

23/04/25

Lecture 27

Spin weight functions & Generalization to Momentum Space RG

In the usual Ising model, the only allowed values of spin are ± 1 . We can think of this in the following manner. Here if the spin variable can take any value b/w $(-\infty, +\infty)$, but there is a weight function associated for every spin value.

So for Ising model we can write

$$W(S) = \delta(S+1) + \delta(S-1) = 2 \delta(S^2 - 1)$$

$$\begin{aligned} \Rightarrow Z_{\text{ising}} &= \sum_{\{S_i\}} e^{K \sum_{\langle i,j \rangle} S_i S_j} \\ &= \prod_n \sum_{S_n = \pm 1} e^{K \sum_{\langle i,j \rangle} S_i S_j} \end{aligned}$$

$$\sum_{S_n = \pm 1} = \int_{-\infty}^{\infty} dS_n \left[\delta(S-1) + \delta(S+1) \right]$$

$$\Rightarrow Z_{\text{ising}} = \prod_n \int_{-\infty}^{\infty} dS_n \, 2\beta(S^2 - 1) e^{-\beta H}$$

Thus above is another way to write the ising model incorporating continuous spin but an appropriate weight func. of $W(S) = 2\beta(S^2 + 1)$

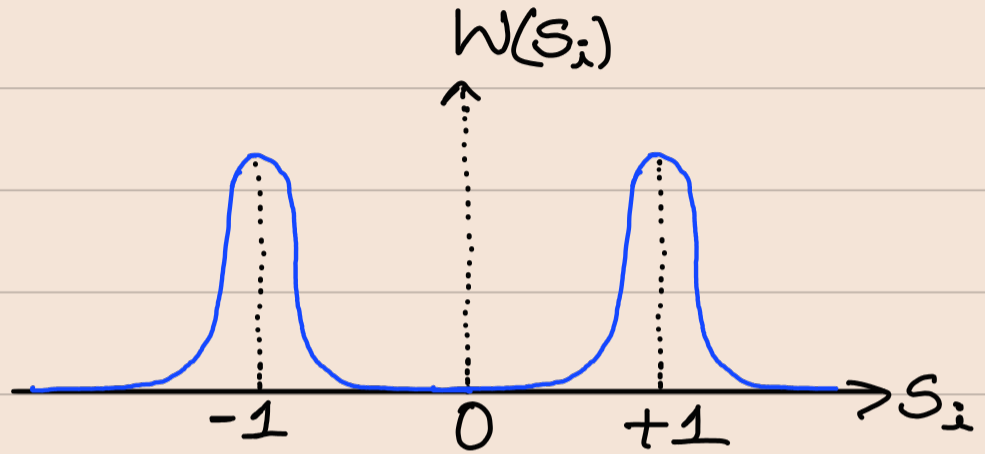


We can generalize this idea by exploring various other weight functions such as double gaussian / gaussian, S^4 term etc.

In this theory we have spins S_i at discrete lattice points 'i' but S_i can choose any value in $(-\infty, +\infty)$ with an appropriate weight.

Eg 1: Soft Spin model

$$W(s) = \frac{1}{\sigma\sqrt{2\pi}} \left[e^{-\frac{(s+1)^2}{2\sigma^2}} + e^{-\frac{(s-1)^2}{2\sigma^2}} \right]$$



Eg 2: Gaussian Model

$$W(s) = e^{-\frac{1}{2}bs^2}$$

(Spin peaks at zero)

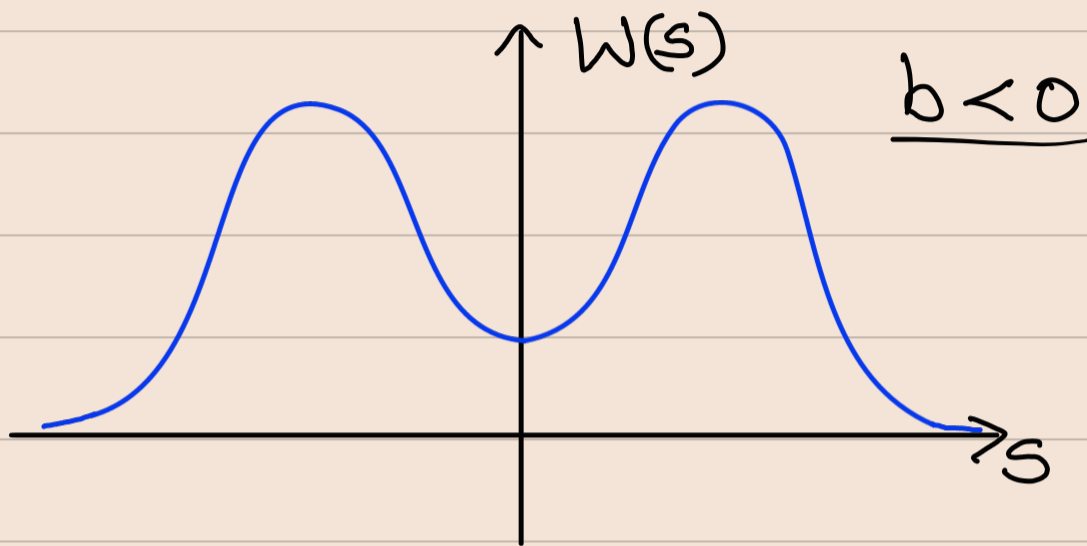


$b \rightarrow$ controls the sharpness
higher $b \rightarrow$ more sharp.

Eg 3: S^4 model

$$W(s) = e^{-\frac{1}{2}bs^2 - ws^4}$$

$w > 0$, $b \rightarrow \pm ve$



Multiplying the above weight function by a constant $\sqrt{\frac{w}{\pi}} e^{-w}$

& setting $b = -4w$ makes it equal to Soft Spin (double Gaussian) model.

& if $w \rightarrow \infty$, then we recover Ising model.

S^4 model is a powerful tool to study phase transitions using RG.

Partition function in Momentum Space

For any spin model the partition function can be written as

$$Z = \int \left[\prod_m dS_m W(S_m) \right] e^{\tilde{K}(\{S_m\})}$$

where $W(S_m)$ = weight function

$$\tilde{K} = -\beta \mathcal{H} \quad \mathcal{H} \Rightarrow \text{Hamiltonian}$$

$\tilde{K} \Rightarrow$ dimensionless hamiltonian

$$\mathcal{H} = -\frac{1}{2} \sum_{i,j} S_i J_{ij} S_j \rightarrow \text{General Ising model.}$$

$J_{ij} \Rightarrow$ Interaction matrix (Symmetric)

$$\Rightarrow \tilde{K} = \frac{1}{2} \sum_{i,j} S_i A_{ij} S_j \quad (A_{ij} = A_{ji})$$

$$\Rightarrow Z = \int \left[\prod_m dS_m W(S_m) \right] e^{\frac{1}{2} \sum_{i,j} S_i A_{ij} S_j}$$

If we choose $W(S) = e^{-\frac{1}{2}bS^2 - \lambda S^4}$, then

$$Z = \int_{-\infty}^{\infty} \left[\prod_m dS_m e^{-\frac{1}{2}bS_m^2 - \lambda S_m^4} \right] e^{\frac{1}{2} \sum_{i,j} S_i A_{ij} S_j}$$

$$= \int_{-\infty}^{\infty} \left[\prod_m dS_m \right] e^{-\frac{1}{2} \sum_m (bS_m^2 + \lambda S_m^4)} e^{\frac{1}{2} \sum_{i,j} S_i A_{ij} S_j}$$

$$= \int_{-\infty}^{\infty} \left[\prod_m dS_m \right] e^{\left[\frac{1}{2} \sum_{i,j} S_i A_{ij} S_j - \sum_m \left(\frac{1}{2} b S_m^2 + \lambda S_m^4 \right) \right]}$$

$K(\{S_i\}) = \frac{1}{2} \sum_{i,j} S_i A_{ij} S_j - \sum_m \left(\frac{1}{2} b S_m^2 + \lambda S_m^4 \right)$ is the generalized hamiltonian which includes the weight function built into it. Thus the partition function for S^4 model can be written as

$$Z = \int_{-\infty}^{\infty} \left[\prod_m dS_m \right] e^{K(\{S_i\})}$$

where $K(\{S_i\}) = \frac{1}{2} \sum_{i,j} S_i A_{ij} S_j - \sum_m \left(\frac{1}{2} b S_m^2 + \lambda S_m^4 \right)$

If $\lambda = 0$ it becomes gaussian model.

Now we can go to momentum space using discrete fourier transform

$$\sigma_{\vec{q}} = \sum_i S_i e^{-i\vec{q} \cdot \vec{R}_i} \Rightarrow S_m = \frac{1}{N} \sum_{\vec{q}} \sigma_{\vec{q}} e^{i\vec{q} \cdot \vec{R}_m}$$

$$A(\vec{q}) = \sum_{\vec{R}_{ij}} A_{ij} e^{-i\vec{q} \cdot \vec{R}_{ij}}$$

where $\vec{R}_{ij} = \vec{R}_i - \vec{R}_j$

$$\rightarrow A_{ij} = \frac{1}{N} \sum_{\vec{q}} A(\vec{q}) e^{i\vec{q} \cdot \vec{R}_{ij}}$$

$$\Rightarrow \frac{1}{2} \sum_{i,j} S_i A_{ij} S_j$$

$$= \frac{1}{2} \sum_{i,j} \sum_{\vec{q}, \vec{q}', \vec{q}''} \frac{\sigma_{\vec{q}} \sigma_{\vec{q}'} A(\vec{q}'')}{N^3} e^{i[\vec{q} \cdot \vec{R}_i + \vec{q}' \cdot \vec{R}_j + \vec{q}'' \cdot (\vec{R}_i - \vec{R}_j)]}$$

$$= \frac{1}{2} \sum_{\vec{q}, \vec{q}', \vec{q}''} \frac{\sigma_{\vec{q}} \sigma_{\vec{q}'} A(\vec{q}'')}{N^3} \sum_{i,j} e^{i[(\vec{q} + \vec{q}'') \cdot \vec{R}_i + (\vec{q}' - \vec{q}'') \cdot \vec{R}_j]}$$

$$= \frac{1}{2} \sum_{\vec{q}, \vec{q}', \vec{q}''} \frac{\sigma_{\vec{q}} \sigma_{\vec{q}'} A(\vec{q}'')}{N} \delta(\vec{q} + \vec{q}'') \delta(\vec{q}' - \vec{q}'')$$

$$\Rightarrow \frac{1}{2} \sum_{i,j} S_i A_{ij} S_j = \frac{1}{2} \sum_{\vec{q}, \vec{q}'} \frac{\sigma_{\vec{q}} \sigma_{\vec{q}'}}{N} A(\vec{q}) \delta(\vec{q} + \vec{q}')$$

$$= \frac{1}{2N} \sum_{\vec{q}'} \sigma_{-\vec{q}'} \sigma_{\vec{q}'} A(\vec{q}') \\ (\text{replace } \vec{q}' \rightarrow \vec{q})$$

$$\Rightarrow \frac{1}{2} \sum_{i,j} S_i A_{ij} S_j = \frac{1}{2N} \sum_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}} A(\vec{q})$$

Similarly we can fourier transform

$$\frac{1}{2} \sum_m b S_m^2 = \frac{1}{2} \sum_m b \frac{1}{N^2} \sum_{\vec{q}, \vec{q}'} \sigma_{\vec{q}} \sigma_{\vec{q}'} e^{i(\vec{q} + \vec{q}') \cdot \vec{R}_m}$$

$$= \frac{b}{2N} \sum_{\vec{q}, \vec{q}'} \sigma_{\vec{q}} \sigma_{\vec{q}'} \delta(\vec{q} + \vec{q}')$$

$$\Rightarrow \frac{1}{2} \sum_m b S_m^2 = \frac{b}{2N} \sum_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}}$$

Then lastly $\frac{u}{2} \sum_m S_m^4$ can be fourier transformed as follows.

$$\omega \sum_m S_m^4 = \omega \sum_{q_1, q_2, q_3, q_4} \frac{\sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{\vec{q}_4} \delta(q_1 + q_2 + q_3 + q_4)}{N^3}$$

$$\Rightarrow \omega \sum_m S_m^4 = \frac{\omega}{N^3} \sum_{q_1, q_2, q_3} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3}$$

Thus by combining them we get the Hamiltonian in fourier space as,

$$\mathcal{H}(\sigma) = \frac{1}{2N} \sum_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}} (A(\vec{q}) - b) - \frac{\omega}{N^3} \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3}$$

Now we are left with the fourier transform of $\prod_i dS_i$

Since fourier transform is unitary transform, the jacobian is 1.

$$\Rightarrow \prod_i dS_i = \prod_{\vec{q}} d\sigma_{\vec{q}}$$

The spin S_i is real. This puts a restriction on $\{\vec{q}\}$. Consider

$$\sigma_{\vec{q}} = \frac{1}{N} \sum_i S_i e^{-i\vec{q} \cdot \vec{R}_i}$$

$$\Rightarrow \sigma_{\vec{q}}^* = \frac{1}{N} \sum_i S_i e^{i\vec{q} \cdot \vec{R}_i}$$

$$\Rightarrow \underbrace{\sigma_{\vec{q}}^*}_{\sigma_{-\vec{q}}} = \sigma_{-\vec{q}}$$

Thus the fourier components of S_i $\sigma_{\vec{q}}$ & $\sigma_{-\vec{q}}$ are not independent but rather they are complex conjugates of each other.

Hence for every mode $\vec{q}$ there exists a mode $-\vec{q}$. $\Rightarrow$ Set of allowed values of $\vec{q}$ are $\{\vec{q}, -\vec{q}\}$.

$$\Rightarrow \prod_i dS_i = \prod_{\vec{q}} d\sigma_{\vec{q}} = d\sigma_{\vec{q}=0} \prod_{\vec{q}>0} d\sigma_{\vec{q}} d\sigma_{-\vec{q}}$$

where $\vec{q}>0$ is just a label to choose half of the modes.

Consider any spin mode $\sigma_{\vec{q}}$

$$\sigma_{\vec{q}} = \sigma_{\vec{q}}^{(R)} + i\sigma_{\vec{q}}^{(I)} \Rightarrow \sigma_{\vec{q}}^* = \sigma_{\vec{q}}^{(R)} - i\sigma_{\vec{q}}^{(I)} = \sigma_{-\vec{q}}$$

$$\Rightarrow \sigma_{-\vec{q}}^{(R)} + i\sigma_{-\vec{q}}^{(I)} = \sigma_{\vec{q}}^{(R)} - i\sigma_{\vec{q}}^{(I)}$$

$$\Rightarrow \sigma_{\vec{q}}^{(R)} = \sigma_{-\vec{q}}^{(R)} \quad \& \quad \sigma_{\vec{q}}^{(I)} = -\sigma_{-\vec{q}}^{(I)}$$

Also we get $\sigma_{\vec{q}=0}^{(I)} = -\sigma_{\vec{q}=0}^{(I)} \Rightarrow \sigma_{\vec{q}=0}^{(I)} = 0$
Thus $\sigma_{\vec{q}=0}$ is real.

So we can write,

$$\& \sigma_{\vec{q}} = \sigma_{\vec{q}}^{(R)} + i\sigma_{\vec{q}}^{(I)}, \quad \sigma_{-\vec{q}} = \sigma_{\vec{q}}^{(R)} - i\sigma_{\vec{q}}^{(I)}$$

is a change of variables with the $|\text{Jacobian}| = 2$. So for $\vec{q} > 0$, we write,

$$\underline{d\sigma_{\vec{q}} d\sigma_{-\vec{q}} = 2 d\sigma_{\vec{q}}^{(R)} d\sigma_{\vec{q}}^{(I)}}$$

$$\Rightarrow \prod_{\vec{q}} d\sigma_{\vec{q}} = d\sigma_{\vec{q}=0} \prod_{\vec{q}>0} d\sigma_{\vec{q}} d\sigma_{-\vec{q}}$$

$$\int \prod_{\vec{q}} d\sigma_{\vec{q}} = \int_{\vec{q}} = d\sigma_{\vec{q}=0} \prod_{\vec{q}>0} 2 d\sigma_{\vec{q}}^{(R)} d\sigma_{\vec{q}}^{(I)}$$

In thermodynamic limit, $N \rightarrow \infty$.

$$\Rightarrow \frac{1}{N} \sum_{\vec{q} \in 1\text{B.Z.}} = \frac{1}{(2\pi)^d} \int d^d q = \int_{\vec{q}}$$

where $-\pi < q_i < \pi \Rightarrow 1^{\text{st}}$ Brillouin zone.

Then the generalized Hamiltonian is,

$$\begin{aligned} \mathcal{H}(\sigma) &= \frac{1}{2N} \sum_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}} (A(\vec{q}) - b) \\ &\quad - \frac{\omega}{N^3} \sum_{\vec{q}_1, \vec{q}_2, \vec{q}_3} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \\ &= \frac{1}{2} \int_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}} (A(\vec{q}) - b) - \omega \iiint_{q_1, q_2, q_3} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \end{aligned}$$

Now we do the long wavelength approximation where for a cubic lattice in d -dimensions, $A(\vec{q})$ is given by,

$$A(\vec{q}) = A(0) [1 - \alpha |\vec{q}|^2]$$

(Similar to Ornstein Zernike theory)

$$\begin{aligned}
\Rightarrow \mathcal{H}(\sigma) &= \frac{1}{2} \int_{\mathcal{V}} \sigma_{\vec{r}} \sigma_{-\vec{r}} (A(0) [1 - \alpha q^2] - b) \\
&\quad - \omega \iiint_{q_1 q_2 q_3} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \sigma_{\vec{r}_3} \sigma_{-\vec{r}_1 - \vec{r}_2 - \vec{r}_3} \\
&= \frac{1}{2} \int_{\mathcal{V}} \sigma_{\vec{r}} \sigma_{-\vec{r}} (A(0) - b - A(0) \alpha q^2) \\
&\quad - \omega \iiint_{q_1 q_2 q_3} \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \sigma_{\vec{r}_3} \sigma_{-\vec{r}_1 - \vec{r}_2 - \vec{r}_3} \\
&= -\frac{1}{2} \int_{\mathcal{V}} \sigma_{\vec{r}} \sigma_{-\vec{r}} \alpha A(0) \left(\frac{b - A(0)}{\alpha A(0)} + q^2 \right) \\
&\quad - \frac{\omega}{[\alpha A(0)]^2} \iiint_{q_1 q_2 q_3} [\alpha A(0)]^2 \sigma_{\vec{r}_1} \sigma_{\vec{r}_2} \sigma_{\vec{r}_3} \sigma_{-\vec{r}_1 - \vec{r}_2 - \vec{r}_3}
\end{aligned}$$

By rescaling σ 's such that

$$\alpha A(0) \sigma_{\vec{r}} \sigma_{-\vec{r}} \longrightarrow \sigma_{\vec{r}} \sigma_{-\vec{r}}$$

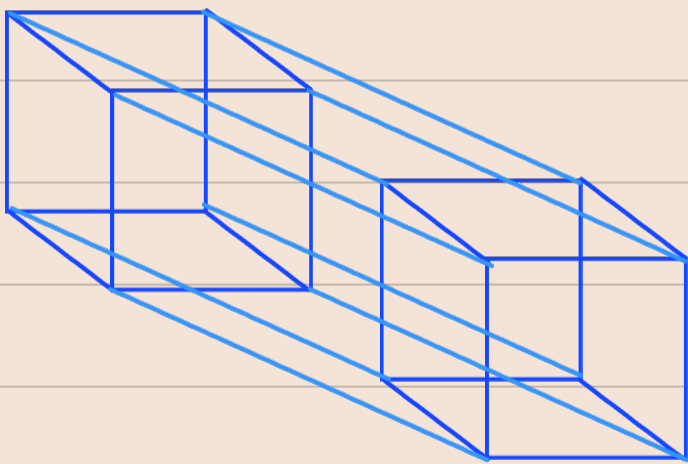
& by redefining $\frac{b - A(0)}{\alpha A(0)} = \gamma$ & $\frac{\omega}{[\alpha A(0)]^2} = \mathcal{U}$

we get standard form of hamiltonian of S^4 model.

$$\mathcal{K}(\sigma) = -\frac{1}{2} \int_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}} (\gamma + q^2) - \mathcal{U} \left(\prod_{\vec{q}_1, \vec{q}_2, \vec{q}_3} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3} \right)$$

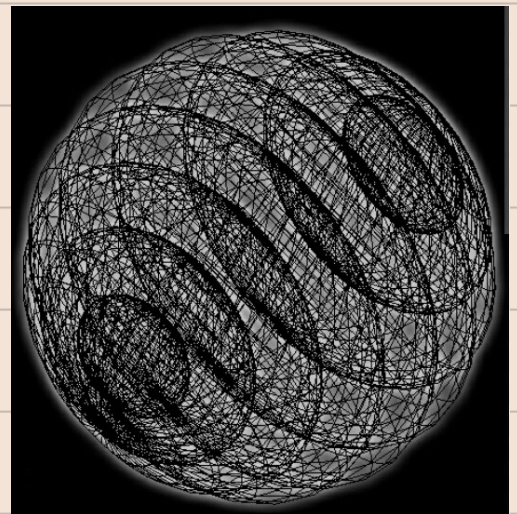
Then we change the integration limit of $-\pi < q_i < \pi$, that represents the d -dimensional cube which is the 1st Brillouin zone to a sphere of radius 1 i.e. $0 < |\vec{q}|^2 < 1$.

(d -dimensional cube)



$$-\pi < q_i < \pi$$

(d -dim Sphere)



$$0 \leq |\vec{q}|^2 \leq 1$$

Thus now $\int_{\vec{q}}$ represents $\frac{1}{(2\pi)^d} \int d^d q$

where q_i 's are such that $q = |\vec{q}|$ lies between $0 \neq 1$.

The change in integrating limits of $q_i = \{q_x, q_y, q_z, \dots\}$ conceptually changes the interpretation of $\sigma_{\vec{q}}$'s, which were initially, variables that were just the Fourier modes of S_i 's. But now, $\sigma_{\vec{q}}$'s are functional variables which makes the partition function in the momentum space as a functional integral over $\sigma_{\vec{q}}$ instead of ordinary integrals over S_i .

$$\Rightarrow \text{We can write } \underbrace{\int \prod_{\vec{q}} d\sigma_{\vec{q}}}_{\text{functional measure}} = \int_{\sigma_{\vec{q}}}$$

So partition function becomes,

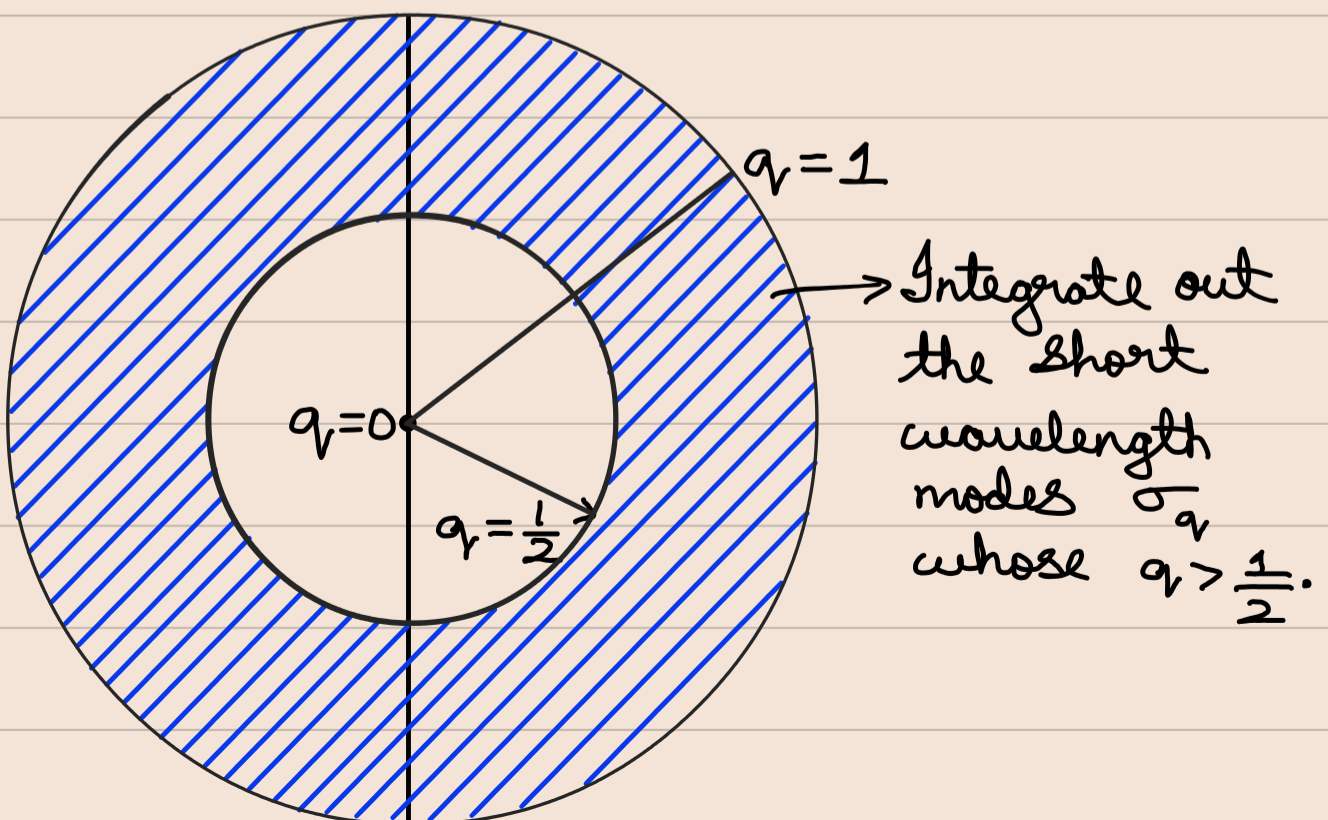
$$Z = \int_{\sigma_{\vec{q}}} e^{\mathcal{K}(\sigma_{\vec{q}})} \Rightarrow \text{Functional Integration over } \sigma_{\vec{q}}\text{'s}$$

where $0 \leq |\vec{q}| \leq 1$ &

$$\mathcal{K}(\sigma) = -\frac{1}{2} \int_{\vec{q}} \sigma_{\vec{q}} \sigma_{-\vec{q}} (\chi + q^2) - \mathcal{U} \iiint_{q_1 q_2 q_3} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{\vec{q}_3} \sigma_{-\vec{q}_1 - \vec{q}_2 - \vec{q}_3}$$

The basic Steps in Momentum space
RG is as follows.

1) We treat $\vec{q}$ whose $0 \leq |\vec{q}| \leq \frac{1}{2}$ as long wavelength modes which represents large scale fluctuations of S_i & those $\vec{q}$ whose $|\vec{q}| > \frac{1}{2}$ are short wavelength modes. Since in RG we are interested in large scale phenomena, we integrate out the fourier modes $\sigma_{\vec{q}}$ whose $|\vec{q}| > \frac{1}{2}$ & leaving $\sigma_{\vec{q}}$ for $|\vec{q}| < \frac{1}{2}$ as it is.



This process is the course-graining step in momentum space R_G , which is analogous to the Kadanoff's block spin method in real space R_G where we course grain & integrate out the small scale interaction to see the large scale behaviour. The long wavelength modes $\sigma_{\vec{q}}$ ($q < \frac{1}{2}$) are like block spins.

2) The second step is rescaling where we redefine $\vec{q}$ to get a unit sphere in the scaled momenta $\vec{q}'$. This step is rescaling.

3) Redefine spin modes & parameters to make the expression of K' similar to the old one where

$$K(u, \tau) \rightarrow K(u', \tau')$$
$$\& \sigma_{\vec{q}} \rightarrow \sigma_{\vec{q}'} \quad (\vec{q}' = 2\vec{q})$$

This gives us the recursion eqⁿ for the parameters.

4) Find interaction relations & their fixed points, then by performing linear stability analysis we can arrive at scaling forms of the thermodynamic observables & get the T_c & critical exponents.

Note

(i) The parameter ' χ ' is equivalent to the reduced temperature

(ii) The change of limits of ' q ' from 1st B.Z. to a sphere of radius '1' is justified because the physics of interest is in high wavelength near $\vec{q} \sim 0$, thus neglecting higher ' q ' will not matter as it is integrated out anyway, adding only to the total free energy which is not our interest.