

19/03/25

Lecture 21

Physical Problems on Percolation

In this lecture we will focus on various versions of percolation & some Physical problems on percolation which can model disordered Systems.

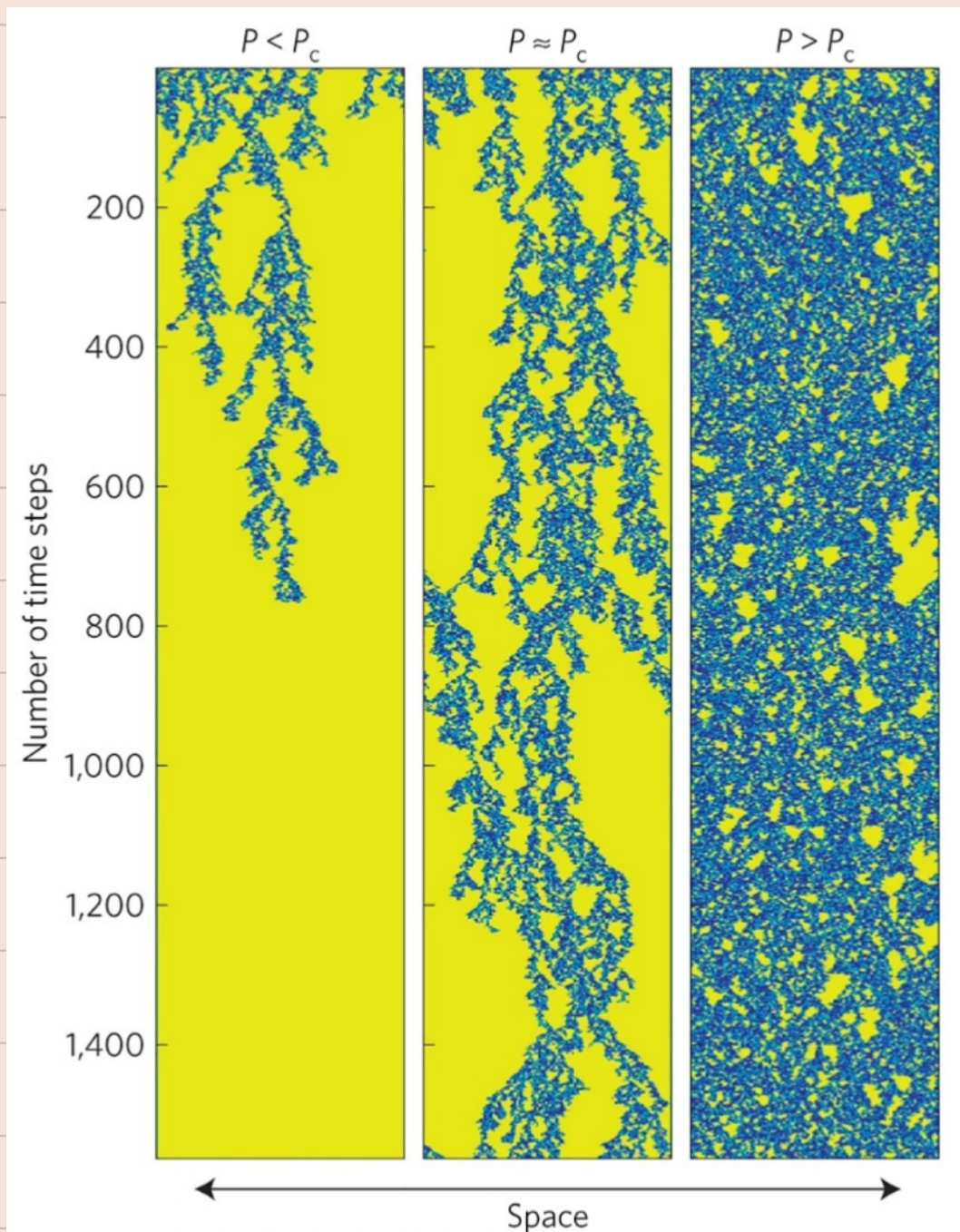
Other types of Percolation

a) Directed Percolation

This is an anisotropic version of the usual simple percolation. Here the connectivity flows in a preferred direction, where the clusters grow asymmetrically.

It can model phenomena like epidemic spreading, fluid flow along a porous media due to gravity, stochastic dynamics with an arrow of time.

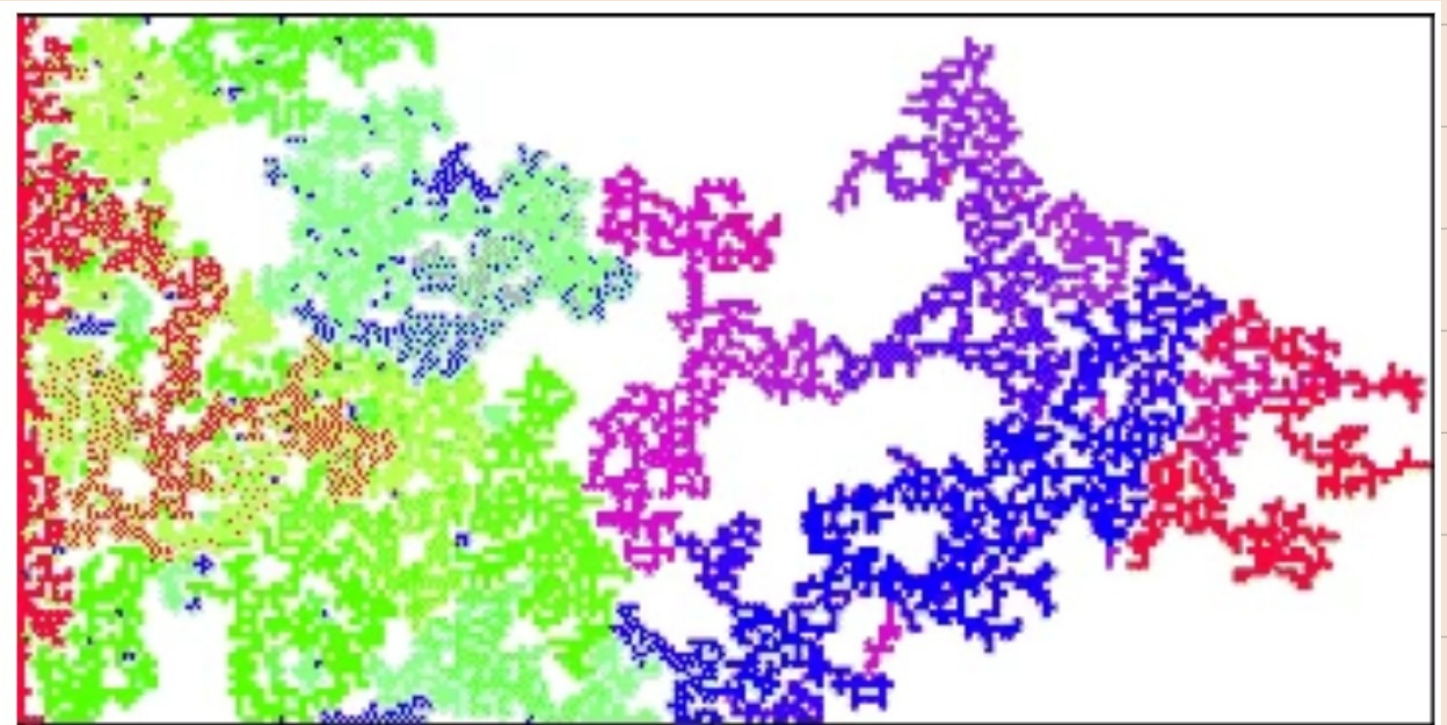
Eg: (i) A 1D lattice & its evolution over time, here sites at time ' t ' units can affect sites at time $t+1$ units only.
(ii) If in a 2D percolation a site (x_i, y_i) can connect only to site at (x_i+1, y_i) or (x_i, y_i+1) , forward in space.



Directed Percolation ↑

b) Invasive Percolation

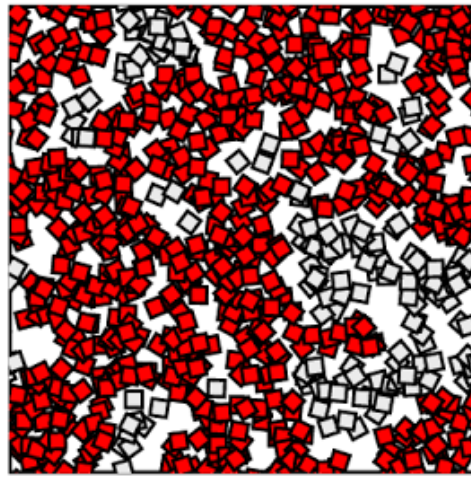
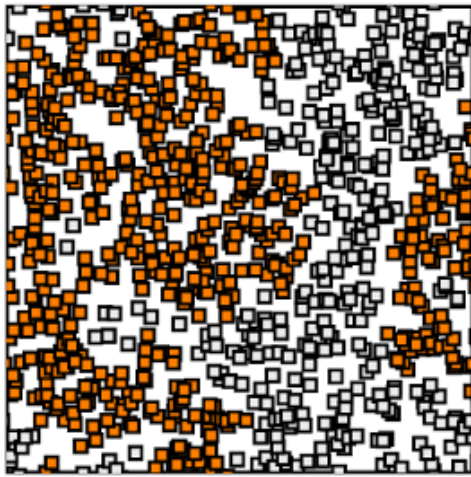
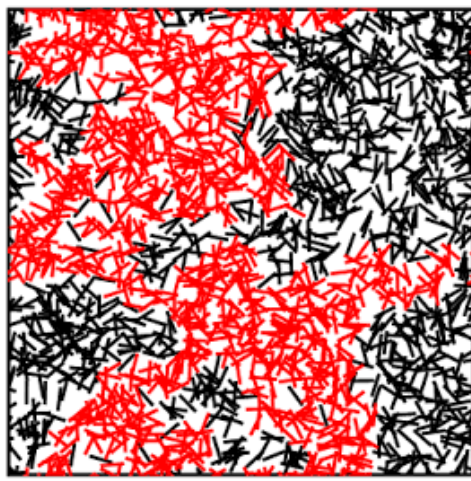
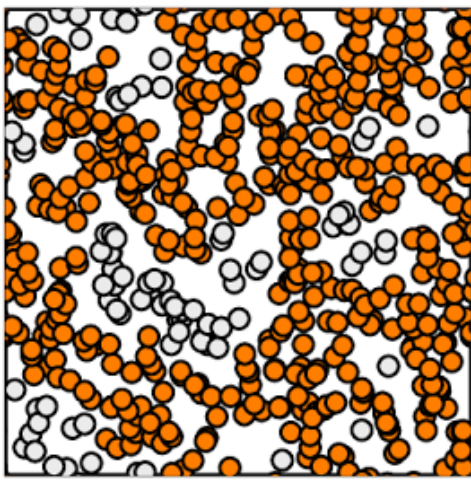
It is a type of Percolation process where bonds progress in a medium not randomly but by following a dynamical rule that dictates how the clusters (called invasions) propagates. Here the cluster formation can be controlled by competing growth mechanisms such as flow resistances, pressure differences, energy minimization. Here the clusters looks like trees. Eg: Fracture propagation, etc.



(Invasive percolation)

c) Continuum Percolation

This is a generalization of classical percolation theory to continuous space, rather than being restricted to discrete lattice. Here instead of bonds/sites on the grid, the system consists of random objects like disks (2D) or sphere (3D) or even more complicated shapes in a continuous domain. These objects can connect/overlap or interact which lead to formation of clusters, usually we define a connectivity radius, where if two objects are within this radius, then they are connected. The key parameter here is the intensity ' λ ' which is the average no: of objects per unit volume (d -dim volume). ' λ ' plays a role analogous to that of occupation probability ' p ' of the usual version of percolation. There exists a critical ' λ_c ' above which infinite clusters appear.



Examples of some continuum percolation in 2D.

After the formation of ∞ -clusters i.e. for $\lambda > \lambda_c$ the structure is like blobs of clusters spread

out with some cavity region. This is usually referred to as "Swiss cheese".

d) Explosive Percolation

Here the growth of large clusters is delayed by a competitive process by putting a bias in choosing where new sites/bonds are added. This delays the formation of ∞ -clusters, & forces a sudden transition, which resembles an explosion, models a 1st order transition.

Physical problems on Percolation

- (i) Random Resistor Network
- (ii) Dilute Magnets
- (iii) Diffusion process
- (iv) Fluid flow problem

Random Resistor Network

A random resistor network is a model used to study electrical conduction in disordered media. It consists of a lattice where each edge represents a resistor & they are randomly assigned a state on/off (conducting/insulating). Each bond can be assigned a conductance ' g ' where the resistor is in on state, with some probability ' p ' (occupation prob.) or an insulating (off state) with a probability of $1-p$. In this model there exists a threshold probability ' p_c '.

For $p < p_c$: There is no conductance in the network of resistors in lattice. This phase is analogous to the absence of ∞ -clusters in usual percolation theory.

At $p = p_c$: There is a phase transition from an insulating state to conducting state where ∞ -clusters start to emerge.

For $p > p_c$: There are ∞ -clusters that lead to a macroscopic conductance in the network.

Back-Bone of Percolation cluster

In percolation theory, the back bone of a percolating cluster represents the substructure through which transport such as electric current / fluid flow occurs, excluding all dead ends or dangling bonds / branches.

The back bone contains bonds that participates in atleast one path connecting two sides of the system. The dangling bonds/branches that do not contribute to connectivity directly are removed.

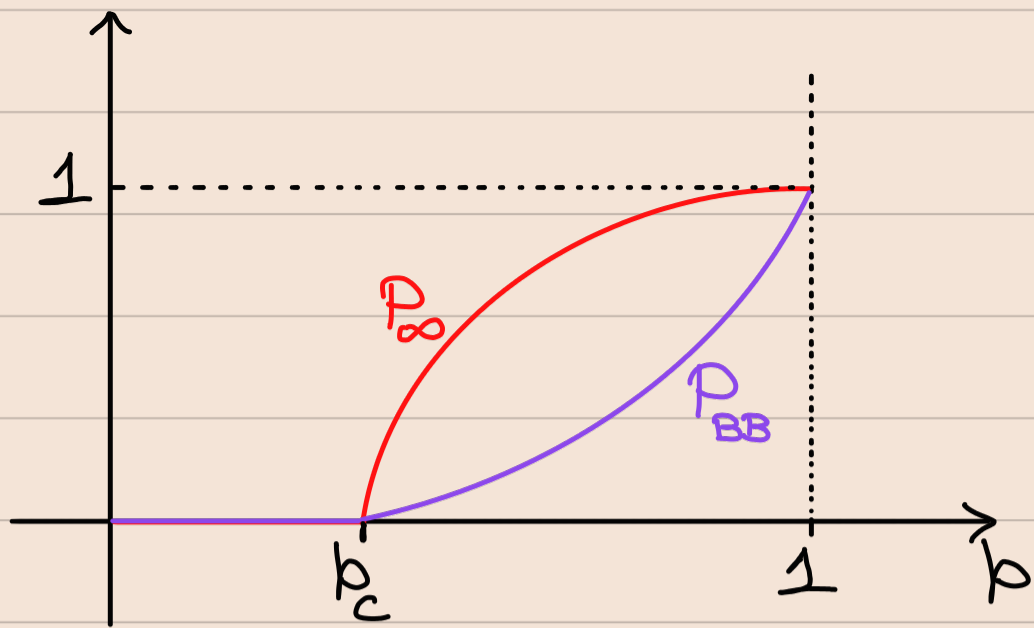
$$\Rightarrow \infty\text{-cluster} = \text{Back bone} + \text{Branches}$$

Probability that a site/bond belong to the back-bone is P_{BB} which depends on occupation probability (p). P_{BB} is the fraction of bonds that belong to the back bone & actively participate in the transport process.

P_{BB} is always less than or equal to the probability P_{∞} . And the critical exponent of P_{BB} is different from that of P_{∞} .

$$P_{BB} \propto (p - p_c)^{B'}$$

$$\& P_{\infty} \propto (p - p_c)^B \quad \underline{\underline{B' \neq B}}$$



The back bone of ∞ -cluster at p_c is a fractal object with a fractal dimension of d_B which is given by

$$d_B = d - \frac{\beta'}{2}$$

$d \Rightarrow$ System's spatial dimension.

$\beta' \Rightarrow$ correlation exponent

Fractal dimension is an index which characterizes fractal patterns or sets by quantifying their complexity as a ratio of the change in detail to the change in scale. In 2D, $d_B = \frac{91}{48}$ for a full percolating back bone.

For bethe lattice we have

$P_{\infty} = 1 - Q^z$ as the probability of a site to be a part of ∞ -cluster.
And $1 - P_{\infty} = Q^z$ is the prob. of a site to not be connected to ∞ -cluster.

A site is dangling if it has exactly one neighbour connecting to ∞ -cluster & $z-1$ other branches not connected to ∞ -cluster.

$\Rightarrow$ Prob. for a site to be a dangling site is

$$P_{\text{dangling}} = P_{\infty} \underbrace{(1 - P_{\infty})^{z-1}}_{\substack{\text{Prob. of } z-1 \\ \text{neighbour} \\ \text{sites to not} \\ \text{connect to} \\ \text{'}\infty\text{'}}}}$$

$\downarrow$
Prob. of a site to connect to ∞

$$\Rightarrow P_{BB} = P_{\infty} - P_{\text{dangling}}$$

$$\Rightarrow P_{BB} = P_{\infty} - P_{\infty} (1 - P_{\infty})^{z-1}$$

$$P_{BB} = P_{\infty} \left[1 - (1 - P_{\infty})^{z-1} \right]$$

$$(1 - P_{\infty})^{z-1} = \left[1 - (z-1)P_{\infty} + \frac{(z-1)(z-2)}{2} P_{\infty}^2 \dots \right]$$

Near critical point P_{∞} is very small.
thus by neglecting higher orders of P_{∞} ,

$$\Rightarrow P_{BB} \sim P_{\infty} \left[1 - 1 + (z-1)P_{\infty} \right]$$

$$\sim P_{\infty} (z-1) P_{\infty}$$

$$\Rightarrow P_{BB} \sim \frac{1}{k_c} (P_{\infty})^2$$

Since, $P_{\infty} \propto (p - p_c)^1$ near ' p_c '.

$$\Rightarrow \underline{P_{BB} \propto (p - p_c)^2}$$

Thus $B' = 2$ (for Bethe lattice)

Dilute Magnets

The dilute magnet problem concerns magnetic systems in which a fraction of the magnetic sites are removed.

Let N_s be the total number of sites in the lattice. Given a site, it can either carry a magnetic moment with a probability of ' p ' or remain non-magnetic with a probability of $1-p$.

Thus not every site is magnetic. This makes the material non-homogeneous & dilute which introduces quenched disorder that has a profound effect on magnetic ordering and critical behaviour. This is quenched disorder, where the impurities are introduced randomly, but remain static in time, & are frozen (Example: Spin glasses).

Another type of disorder is Annealed disorder where impurities are allowed to evolve.

The dilute quenched disorder ising model Hamiltonian is
$$\underline{H_{\text{diluted}} = -J \sum_{\langle i,j \rangle} \eta_i \eta_j S_i S_j}$$

' η_i ' is the indicator function that takes value '1' if site 'i' has magnetic spin & '0' if not.

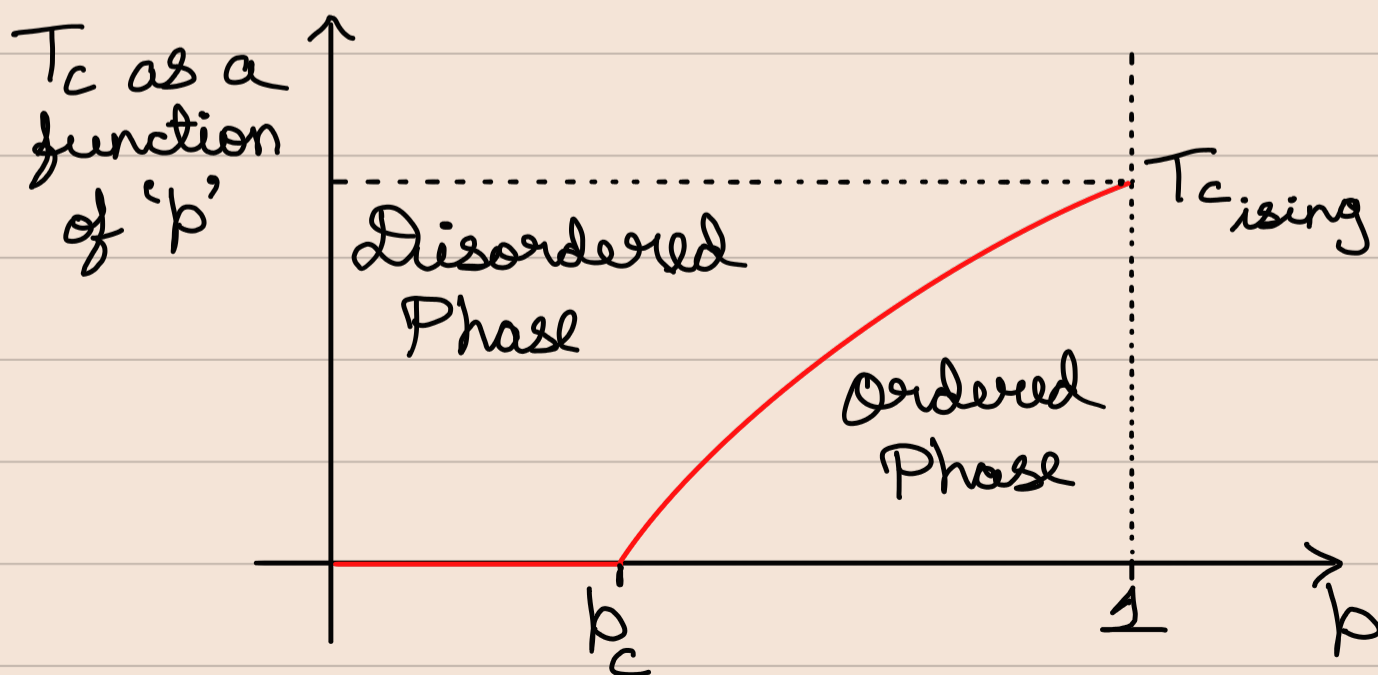
$\langle \eta_i \rangle_{\downarrow} = p \Rightarrow$ controls the long-range ensemble average order in the system.

Only if $p > p_c$ then there are infinite clusters which makes the spins to correlate for long distances. This is because, $\langle S_i S_j \rangle$ is equal to zero if sites 'i' & 'j' belong to two different clusters as there is no possible effect of one spin on the other.

For two spins in same finite cluster to have $\langle S_i S_j \rangle = 1$, it is possible only at $T=0$, as no finite system shows spontaneous ordering in any dimensions ($N \rightarrow \infty$ is needed). Thus for $p < p_c$, $T_c = 0$ (no infinite clusters).

If $p > p_c$, then infinite clusters are present which leads to T_c being greater than zero. For $T < T_c$, we still have,

$$\langle S_i S_j \rangle = \begin{cases} 1, & 'i' \neq 'j' \text{ are in same cluster (finite or } \infty) \\ 0, & \text{otherwise} \end{cases}$$



There is quenched disorder if $p < 1$.

In disordered systems like dilute magnets, every value of 'p' will define a different system with varying concentration/dilution of spins, with different $T_c(p)$ values. To find the ensemble average of the system at a given value of 'p', we need to take all possible configuration of spin occupations & spin value configurations.

The two types of configurations here are the spin occupation configuration $\mathcal{I}(p): \{\eta_i\}$, where η_i is the indicator function that takes value '1' if site 'i' has magnetic spin & '0' if not. $\sum_i \eta_i = N$ is the no: of spins occupied & for a given 'p', it has to satisfy $N = pN_s$. The probability of a specific configuration $\mathcal{I}(p): \{\eta_i\}$ is given by,

$$W(\mathcal{I}) = \prod_i \underbrace{p^{\eta_i}}_{\text{Prob. of 'i' to be occupied}} \underbrace{(1-p)^{1-\eta_i}}_{\text{Prob. of i not to be occupied}}$$

Prob. of 'i' to be occupied Prob. of i not to be occupied

$$\Rightarrow W(\mathcal{I}) = p^{\sum_i \eta_i} (1-p)^{\sum_i (1-\eta_i)}$$

$$\sum_i (1) = N_s \quad \sum_i \eta_i = N$$

$$\Rightarrow \underline{W(\mathcal{I}) = p^N (1-p)^{N_s - N}}$$

Here every configuration has equal prob. to exist at a given 'p'.

For a given occupation configuration $\mathcal{I}(p)$, there are many possible configurations of spin values $\{S_i\}$, where each 'i' is occupied by a spin S_i with a probability 'p', or unoccupied with a probability '1-p'.

The probability of a particular spin configuration, given $\{\eta_i\}$ is given by the boltzmann weight,

$$\underbrace{\text{Prob.}(\{S_i\}|\{\eta_i\})}_{\text{conditional prob.}} = \frac{e^{-\beta H(\{S_i\}, \{\eta_i\})}}{\sum_{\{S_i\}} e^{-\beta H(\{S_i\}, \{\eta_i\})}}$$

The joint probability of a spin dilution and spin configuration is,

$$\text{Prob.}(\{S_i\}, \{\eta_i\}) = W(\mathcal{A}) \frac{e^{-\beta H(\{S_i\}, \{\eta_i\})}}{\sum_{\{S_i\}} e^{-\beta H(\{S_i\}, \{\eta_i\})}}$$

The net magnetization in a particular configuration is given by,

$$M(\{S_i\}, \{\eta_i\}) = \sum_i \eta_i S_i$$

In a fixed realization $\{\eta_i\}$, the average over all spin configurations $\{S_i\}$ of magnetization is $\langle M \rangle_{\mathcal{A}}$

$$\rightarrow \langle M \rangle_{\mathcal{A}} = \sum_{\{S_i\}} \frac{M(\{S_i\}, \{\eta_i\}) e^{-\beta H(\{S_i\}, \{\eta_i\})}}{\sum_{\{S_i\}} e^{-\beta H(\{S_i\}, \{\eta_i\})}}$$

The disorder average of a spin $\{S_i\}$ averaged $\langle M \rangle_{\mathcal{A}}$ is given by,

$$\langle M \rangle = \sum_{\{\eta_i\}} W(\{\eta_i\}) \langle M \rangle_{\mathcal{A}}$$

But $\overline{\langle S_i \rangle} = 0$ as the occupation of sites is random, and over many disorder realizations, there is no any preferred global spin orientation.

Magnetic Susceptibility ($\overline{\chi}$)

$$\chi_i = \beta \sum_{\gamma} \langle S_{i+\gamma} S_{\gamma} \rangle - \langle S_i \rangle \langle S_{i+\gamma} \rangle$$

$\rightarrow$ for $p=1$ (Pure Ising system)
($\chi_i \sim \chi$)

But when $p < 1$ (dilute magnets), the magnetic susceptibility of the disordered system's is $\overline{\chi}$, which is an ensemble average over many occupation configurations $\{\mathcal{C}(p)\}$, with a fixed 'p'.

$$\Rightarrow \overline{\langle \chi \rangle} = \beta \sum_{\mathcal{C}} W(\mathcal{C}) \sum_{\gamma} \langle S_i S_{i+\gamma} \rangle_{\mathcal{C}}$$

$W(\mathcal{C}) = p^N (1-p)^{N_s - N}$ is the weight of a configuration $\{\eta_i\}$ & $\langle S_i S_{i+\gamma} \rangle_{\mathcal{C}}$ is the average over spin config. $\{S_i\}_{\mathcal{C}}$ of ' $S_i S_{i+\gamma}$ '

Now
$$\sum_{\mathcal{C}} W(\mathcal{C}) = \sum_{\{\eta_i\}} p^N (1-p)^{N_s-N}$$

$$= \sum_{N=1}^{N_s} \underbrace{N_s C_N}_N p^N (1-p)^{N_s-N}$$

given by binomial distribution

$$\Rightarrow \sum_{\mathcal{C}} W(\mathcal{C}) = [p + (1-p)]^N = 1 //$$

Similarly,
$$\langle \eta_i \rangle = \sum_{\mathcal{C}} W(\mathcal{C}) \eta_i = p$$

As occupancy probability of spin at i^{th} site to be occupied in all $\mathcal{C}: \{\eta_i\}$ configurations is 'p'.

If site 'i' & 'x' are occupied in the same cluster, $\langle S_i S_{i+x} \rangle = 1$ $[T < T_c(p)]$

Let $n(\mathcal{C})$ be the number of times site 'i' & 'x' to occupy same cluster in all $\mathcal{C}: \{\eta_i\} \Rightarrow \sum_{\mathcal{C}} \langle S_i S_{i+x} \rangle_{\mathcal{C}} = n_{\mathcal{C}}$

Then
$$\langle \bar{X} \rangle = \beta \sum_{\mathcal{C}} W(\mathcal{C}) p n(\mathcal{C})$$

$$\Rightarrow \underline{\langle \bar{X} \rangle = \beta p \langle \bar{n} \rangle}$$

Another approach is by considering the magnetization per spin, which is given by,

$$m_i = \tanh(\langle n \rangle_i B h)$$

$$\Rightarrow \langle \bar{m} \rangle = \sum_i p W(i) m_i$$

$$\Rightarrow \langle \bar{m} \rangle = \sum_i p W(i) \tanh(\langle n \rangle_i B h)$$

$$\Rightarrow \bar{\chi} = \left. \frac{\partial \langle \bar{m} \rangle}{\partial h} \right|_{h \rightarrow 0^+}$$

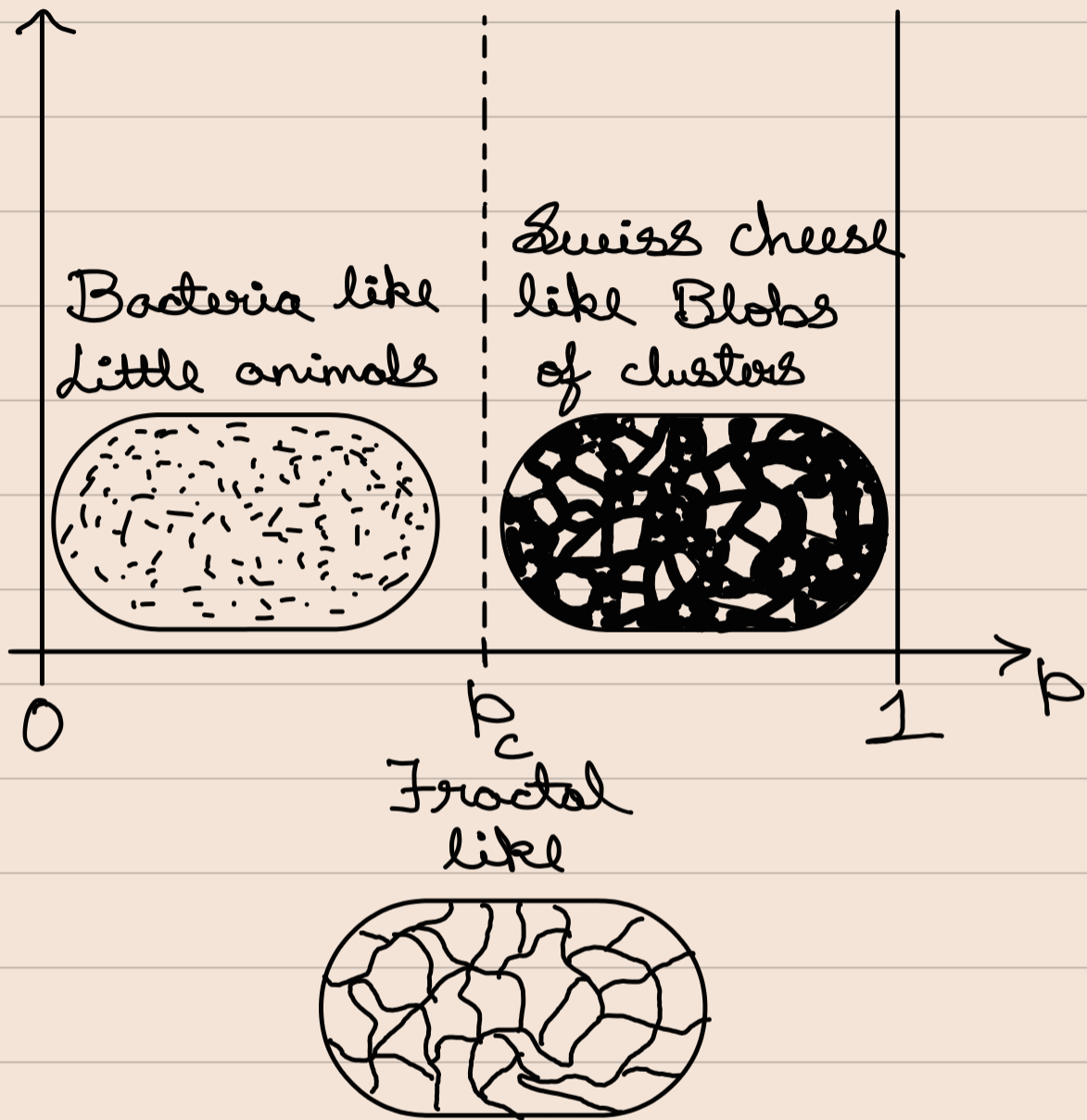
$$= \lim_{h \rightarrow 0} \sum_i p W(i) \langle n \rangle_i B \operatorname{sech}^2(\langle n \rangle_i B h)$$

$$= \sum_i p W(i) \langle n_i \rangle B$$

$$= B p \sum_i W(i) \langle n_i \rangle$$

$$\Rightarrow \underline{\bar{\chi} = B p \langle \bar{n}_i \rangle}$$

Morphology of Percolating Systems



At " p_c ", the system looks like a fractal, where in any length scale, the system looks the same. This is because the characteristic length scale of the system " ξ " itself has become infinity, hence the fractal structure.

Let ' γ ' be the length scale with which we measure the thermodynamic quantities of the system.

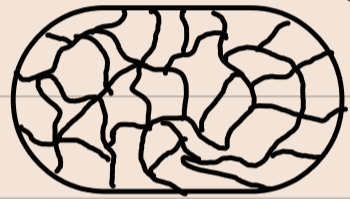
At $p = p_c$ & $\gamma < \xi$ $\Rightarrow$ System looks like fractal

At $p > p_c$ & $\gamma \ll \xi$ $\Rightarrow$ System looks like fractal

Since $G(\gamma) \sim \frac{e^{-\gamma/\xi}}{\gamma^{2-d+\eta}}$

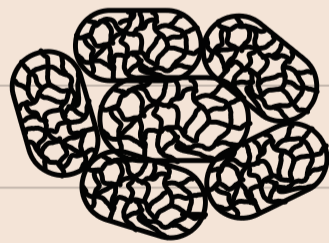
$\gamma \ll \xi \Rightarrow e^{-\gamma/\xi} \sim 1$

$\Rightarrow G(\gamma) \sim \frac{1}{\gamma^{2-d+\eta}} \Rightarrow$ as if the system is at p_c



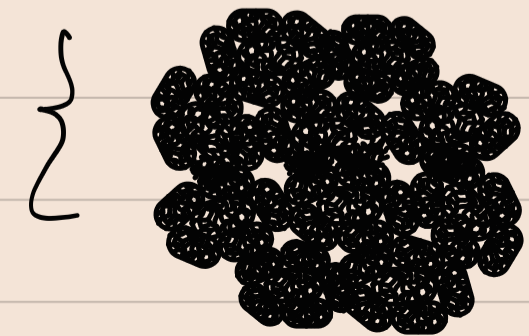
But for $\gamma > \xi$ $\Rightarrow$ System looks like blobs

($p \gtrsim p_c$)
(p slightly more than p_c)



These go on to become Swiss cheese

(p significantly higher than p_c)



$\gamma > \xi$

When $p \geq p_c$ & $\sigma \gg \varepsilon_g$

Here $G(\sigma) \sim \frac{e^{-\sigma/\varepsilon_g}}{\sigma^{d-2+\eta}}$
negligible effect by denominator

$\rightarrow G(\sigma) \sim e^{-\sigma/\varepsilon_g}$ for $\sigma \gg \varepsilon_g$
when $p \geq p_c$.

Thus System looks as if $p < p_c$

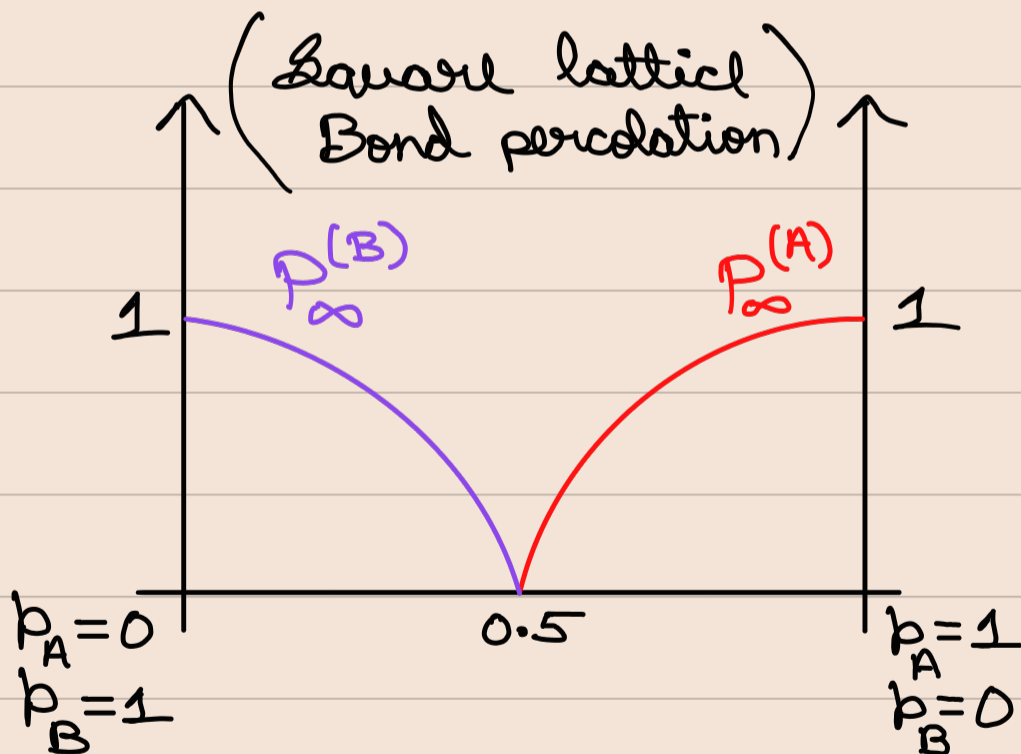
Hence Scaling form of $G(\sigma)$
is given by

$$G(\sigma) \sim \frac{1}{\sigma^{d-2+\eta}} \Upsilon\left(\frac{\sigma}{\varepsilon_g}\right)$$

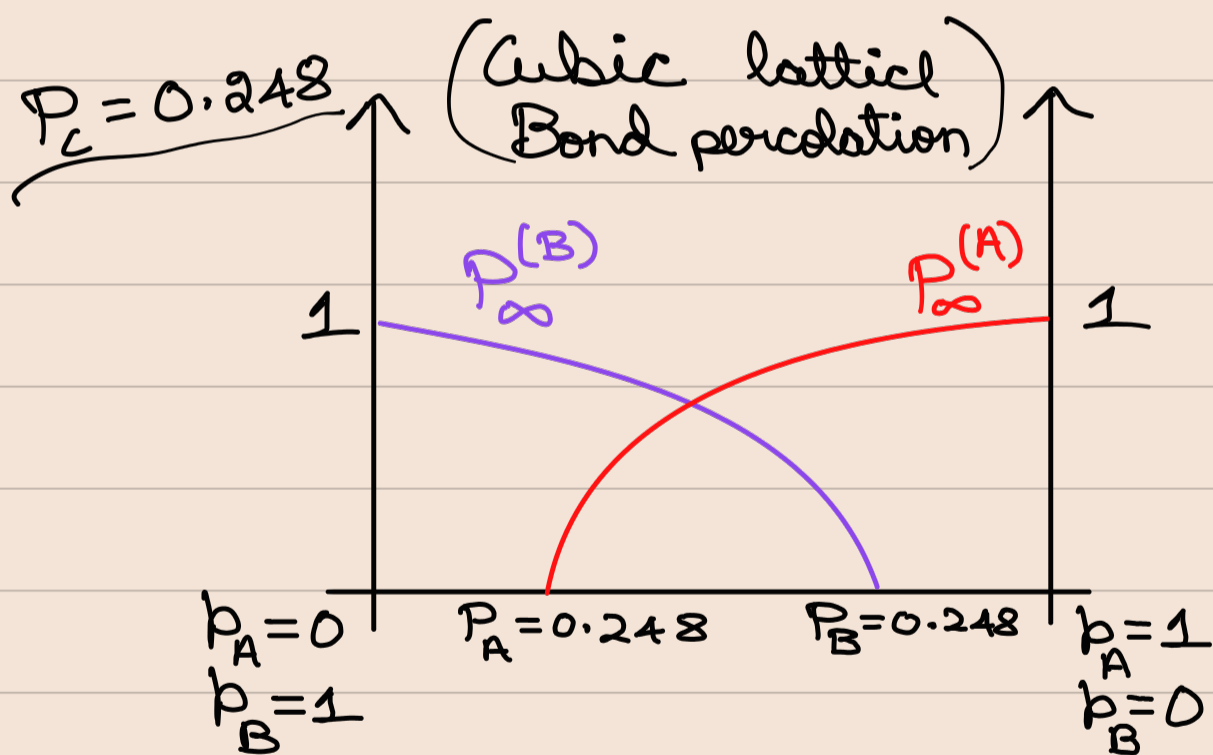
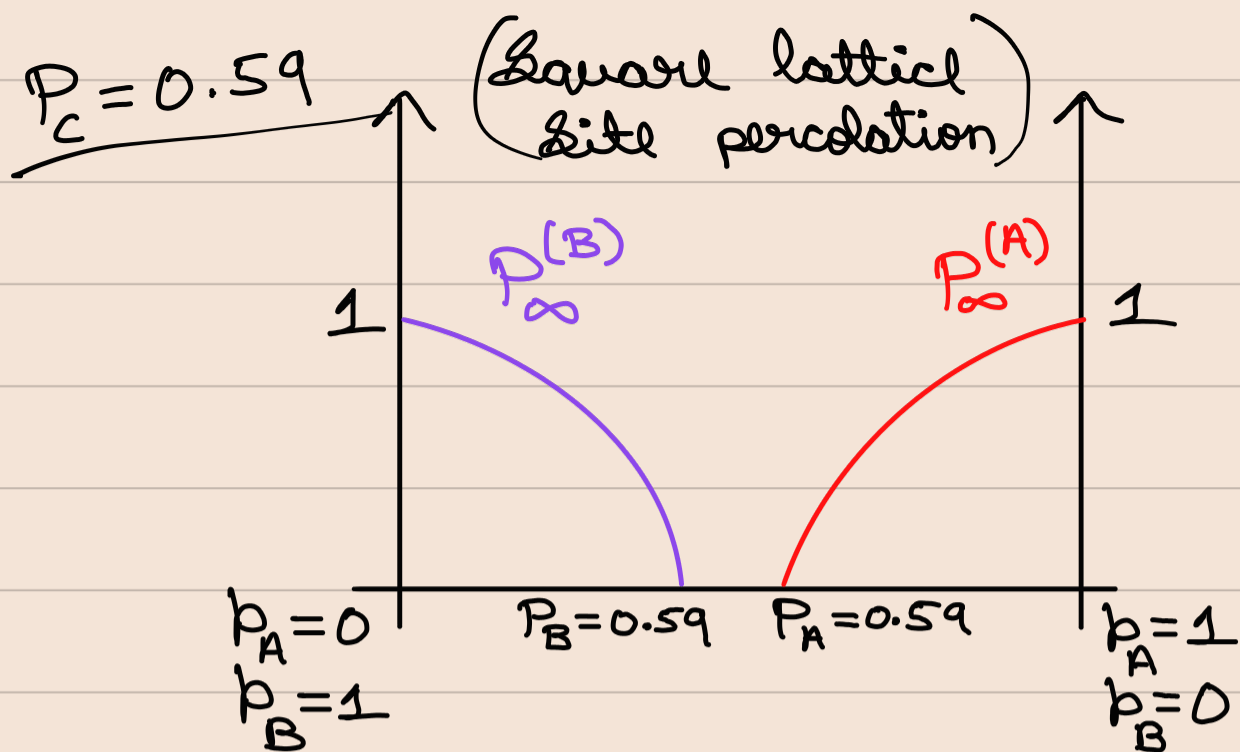
Existence of two different bonds or sites in a percolation model

Consider two types of bonds 'A' & 'B' in a lattice. So, clusters are also of two types.

In a square lattice, $p_c = 0.5$. So, for both p_A & p_B to be more than 0.5 is not possible as they have to occupy the available sites in the same lattice.



Thus we don't simultaneously see both A & B type ∞ -clusters at any 'p'. Same happens for site percolation of A & B.



Thus for those lattices whose $p_c < 0.5$, they can contain infinite clusters of two types at same instance.