

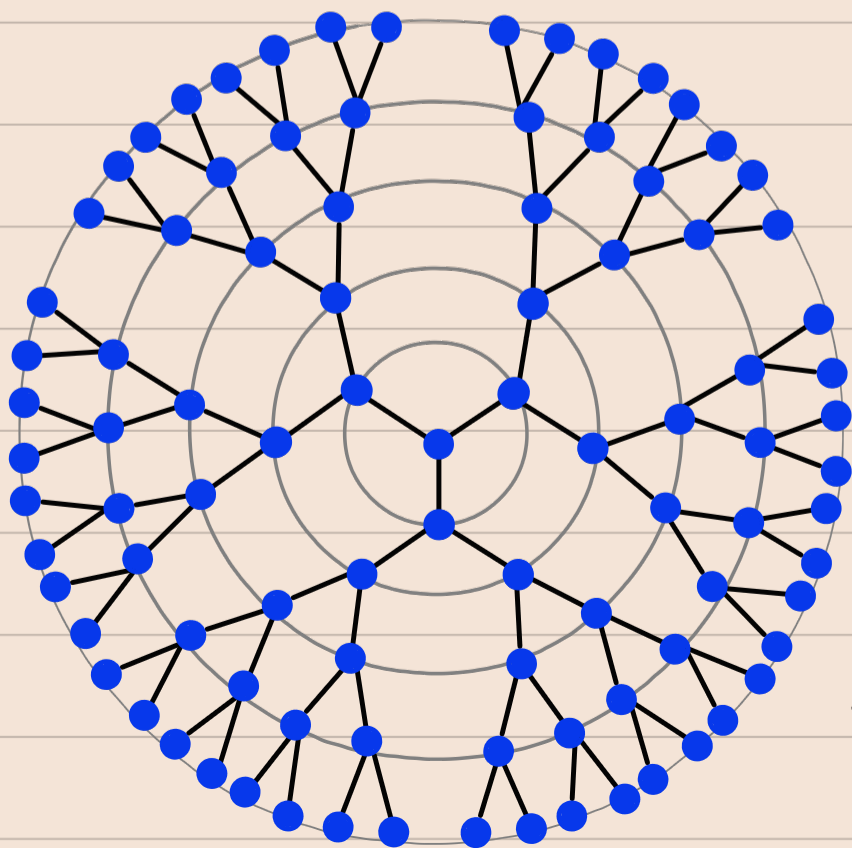
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Lecture 20

Cayley Tree & Bethe Lattice

Bethe lattice is a graph with an infinite number of points, each connected to a fixed number of neighbours ' z ' (coordination number), such that no loops/circuits exist. Usual Bethe lattices have $z \geq 3$.

A finite type of the above graph is called a 'Cayley Tree'.



$$z = 3$$



Cayley tree with
1 central node
(root) & 5 shells
(generations)

with $N = 94$
sites (vertices)

& 93 bonds (edges)

A Cayley tree with one root (node) can be characterized by the coordination number ' z ' & no: of generations ' g '.

where $N(z, g) = 1 + b(z, g)$
no: of sites (vertices) no: of bonds (edges)

$$\& b(z, g) = z \sum_{n=0}^{g-1} (z-1)^n$$

Bethe lattice is an infinitely big Cayley tree where $g \rightarrow \infty \Rightarrow N \rightarrow \infty$. Cayley tree & Bethe lattice has no well defined spatial (euclidian) dimensions as the Cayley tree grows exponentially instead of growing polynomially with the distance from the root node. For usual lattices $N(g) \sim g^d$ $\Rightarrow$ euclidian dimension. But for Cayley tree $N(g)$ varies exponentially, $N(g) \sim (z-1)^g$ which implies that the euclidian dimension of space where Cayley tree is embedded has to grow exponentially with ' g '.

Cayley tree has a root vertex & has a well defined boundary 'g' which is the number of generations. Every vertex except the ones at the boundary (g^{th} generation) have Coordination number 'z'. For large 'z', most vertices lie at the boundary (Surface).

For Bethe lattice, $g \rightarrow \infty \Rightarrow N \rightarrow \infty$, so no finite euclidean dimensional Space can be used to embed Bethe lattice. Thus Bethe lattice is an infinite dim. lattice. It has no natural boundary, it is homogenous & isotropic, every vertex has z-number of neighbours with no loops. 'z' is the fundamental property of a given Bethe lattice. Its topology in a loose sense is hyperbolic with a negative curvature, where the distance between two branches grow exponentially same as in hyperbolic Space where two geodesics exponentially go far away.

Consider a Cayley tree $(\mathbb{Z}, g)$, the total number of sites is given by

$$\begin{aligned}
 N(\mathbb{Z}, g) &= 1 + \mathbb{Z} \sum_{n=0}^{g-1} (\mathbb{Z}-1)^n \\
 &= 1 + \mathbb{Z} \left[\frac{(\mathbb{Z}-1)^g - 1}{(\mathbb{Z}-1) - 1} \right] = 1 + \frac{[\mathbb{Z}(\mathbb{Z}-1)^g - \mathbb{Z}]}{\mathbb{Z}-2} \\
 &= \frac{\mathbb{Z}-2 + \mathbb{Z}(\mathbb{Z}-1)^g - \mathbb{Z}}{\mathbb{Z}-2} \\
 \Rightarrow N(\mathbb{Z}, g) &= \frac{\mathbb{Z}(\mathbb{Z}-1)^g - 2}{(\mathbb{Z}-2)}
 \end{aligned}$$

The number of surface sites is $\mathbb{Z}(\mathbb{Z}-1)^{g-1}$.

$$\begin{aligned}
 \Rightarrow N &= \frac{(\mathbb{Z}-1) N_{\text{surf}} - 2}{\mathbb{Z}-2} \\
 \Rightarrow (\mathbb{Z}-2) N &= (\mathbb{Z}-1) N_{\text{surf}} - 2
 \end{aligned}$$

$$\Rightarrow \frac{N_{\text{surf}}}{N} = \frac{\mathbb{Z}-2}{\mathbb{Z}-1} + \frac{2}{(\mathbb{Z}-1)N}$$

Thus for large N limit or $g \rightarrow \infty$
(Bethe lattice)

$$\boxed{\frac{N_{\text{surface}}}{N} \sim \frac{z-2}{z-1}}$$

Now, $N_{\text{interior}} = N - N_{\text{surface}}$

$$\Rightarrow \frac{N_{\text{int.}}}{N} = 1 - \frac{N_{\text{surface}}}{N}$$

$$\Rightarrow \frac{N_{\text{int.}}}{N} = 1 - \frac{z-2}{z-1} = \frac{z-1-(z-2)}{z-1}$$

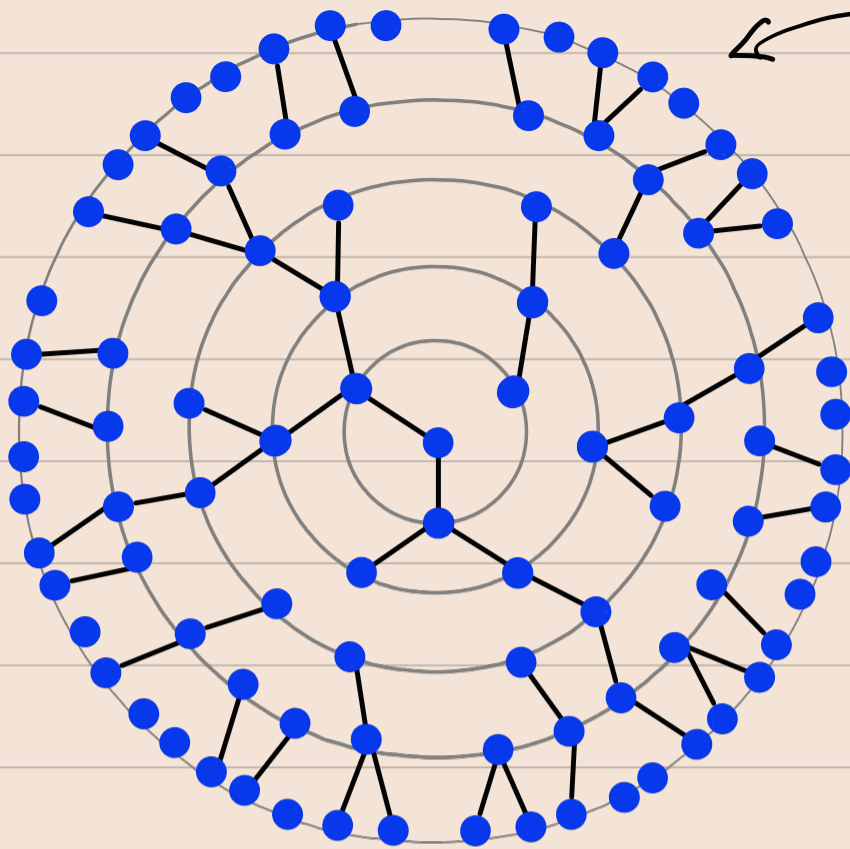
$$\Rightarrow \frac{N_{\text{int.}}}{N} = \frac{1}{z-1} \quad \& \quad \frac{N_{\text{int.}}}{N_{\text{surf}}} = \frac{1}{z-2}$$

z	$\frac{N_{\text{surf}}}{N}$	$\frac{N_{\text{int}}}{N}$	$\frac{N_{\text{int}}}{N_{\text{surf.}}}$
3	$1/2$	$1/2$	1
4	$2/3$	$1/3$	$1/2$
5	$3/4$	$1/4$	$1/3$
6	$4/5$	$1/5$	$1/4$
7	$5/6$	$1/6$	$1/5$

Percolation in Bethe lattice

Percolation problem can be exactly solved for $d=1$ (1D lattice) & for $d=\infty$ (Bethe lattice), because they have no loops / cycles which simplifies the calculations. Surprisingly, if $z=2$ in Bethe lattice, it is topologically equal to 1D lattice. Thus in few results of Bethe lattice, if we put $z=1$ (if allowed) it will be equal to that of 1D lattice.

Consider the Bethe lattice & for every bond, remove it with a probability ' q '. Thus ' q ' fraction of bonds will be removed at random. This is equivalent to a bond being occupied with a probability ' p ' & then finally having ' p ' fraction of occupied bonds in total. This is bond percolation in Bethe lattice.



Here q -fraction of bonds are removed. And $p = 1 - q$ fraction of bonds are present. This is one of the many allowed Config.

$p \Rightarrow$ occupation probability.

p_c is that critical probability where the infinite clusters are possible in an infinite Bethe lattice.

Starting from any site, & when we go outwards, we encounter $z-1$ new nearest neighbours. Thus on an average we have ' $p(z-1)$ ' new ways to continue the path, thus if $p(z-1) = 1$, then we have at least one way to go to infinity.

Thus the probability at which $p(z-1)=1$ is ' p_c '.

$$\Rightarrow \boxed{p_c = \frac{1}{z-1}}$$

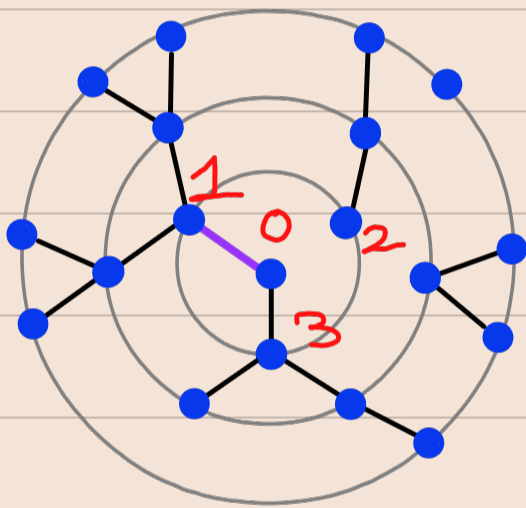
$\Rightarrow$ Threshold probability for infinite cluster.

For $z=3$ as in the graph shown above, $p_c = 1/2$. Thus only if half the fraction of bonds are filled, there is a non-zero probability to have an infinite cluster.

For $z=2$, we arrive at a 1D lattice. This results in $p_c = 1$, a correct value. Because in 1D chain, only if every bond is occupied, we can have an infinite cluster. Even if one bond is not occupied, it will not be an infinite cluster.

Disconnection Probability (Q)

The quantity Q represents the probability that an arbitrary Site is NOT connected to the infinite cluster through a particular Branch extending from that Site.



Consider an arbitrary Site '0' (equivalent to every other site)

' Q ' is given by the probability that the branch through Site '1' is not connected to infinity.

The branch can lead to an immediate disconnection if the site '1' is not occupied. But if it is occupied, no sub-branch of '1' should lead to infinity. These two cases are mutually exclusive (but not exhaustive).

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$$\Rightarrow Q = \text{Prob.} ['1' \text{ is not occupied}] \\ + \text{Prob.} ['1' \text{ is occupied}] \\ \times \text{Prob.} \left[\begin{array}{l} \text{No sub-branch} \\ \text{of '1' lead to } \infty \end{array} \right]$$

Now we need to account for the other ' $z-1$ ' branches of ' 1 ' & we don't need them leading to infinity. Probability of each branch not leading to infinity is ' Q '. Then the probability that the $z-1$ sub-branches not leading to infinity is Q^{z-1} . This is true as all sites & branches of the Bethe lattice are all equivalent.

$$\Rightarrow Q = \underbrace{(1-p)}_{\text{Prob. that Site '1' is not occupied}} + p \underbrace{Q^{z-1}}_{\text{Prob. that Site '1' is occupied}} \left. \begin{array}{l} \text{Prob. that } z-1 \\ \text{sub branches} \\ \text{don't lead to } \infty \end{array} \right\}$$

Note: The above results were for site percolation, but even for bond percolation the logic & results are same.

The eqⁿ $Q = (1-p) + pQ^{z-1}$ is an exact relation which can be solved iteratively or analytically if possible.

Eg: For $z=2$, $Q = 1-p + pQ$

(1D lattice) $\Rightarrow Q(1-p) = 1-p$

$\Rightarrow Q = 1$ if $p \neq 1$ which implies that there is no infinite cluster. When $p=1$, every site (bond) is occupied with certainty. Hence $Q=0$ when $p=1$.

For $z=3$: $Q = 1-p + pQ^2$

$\Rightarrow Q_{\pm} = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p}$

$\Rightarrow Q_{\pm} = \frac{1 \pm \sqrt{4p^2 - 4p + 1}}{2p} = \frac{1 \pm |(2p-1)|}{2p}$

$\Rightarrow Q = 1$ or $\frac{1-p}{p}$. when $p < p_c$,

there are no ∞ -clusters, thus $Q=1$.

And for $p > p_c$ there can be ∞ -clusters.

So $Q = \frac{1-p}{p}$ for $p > p_c$.

The critical p_c value is such that the infinite clusters just begin to appear. So at p_c , $Q = 1 = \frac{1-p_c}{p_c}$

This gives us a way to find " p_c "
 $\Rightarrow p_c = 1 - p_c \Rightarrow 2p_c = 1 \Rightarrow p_c = \frac{1}{2}$
 for $z=3$.

For $z=4$: $Q = 1 - p + pQ^3$

$$\Rightarrow pQ^3 - Q + (1-p) = 0$$

$Q = 1$ is a possible solution.

$$\begin{array}{r}
 \Rightarrow Q - 1 \overline{) \begin{array}{l} pQ^2 + pQ + (p-1) \\ pQ^3 + 0Q^2 - Q + (1-p) \\ (-) pQ^3 - pQ^2 \\ \hline pQ^2 - Q \\ (-) pQ^2 - pQ \\ \hline (p-1)Q + 1 - p \\ (-) (p-1)Q - (p-1) \\ \hline 0 \end{array}}
 \end{array}$$

$$\Rightarrow (Q-1)(pQ^2 + pQ + p-1) = 0$$

$$\Rightarrow Q=1 \text{ or } \frac{-p \pm \sqrt{4p-3p^2}}{2p}$$

Q can not be negative. Thus there are two allowed values.

Similarly we can argue that $Q=1$ for $p < p_c$ & $Q = \frac{-p + \sqrt{4p-3p^2}}{2p}$ for $p > p_c$.

$$\text{At } p_c \quad Q=1 = \frac{-p_c + \sqrt{4p_c-3p_c^2}}{2p_c}$$

$$\Rightarrow 2p_c = -p_c + \sqrt{(4p_c-3p_c^2)}$$

$$(3p_c)^2 = 4p_c - 3p_c^2$$

$$\Rightarrow 12p_c^2 = 4p_c$$

$$4p_c(3p_c-1) = 0$$

$$\Rightarrow \underline{p_c = \frac{1}{3}} \text{ for } z=4$$

To find p_c for a general z

Let $Q = (1-p) + pQ^{z-1}$ be written as $Q = f(Q)$. The solution for the eqⁿ is given by the fixed point $Q^* = f(Q^*)$.

When $p < p_c$, only $Q^* = 1$ should be allowed. & for $p > p_c$ $Q = 1$ should not be allowed, i.e., $Q = 1$ should become an unstable fixed point & will lead to another fixed point.

To check the stability of Q^* we resort to linear perturbation & check, when the perturbation will decay (stable) or grow (unstable)

$$\Rightarrow (Q^* + \delta Q) = f(Q^* + \delta Q)$$

δQ is the perturbation.

Performing Taylor expansion, we can tell if δQ grows / decays.

$$\underline{Q^*} + \delta Q = \underline{f(Q^*)} + \left. \frac{\partial f}{\partial Q} \right|_{Q^*} \delta Q + \dots$$

$$\Rightarrow \delta Q = \left. \frac{\partial f}{\partial Q} \right|_{Q^*} \delta Q + O(\delta Q^2)$$

In an iterative analysis

$$\delta Q_{n+1} = \left. \frac{\partial f}{\partial Q} \right|_{Q^*} \delta Q_n$$

where $n \Rightarrow$ iteration index.

$$\text{If } \left. \frac{\partial f}{\partial Q} \right|_{Q^*} < 1, \quad \lim_{n \rightarrow \infty} |\delta Q_{n+1} - \delta Q_n| \rightarrow 0$$

Thus perturbations decay over many iterations. Then the fixed point is stable.

Similarly, if $\left. \frac{\partial f}{\partial Q} \right|_{Q^*} > 1$, the fixed

point is unstable. Thus at ' p_c ' we need $\left. \frac{\partial f}{\partial Q} \right|_{Q^*} = 1$ for $Q^* = 1$ to

transit from stable to an unstable fixed point.

$$\frac{\partial f}{\partial Q} = p(z-1)Q^{z-1} \quad \& \quad \left. \frac{\partial f}{\partial Q} \right|_{Q=1, p=p_c} = 1$$

$$\Rightarrow \left. \frac{\partial f}{\partial Q} \right|_{Q=1, p=p_c} = p_c(z-1)$$

$$\Rightarrow p_c(z-1) = 1$$

$$\Rightarrow \boxed{p_c = \frac{1}{z-1}}$$

Behaviour of 'Q' near $p > p_c$

Here we need to see the functional form of 'Q', near p_c , which can be achieved through Taylor's expansion of $f(Q)$

$$Q = (1-p) + pQ^{z-1} = f(Q)$$

expand $f(Q)$ about $Q=1$, & for $Q \neq 1$ we get its behaviour near ' p_c '.

$$\Rightarrow f(Q) = f(1) + \left. \frac{\partial f}{\partial Q} \right|_{Q=1} (Q-1) + \frac{\left. \partial^2 f}{\partial Q^2} \right|_{Q=2} \frac{(Q-1)^2}{2!} + \dots$$

$$\Rightarrow Q = 1 + p(z-1)(Q-1) + \frac{p(z-1)(z-2)(Q-1)^2}{2!}$$

(keep terms upto 2nd order)

$$\Rightarrow (Q-1) = \left[p(z-1) + \frac{p(z-1)(z-2)(Q-1)}{2} \right] (Q-1)$$

For the solution $Q \neq 1$ we get,

$$p(z-1) + \frac{p(z-1)(z-2)(Q-1)}{2} = 1$$

$$\Rightarrow p(z-1) \left[1 + \frac{(Q-1)(z-2)}{2} \right] = 1$$

$$\Rightarrow (Q-1) = \frac{2}{z-2} \left[\frac{1}{p(z-1)} - 1 \right]$$

Thus when p is very close to ' p_c '

$$Q = \begin{cases} 1, & (p < p_c) \\ 1 - \frac{2[p(z-1)-1]}{p(z-1)(z-2)}, & (p > p_c) \end{cases}$$

At p_c we need ' Q ' to be Continuous, thus $p_c(z-1)-1=0$ is needed. Hence ' $p_c = \frac{1}{z-1}$ ', we

recovered ' p_c ' even here.

Percolation Probability (P_∞)

P_∞ is defined as the probability that an occupied site, chosen at random, belongs to ∞ -cluster. It measures the fraction of sites (or bonds), that are part of ∞ -cluster.

$P_\infty = \text{Prob. at least one of its branch leading to } \infty$

Prob. that none of the branch lead to infinity is given by Q^z as ' Q ' prob. of each branch to not to lead to infinity $\neq$ Since each site has z -branches.

$$\Rightarrow \underline{P_\infty = 1 - Q^z}$$

Behaviour of P_∞ near p_c

To get P_∞ near p_c we should find how Q behaves near p_c . i.e. when $Q \rightarrow 1$ but $Q \neq 1$. We shall use the recurrence relation,

$$Q = 1 - p + pQ^{z-1}$$

$$\text{Let } \underline{Q = 1 - \delta} \quad (0 < \delta < 1)$$

$$\Rightarrow 1 - \delta = 1 - p + p(1 - \delta)^{z-1}$$

$$\Rightarrow \delta = p [1 - (1 - \delta)^{z-1}]$$

$$\frac{\delta}{p} = 1 - \left[1 + (z-1)(-\delta) + \frac{(-\delta)^2}{2}(z-1)(z-2) \right]$$

(2nd order Binomial expansion)

$$\frac{\delta}{p} = (z-1)\delta \left[1 - \frac{\delta(z-2)}{2} \right]$$

$\beta = 0$ corresponds to $Q = 1$
which is when $p < p_c$. But for
 $Q > 1$, $\beta \neq 0$ rather $\beta > 0$.

$$\Rightarrow p(z-1) \left[1 - \frac{\beta}{2} (z-2) \right] = 1$$

$$\Rightarrow \frac{\beta}{2} (z-2) = 1 - \frac{1}{p(z-1)}$$

$$\Rightarrow \beta = \frac{2[p(z-1) - 1]}{p(z-1)(z-2)}$$

In terms of p_c , $p_c = \frac{1}{z-1}$

$$\Rightarrow \beta = 2 \left[\left(\frac{p}{p_c} \right) - 1 \right] \left(\frac{p_c}{p} \right) \frac{1}{\left[\frac{1}{p_c} - 1 \right]}$$

$$\beta = 2 \left[\frac{p - p_c}{p_c} \right] \frac{p_c}{p} \frac{p_c}{(1 - p_c)}$$

$$z = \frac{2p_c}{1-p_c} \left[\frac{p-p_c}{p} \right]$$

$$p = p_c \left[\frac{(p-p_c)}{p_c} + 1 \right]$$

$$\Rightarrow z = \frac{2p_c^2}{(1-p_c)} \left[\frac{p-p_c}{p_c} \right] \left(\frac{1}{p_c} \right) \frac{1}{1 + \left[\frac{p-p_c}{p_c} \right]}$$

$$\frac{1}{1 + \left(\frac{p-p_c}{p_c} \right)} = 1 - \left(\frac{p-p_c}{p_c} \right) + \left(\frac{p-p_c}{p_c} \right)^2 - \dots$$

$$\Rightarrow z = \frac{2p_c}{1-p_c} \left(\frac{p-p_c}{p_c} \right) \left[1 - \left(\frac{p-p_c}{p_c} \right) + \left(\frac{p-p_c}{p_c} \right)^2 - \dots \right]$$

Upto 1st order of $\frac{p-p_c}{p_c}$ we have

$$z \sim \frac{2}{1-p_c} (p-p_c)$$

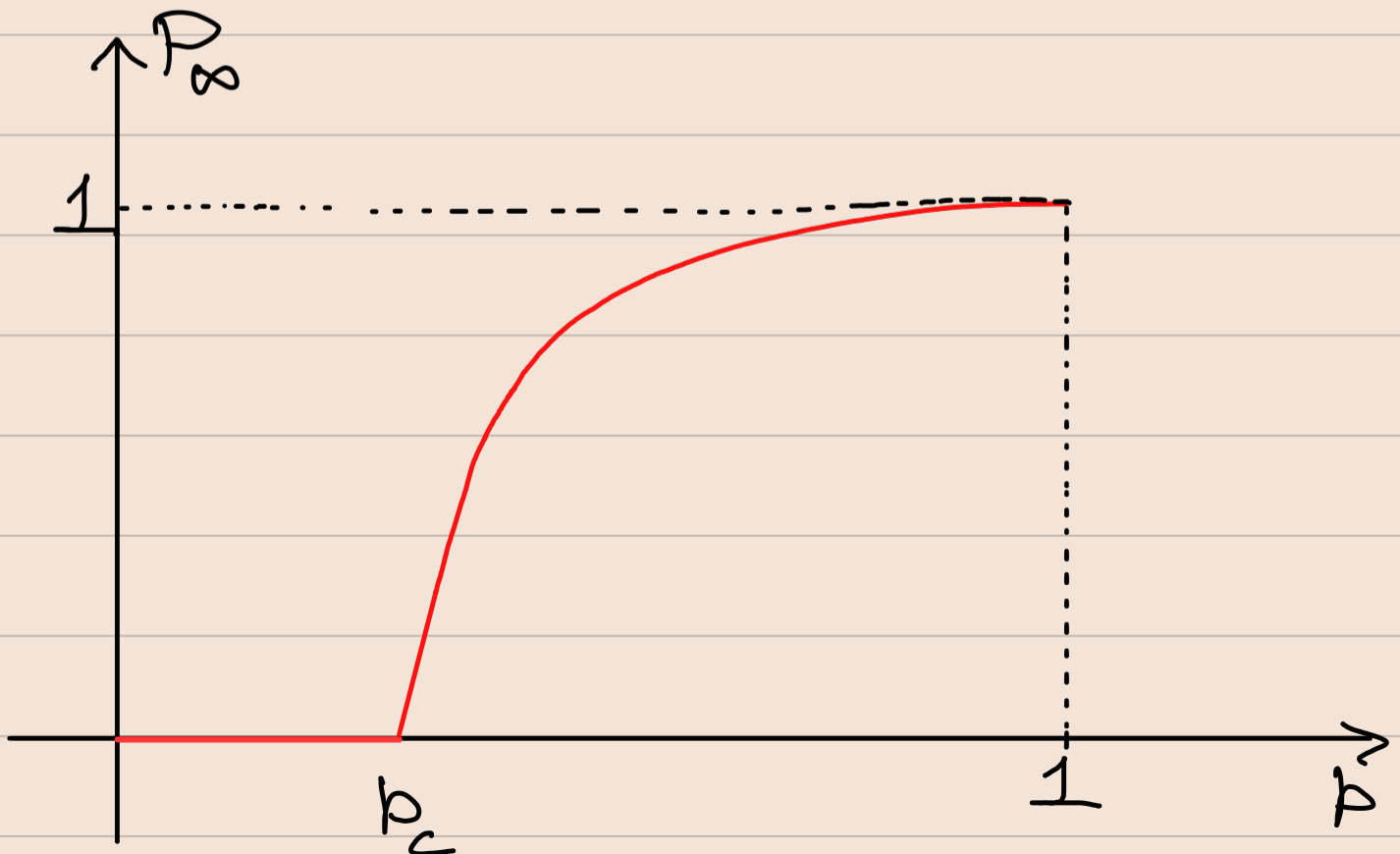
$$\text{Now } P_{\infty} = 1 - Q^z$$

$$\Rightarrow P_{\infty} = 1 - (1-z)^z \sim 1 - 1 + z$$

$$\Rightarrow P_{\infty} \sim \frac{2z}{1-p_c} (p - p_c)$$

$$\Rightarrow P_{\infty} \propto (p - p_c) \quad \underline{\text{when } p \rightarrow p_c}$$

Thus the critical exponent ' β ' is exactly given by '1'.



Mean Size of finite clusters

Every site of Bethe lattice are equivalent. So, to find the mean size of finite clusters, we can just choose the origin & find the average size of the finite cluster containing it.

Let η_{om} be an indicator function to indicate the connection between site 'o' & site 'm'.

$$\eta_{oi} = \begin{cases} 1, & \text{'o' \& 'm' are connected} \\ 0, & \text{'o' \& 'm' are not connected} \end{cases}$$

Then size of cluster containing 'o' is

$$S = \sum_m \eta_{om}$$

$S \Rightarrow$ counts the no: of sites where 'o' is connected to.

This definition is possible since there are no loops in lattice. If a particular site is not connected, then that entire sub-branch is disconnected from the origin.

$$\text{So, } S = \sum_m \eta_{om} \Rightarrow \langle S \rangle = \sum_m \langle \eta_{om} \rangle$$

In Bethe lattice, the number of sites at 'm' steps away is given by ' $z(z-1)^{m-1}$ ' as seen in the properties of Cayley tree. Now each site in the path from 1 to 'm' should be occupied, so when $p \rightarrow p_c$ & $p < p_c$, we get

$$\langle \eta_{om} \rangle = p^m z(z-1)^{m-1}$$

$$\Rightarrow \langle S \rangle = \sum_m p^m \left(\frac{z}{z-1} \right) (z-1)^m$$

$$\Rightarrow \langle S \rangle = \frac{z}{z-1} \sum_m [p(z-1)]^m$$

$$\Rightarrow \langle S \rangle = \frac{z}{z-1} \left[\frac{1}{1 - \frac{p}{p_c}} \right]$$

$$\Rightarrow \langle S \rangle = \frac{z}{z-1} \frac{p_c}{(p_c - p)} \quad (p < p_c)$$

$$\Rightarrow \langle S \rangle \propto (p_c - p)^{-1} \quad (p < p_c)$$

' γ ' is defined as $\langle S \rangle \propto |p - p_c|^{-\gamma}$

Thus $\gamma = 1$ for Bethe lattice.