

12/03/25

Lecture 19

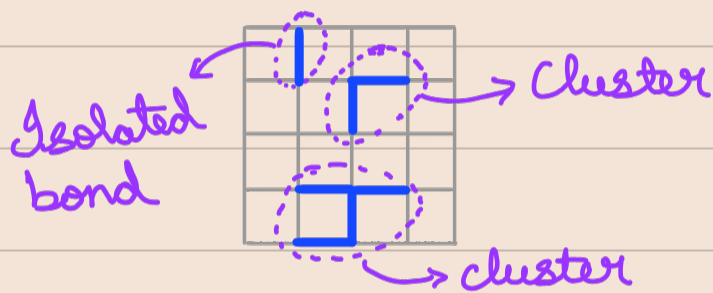
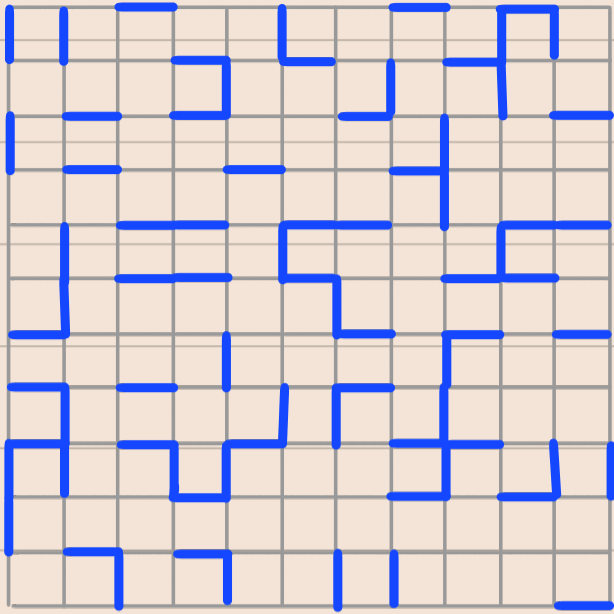
Percolation Theory

Consider a Square lattice which is infinitely extended over all directions.

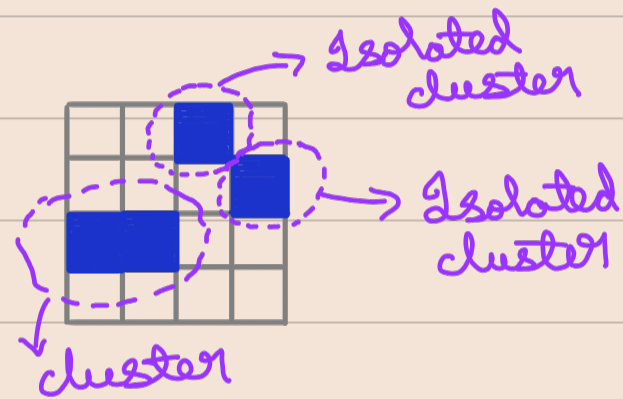
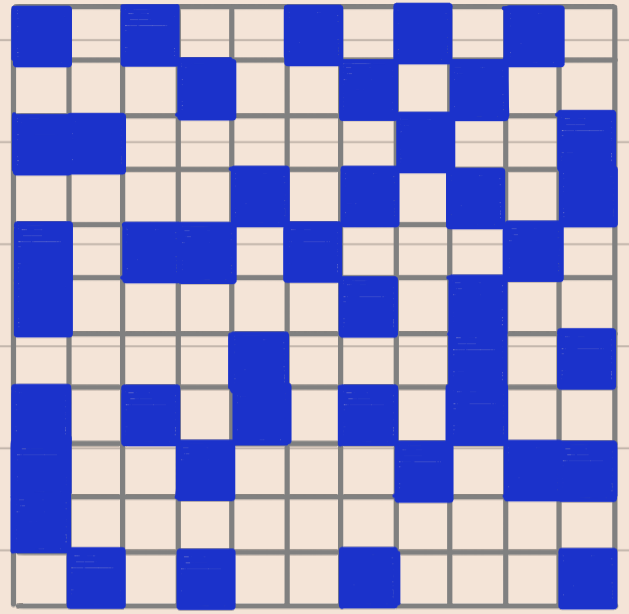
Bond percolation: Here one views the lattice as a graph consisting of horizontal & vertical edges, where each edge is occupied with a bond with a probability ' p '. A cluster is defined as a collection of connected bonds.

Site percolation: Here one views the lattice as a rectangular array of Squares where each Square is occupied with a box (or filled) with a probability ' p '. A cluster is defined as a collection of nearest neighbour connected boxes.

Bond Percolation



Site Percolation



The probability " p " is also equal to the fraction of edges being occupied in bond percolation or it is the fraction of sites being filled in the site percolation.

An infinite cluster is that cluster which connects one end of the infinite square lattice to the opposite end of the square lattice.

Now, the question is, given a probability ' p ', is there a connection from one end to the other end of square lattice.

For relatively low fill fraction ' p ' the answer is No, & for a relatively high fill fraction ' p ' the answer is Yes. But as the ' p ' value is increased from low (~ 0) to a high value (~ 1) there will be some intermediate critical value ' p_c ' where the answer to the question turns from 'no' to 'yes'. To quantify this we can define a term ' P_∞ ' as the Probability of a given random site connected to infinity (end of lattice). Then ' p_c ' can be defined by the following relation.

$$P_\infty = 0, \text{ if } p < p_c \text{ \& } P_\infty \neq 0, \text{ if } p > p_c$$

This process can be thought of as a geometric phase transition that occurs from a $P_\infty = 0$ phase to $P_\infty \neq 0$ phase at a critical probability ' p_c '.



$p \sim \text{small}$



$p \rightarrow p_c$

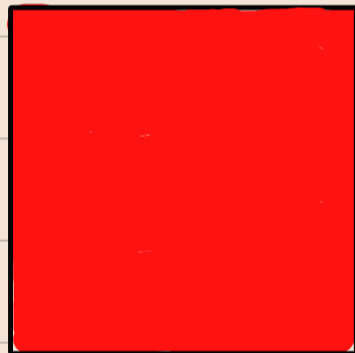


$p = p_c$

(Fractal like structure)



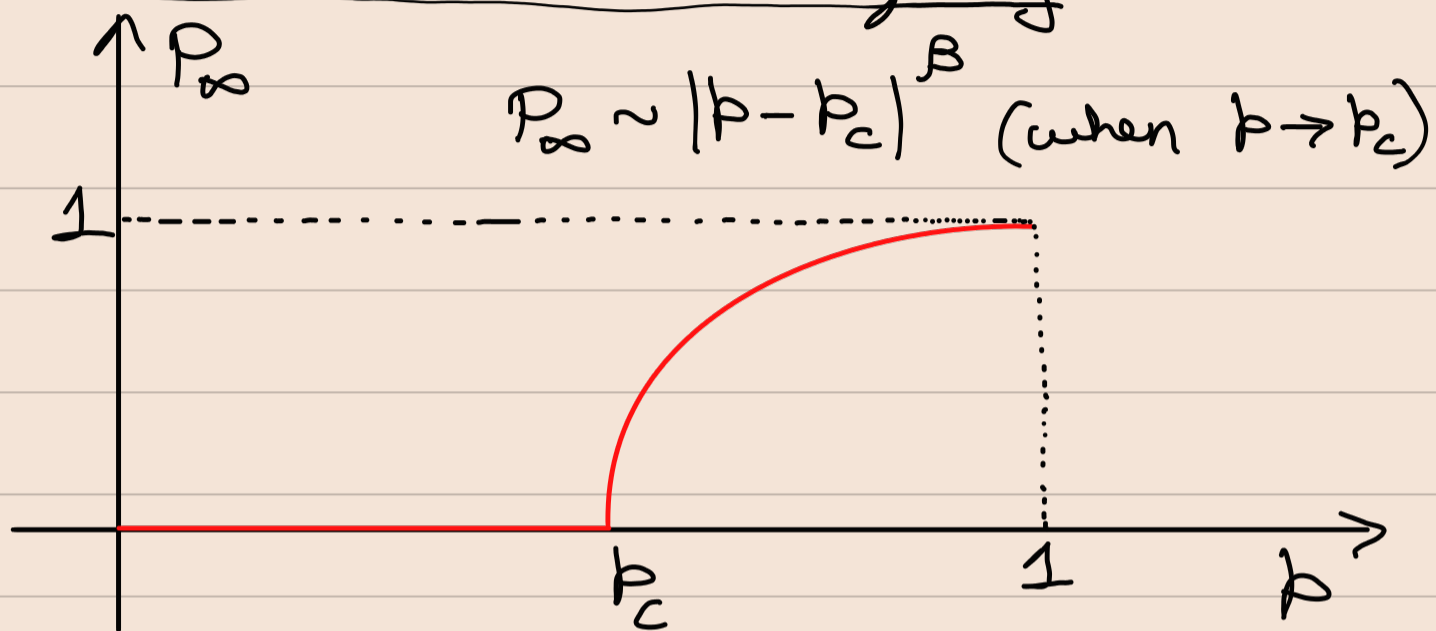
$p > p_c$



$p = 1$

Thus ' P_∞ ' acts as an indicator for Percolating phase ($P_\infty \neq 0$) vs a non-percolating phase ($P_\infty = 0$). Hence P_∞ is the order parameter for this geometric phase transition.

P_∞ is also equal to the fraction of sites which belong to ∞ -cluster & the equivalent statement is the Probability of a given random site Connected to infinity (lattice end).



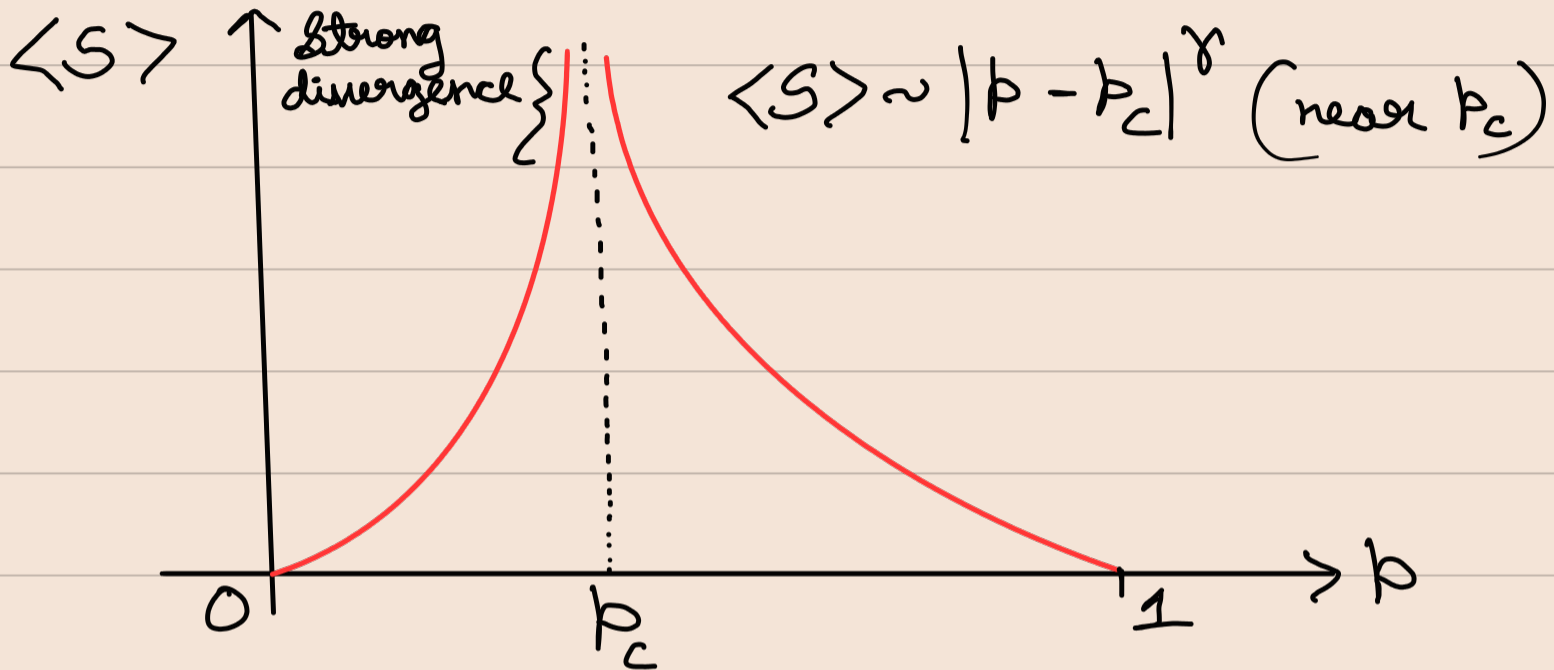
Mean Size of finite clusters $\langle S \rangle$

The existence of finite clusters is actually an indication of fluctuations in the number of sites per finite cluster. Thus the mean size of finite clusters is analogous to the magnetic susceptibility of spin systems.

$$\langle S \rangle = \sum_{\{S\}} S f(S, p)$$

$S \Rightarrow$ Size of finite cluster

$f(S, p) \Rightarrow$ Probability of having a cluster of size S at a given ' p '.

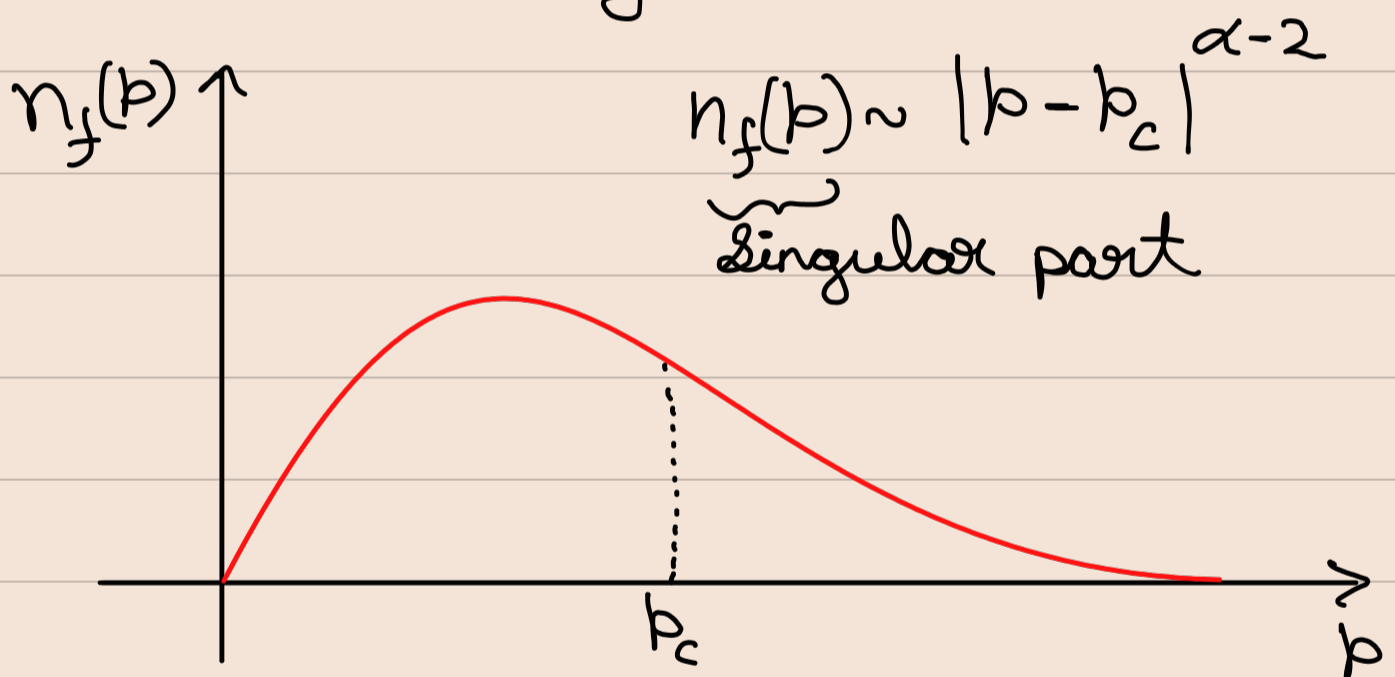


Average number of finite clusters $n_f(p)$

This is the percolation analogue of free energy. It counts how many distinct finite clusters exist in the system which describes the structure of fluctuations, same like how free energy has information about fluctuation.

$$n_f(p) = \frac{\text{Total number of finite clusters}}{\text{Total number of sites}}$$

$n_f(p)$ counts the number of distinct finite clusters, describing how they grow and shrink as p changes. This is nothing but the fluctuations in the percolating system.



The quantity $n_f(p)$ is continuous & smooth, but its first derivative is continuous but not smooth. And its second derivative is discontinuous & diverge. Thus percolation theory models a second order phase transition.

Correlation Function $G(r)$

Correlation Function b/w two occupied sites is the probability that the two occupied sites at a distance ' r ' away will belong to the same finite cluster. Thus $G(r)$ represents the Connectedness between two clusters.

$$G(r) = \sum_S P_S(r)$$

where $P_S(r)$ is the probability of two occupied sites at a distance ' r ' are in the same cluster of size ' S '.

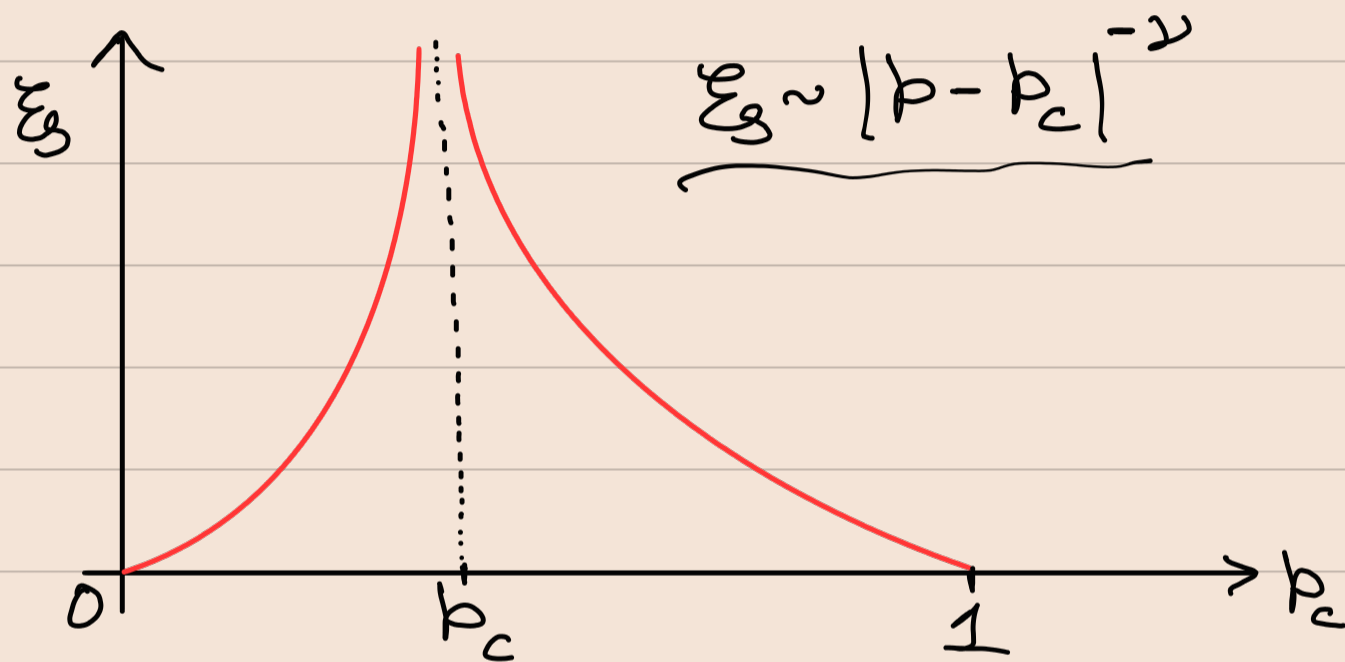
$$\text{For } p < p_c, \quad G(r) \sim e^{-r/\xi_s}$$

$$\text{For } p = p_c, \quad G(r) \sim \frac{1}{r^{d-2+\eta}}$$

For $p > p_c$, $G(r) \sim e^{-r/\xi_s}$ decays exponentially because of appearance of infinite clusters where finite clusters becomes rare.

Correlation Length (ξ_s)

It is a measure of connectivity b/w finite clusters & defines the characteristic length scales of the system. It determines how the correlation function behaves.



ξ_s is small for $p < p_c$ since the finite clusters are far apart & hence $G(r)$ decays rapidly which implies ξ_s is finite. At $p = p_c$, $\xi_s = \infty$, which implies there is no characteristic length scale. The system looks exactly the same in small or large lengths, hence it mimics a fractal.

For $p > p_c$, the number of finite clusters decrease & even the size reduces. Infinite clusters start to dominate. As the finite cluster starts to become smaller, the characteristic length scale also will decrease & become finite. At $p=1$, no finite cluster exist, the system looks homogenous & every site belongs to one giant infinite cluster, hence $E_s=0$ at $p=1$.

Some threshold p_c values

Square lattice :-

- Bond percolation : $p_c = 0.5$
- Site percolation : $p_c = 0.592746$

Simple cubic lattice :-

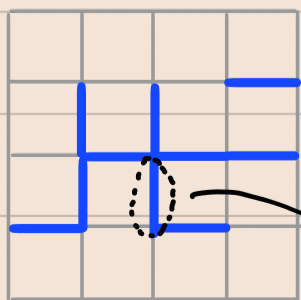
- Bond percolation : $p_c = 0.24$
- Site percolation : $p_c = 0.31$

Triangular lattice :-

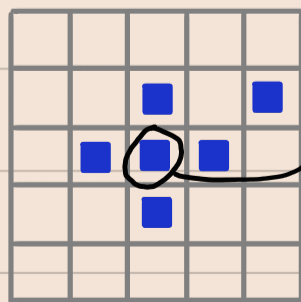
→ Bond percolation: $p_c = 0.3472$

→ Site percolation: $p_c = 0.5$

Thus usually it is easier to have geometric phase transition in bond percolation than in site percolation because in site percolation if a particular site is not occupied, then all four of its neighbouring site will not be connected leading to more possibility of breaking the cluster completely. But in bond percolation, even if one bond is not occupied, there is possibility of other three bonds to maintain connectivity w.r.t a particular site.



If this bond is not occupied, still cluster is not broken



If this site is not occupied then entire cluster is disrupted

Table of Critical exponents for Square / Simple Cubic lattice

d	$p_c(\text{bond})$	$p_c(\text{site})$	β	γ	ν	d_f
2	0.5	0.5927	$5/36$	$43/18$	$4/3$	$91/48$
3	0.248	0.3116	0.41	1.80	0.875	2.53
4	0.1601	0.1968	0.49	1.43	0.68	3
5	0.1182	0.1408	1	1	0.5	4
≥ 6	0.094	0.0978	1	1	0.5	4

The critical exponents & $p_c(\text{bond})$ for Square lattice can be got exactly. By doing the duality mapping as seen previously in q -potts model.

The duality transform for bond percolation in square lattice is another bond percolation in dual lattice.

Also the critical exponents for bond & site percolation will be same for a particular dimension as they belong to the same universality class.

Relating q -Potts random cluster model with Percolation Theory

Percolation problem is a Special Case of q -potts model when $q \rightarrow 1$, and is also called the Random structure model.

The q -potts random cluster model partition function is given by,

$$Z_{RCM} = \sum_{\{G_i\}} q^{n(G_i)} v^{b(G_i)} \quad (v = e^{\beta J} - 1)$$

$\{G_i\}$ = set of all possible configuration of graphs(G_i)

Put $q = 1 + \epsilon$ & keep upto 1st order in ' ϵ '.

$$\Rightarrow Z_{RCM}(q=1+\epsilon) = \sum_{\{G_i\}} (1+\epsilon)^{n(G_i)} v^{b(G_i)}$$

$$\Rightarrow Z_{RCM}(\epsilon \rightarrow 0) \sim \sum_{\{G_i\}} [1 + n(G_i)\epsilon] v^{b(G_i)}$$

$$Z_{RCM}(\epsilon \rightarrow 0) = \sum_{\{G_i\}} [v^{b(G_i)} + n(G_i)\epsilon v^{b(G_i)}]$$

This is a summation over all the possible bond configuration which has $b(G)$ number of occupied bonds with a weight of ' v ' for each bond.

The total number of possible bonds is equal to the number of edges (E) in the square lattice.

Since each edge can be either present or absent, there are a total of 2^E configurations. For 2d-square lattice in periodic boundary conditions $E = 2N$ & in non-periodic boundary, $E = 2N - N_x - N_y$ where $N \rightarrow$ total number of sites in the square lattice.

$\Rightarrow \{G\} : \{b=0, b=1, b=2, \dots, b=E\}$
where each element of above set represents all configurations with a specific value ' b '.

$$\Rightarrow \sum_{\{G\}} v^{b(G)} = \sum_{b=0}^E \left(\begin{array}{c} \text{no. of Config.} \\ \text{with } b\text{-bonds} \end{array} \right) \cdot v^b$$

Since each bond is independently placed, total no. of ways to choose 'b' bonds from E is ${}^E C_b$

$$\Rightarrow \sum_{\{G\}} v^{b(G)} = \sum_{b=0}^E {}^E C_b v^b = (1+v)^E$$

$$\Rightarrow \mathcal{Z}_{RCM}(q \rightarrow 1) = (1+v)^E + \varepsilon \sum_{\{G\}} n(G) v^{b(G)}$$

$$\Rightarrow \mathcal{Z}_{RCM}(q \rightarrow 1) = (1+v)^E \left[1 + \frac{\varepsilon}{(1+v)^E} \sum_{\{G\}} n(G) v^{b(G)} \right]$$

Put $v = \frac{p}{1-p} \Rightarrow p = \frac{v}{1+v} \Rightarrow 1+v = \frac{1}{1-p}$

where $v = e^{BJ} - 1 \Rightarrow p = 1 - e^{-BJ}$

Then we get $\mathcal{Z}_{RCM}(q \rightarrow 1)$ as

$$\mathcal{Z}_{RCM}(q \rightarrow 1) = \frac{1}{(1-p)^E} \left[1 + \varepsilon (1-p)^E \sum_{\{G\}} \left(\frac{p}{1-p} \right)^{b(G)} n(G) \right]$$

In Percolation theory a bond will be occupied with a probability ' p ' on every edge of the square lattice.

Let $G: \{\text{all possible configurations}\}$ & $1-p = \text{probability for a bond to be not occupied}$. Thus the probability weight of a particular configuration with ' $b(G)$ ' number of occupied bonds & ' $E - b(G)$ ' number of unoccupied bonds is given by

$$\omega(G) = p^{b(G)} (1-p)^{E-b(G)}$$

There is no any Hamiltonian governing the model, it is completely random.

Thus
$$\mathcal{Z}_{\text{Percolation}} = \sum_{\{G\}} \omega(G)$$

$$\sum_{\{G\}} = \sum_{b=0}^{\infty} E C_b \text{ as seen earlier.}$$

$$\Rightarrow \mathcal{Z}_{\text{Percolation}} = \sum_{b=0}^{\infty} E C_b p^b (1-p)^{E-b} = 1$$

Thus the partition function for the percolation model is trivially '1', reflecting that probabilities sum over to 1 over all possible bond config.

Even though partition function for the Percolation model is trivially '1', but still the quantity analogous to the free energy (F) can be found by calculating the average number of finite clusters $\langle n_f(p) \rangle$ which is analogous to F .

$$\langle n_f(p) \rangle = \sum_{\{G\}} W(G) n(G)$$

$$= \sum_{\{G\}} p^{b(G)} (1-p)^{E-b(G)} n(G)$$

$$= \sum_{\{G\}} \left(\frac{p}{1-p} \right)^{b(G)} (1-p)^E n(G)$$

$$\Rightarrow \langle n_f(p) \rangle = (1-p)^E \sum_{\{G\}} \left(\frac{p}{1-p} \right)^{b(G)} n(G)$$

Compare this with the $Z_{RCM}(q \rightarrow 1)$ expression which is

$$Z_{RCM}(q \rightarrow 1) = \frac{1}{(1-p)^E} \left[1 + \varepsilon (1-p)^E \sum_{\{G\}} \left(\frac{p}{1-p} \right)^{b(G)} n(G) \right]$$

$$\Rightarrow Z_{RCM}(q \rightarrow 1) = \frac{1}{(1-p)^E} (1 + \varepsilon \langle n_f(p) \rangle)$$

Sometimes $\langle n_f(p) \rangle = (1-p)^E \sum_{\{G\}} \left(\frac{p}{1-p} \right)^{b(G)} n(G)$

is chosen as the useful partition function analogue in the Fortuin-Kasteleyn (FK) theory of percolation model.

$$Z_{FK} = \sum_{b=0}^E C_b p^b (1-p)^{E-b} n(b)$$

This form of Z_{FK} is valid for any lattice & any dimension where $n(G)$ will change accordingly.

We can also get an expression to find the order parameter P_∞ as follows.

$$P_\infty = \sum_{\{G\}} W(G) \eta_{I.C}(G)$$

where $\eta_{I.C} = \begin{cases} 1, & \text{if 'G' has a } \infty\text{-cluster} \\ 0, & \text{if 'G' has no } \infty\text{-cluster} \end{cases}$

where $W(G) = p^{b(G)} (1-p)^{E-b(G)}$

$$\Rightarrow P_\infty = \langle \eta_{I.C}(G) \rangle$$

$$P_\infty = \sum_{b=0}^{\infty} C_b p^b (1-p)^b \eta_{I.C}(b)$$

This is also valid for any lattice & any dimension where $\eta_{I.C}(b)$ will change accordingly.

Spanning tree Problem

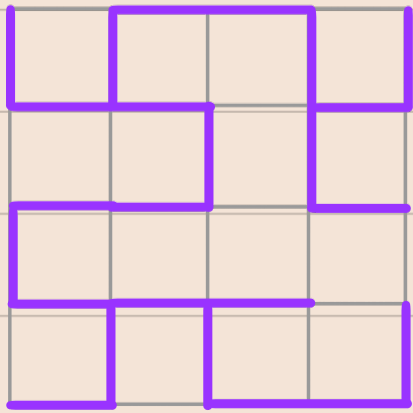
A Spanning tree is that configuration of bonds, where every site is a part of one big connected cluster ($n_G = 1$) with no loops (or circuits) ($C_G = 0$). This configuration is also called as uniform spanning tree.

Thus a Spanning tree configuration has $C(G) = 0$ & $n(G) = 1$.

Using Euler relation $\underbrace{b(G)}_{\text{no: of bonds}} + \underbrace{n(G)}_{\text{no: of clusters}} = \underbrace{N}_{\text{No: of sites}} + \underbrace{C(G)}_{\text{no: of circuits}}$

we get $b(G) = N - 1$ (no: of bonds)

Eg:



$N = 25$
 $b = 24$
This is one of the Spanning tree Config. for 5×5 Square lattice

Consider the q -potts random cluster model partition function.

$$Z_{RCM} = \sum_{\{G\}} v^{b(G)} q^{n(G)}$$

$$\text{If } v = q^\alpha \text{ (for some } \alpha) \Rightarrow e^K - 1 = q^\alpha \quad (K = \beta J)$$

$$\Rightarrow \beta J = \ln[1 + q^\alpha] \Rightarrow T = \frac{J}{K_B \ln[1 + q^\alpha]}$$

i.e. at this temperature of q -potts model, we get

$$Z_{RCM} = \sum_{\{G\}} q^{[n(G) + \alpha b(G)]}$$

Now as $q \rightarrow 0$, the values $q^{[n(G) + \alpha b(G)]}$ will be very negligible for high values of $n(G)$. The minimum value of $n(G) = 1$ for any graph. Thus if $n(G) = 1$, then by using Euler's formula, $b(G) + n(G) = N + C(G)$
 $\Rightarrow b(G) = N - n(G) + C(G)$

$$\Rightarrow \lim_{q \rightarrow 0} Z_{RCM} = \lim_{q \rightarrow 0} \sum_{\{G_i\}} q^{n(G_i) + \alpha[N - n(G_i) + C(G_i)]}$$

Those configurations which have most minimal value of the exponent of 'q' will be dominant.

$\Rightarrow \alpha N + n(G_i)[1 - \alpha] + C(G_i)$ should be as small as possible. Now $C(G_i)$ can be zero in some configurations, thus those configurations where $C(G_i) = 0$, will be dominant $\Rightarrow \underline{b(G_i) + n(G_i) = N}$

$$n(G_i) = 1 \quad (\text{as small as possible})$$

$$\Rightarrow \underline{b(G_i) = N - 1}$$

$$\Rightarrow \lim_{q \rightarrow 0} Z_{RCM} = \lim_{q \rightarrow 0} \sum_{\{G_i\}} q^{1 + \alpha(N-1)}$$

$\left(\begin{matrix} C(G_i) = 0 \\ n(G_i) = 1 \end{matrix} \right) \rightarrow \text{restriction}$

$$\Rightarrow \lim_{q \rightarrow 0} Z_{RCM} = \lim_{q \rightarrow 0} \sum_{\{ST\}} q^{\alpha N} q^{1 - \alpha}$$

$\{ST\} \rightarrow \text{Spanning tree}$

$$\{S.T.\} = \{G_i : \text{Such that } C(G_i) = 0, n(G_i) = 1\}$$

Now in thermodynamic limit we need a non zero partition function. If $\alpha \neq 1$ then as $q \rightarrow 0$, Z_{RCM} will either become zero if $\alpha < 1$ or become infinity if $\alpha > 1$. Thus for physically allowed partition function we need $\alpha = 1$.

$$\Rightarrow \lim_{q \rightarrow 0} Z_{RCM} = \sum_{\{S, T\}} \lim_{q \rightarrow 0} q^N$$

Now in thermodynamic limit $N \rightarrow \infty$. Thus the RHS $\neq 0$.

$\Rightarrow \lim_{\{q \rightarrow 0, N \rightarrow \infty\}} q^N$ is some non-zero number. If the limits are taken in balanced way, we can set it to '1'.

$$\lim_{\substack{q \rightarrow 0 \\ N \rightarrow \infty}} Z_{RCM} = \sum_{\{S, T\}} (1) = \text{No: of possible spanning trees}$$

Thus the resultant is a uniform weight spanning tree partition function.

The corresponding potts temperature for Spanning tree limit is given by

$$T = \lim_{q \rightarrow 0} \frac{T}{K_B \ln[1 + q^\alpha]} \Rightarrow \begin{matrix} T \rightarrow \infty \\ \& \& q \rightarrow 0 \end{matrix}$$

This doesn't mean the potts model is at infinite temperature. Rather this has to be interpreted as a completely randomized state with unit weight for all allowed configurations.

We can also relate this to the critical fill fraction p_c of the percolation model. When $q \rightarrow 0$, Spanning trees are the dominant configuration. This is where the infinite cluster is barely possible with minimal no. of bonds. Thus this state correspond to percolation at phase transition " p_c ". For square lattice we have $p_c = 0.5$ (bond). Here $b = N - 1$
 $E \sim 2N$ (total edges) $\Rightarrow p = \underline{b/E} \sim 0.5 \Rightarrow \underline{\underline{p_c}}$