

26/03/25

Lecture 22

Renormalization Group

Renormalization group is a powerful conceptual & mathematical framework in statistical mechanics (and quantum field Theory) used to understand system with many degrees of freedom especially near phase transition and critical points.

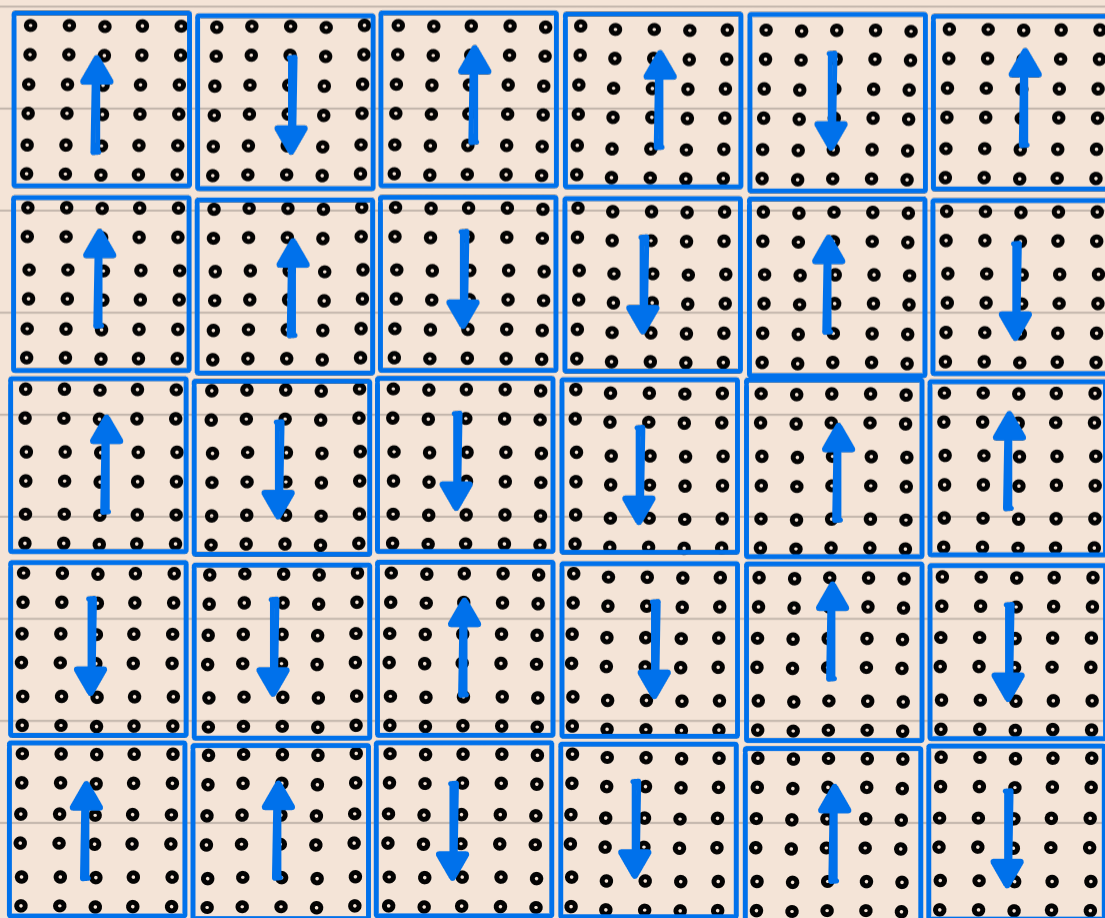
We have seen that the behaviour of the system near the critical point or phase transition will depend only on long distances fluctuation since the characteristic length scale ' ξ ' specified by 'correlation length' will diverge to infinity. Thus the irrelevant degrees of freedom should be discarded or integrated out as they do not affect the long distance behaviour near critical point.

The process of integrating out the short range degrees of freedom is called as 'Coarse-graining'. Once this is done the system is rescaled such that system looks the same size as before & the length scale is restored. This results in a new parameters of the system (like coupling constants and the temperature) & a new Hamiltonian.

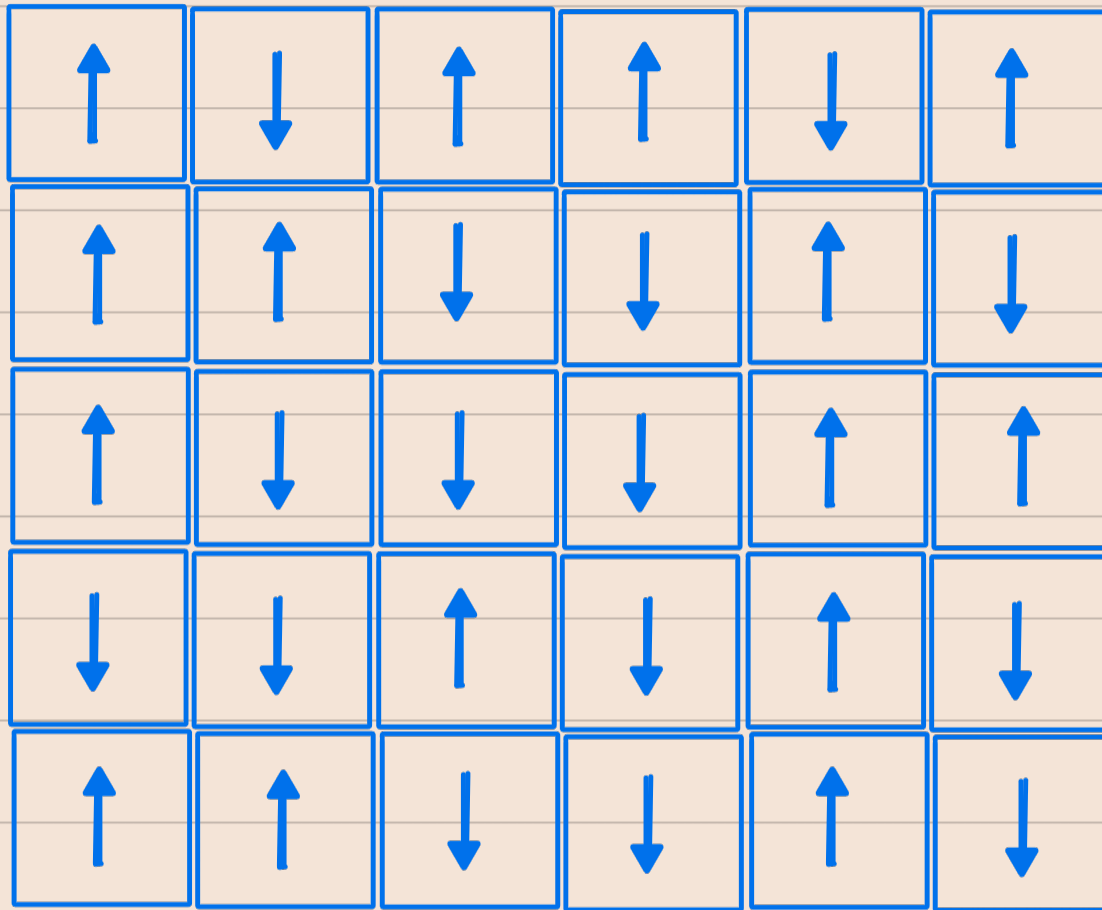
Thus in this process of Coarse-graining & Rescaling, we have removed the short-ranged degrees of freedom & got a new effective Hamiltonian with new coupling constants & temperature. By repeatedly doing this procedure we can track how the parameters of the system evolve - This is called as R-G flow, which can eventually reach a fixed point which resembles a system with an infinite characteristic length scale.

Steps in RG procedure (Eg: Spin System)

- 1) Consider blocks of size ' b ' in the original lattice where each block has b^d spins $\{S\}$ ($d \rightarrow \text{dim}$)
- 2) For each block, assign a spin S' called as Block Spins based on the majority rule of the original spins inside each block, this results in a set of block spin variables $\{S'\}$.



3) From the result, deduce the new effective Hamiltonian H' , which describes the couplings b/w $\{S'\}$.



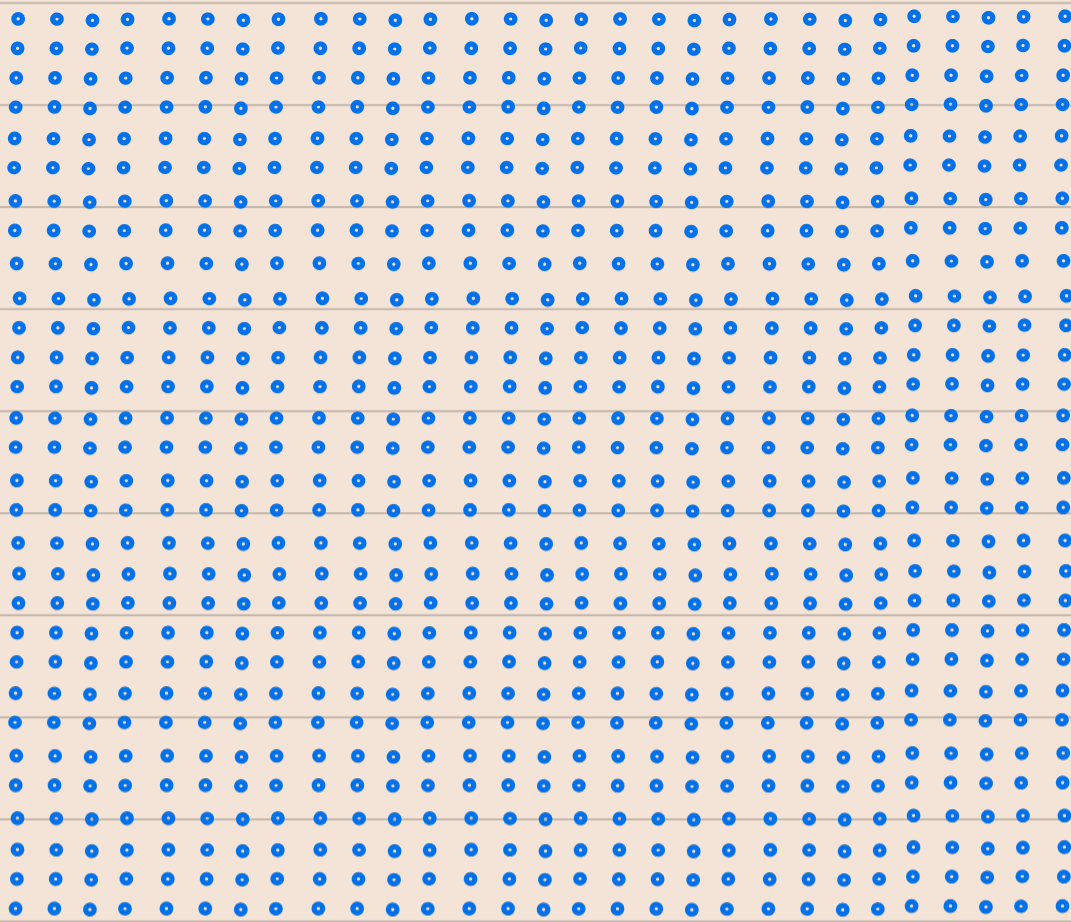
} Only Coarse grained Spins
($\uparrow \rightarrow$ Block Spin)

4) Now rescale the length scale such that the block spins will span a lattice of original size, but with a new Hamiltonian H' .

$$\xi_S \rightarrow \xi_{S'} = \frac{\xi_S}{b} \Rightarrow (\text{Correlation length})$$

$$H \rightarrow H' \Rightarrow (\text{Hamiltonian})$$

$$\{K\} \rightarrow \{K'\} \Rightarrow (\text{Coupling Constants})$$



} System
after
rescaling
comes
back to
original
size

[By changing
length
scale]

• → Block
Spin

In a space of hamiltonians, in one action of renormalization, we went from one point to another. This is like a flow in the space of $\{H\}$.

5) Now perform steps '1 to 4' again
& again, $H^{(0)} \rightarrow H^{(1)} \rightarrow H^{(2)} \rightarrow \dots H^{(k)}$
Then under this repeated application
of the procedure, the system goes
to a fixed point in the RG flow

The procedure of coarse graining and rescaling will define a transformation denoted by $\hat{R}$.

$$\Rightarrow \hat{R} H^{(0)} \longrightarrow H^{(1)}$$

$$\neq \hat{R} \xi \longrightarrow \xi'$$

$$T \longrightarrow T'$$

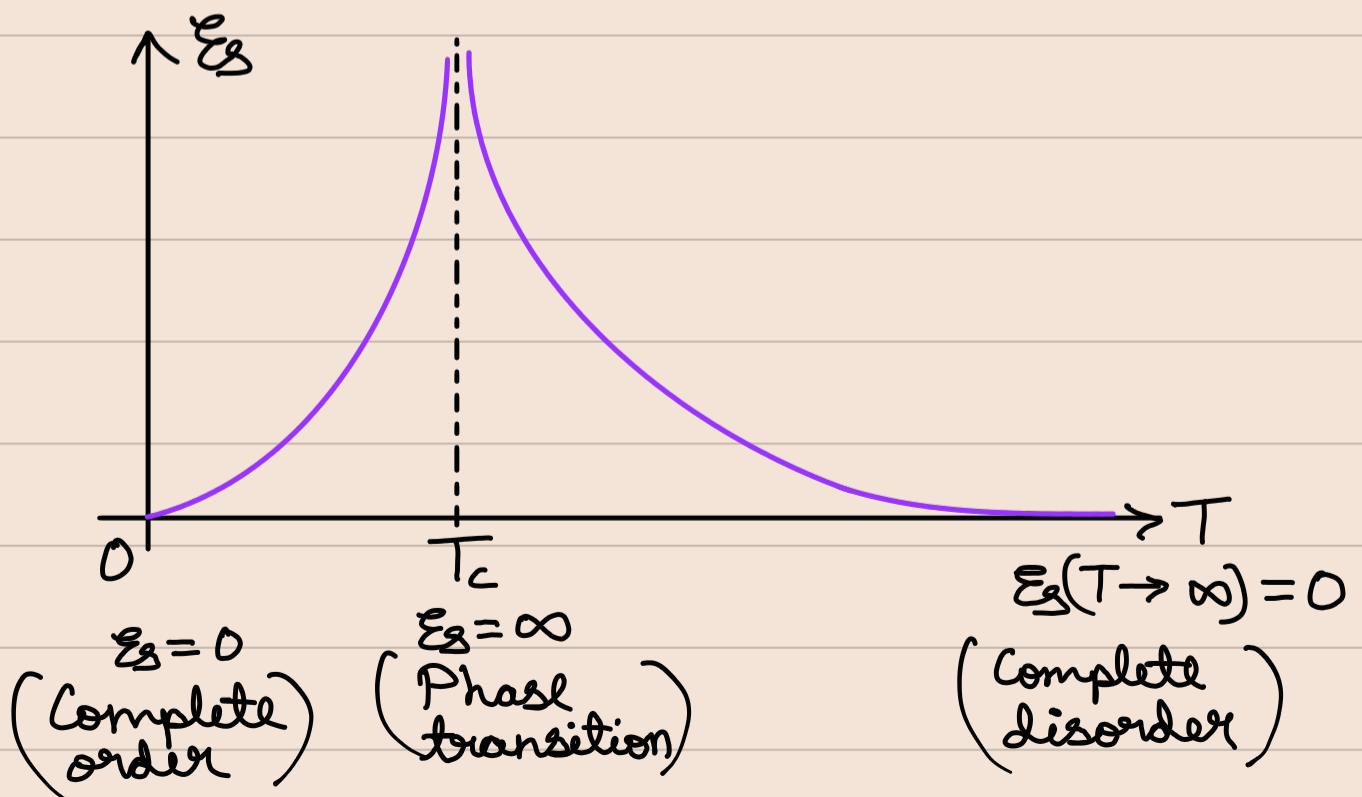
$$\hat{R} \{K\} \longrightarrow \{K'\} \text{ (set of coupling Parameters)}$$

Under repeated applications of $\hat{R}$ we get a fixed point in space of Hamiltonians whose couplings are $\{K^*\}$ such that $\hat{R} \{K^*\} = \{K^*\} \Rightarrow$ Fixed point

$$\begin{aligned} \rightarrow \hat{R} \xi^* &\longrightarrow \xi^* \Rightarrow \frac{\xi^*}{b} = \xi^* \\ &\Rightarrow \xi^* = 0 \text{ or } \xi^* = \infty \end{aligned}$$

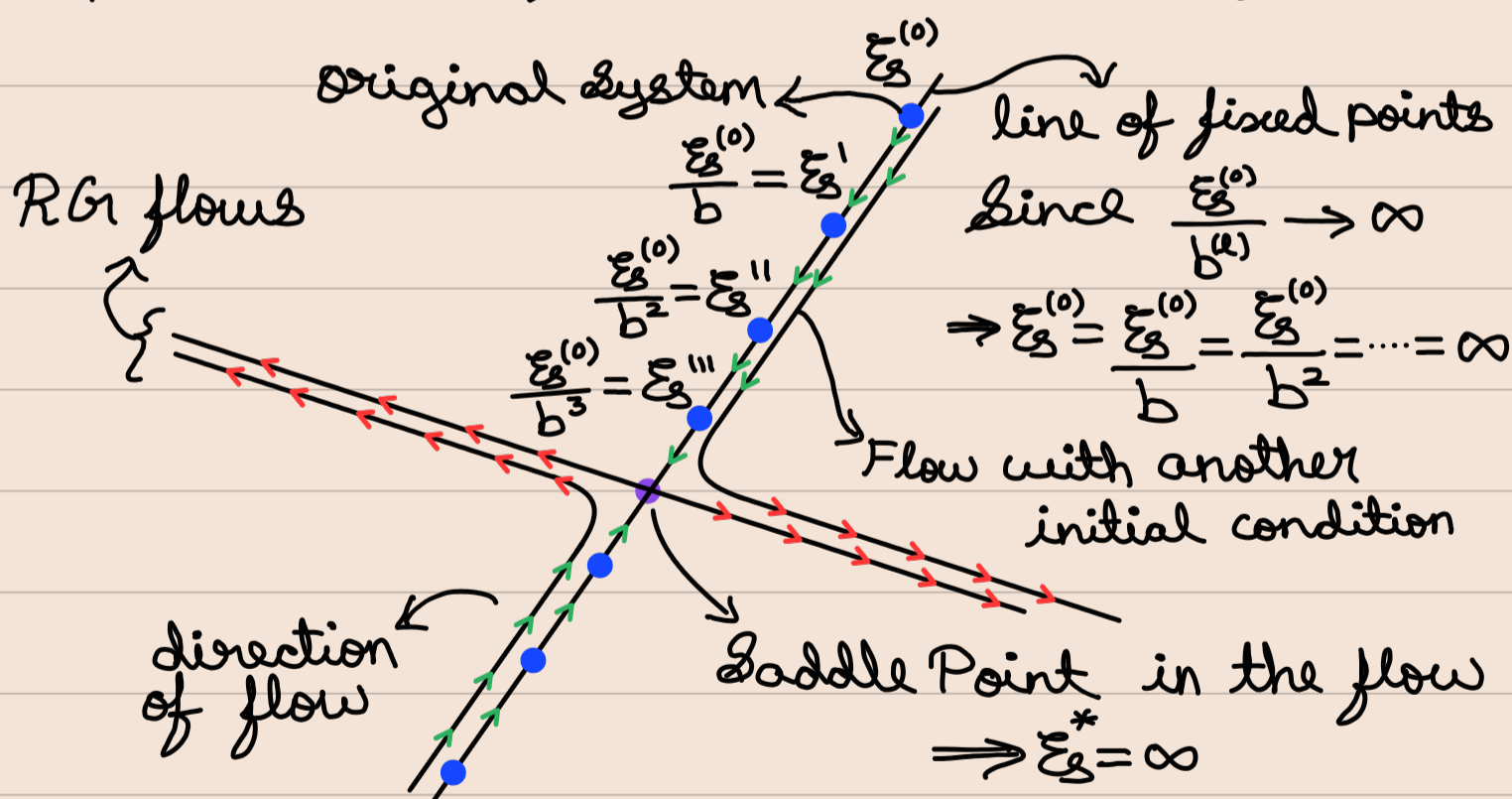
Thus $\{K^*\}$ describes a system with either complete order ($\xi = 0$) or zero order ($\xi = 0$) or at critical point/phase transition with $\xi = \infty$.

The correlation length is a measure of length scale over which fluctuations of order parameter are correlated. At perfect order, there are no fluctuations at all in system, thus $\xi_s = 0$. And in a zero order (complete disorder) state, the fluctuations are not all correlated, thus $\xi_s = 0$. But at critical point the correlations between fluctuations is ∞ , hence $\xi_s = \infty$.



RG is a better procedure than mean field theory because, in RG, the spin correlation is not removed. (Where as in mean field theory, the spin-spin interaction is neglected. Hence the critical exponents measured using this theory will be far from real value. But in RG, we try to get true values of critical exponents.

Example of Renormalization flow



This is how critical point is found in the RG flow by analyzing the systems in the neighbourhood of saddle point.

Mathematical Procedure of RG

- Fix the block spin length 'b' & the blocks with b^d spins.
- Assign block spins S' using a majority rule, where

$$S'_{(\text{BLOCK})} = \text{sgn} \left[\sum_{i \in \text{BLOCK}} S_i \right] = \begin{cases} 1, & \text{majority up} \\ -1, & \text{majority down} \end{cases}$$

$\text{sgn} \rightarrow \text{Sign function}$

- Let $\hat{T}$ be an operator which ensures the majority rule in each block during coarse graining.

$$\hat{T}(S'_{\text{BLOCK}}; S_i, i \in \text{BLOCK}) = \begin{cases} 1, & \text{if } S'_{(\text{BLOCK})} \times \text{sgn} \left[\sum_{i \in \text{BLOCK}} S_i \right] > 0 \\ 0, & \text{if } S'_{(\text{BLOCK})} \times \text{sgn} \left[\sum_{i \in \text{BLOCK}} S_i \right] < 0 \end{cases}$$

- Let $\tilde{H}\{S_i\} = \sum_{\{S_i\}} [J_{ij} S_i S_j + J_{ijkl} S_i S_j S_k S_l + \dots]$

be the original Hamiltonian.

We define $B\tilde{\mathcal{H}}\{S_i\} = \mathcal{H}\{S_i\}$ as the dimensionless hamiltonian.

$$\Rightarrow \mathcal{H}\{S_i\} = \sum_{\{S_i\}} [K_{ij} S_i S_j + K_{ijkl} S_i S_j S_k S_l + \dots]$$

$$K_{ij} = B J_{ij} \text{ and } K_{ijkl} = B J_{ijkl}, \dots \text{ so on.}$$

These K_{ij} , K_{ijkl} , so on... will define the coupling constants of the original system. They can contain interactions of all possible types that can be possible.

- Let $\vec{K} = \{K_1, K_2, K_3, \dots\}$ be the set of all coupling constants of the original system. They are the ones which define the system. Under the action of $\hat{R}$, every coupling constant will change to $\vec{K}' = \{K'_1, K'_2, K'_3, \dots\}$

which defines a new hamiltonian with different couplings & temperature.

- The partition function of the system after the application of $\hat{R}$ is,

$$e^{-\mathcal{H}\{S'\}} = \text{Trace}_{\{S\}} \left[\prod_{\text{BLOCK}} \hat{T}(S'; S_{i \in \text{BLOCK}}) \right] e^{-\mathcal{H}\{S\}}$$

$\nearrow$ BLOCK SPINS
 $\nwarrow$ original spins

This ensures only those configurations of S' which follows majority rule.

Even if one block doesn't follow the majority rule, then it won't contribute to the partition function.

Then we can evaluate the new partition function

$$\underline{Z' = \text{Trace}_{\{S'\}} [e^{-\mathcal{H}\{S'\}}]}$$

Until this step it is called as Coarse graining.

Under 'l' applications of $\hat{R}$

$$\hat{R}^l \{\vec{R}\} \longrightarrow \{\vec{R}^{(l)}\}$$

$$\mathcal{H}\{\vec{R}\} \longrightarrow \mathcal{H}^{(l)}\{\vec{R}^{(l)}\} + \underbrace{\text{Constants}}_{\text{Can't be neglected}}$$

$$N \longrightarrow N^{(l)} = \frac{N}{b^d l}$$

$$\mathcal{E}_S\{\vec{R}\} \longrightarrow \mathcal{E}_S^{(l)}\{\vec{R}^{(l)}\} = \frac{\mathcal{E}_S\{\vec{R}\}}{b^d}$$

(Constants can be neglected)

$$\left(\begin{array}{l} \text{free energy} \\ \text{per spin} \end{array} \right) f \longrightarrow f^{(l)} = b^{dl} f$$

(Constants can't be neglected)

Constants arise during rescaling.
& usually depend on 'b' & 'l'.