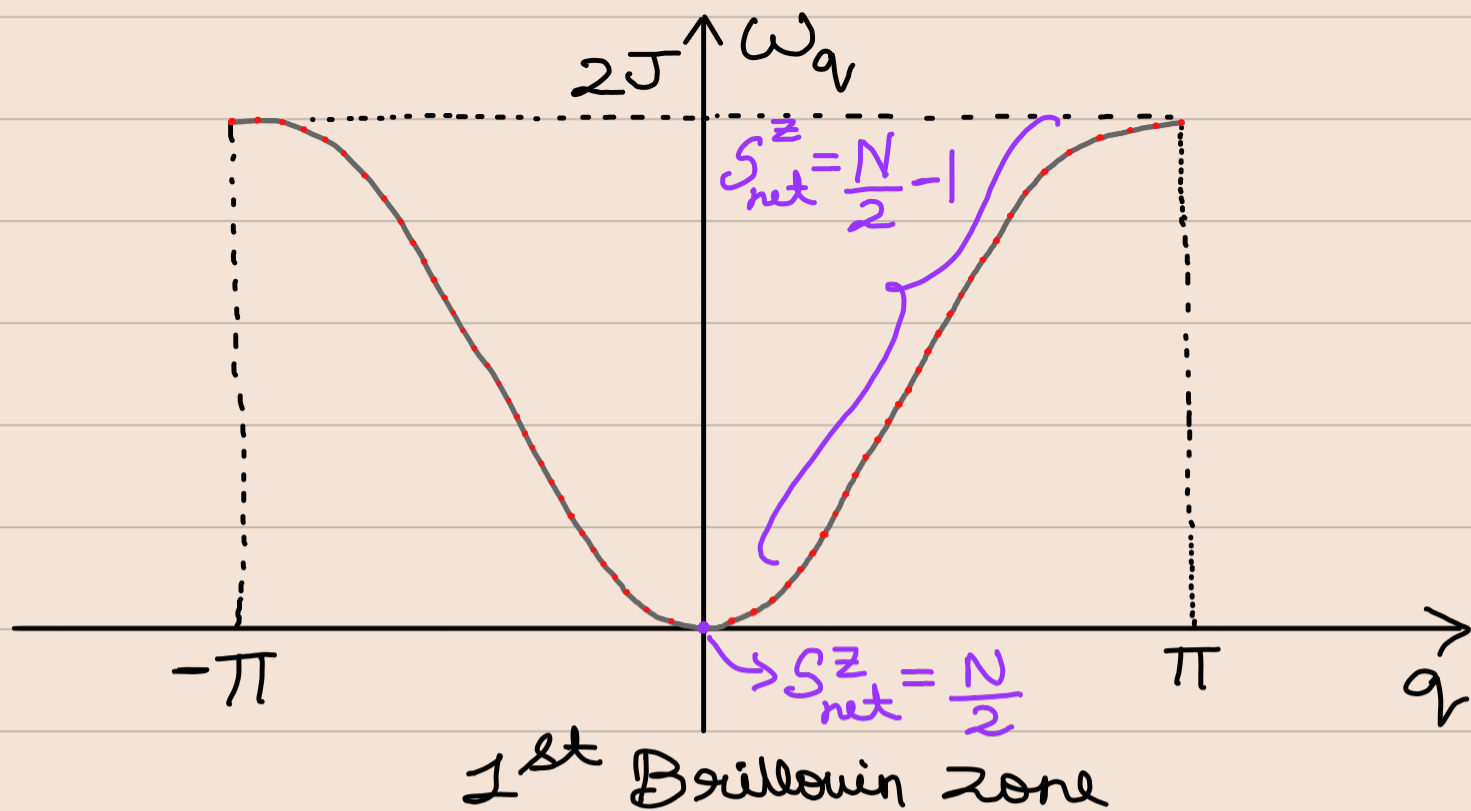
$\hookrightarrow$ To visualize $\omega_{\vec{q}}$ in 3D.

For a 1D chain it becomes

$$\omega_q = J(1 - \cos(qr)) \quad q \rightarrow q_x$$

$-\pi < q < \pi$

$$\Rightarrow \omega_q = 2J \sin^2\left(\frac{q}{2}\right)$$



Since $N + \gamma = \gamma$ in 1D periodic chain boundary condition,

$$|q\rangle = \sum_i e^{iq\gamma} |\gamma\rangle = \sum_i e^{iq(\gamma+N)} |\gamma\rangle$$

$$\Rightarrow |q\rangle = e^{iqN} |q\rangle$$

'q' has to satisfy the above eqⁿ.

Thus the allowed q-values are given by $qN = 2m\pi$ $m = \{\pm 1, \pm 2, \dots, \pm \frac{N}{2}\}$

<https://www.desmos.com/3d/fuj86dvd>

$$\Rightarrow q = \left\{ \pm \frac{2\pi}{N}, \pm \frac{4\pi}{N}, \dots, \pm \pi \right\}$$

These are the possible values of non zero 'q' which are N in number. Thus for N spins there are N modes of 'q'.

In ground state the net spin is given by $S_{\text{net}}^z = \frac{N}{2}$. But in the presence of a spin wave of $q \neq 0$, the net spin is given by $\frac{N}{2} - 1$. Thus we can think of spin waves to have a integer spin '1'.

The collective excitation of the lattice spins structure has a quasi-particle associated with it which are called as Magnons. They are the quantization of spin waves, which carry a fixed amount of energy & crystal momentum whose spin quantum number is '1'.

Commutation relations for magnon creation & annihilation operators

$$\begin{aligned}\hat{S}_{\vec{q}}^{(-)} &= \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{q} \cdot \vec{r}} \hat{S}_{\vec{r}}^{(-)} \\ \Rightarrow \hat{S}_{\vec{q}}^{(-)} |G\rangle &= \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{i\vec{q} \cdot \vec{r}} \underbrace{\hat{S}_{\vec{r}}^{(-)} |G\rangle}_{|\vec{r}\rangle} \\ \Rightarrow \hat{S}_{\vec{q}}^{(-)} |G\rangle &= |\vec{q}\rangle\end{aligned}$$

Thus $\hat{S}_{\vec{q}}^{(-)}$ acts as a magnon creation operator of momentum ' $\vec{q}$ '.

$$\& \left[\hat{S}_{\vec{q}}^{(-)}, \hat{S}_{\vec{q}'}^{(-)} \right] = \frac{1}{N} \sum_{\vec{r}, \vec{r}'} e^{i\vec{q} \cdot (\vec{r} + \vec{r}')} \underbrace{\left[S_{\vec{r}}^{(+)}, S_{\vec{r}'}^{(-)} \right]}_{\text{always commute}}$$

$$\rightarrow \left[\hat{S}_{\vec{q}}^{(-)}, \hat{S}_{\vec{q}'}^{(-)} \right] = 0$$

Similarly $\left[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(+)} \right] = 0$

where $\hat{S}_{\vec{q}}^{(+)} = \frac{1}{\sqrt{N}} \sum_{\vec{r}} e^{-i\vec{q} \cdot \vec{r}} \hat{S}_{\vec{r}}^{(+)}$

$\& \hat{S}_{\vec{r}}^{(+)} |\vec{r}\rangle = |G\rangle \Rightarrow \hat{S}_{\vec{q}}^{(+)} |\vec{q}\rangle = |G\rangle$

Thus $\hat{S}_{\vec{q}}^{(+)}$ acts as a magnon annihilation operator of momentum ' $\vec{q}$ '.

$$\left[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)} \right] = \frac{1}{N} \sum_{\vec{r}, \vec{r}'} e^{-i\vec{q} \cdot \vec{r}} e^{i\vec{q}' \cdot \vec{r}'} \left[\hat{S}_{\vec{r}}^{(+)}, \hat{S}_{\vec{r}'}^{(-)} \right]$$

$$= \frac{1}{N} \sum_{\vec{r}, \vec{r}'} e^{-i\vec{q} \cdot \vec{r}} e^{i\vec{q}' \cdot \vec{r}'} (2\delta_{\vec{r}\vec{r}'}) \hat{S}_{\vec{r}}^{(z)}$$

$$\left[\hat{S}_{\vec{q}}^{(+)} , \hat{S}_{\vec{q}'}^{(-)} \right] = \frac{2}{N} \sum_{\vec{r}} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}} \hat{S}_{\vec{r}}^{(z)}$$

Now $\hat{S}_{\vec{r}}^{(z)}$ has eigen values of $\pm \frac{1}{2}$,
 So whenever the Commutator acts on an
 eigen state of $\hat{S}_{\vec{r}}^{(z)}$ we get,

$$\begin{aligned} \left[\hat{S}_{\vec{q}}^{(+)} , \hat{S}_{\vec{q}'}^{(-)} \right] &= \frac{2}{N} \left(\pm \frac{1}{2} \right) \sum_{\vec{r}} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}} \\ &= \pm 1 \underbrace{\frac{1}{N} \sum_{\vec{r}} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}}}_{\delta_{\vec{q}, \vec{q}'}} \end{aligned}$$

$$\Rightarrow \underline{\left[\hat{S}_{\vec{q}}^{(+)} , \hat{S}_{\vec{q}'}^{(-)} \right] = \pm \delta_{\vec{q}, \vec{q}'}}$$

In our formalism we usually are
 concerned with the action of these
 Commutator relations on ground state
 $|\phi\rangle$ or $|\chi\rangle$ or $|\vec{q}\rangle$. We can
 check that $\left[\hat{S}_{\vec{q}}^{(+)} , \hat{S}_{\vec{q}'}^{(-)} \right] |\phi\rangle = \delta_{\vec{q}, \vec{q}'}$

$$\& \left[\hat{S}_{\vec{q}}^{(-)} , \hat{S}_{\vec{q}'}^{(-)} \right] = 0$$

$$\& \left[\hat{S}_{\vec{q}}^{(+)} , \hat{S}_{\vec{q}'}^{(+)} \right] = 0$$

Thus Magnons are bosonic particles
 Since, $S_{\vec{q}}^{(\pm)}$ operators act as the
 creation / annihilation operators for a magnon
 of momentum $\vec{q}$ & they commute
 for different $\vec{q}'$, i.e. $[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(+)}] = 0$. Thus
 under particle exchange, magnons do not
 acquire any phase. Hence they follow
 Bosonic exchange statistics.

i.e. when you create magnons
 it doesn't matter if $\vec{q}$ or $\vec{q}'$
 bosons are first or not.

$$\text{i.e. } [\hat{S}_{\vec{q}}^{(-)}, \hat{S}_{\vec{q}'}^{(-)}] |G\rangle = 0$$

$$\Rightarrow \hat{S}_{\vec{q}}^{(-)} \hat{S}_{\vec{q}'}^{(-)} |G\rangle = \hat{S}_{\vec{q}'}^{(-)} \hat{S}_{\vec{q}}^{(-)} |G\rangle = 0$$

$$\Rightarrow |\vec{q}', \vec{q}\rangle - |\vec{q}, \vec{q}'\rangle = 0$$

$$\Rightarrow |\vec{q}', \vec{q}\rangle = |\vec{q}, \vec{q}'\rangle$$

This repⁿ tells you about how many
 magnons of what momentum are present
 in the system. Above is a two Magnon
 repⁿ.

We can calculate the action of $[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}]$ on $|\vec{r}'\rangle$ & $|\vec{r}''\rangle$ as follows,

$$\begin{aligned} [\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] |\vec{r}'\rangle &= \frac{2}{N} \sum_{\vec{r}} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}} \hat{S}_{\vec{r}}^{(+)} |\vec{r}'\rangle \\ &= \sum_{\vec{r} \neq \vec{r}'} \frac{1}{N} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}} |\vec{r}'\rangle - \frac{1}{N} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}'} |\vec{r}'\rangle \\ &= \frac{1}{N} \sum_{\vec{r}} (1 - 2\delta_{\vec{r}, \vec{r}'}) e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}} |\vec{r}'\rangle \end{aligned}$$

$$[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] |\vec{r}'\rangle = \delta_{\vec{q}, \vec{q}'} |\vec{r}'\rangle - \frac{2}{N} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}'} |\vec{r}'\rangle$$

$$[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] |\vec{r}''\rangle = [\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] |\vec{r}''\rangle$$

$$= [\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] \frac{1}{\sqrt{N}} \sum_{\vec{r}_1} e^{i\vec{q}'' \cdot \vec{r}_1} |\vec{r}_1\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{\vec{r}_1} e^{i\vec{q}'' \cdot \vec{r}_1} [\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] |\vec{r}_1\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_{\vec{r}_1} e^{i\vec{q}'' \cdot \vec{r}_1} \left[\delta_{\vec{q}, \vec{q}'} |\vec{r}_1\rangle - \frac{2}{N} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}_1} |\vec{r}_1\rangle \right]$$

$$= \delta_{\vec{q}, \vec{q}'} |\vec{r}''\rangle - \frac{2}{N^{3/2}} \sum_{\vec{r}_1} e^{i\vec{q}'' \cdot \vec{r}_1} e^{-i(\vec{q}-\vec{q}') \cdot \vec{r}_1} |\vec{r}_1\rangle$$

$$[\hat{S}_{\vec{q}}^{(+)}, \hat{S}_{\vec{q}'}^{(-)}] |\vec{r}''\rangle = \delta_{\vec{q}, \vec{q}'} |\vec{r}''\rangle - \frac{2}{N} |\vec{q}'' + \vec{q}' - \vec{q}\rangle$$

Angular momentum Conservation

Thermodynamic Consequences of Magnons

Magnons are massless bosons, hence they don't have chemical potential.

In momentum space $\hat{H}$ becomes diagonal using $\hat{S}_q^{(-)} \rightarrow \hat{a}_q^+$ as the creation operator, $\hat{S}_q^{(+)} \rightarrow \hat{a}_q$ as the annihilation operator, we get

$$\hat{H} = -J \sum_{\langle i,j \rangle} \left[\hat{S}_i^{(z)} \hat{S}_j^{(z)} + \frac{1}{2} \left(\hat{S}_i^{(+)} \hat{S}_j^{(-)} + \hat{S}_i^{(-)} \hat{S}_j^{(+)} \right) \right]$$

from this we can get

$$\Rightarrow \hat{H}_{\text{magnon}} = \sum_{\vec{q}} \omega_{\vec{q}} \hat{a}_{\vec{q}}^+ \hat{a}_{\vec{q}}$$

$$\text{where } \omega_{\vec{q}} = J \sum_{n=1}^d [1 - \cos(q_n)]$$

The other part in the Hamiltonian corresponds to a constant E_g which don't have any dynamical significance.

$$\underline{\hat{H}_{\text{magnon}} |\vec{q}\rangle = \omega_{\vec{q}} |\vec{q}\rangle}$$

we define number operator

$$\hat{n}_q = \hat{a}_q^\dagger \hat{a}_q \quad (\text{in } q\text{-space})$$

And $\hat{n}_{\vec{r}} = \hat{a}_{\vec{r}}^\dagger \hat{a}_{\vec{r}} \quad (\text{in physical space})$

$$\hat{a}_{\vec{r}}^\dagger = \frac{1}{\sqrt{N}} \sum_{\vec{q}} e^{-i\vec{q} \cdot \vec{r}} \hat{a}_q^\dagger$$

$$\& \hat{a}_{\vec{r}} = \frac{1}{\sqrt{N}} \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} \hat{a}_q$$

$$\Rightarrow \hat{n}_{\vec{r}} = \frac{1}{N} \sum_{\vec{q}, \vec{q}'} e^{i(\vec{q}' - \vec{q}) \cdot \vec{r}} \hat{a}_q^\dagger \hat{a}_q$$

$$\begin{aligned} \Rightarrow \langle \hat{n}_{\vec{r}} \rangle &= \frac{1}{N} \sum_{\vec{q}, \vec{q}'} e^{i(\vec{q}' - \vec{q}) \cdot \vec{r}} \langle \hat{a}_q^\dagger \hat{a}_q \rangle \\ &\quad \text{w.r.t } \downarrow \text{ } |q\rangle \text{ states} \\ &= \frac{1}{N} \sum_{\vec{q}, \vec{q}'} e^{i(\vec{q}' - \vec{q}) \cdot \vec{r}} \delta_{qq'} \langle n_{\vec{q}} \rangle \\ &\quad \text{as } \hat{H} \text{ is diagonal} \end{aligned}$$

$$\Rightarrow \boxed{\langle \hat{n}_{\vec{r}} \rangle = \frac{1}{N} \sum_{\vec{q}} \langle n_{\vec{q}} \rangle}$$

Magnons are massless bosons, hence they don't have chemical potential. Hence the average $\langle n_{\vec{q}} \rangle$ is given by Bose statistics as

$$\langle \hat{n}_{\vec{q}} \rangle = \frac{1}{e^{B\omega_{\vec{q}}} - 1} \quad \left(\text{without chemical Potential} \right)$$

$$\Rightarrow \langle \hat{n}_{\vec{q}} \rangle = \frac{1}{N} \sum_{\vec{q}} \frac{1}{e^{B\omega_{\vec{q}}} - 1}$$

We can convert the sum to an integral where $\vec{q} \in \text{first Brillouin Zone}$

i.e. $-\pi < q_n < \pi$ for all components of ' $\vec{q}$ '.

$$\frac{1}{N} \sum_{\vec{q}} \rightarrow \int_{1^{\text{st}} \text{ B.Z.}} \frac{d^d q}{(2\pi)^d}$$

$$\Rightarrow \langle \hat{n}_{\vec{q}} \rangle = \int_{1^{\text{st}} \text{ B.Z.}} \frac{d^d q}{(2\pi)^d} \frac{1}{(e^{B\omega_{\vec{q}}} - 1)}$$

In long wavelength limit,

$$\cos(qr_n) \sim 1 - \frac{q^2 r_n^2}{2} \Rightarrow 1 - \cos(qr_n) \sim \frac{q^2 r_n^2}{2}$$

$$\omega_q = J \sum_{n=1}^d [1 - \cos(qr_n)] \sim J \sum_{n=1}^d \frac{q^2 r_n^2}{2}$$

$$\omega_q \sim \frac{J}{2} |\vec{q}|^2 = \frac{Jq^2}{2}$$

$$\Rightarrow \langle \hat{n}_{\vec{x}} \rangle = \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)}$$

In d -dimensional Spherical polar Coordinates

$$d^d q = dq_1 dq_2 \dots dq_d = q^{d-1} dq d\Omega_d$$

where $d\Omega_d$ is the Solid angle element in d -dimensions.

$$d\Omega_d = d\phi \prod_{i=1}^{d-2} \sin^{(d-i-1)}(\theta_i) d\theta_i$$

where $0 < \phi < 2\pi$ $(0 < q < \infty)$
 $\& \quad 0 < \theta_i < \pi$

$$\Rightarrow \langle \hat{n}_{\vec{r}} \rangle = \int d\Omega_d \int_0^\infty \frac{dq}{(2\pi)^d} \frac{q^{d-1}}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)}$$

(Surface of 'd' dim. Hypersphere)

$$\int d\Omega_d = \int_0^{2\pi} d\phi \prod_{i=1}^{d-2} \int_0^\pi d\theta_i \sin^{(d-i-1)}(\theta_i)$$

(Surface of 'd' dim. Hypersphere)

$$= \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

→ gamma function

$$\Rightarrow \langle \hat{n}_{\vec{r}} \rangle = \frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \int_0^\infty dq \frac{q^{d-1}}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)}$$

In $d \geq 2$, it can be shown to be convergent.

$$\int_0^\infty dq \frac{q^{d-1}}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)} = 2^{\frac{d}{2}-1} (\beta J)^{-\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \zeta(d/2)$$

$$\Rightarrow \langle n_{\vec{x}} \rangle = \frac{2^{d/2-1} \zeta(d/2)}{2^{d-1} \pi^{d/2} (\beta J)^{d/2}}$$

$$\zeta(z) = \sum_{m=1}^{\infty} \frac{1}{m^z}$$

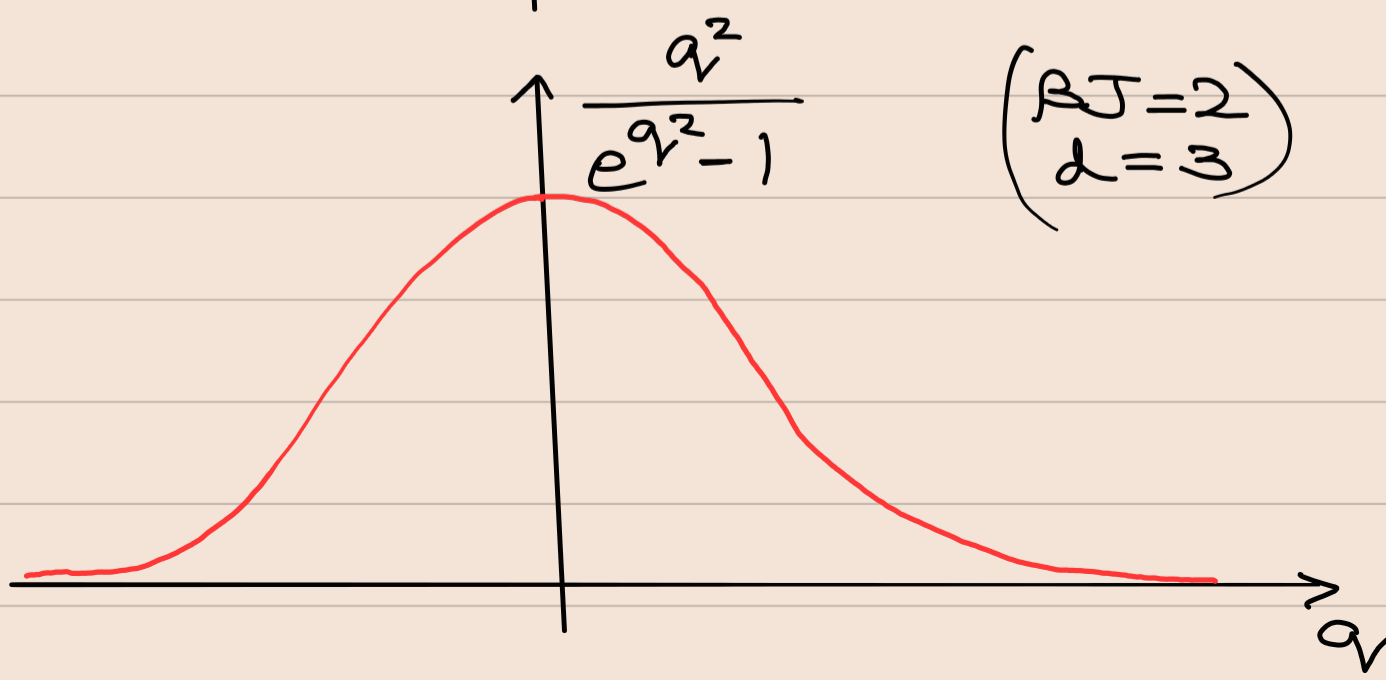
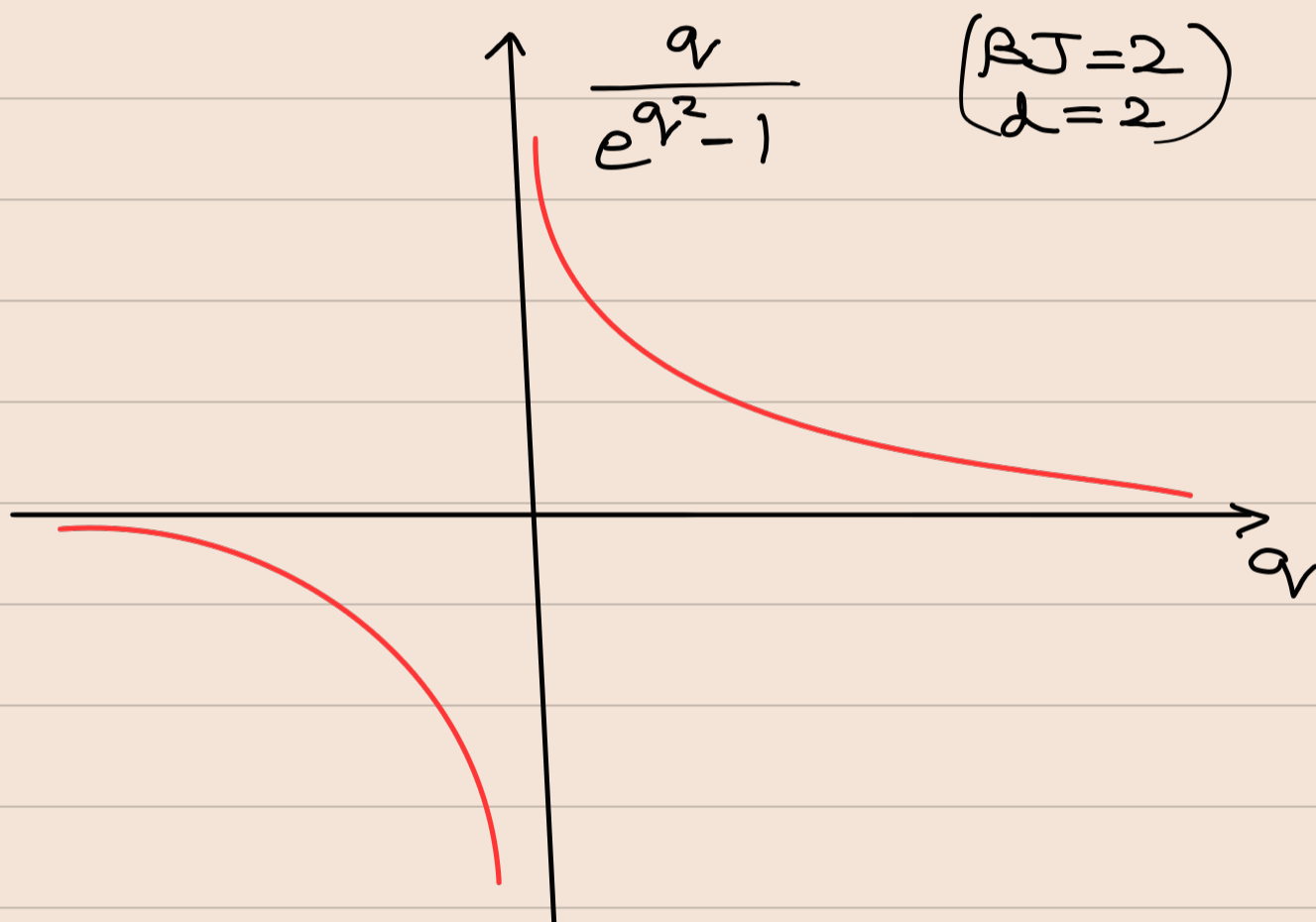
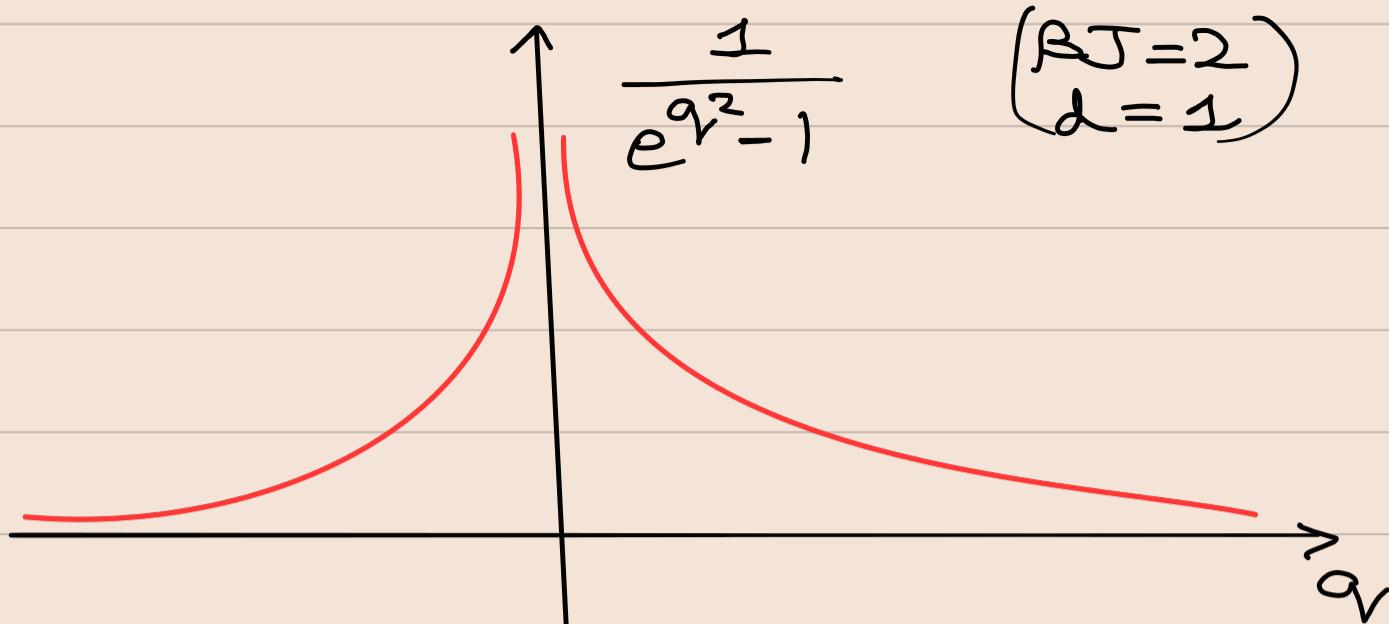
Riemann-Zeta function

$$\langle n_x \rangle = \frac{\zeta(d/2)}{(2\pi)^{d/2} (\beta J)^{d/2}} \quad \underline{\underline{d > 2}}$$

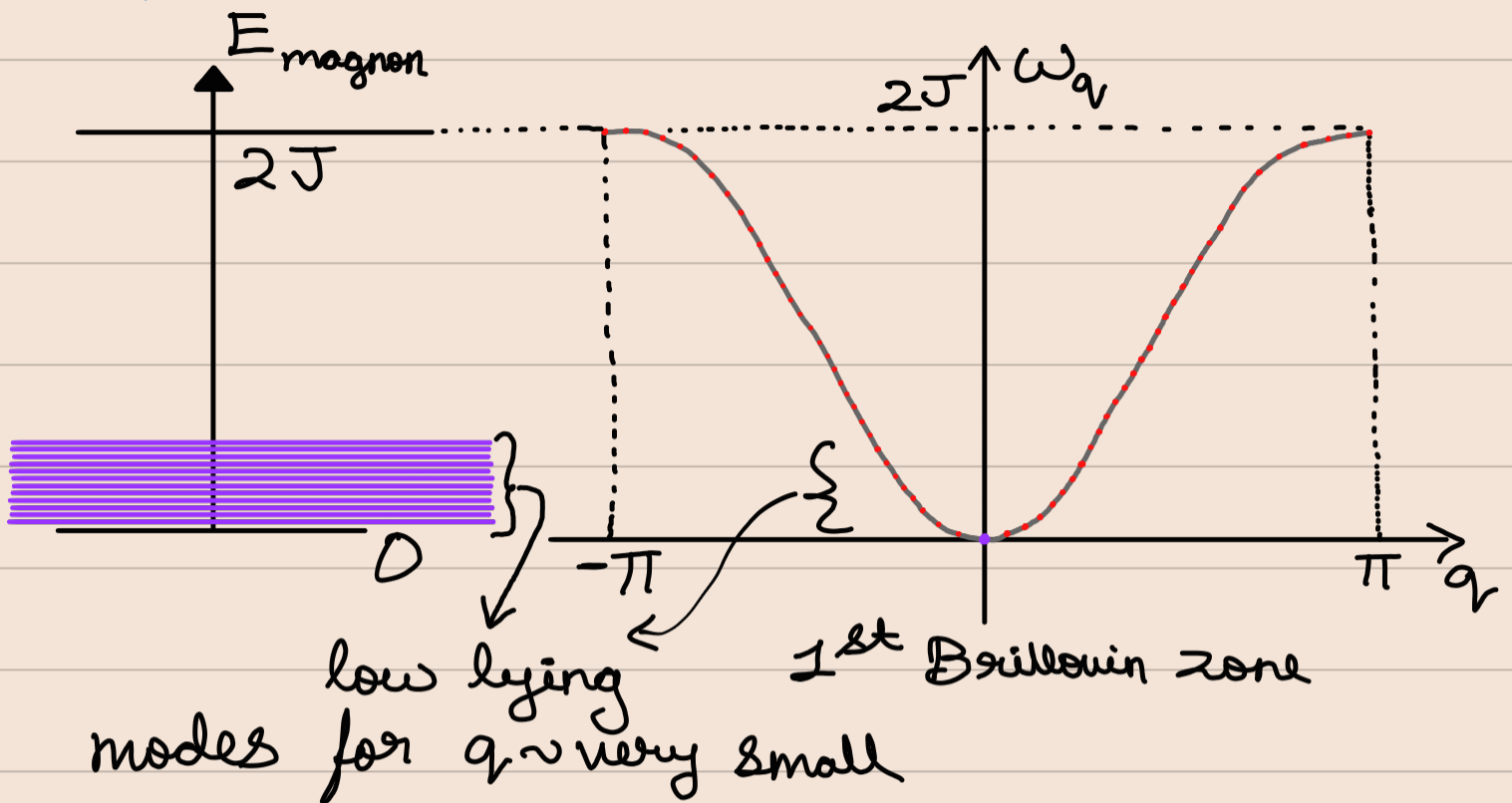
For $d=1, 2$ the integral is divergent because of the singularity at ' $q=0$ '.

$$\langle \hat{n}_{\vec{x}} \rangle_{(d=1)} = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \frac{1}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)} \Rightarrow \text{Diverging Integral}$$

$$\begin{aligned} \text{If } d=2, \quad \langle \hat{n}_{\vec{x}} \rangle &= \iint \frac{d^2 q}{(2\pi)^2} \frac{1}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)} \\ &= \frac{1}{2\pi} \int_0^{\infty} dq \frac{q}{\left(e^{\frac{\beta J q^2}{2}} - 1 \right)} \Rightarrow \text{diverging integral} \end{aligned}$$



Spin waves Goldstone Modes



These modes are called Goldstone modes which arise due to the Spontaneous Symmetry breaking of the Continuous Symmetry present in the governing Hamiltonian.

These modes are gapless i.e. the energy of few modes can go to Zero.

In Heisenberg model, the Hamiltonian has Continuous rotational Symmetry, as it commutes with $\hat{R}(\hat{n}, \theta)$. But the ground state breaks this Symmetry by choosing some preferential axis.

Thus the excitations from ground state will be gapless & can lie very close to ground state. For the Heisenberg model, these excitations are called magnons which are the low lying goldstone modes of the system.

Magnons are quantized spin waves arising from small deviation of spins from their positions. They are collective precessions of the deviated spins which propagate through the lattice.