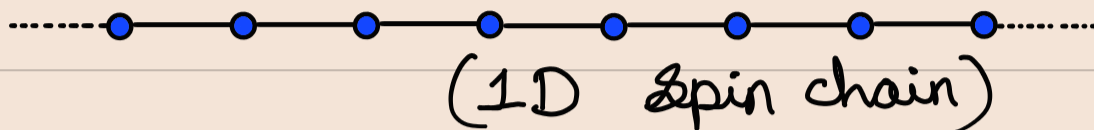


16/04/25

Lecture 25

RG on 1D Ising Model



The 1D dimensionless Ising Hamiltonian is given by

$$\mathcal{H} = -K \sum_i S_i S_{i+1} - L \sum_i S_i - Nc$$

usually neglected

$$K = \beta J, \quad L = \beta h, \quad c = \beta \mu$$

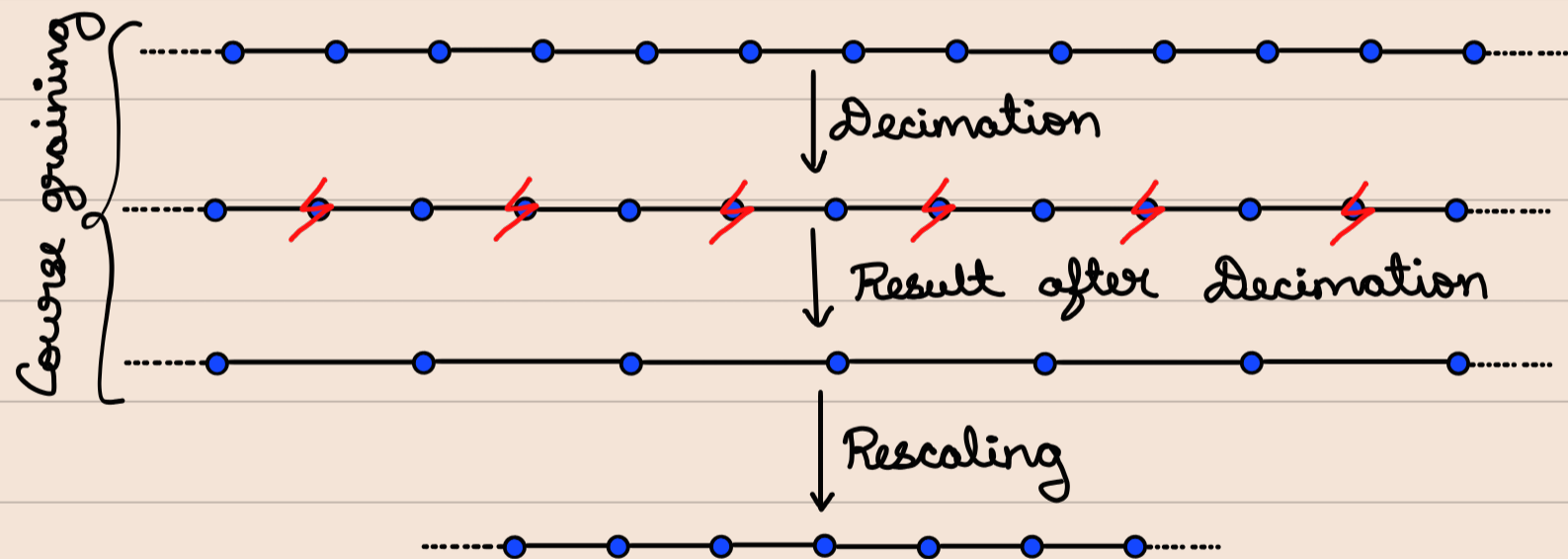
$$\mathcal{H} = -K \sum_i S_i S_{i+1} - \frac{L}{2} \sum_i (S_i + S_{i+1}) - Nc$$

$$\mathcal{H} = - \sum_i \left[K S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1}) \right] - Nc$$

$$\mathcal{Z} = \sum_{\{S_i\}} \exp[-\mathcal{H}]$$

$$\mathcal{Z} = \sum_{\{S_i\}} \exp \left[\sum_i \left[K S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1}) \right] + Nc \right]$$

For such a 1D chain the easiest way for Coarse graining is by a procedure called decimation. Here we perform a partial trace over every alternate spin variable along the chain, leaving others unaffected. Then we arrive at a half the number of original spins.



Here the renormalizing scale factor is $b=2$ as under one application of Coarse-graining, the number of spins reduced by half & the spacing between spins got doubled.

The partition function for original system is,

$$\begin{aligned}
 Z(K, L, N) &= \sum_{\{S_i\}} \exp \left[\sum_i \left[K S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1}) \right] + N C \right] \\
 &= \sum_{\{S_i\}} e^{N C} \prod_i e^{[K S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1})]} \\
 &= e^{N C} \sum_{\{S_i\}} e^{[K S_1 S_2 + \frac{L}{2} (S_1 + S_2)]} e^{[K S_2 S_3 + \frac{L}{2} (S_2 + S_3)]} \dots \\
 &= e^{N C} \sum_{\{S_i\}} \left[e^{K(S_1 S_2 + S_2 S_3)} e^{[L(\frac{S_1 + S_2 + S_3}{2})]} \right. \\
 &\quad \left. \times e^{K(S_3 S_4 + S_4 S_5)} e^{[L(\frac{S_3 + S_4 + S_5}{2})]} \dots \right]
 \end{aligned}$$

Thus the partition function $Z(K, L, N)$ can be re-written as,

$$e^{N C} \sum_{\{S_i\}} \prod_{n=1}^{N/2} e^{K(S_{2n-1} S_{2n} + S_{2n} S_{2n+1})} e^{[L(\frac{S_{2n-1} + S_{2n} + S_{2n+1}}{2})]}$$

Then the summation can be split to odd 'i' & even 'i' values.

Without any loss of generality, let us choose to decimate 'even' spins & keep odd spins unchanged.

$$\Rightarrow Z(K, L, N) = \sum_{\{S_i\}_{i \in \text{odd}}} \sum_{\{S_i\}_{i \in \text{even}}} \exp[-\mathcal{H}]$$

(neglect e^{N_C} in the below calculation)

$\sum_{\{S_i\}_{i \in \text{even}}} \exp[-\mathcal{H}]$ can be found as follows

$$\begin{aligned} & \sum_{\{S_i\}_{i \in \text{even}}} \prod_{n=1}^{N/2} e^{K(S_{2n-1}S_{2n} + S_{2n}S_{2n+1})} e^{L\left(\frac{S_{2n-1} + S_{2n} + S_{2n+1}}{2}\right)} \\ &= \prod_{n=1}^{N/2} \left[e^{K(S_{2n-1} + S_{2n+1})} e^{L\left(\frac{S_{2n-1} + S_{2n+1}}{2} + 1\right)} + e^{-K(S_{2n-1} + S_{2n+1})} e^{L\left(\frac{S_{2n-1} + S_{2n+1}}{2} - 1\right)} \right] \\ &= \prod_{n=1}^{N/2} \left(e^{[K(S_{2n-1} + S_{2n+1}) + L]} + e^{-[K(S_{2n-1} + S_{2n+1}) + L]} \right) \\ & \quad \times e^{L\left(\frac{S_{2n-1} + S_{2n+1}}{2}\right)} \end{aligned}$$

$$\sum_{\substack{\{S_i\} \\ i \in \text{even}}} \exp[-\mathcal{H}] = \prod_{n=1}^{N/2} 2 \cosh[K(S_{2n-1} + S_{2n+1}) + L] \times e^{\left[L \left(\frac{S_{2n-1} + S_{2n+1}}{2} \right) \right]}$$

$$\Rightarrow Z(K, L, N) = e^{NC} \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{n=1}^{N/2} 2 \cosh[K(S_{2n-1} + S_{2n+1}) + L] \times e^{\left[L \left(\frac{S_{2n-1} + S_{2n+1}}{2} \right) \right]}$$

Then the partition function $Z(K, L, N)$ finally takes the form

$$e^{NC} \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{(i \in \text{odd})} 2 \cosh[K(S_i + S_{i+2}) + L] e^{\left[L \left(\frac{S_i + S_{i+2}}{2} \right) \right]}$$

The above is concerned with the configuration of spins in the odd site of 1D chain. Then the new system also has its partition function which we expect to have the same form as before.

$$\Rightarrow Z(K, L, N) = Z(K', L', N')$$

$$\Rightarrow Z(K', L', N') = e^{N'c'} \sum_{\{S'_i\}} e^{\left(\sum_i \left[K' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) \right] \right)}$$

$$\Rightarrow Z(K', L', N') = e^{N'c'} \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{i \in \text{odd}} e^{\left[K' S_i S_{i+2} + \frac{L'}{2} (S_i + S_{i+2}) \right]}$$

$$N' = \frac{N}{2} \quad \& \quad \text{length} \left[\{S_i\}_{i \in \text{odd}} \right] = \frac{N}{2}$$

$$\Rightarrow Z(K', L', N') = \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{i \in \text{odd}} e^{c'} e^{\left[K' S_i S_{i+2} + \frac{L'}{2} (S_i + S_{i+2}) \right]}$$

Note: S'_i are spins remaining after coarse graining which are nothing but $S_i (i \in \text{odd})$.

The partition function in terms of K, L & N (old parameters) is,

$$e^{Nc} \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{i \in \text{odd}} 2 \cosh [K(S_i + S_{i+2}) + L] e^{\left[L \left(\frac{S_i + S_{i+2}}{2} \right) \right]}$$

[since, $e^{Nc} = \prod_{i \in \text{odd}} e^{2c}$]

$$= \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{i \in \text{odd}} 2 e^{2c} \cosh [K(S_i + S_{i+2}) + L] e^{\left[L \left(\frac{S_i + S_{i+2}}{2} \right) \right]}$$

$$\text{But } Z(K, L, N) = Z(K', L', N')$$

$$\begin{aligned} \Rightarrow \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{(i \in \text{odd})} 2e^{2c} \cosh[K(S_i + S_{i+2}) + L] e\left[L\left(\frac{S_i + S_{i+2}}{2}\right)\right] \\ = \sum_{\substack{\{S_i\} \\ i \in \text{odd}}} \prod_{(i \in \text{odd})} e\left[K'S_i S_{i+2} + \frac{L'}{2}(S_i + S_{i+2}) + c'\right] \end{aligned}$$

Thus each term in the above eqⁿ should be equal.

$$\begin{aligned} \Rightarrow 2e^{2c} \cosh[K(S_i + S_{i+2}) + L] e\left[L\left(\frac{S_i + S_{i+2}}{2}\right)\right] \\ = e\left[K'S_i S_{i+2} + \frac{L'}{2}(S_i + S_{i+2}) + c'\right] \end{aligned}$$

The above relation should hold for all possible values of S_i & S_{i+2}

$$\begin{aligned} \Rightarrow e^{2c} \left(e^{[K(S_i + S_{i+2}) + L]} + e^{-[K(S_i + S_{i+2}) + L]} \right) e\left[L\left(\frac{S_i + S_{i+2}}{2}\right)\right] \\ = e\left[K'S_i S_{i+2} + \frac{L'}{2}(S_i + S_{i+2}) + c'\right] \end{aligned}$$

For $S_i = 1$ $S_{i+2} = 1$,

$$e^{2c} [e^{2(k+L)} + e^{-2k}] = e^{k' + L' + c'}$$

$$\Rightarrow e^{2c} e^L [e^{2k+L} + e^{-(2k+L)}] = e^{k' + L' + c'}$$

$$\Rightarrow 2e^{2c} e^L \cosh(2k+L) = e^{k' + L' + c'} \quad \xrightarrow{\quad} (1)$$

For $S_i = 1$, $S_{i+2} = -1$ or $S_i = -1$, $S_{i+2} = 1$

$$e^{2c} [e^L + e^{-L}] = 2 \cosh(L) = e^{(-k' + c')}$$

$$\Rightarrow e^{c'} = 2e^{2c} e^{k'} \cosh(L) \rightarrow (2)$$

For $S_i = -1$ $S_{i+2} = -1$,

$$e^{2c} [e^{-2k} + e^{2(k-L)}] = e^{k' - L' + c'}$$

$$\Rightarrow 2e^{2c} e^{-L} \cosh(2k-L) = e^{k' - L' + c'} \quad \xrightarrow{\quad} (3)$$

Substituting eqⁿ (2) into (1) & (2),

$$e^L \cosh(2K+L) = e^{K'+L'} e^{K'} \cosh(L)$$

$$e^{-L} \cosh(2K-L) = e^{K'-L'} e^{K'} \cosh(L)$$

Multiplying the above two eqⁿ,

$$e^{4K'} = \frac{\cosh(2K+L) \cosh(2K-L)}{\cosh^2(L)}$$

Dividing these two eqⁿ we get,

$$e^{2L'} = e^{2L} \frac{\cosh(2K+L)}{\cosh(2K-L)}$$

Substituting these final eqⁿ in eqⁿ (2) we get eqⁿ for $e^{c'}$.

$$e^{c'} = 2 e^{2c} e^{K'} \cosh(L)$$

$$\Rightarrow e^{4c'} = 16 e^{8c} e^{4K'} \cosh^4(L)$$

$$\Rightarrow e^{4c'} = 16 e^{8c} \cosh(2K+L) \cosh(2K-L) \cosh^2(L)$$

The new partition function is

$$Z(K', L', N' = \frac{N}{2}) = \sum_{\{S'_i\}} e^{\left[\sum_i \left[K' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) \right] + N' C' \right]}$$

$i \in \{1, 2, 3, \dots, \frac{N}{2}\}$

$$\& Z(K, L, N) = Z(K', L', N' = \frac{N}{2})$$

In each iteration, the number of spins reduce by half & the lattice spacing doubles, thus ' $\epsilon_S' = \epsilon_S/2$ ' $\Rightarrow$ Rescaling.

The resulting hamiltonian after one procedure of RG is,

$$\mathcal{H}' = \sum_i \left[K' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) \right] + N' C'$$

It has new coupling K' & field L' which also describes ising spins $\{S'\}$ with nearest neighbour interaction.

Variation of free energy under RG

$$\text{Since } f(K, L, N) = -\frac{k_B T}{N} \ln [Z(K, L, N)]$$

$$= -\frac{k_B T}{N} \ln \left[\sum_{\{S_i\}} \prod_i e^{\left[K S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1}) + c \right]} \right]$$

$$f = -k_B T c - \frac{k_B T}{N} \ln \left[\sum_{\{S_i\}} \prod_i e^{\left[K S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1}) \right]} \right]$$

Under RG, it transforms as,

$$f'(K', L', N') = -\frac{k_B T}{N'} \ln [Z'(K', L', N')]$$

$$= -\frac{k_B T}{N'} \ln \left[\sum_{\{S'_i\}} \prod_i e^{\left[K' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) + c' \right]} \right]$$

$$= -\frac{k_B T}{N'} \ln \left[e^{N' c'} \sum_{\{S'_i\}} \prod_i e^{\left[K' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) \right]} \right]$$

$$f' = -k_B T c' - \frac{k_B T}{N'} \ln \left[\sum_{\{S'_i\}} \prod_i e^{\left[K' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) \right]} \right]$$

Now, by construction,

$$\begin{aligned} \ln \left[\sum_{\{S_i\}} \prod_i e^{\left[k S_i S_{i+1} + \frac{L}{2} (S_i + S_{i+1}) \right]} \right] \\ = \ln \left[\sum_{\{S'_i\}} \prod_i e^{\left[k' S'_i S'_{i+1} + \frac{L'}{2} (S'_i + S'_{i+1}) \right]} \right] \\ = [f' + k_B T c'] \left[\frac{-N'}{k_B T} \right] \end{aligned}$$

$$\Rightarrow f = -k_B T c + \frac{k_B T}{N} \left[f' + k_B T c' \right] \left[\frac{N'}{k_B T} \right]$$

$$\Rightarrow f = \frac{N'}{N} f' + k_B T (c' - c)$$

$$\Rightarrow f' = \frac{N}{N'} [f + k_B T (c - c')]$$

Since $N' = \frac{N}{b} \Rightarrow \frac{N}{N'} = b$

$$\Rightarrow f' = b f + k_B T b (c - c')$$

or $f^{(l)} = b^l f + \text{Constant}$

(Singular part)

$d=1$
 $\& b=2$ } In our case

These are the recursion relations.

$$e^{4K'} = \frac{\cosh(2K+L) \cosh(2K-L)}{\cosh^2(L)}$$

$$e^{2L'} = e^{2L} \frac{\cosh(2K+L)}{\cosh(2K-L)}$$

$$e^{4C'} = 16e^{8C} \cosh(2K+L) \cosh(2K-L) \cosh^2(L)$$

We define

$$\begin{aligned} x &= e^{-4K} & , & \quad x' = e^{-4K'} \\ y &= e^{-2L} & , & \quad y' = e^{-2L'} \\ z &= e^{-4C} & , & \quad z' = e^{-4C'} \end{aligned}$$

$$\Rightarrow e^{4K'} = \frac{\cosh(2K+L) \cosh(2K-L)}{\cosh^2(L)}$$

$$= \frac{\left[e^{2K+L} + e^{-2K-L} \right] \left[e^{2K-L} + e^{-2K+L} \right]}{(e^L + e^{-L})^2}$$

$$\Rightarrow e^{4K'} = \frac{e^{4K} + e^{2L} + e^{-2L} + e^{-4K}}{e^{2L} + e^{-2L} + 2}$$

$$\Rightarrow x' = \frac{y^{-1} + y + 2}{(x^{-1} + y^{-1} + y + x)}$$

$$= \frac{x(y^2 + 2y + 1)}{(y + x + x^2y + xy^2)}$$

$$\Rightarrow \boxed{x' = \frac{x(1+y)^2}{(x+y)(1+xy)}}$$

$$e^{2L'} = e^{2L} \frac{\cosh(2K+L)}{\cosh(2K-L)}$$

$$= e^{2L} \left[\frac{e^{2K+L} + e^{-2K-L}}{e^{2K-L} + e^{-2K+L}} \right]$$

$$= \frac{e^L}{e^{-L}} \left[\frac{e^L + e^{-4K} e^{-L}}{e^{-L} + e^{-4K} e^L} \right]$$

$$\Rightarrow e^{2L'} = \frac{e^{2L} + e^{-4K}}{e^{-2L} + e^{-4K}}$$

$$\Rightarrow y' = \frac{y+x}{y^{-1}+x}$$

$$\Rightarrow \boxed{y' = y \left(\frac{x+y}{1+xy} \right)}$$

$$e^{4C'} = 16e^{8C} \cosh(2K+L) \cosh(2K-L) \cosh^2(L)$$

$$= e^{8C} [e^{2K+L} + e^{-2K-L}] [e^{2K-L} + e^{-2K+L}] [e^L + e^{-L}]^2$$

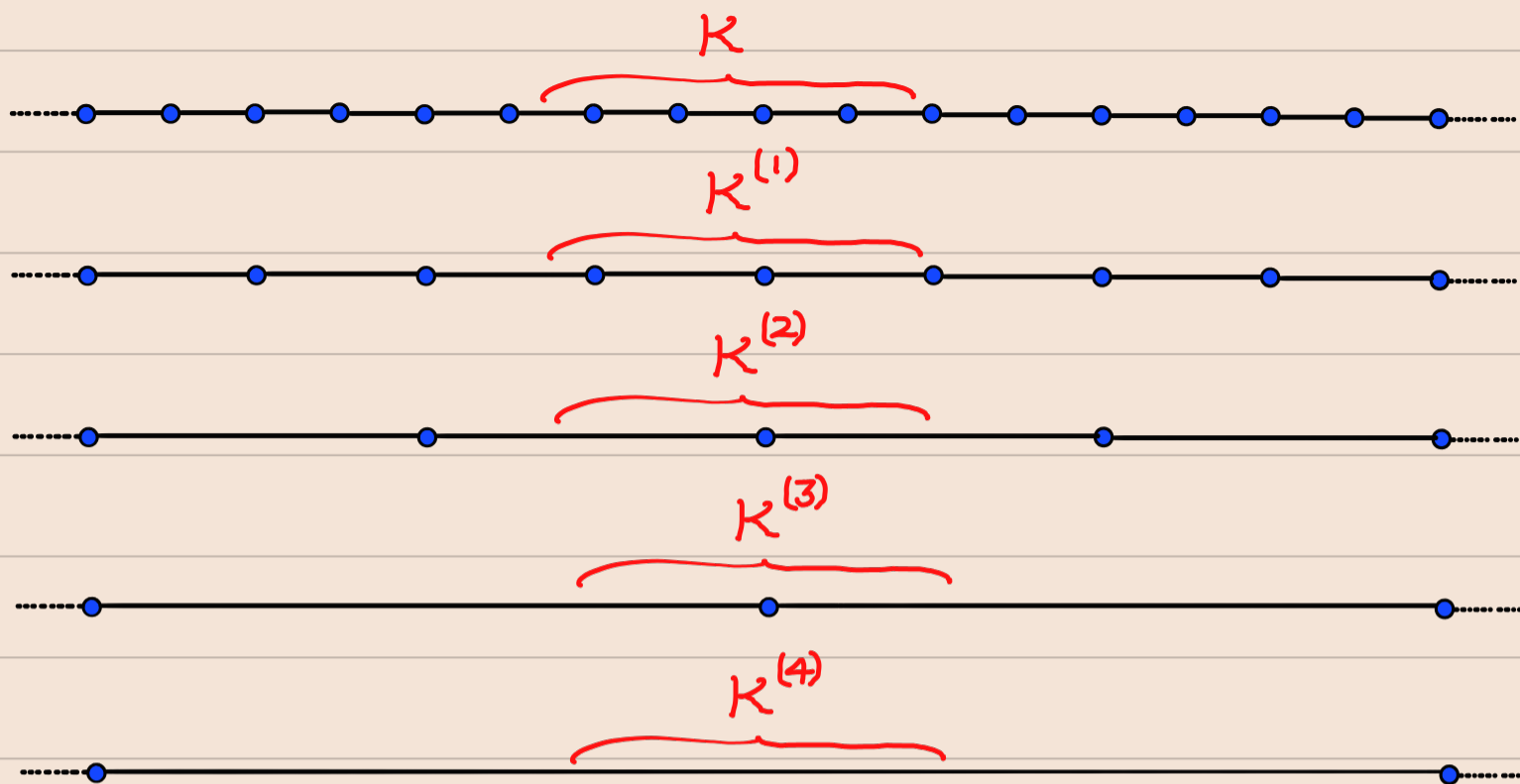
$$= e^{8C} [e^{4K} + e^{2L} + e^{-2L} + e^{-4K}] [e^{2L} + e^{-2L} + 2]$$

$$= z^{-2} (x^{-1} + y^{-1} + y + x) (y^{-1} + y + 2)$$

$$= \frac{(x+y)(1+xy)(1+y)^2}{z^2 xy^2}$$

$$\Rightarrow \boxed{z' = \frac{z^2 xy}{(x+y)(1+xy)(1+y)^2}}$$

Flow eqⁿ & fixed points



The below eqⁿ determine the RG flow.

$$x' = \frac{x (1+y)^2}{(x+y)(1+xy)}$$

$$y' = y \left(\frac{x+y}{1+xy} \right)$$

$$z' = \frac{z^2 xy}{(x+y)(1+xy)(1+y)^2}$$

The fixed points are given by,

$$x^* = \frac{x^* (1 + y^*)^2}{(x^* + y^*)(1 + x^* y^*)} \quad \text{--- (1)}$$

$$y^* = y^* \left(\frac{x^* + y^*}{1 + x^* y^*} \right) \quad \text{--- (2)}$$

$$z^* = \frac{(z^*)^2 x^* y^*}{(x^* + y^*)(1 + x^* y^*)(1 + y^*)^2} \quad \text{--- (3)}$$

The eqⁿ of 'x' & 'y' doesn't have 'z' in it. So they can be solved separately & 'z' follows it.

(I) If $x^* = 0 \Rightarrow y^* = (y^*)^2$
Then $y^* (1 - y^*) = 0$.
 $\Rightarrow \underline{y^* = 0}$ ($y^* = 1$ is dealt next).

But directly we cannot put both $x^* = y^* = 0$ in eqⁿ (1). Thus the recursion is illdefined here.
 $\Rightarrow x^* = y^* = 0$ is a singular point.

When $x=0 \Rightarrow e^{-4K} = 0 \Rightarrow K = \beta J = \infty$
 $\Rightarrow T=0$, Thus $x'=0$.

Then the recursion relation for $y = e^{-2L}$ is $y' = y^2$. If $h > 0 \Rightarrow L > 0 \Rightarrow y < 1$
 $\Rightarrow y' < y < 1 \Rightarrow y'' < y' < y < 1$

Thus the flow reaches $(0,0)$, but RG mapping breaks down at $(x,y)=(0,0)$.

Here along the line $x=1$, we have $E_s = 0$, since $T=0$, no fluctuation is present.

Here h need not be '0'. This represents frozen spins at $h \neq 0$ & it is not spontaneous ordered phase.

Thus $x=y=0$ is a singular point which is a limit of certain flows, but it is not a fixed point as there is no definite RG update here.

$$(II) \text{ If } x^* = 0 \Rightarrow y^* = (y^*)^2$$

$$\text{If } y^* \neq 0 \Rightarrow y^* = 1.$$

Thus $x^* = 0$ & $y^* = 1$ is another fixed point which gives

$$\left. \begin{array}{l} e^{-4K} = 0 \Rightarrow K = \beta J = \infty \\ e^{-2L} = 1 \Rightarrow L = \beta h = 0 \end{array} \right\} \begin{array}{l} \text{If } J \neq 0, \\ \text{then } B = \infty \\ \Rightarrow h = 0 \end{array}$$

- This fixed point represents
- $J \neq 0$ (Spin-Spin Interactions present)
 - $B = \infty$ or $T = 0$
 - $h = 0$ (Zero external field)

This is the Spontaneously ordered phase at $h = 0$ & $T = 0$.

Here $E_g = \infty$.

We can show that this is a critical point by finding the stability of this point (will be done later).

(III) If $x^* = 1$, eq (1) holds, which gives $1 = 1$. & eqⁿ (2) becomes

$$y^* = y^* \left(\frac{1 + y^*}{1 + y^*} \right) = y^*$$

Thus $x^* = 1$ & y^* any value is a fixed point.

$$e^{-4K} = 1 \Rightarrow K = \beta J = 0$$

$$e^{-2L} = \text{any} \Rightarrow L = \beta h = \text{any}$$

$$\text{If } J \neq 0 \Rightarrow \beta = 0 \Rightarrow T = \infty \Rightarrow L = 0$$

$$\Rightarrow x^* = 1 \text{ \& } y^* = 1$$

This represents a high temperature completely disordered phase like a paramagnetic phase with zero correlation & hence $\bar{\epsilon}_s = 0$.

$$\text{If } J = 0 \Rightarrow \beta = \text{any} \text{ \& } h = \text{any.}$$

This represents non-interacting spins or a proper paramagnet which is equivalent to $T = \infty$ phase. where $\bar{\epsilon}_s = 0$.

(IV) If $y^* = 0$, then $eq^n(z)$ holds trivially & $eq(1)$ becomes

$$\Rightarrow \underbrace{e^{-4K} = 1}_{x^* = 1} \Rightarrow \beta J = 0$$

$$e^{-2L} = 0 \Rightarrow \beta h = \infty$$

If $J \neq 0, \Rightarrow \beta = 0 \Rightarrow T = \infty, h = \text{any}$
 $\hookrightarrow$ Paramagnetic phase

Here $E_S = 0$ because of zero correlation.

$$\text{If } \underline{J = 0} \Rightarrow \beta \Rightarrow \text{any} \Rightarrow h = \infty$$

$$\Rightarrow T = \text{any}$$

which represents the infinite field $B_{ext} = \infty$ or forced ordered phase where all spins are forced to align along the applied field.

Here $E_S = 0$ as there is no fluctuation (all are suppressed). This corresponds to line $y^* = 0$ & all points of this line are attracted & flow towards $(x^* = 1, y^* = 1)$ which is a fixed point.

Flow Curves

Thus finally, the fixed points are given by,

$$a) (x^* = 0, y^* = 1) \Rightarrow (u, v) = (0, 0)$$

$$b) (x^* = 1, y^* = \text{any}) \Rightarrow (u = 1, v = \text{any})$$

& a singular point given by,
 $(x^* = 0, y^* = 0) \Rightarrow (u = 0, v = 1)$

The stability of fixed points around (x^*, y^*) can be found by linear perturbation where we define $x = x^* + \delta x$ & $y = y^* + \delta y$ & keep terms only upto first order.

$$x' = X(x, y) \text{ \& \& } y' = Y(x, y)$$

$$\begin{aligned} \Rightarrow x^* + \delta x' &= X(x^* + \delta x, y^* + \delta y) \\ &= \underbrace{X(x^*, y^*)}_{x^*} + \left. \frac{\partial X}{\partial x} \right|_* \delta x + \left. \frac{\partial X}{\partial y} \right|_* \delta y \end{aligned}$$

$$\Rightarrow \delta x' = \left. \frac{\partial X}{\partial x} \right|_* \delta x + \left. \frac{\partial X}{\partial y} \right|_* \delta y$$

$$\Rightarrow \delta y' = \left. \frac{\partial Y}{\partial x} \right|_* \delta x + \left. \frac{\partial Y}{\partial y} \right|_* \delta y$$

$$\rightarrow \begin{bmatrix} \partial x' \\ \partial y' \end{bmatrix} = \left(\begin{array}{c} \text{Jacobian} \\ \text{matrix} \end{array} \right)_{\text{at } *} \begin{bmatrix} \partial x \\ \partial y \end{bmatrix}$$

For $x^* = 0, y^* = 1$
 $\Rightarrow x = \partial x \quad y = 1 - \partial y.$

$$\Rightarrow \partial x' = \frac{\partial x (2 - \partial y)^2}{[1 + \partial x (1 - \partial y)][\partial x + 1 - \partial y]}$$

Upto 1st order,

$$\partial x' = \frac{4\partial x}{1 + 2\partial x - \partial y} = 4\partial x$$

$$\Rightarrow \underline{\partial x' = 4\partial x}$$

$$\& \quad 1 - \partial y' = \frac{(1 - \partial y)(\partial x + 1 - \partial y)}{[1 + \partial x (1 - \partial y)]}$$

$$= \frac{1 + \partial x - \partial y - \partial y}{[1 + \partial x]}$$

$$= (1 + \partial x - 2\partial y)(1 - \partial x)$$

$$= 1 + \partial x - 2\partial y - \partial x$$

$$\Rightarrow \underline{\partial y' = 2\partial y}$$

Thus both δx & δy goes far away from fixed point $x^* = 0, y^* = 1$. This is an unstable fixed point.

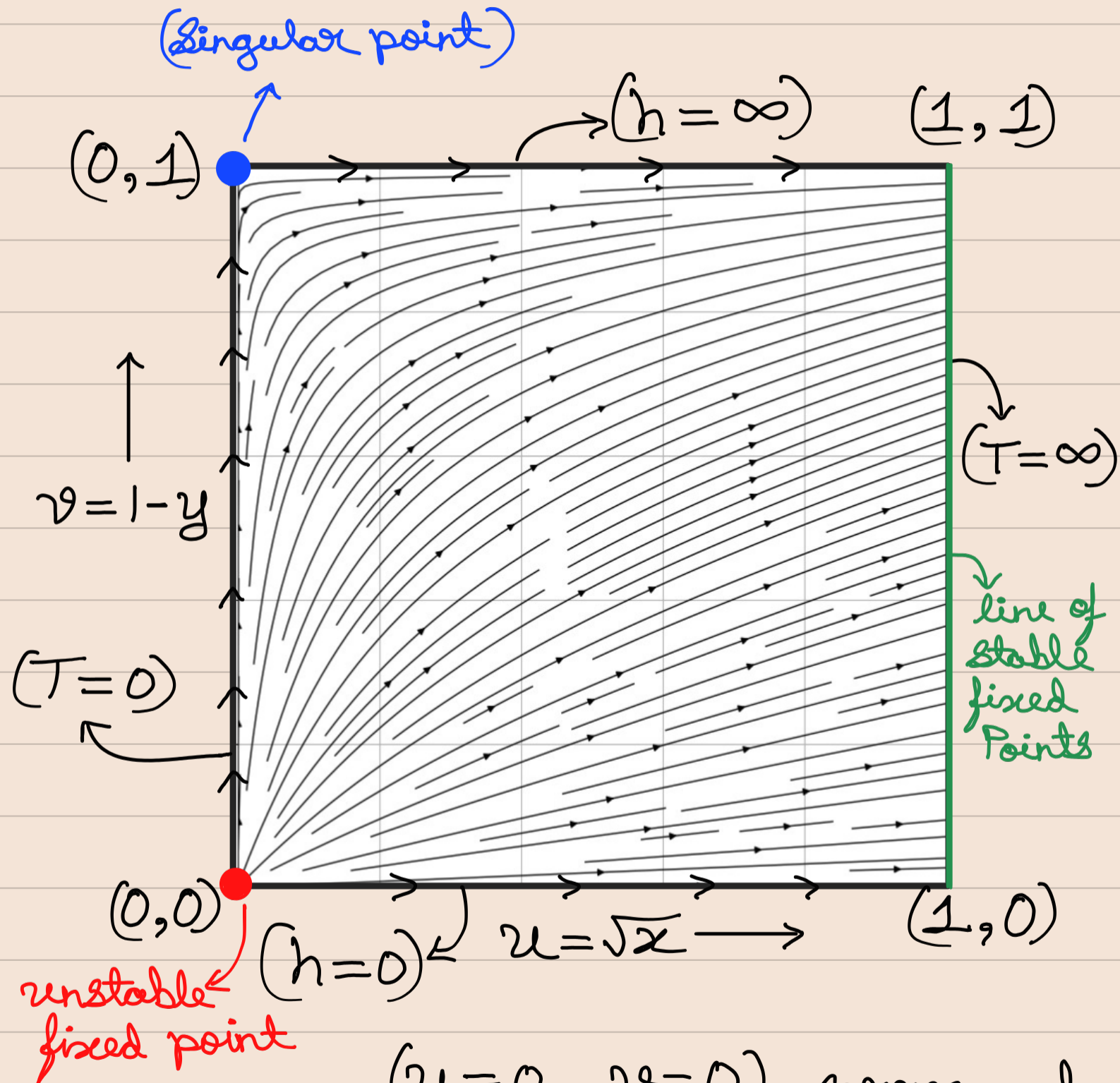
For $x^* = 1, y^* = \text{any}$,
 $\rightarrow x = 1 - \delta x$
 $y = y^* + \delta y$

Here we get $\delta x' = 0$. Thus the flow curves are stable or attracted to the line $x = 1$ along x direction. Any small perturbation is immediately crushed to zero.

$$\delta y' = \frac{y^*(y^*-1)(\delta x) + (\delta y)}{(1+y^*)}$$

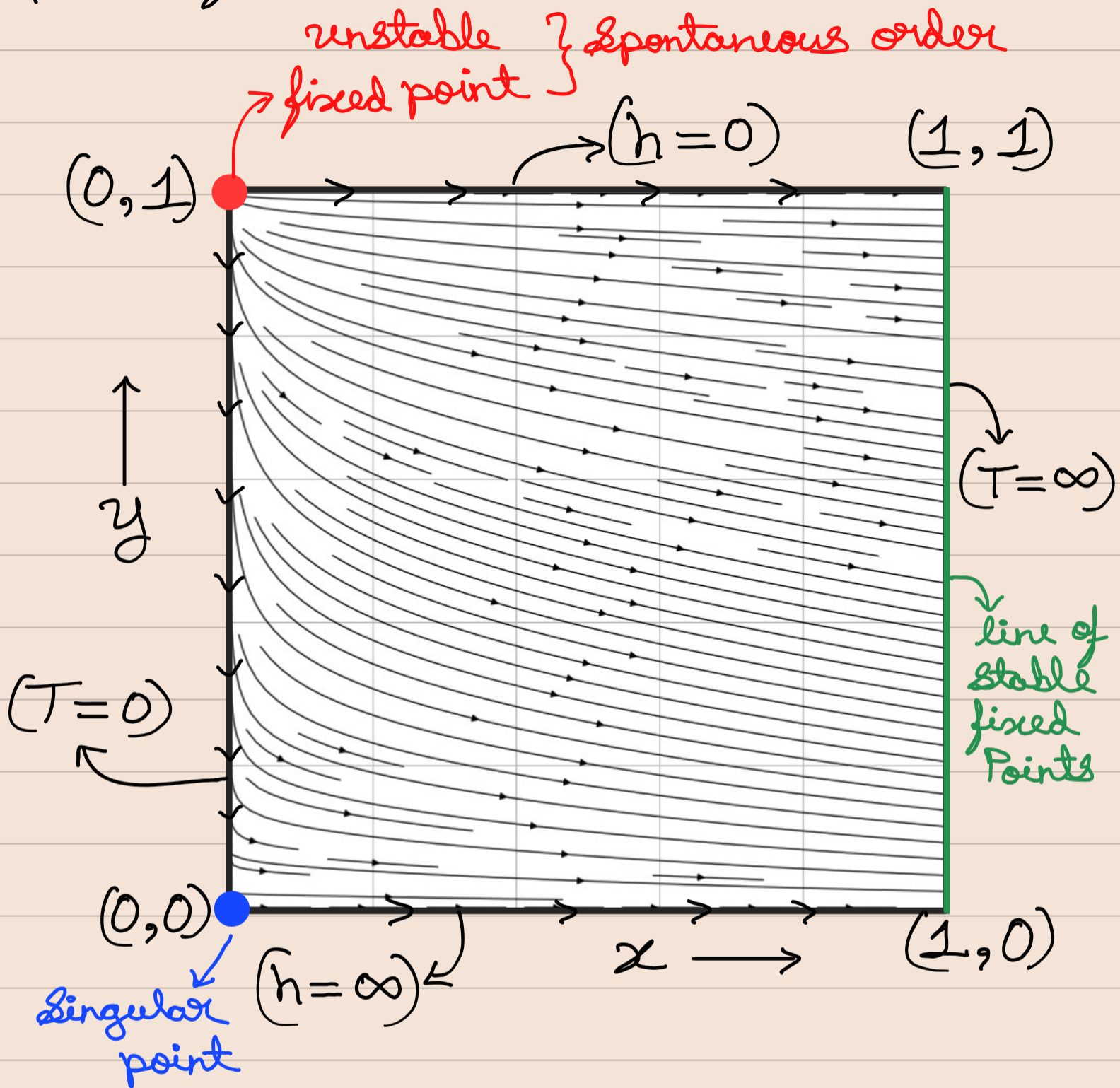
As $0 < y^* < 1$, the coefficients of δx is -ve and less than 1. & coefficient of δy is 1. The perturbations neither grow nor shrink in y -direction (in 1st order)

RG flow curve in $u-v$ plane
 where $u = \sqrt{x}$ & $v = 1-y$



$(u=0, v=0)$ correspond to $T=0, h=0$, which is the spontaneous order phase.

RG flow curve in x - y plane.



Since we got $x=0$, $y=1$ as the critical point of the flow where $\epsilon_g = \infty$ (spontaneous order), we can get the scaling relations.

Scaling relations

Around $x=0$ & $y=1$ we have
 $x = 0 + \delta x$ & $y = 1 - \delta y$

$$\Rightarrow \begin{bmatrix} \delta x' \\ \delta y' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}$$

$$\Lambda_x = 4 \quad \Lambda_y = 2$$

Since $b=2$ & $\Lambda_m = b^{\lambda_m}$

we get $\Lambda_x = 2^2$ & $\Lambda_y = 2^1$

$$\rightarrow \lambda_x = 2 \quad \& \quad \lambda_y = 1$$

Thus both x (Temp) & y (heat)
are relevant variables.

$$\delta x' = 2^2 \delta x \quad \& \quad \delta y' = 2^1 \delta y$$

$$\text{or } \underline{\delta x^{(l)} = 2^{2l} \delta x}, \quad \underline{\delta y^{(l)} = 2^l \delta y}$$

Choose l such that $\delta x^{(l)} = 1$

$$\Rightarrow 2^{2l} = (\delta x)^{-1} \Rightarrow 2^l = (\delta x)^{-\frac{1}{2}}$$

Scaling form of free energy

We know that the singular part of free energy transforms as

$$f^{(L)}(K^{(L)}, L^{(L)}) = b^{dL} f(K, L) \quad \left(\begin{array}{l} d=1 \\ b=2 \end{array} \right)$$

In terms of x & y ,

$$f^{(L)}[x^{(L)}, y^{(L)}] = 2^L f(x, y)$$

$$\Rightarrow f(x, y) = 2^{-L} f^{(L)}[x^{(L)}, y^{(L)}]$$

Near the critical point,

$$x = x^* + \delta x \quad \text{where } x = e^{-4K}$$

$$y = y^* + \delta y \quad \text{where } y = e^{-2L}$$

$$x^* = 0, \quad y^* = 1$$

$$\Rightarrow f(\delta x, \delta y) = 2^{-L} f^{(L)}[\delta x^{(L)}, \delta y^{(L)}]$$

choosing l such that $\partial x^{(l)} = 1$

$$\Rightarrow 2^{2l} = (\partial x)^{-1} \Rightarrow 2^l = (\partial x)^{-\frac{1}{2}}$$

$$\Rightarrow \partial y^{(l)} = 2^l \partial y = (\partial x)^{-\frac{1}{2}} \partial y$$

$$\Rightarrow f(\partial x, \partial y) = (\partial x)^{\frac{1}{2}} f^{(l)} \left[1, (\partial x)^{-\frac{1}{2}} \partial y \right]$$

$$y = 1 - \partial y \rightarrow e^{-2L} = 1 - \partial y$$

Since ∂y is small, it means e^{-2L} is close to 1, thus it implies that L is very small. Thus,

$$e^{-2L} \sim 1 - 2L \Rightarrow \underline{\underline{\partial y = 2L}}$$

$$x = e^{-4K} = x^* + \partial x = \partial x \quad (x^* = 0)$$

$$\Rightarrow f(\partial x, \partial y) = e^{-2K} f^{(l)} \left[1, \frac{2L}{e^{-2K}} \right]$$

$$\Rightarrow \underline{\underline{f(e^{-4K}, 2L) = e^{-2K} \mathcal{F}(e^{2K} L)}}$$

This is the scaling form for free energy in 1D ising model.

Starting from the free energy scaling form, we can get other thermodynamic variable behaviours.

$$\text{We have, } f(x, L) = x^{1/2} F(x^{-1/2} L)$$

$$m(x, L) = -\frac{\partial f}{\partial L} = -x^{1/2} x^{-1/2} F'(x^{-1/2} L)$$

$$\Rightarrow \underline{m(x, L) = M(x^{-1/2} L)}$$

↳ Scaling form for magnetization

The magnetic susceptibility χ at critical point is given by,

$$\chi_s = \lim_{L \rightarrow 0} \left(\frac{\partial m}{\partial L} \right) \propto x^{-1/2} \quad (x = e^{-4K})$$

$$\Rightarrow \underbrace{\chi_s \propto e^{2\beta T}}_{(\text{as } T \rightarrow 0)} \quad \downarrow \quad (\text{dominant contribution})$$

Thus χ_s diverges exponentially as $T \rightarrow 0$ or $\beta \rightarrow \infty$.

Scaling form of Correlation function

$$G_1(R, x, 2L) = \frac{\langle S_i S_{i+R} \rangle = \sum_{\{S_i\}} S_i S_{i+R} e^{-\mathcal{H}}}{\sum_{\{S_i\}} e^{-\mathcal{H}}}$$

After performing one RG-step near the vicinity of $x^* = 0$ & $y^* = 1$

$$G_1(R, \partial x, \partial y) = G_1'(R', \partial x', \partial y')$$

$$= G_1^{(l)} \left[\frac{R}{2^l}, \partial x^{(l)}, \partial y^{(l)} \right]$$

$$= G_1^{(l)} \left[2^{-l} R, \underbrace{2^{2l} \partial x}_1, 2^l \partial y \right]$$

choose $2^l = (\partial x)^{-\frac{1}{2}}$, $(\partial y = 2L)$

$$\Rightarrow G_1(R, \partial x, \partial y) = \mathcal{G} \left[(\partial x)^{\frac{1}{2}} R, (\partial x)^{-\frac{1}{2}} L \right]$$

$\hookrightarrow$ Scaling form of correlation

Thus in the absence of heat,
 $L=0$,

$$\Rightarrow G(R, \beta x) = \mathcal{G} [e^{-2\beta J} R]$$

$$\Rightarrow G(R, \beta x) = \mathcal{G} \left[\frac{R}{e^{2\beta J}} \right]$$

Scaling form of correlation when $h_{\text{ext}}=0$

Thus the correlation length defined as the reciprocal of the coefficient of 'R' in the correlation Scaling form is given by ,
 $\xi_g = e^{2\beta J}$

Thus the correlation length also diverges exponentially as $T \rightarrow 0$.

Note:

(i) If $L = 0$, then the RG for 1D is trivial & the recurrence relation can be found easily. We know that for 1D Ising ferromagnetic model, the correlation b/w 2-spins is given by $\langle S_i S_{i+\sigma} \rangle = \tanh(K)^\sigma$

After RG with $b=2$ we get $\sigma \rightarrow \sigma' = \frac{\sigma}{2}$, $K \rightarrow K'$, $S \rightarrow S'$

$$\langle S'_i S'_{i+\sigma'} \rangle = \tanh(K')^{\sigma'}$$

$$\& \langle S_i S_{i+\sigma} \rangle = \langle S'_i S'_{i+\sigma'} \rangle_{\sigma'}$$

$$\Rightarrow \tanh^\sigma(K) = \tanh(K')^{\sigma'}$$

$$\Rightarrow [\tanh(K)]^\sigma = \tanh(K')^{\frac{\sigma}{2}}$$

$$\Rightarrow \underline{\tanh(K') = \tanh^2(K)}$$

$\hookrightarrow$ Recursion relation

$$\tanh(k') = \left[\frac{e^k - e^{-k}}{e^k + e^{-k}} \right]^2$$

$$\frac{e^{k'} - e^{-k'}}{e^{k'} + e^{-k'}} = \frac{e^{2k} - 2 + e^{-2k}}{e^{2k} + 2 + e^{-2k}}$$

$$\frac{1 - e^{-2k'}}{1 + e^{-2k'}} = \frac{e^{2k} - 2 + e^{-2k}}{e^{2k} + 2 + e^{-2k}}$$

$$\text{Let } x = e^{-2k}, \quad x' = e^{-2k'}$$

$$\Rightarrow \frac{1 - x'}{1 + x'} = \frac{x^{-1} - 2 + x}{x^{-1} + 2 + x}$$

$$\Rightarrow (1 - x') = \left(\frac{1 - 2x + x^2}{1 + 2x + x^2} \right) (1 + x')$$

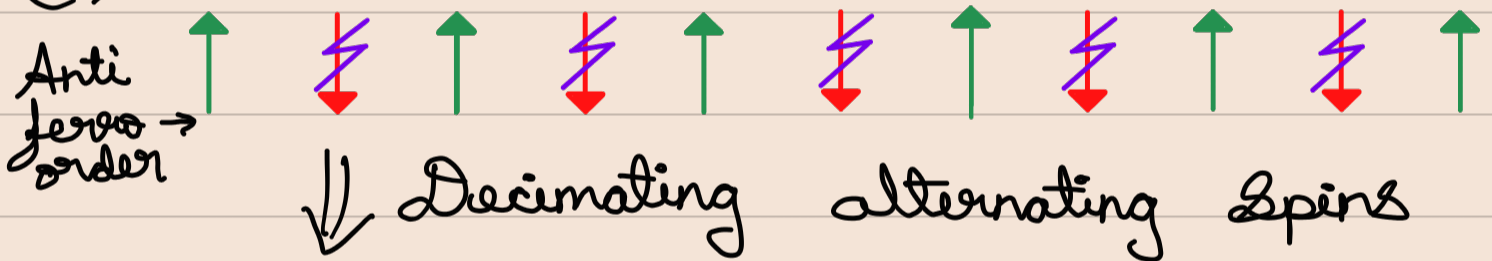
$$\Rightarrow 1 - \left(\frac{1 - 2x + x^2}{1 + 2x + x^2} \right) = \left[1 + \left(\frac{1 - 2x + x^2}{1 + 2x + x^2} \right) \right] x'$$

$$\Rightarrow \boxed{x' = \frac{2x}{1 + x^2}}$$

(ii) RG on 1D anti-ferromagnetic chain

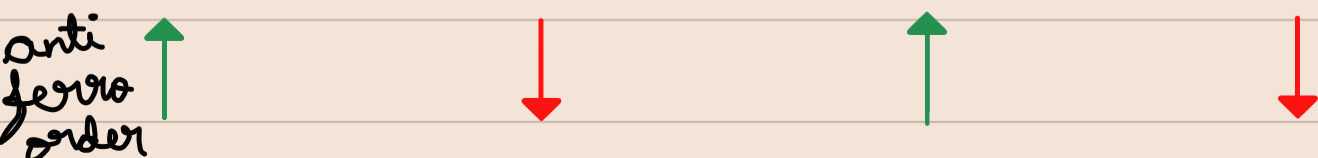
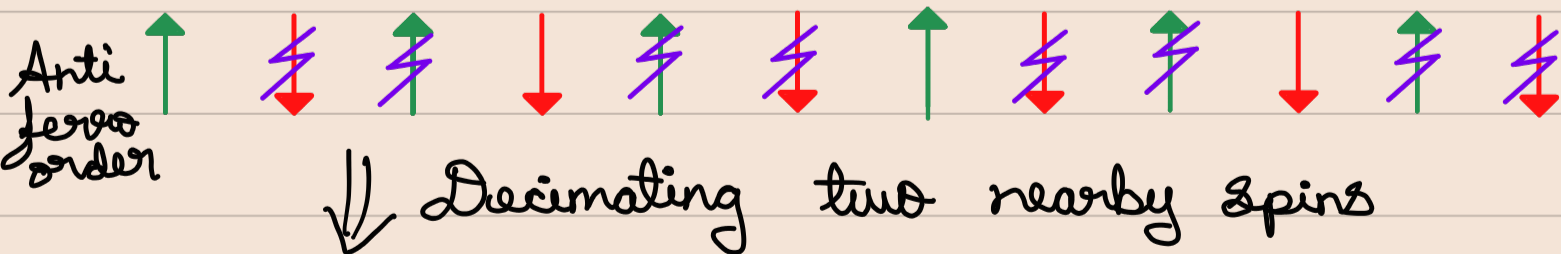
In any RG procedure, when we perform RG on a fixed point it returns back to the same point. i.e. when we do RG on 1D anti-ferromagnetic order then the result also should be an anti-ferromagnetic order.

(a)



The above is not valid RG.

(b)



Thus above is a valid RG.

(iii) The Saddle f.p. governs all different Hamiltonians. That is why we have universality class.

Eg: Heisenberg model with an $[S_i^z]^2$ term.

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - \epsilon \sum_i (S_i^z)^2$$

The ϵ -term is an Ising like term which makes the system to favour z -direction.

For $\epsilon = 0 \Rightarrow$ Heisenberg model f.p.
 & for $\epsilon = \infty \Rightarrow$ Ising like f.p.

