

Relativistic String

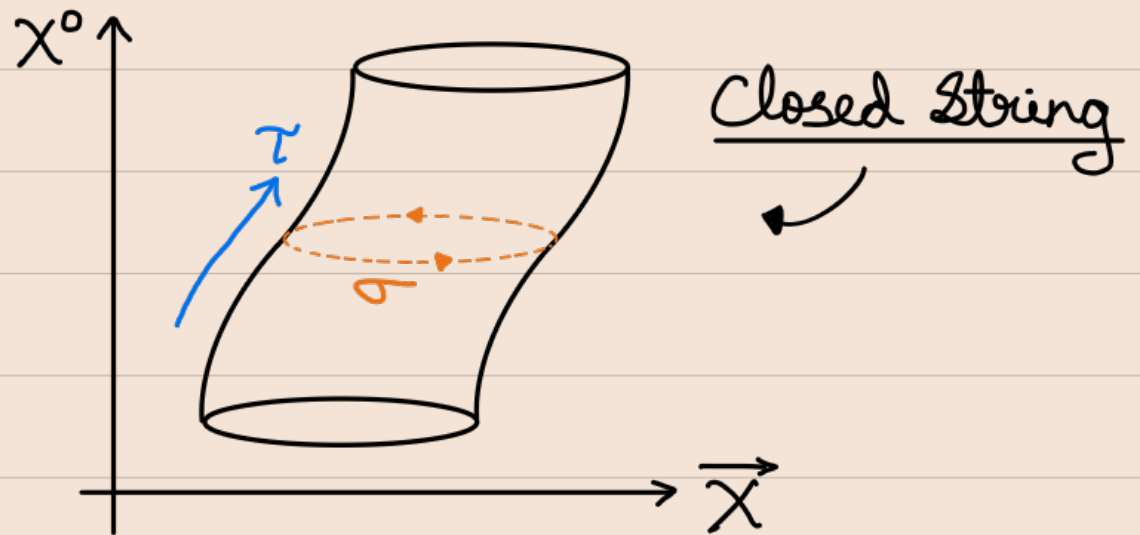
A 1 dim string sweeps out a 2-dim World Sheet embedded in a D-dim Space-time. This is similar to a 0-dim point particle which describes a 1-dim world line in D-dim Space.

So for the world sheet, we shall use two parameters (τ, σ) , where the D-dim Space-time coordinates $X^\mu \rightarrow X^\mu(\tau, \sigma)$ completely describe the string's location in Space time. We shall set ' τ ' to be a time-like parameter and ' σ ' space-like.
 $\rightarrow \sigma^a = (\sigma^0, \sigma^1) = (\tau, \sigma)$

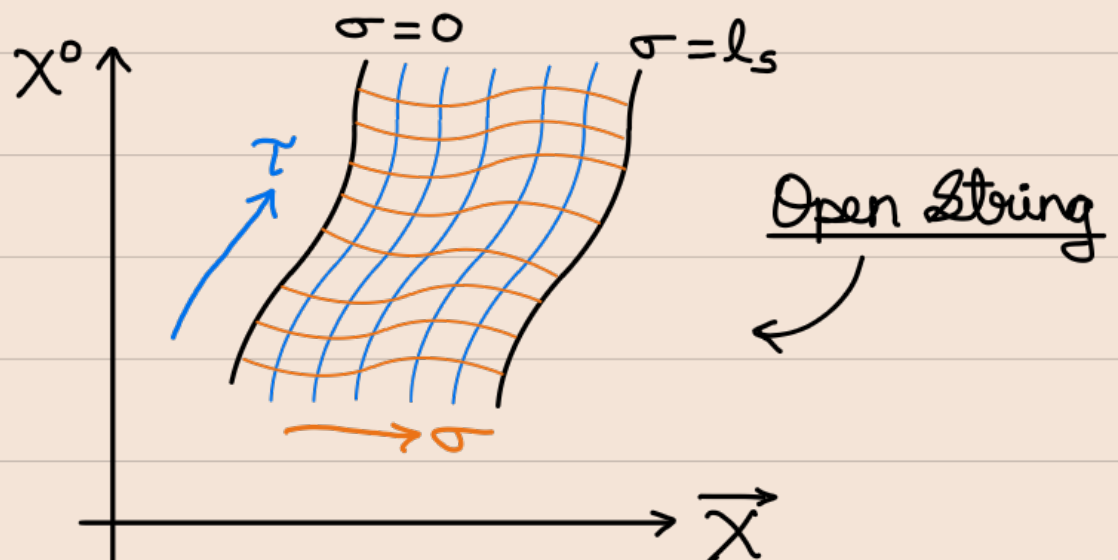
$X^\mu(\tau, \sigma)$ acts as a map to embed the string's world sheet in the D-dim Minkowski Space & (τ, σ) are the world sheet coordinates describing the 2-dim manifold.

For closed string we require $X^\mu(\tau, \sigma)$ to be periodic in ' σ ' with a period of ' l_s ' (length of the string). Thus,

$$\sigma = [0, l_s) \quad \& \quad X^\mu(\tau, \sigma + l_s) = X^\mu(\tau, \sigma)$$



For open strings, $X^\mu(\tau, \sigma)$ need not be periodic in ' σ '. If ' l_s ' is the length of the string, then, $\sigma = [0, l_s]$ $\&$ $X^\mu(\tau, \sigma + l_s) \neq X^\mu(\tau, \sigma)$.



To define the action of such strings, we need to define the induced metric as h_{ab} , which is the pull back of the Minkowski metric $\eta_{\mu\nu}$.

$$h_{ab} = \left(\frac{\partial x^\mu}{\partial \sigma^a} \right) \left(\frac{\partial x^\nu}{\partial \sigma^b} \right) \eta_{\mu\nu}$$

Let $\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau}$ and $\dot{x}^\mu = \left(\frac{\partial x^\mu}{\partial \sigma} \right)$

$$\rightarrow [h_{ab}] = \begin{bmatrix} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} & \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} \\ \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} & \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} \end{bmatrix}$$

This $[h_{ab}]$ defines the manifold of the world sheet. But it is not independent. It completely relies on $x^\mu(\tau, \sigma)$.

The Nambu-Goto Action

Analogous to the point particle, we can write the Nambu-Goto action for the String as,

$$S_{NG} = \frac{-1}{2\pi\alpha'} \int_W d\tau d\sigma \sqrt{-\det(h_{ab})}$$

'W' denotes the world sheet surface. Then the tension (T) of the string is given by,

$$T = \frac{1}{2\pi\alpha'} \quad , \quad \alpha' = \text{Regge Slope}$$

The Nambu-Goto action can be expanded as,

$$S_{NG} = \frac{-1}{2\pi\alpha'} \int_W d\tau d\sigma \sqrt{(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu})^2 - (\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu})(\dot{X}^\alpha \dot{X}^\beta \eta_{\alpha\beta})}$$

This action is proportional to the area of the world sheet defined by $\eta_{\mu\nu}$ metric.

In the non-relativistic limit, we shall choose $\tau = X^0 = t$ (time), $\dot{X}^i = v^i$ is the velocity of string, $\dot{X}^i = u^i$ is the longitudinal velocity of string.

$$\rightarrow \frac{dX^0}{d\sigma} = 0, \quad (\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}) = (\vec{v} \cdot \vec{v}),$$

$$(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}) = -1 + |\vec{v}|^2,$$

$$\& (\dot{X}^\alpha \dot{X}^\beta \eta_{\alpha\beta}) = |\vec{u}|^2$$

$$\begin{aligned} \rightarrow \sqrt{-\det(h_{ab})} &= \sqrt{(\vec{v} \cdot \vec{v})^2 + (1 - |\vec{v}|^2)|\vec{u}|^2} \\ &= |\vec{u}| \sqrt{1 - |\vec{v}|^2 + \frac{(\vec{v} \cdot \vec{v})^2}{|\vec{u}|^2}} \end{aligned}$$

$$\approx |\vec{u}| \left[1 - \frac{1}{2} \left(|\vec{v}|^2 - \frac{(\vec{v} \cdot \vec{v})^2}{|\vec{u}|^2} \right) \right]$$

$$\approx |\vec{u}| \left[1 - \frac{|\vec{v}|^2}{2} + \frac{(\vec{v} \cdot \vec{v})^2}{2|\vec{u}|^2} \right] \quad \text{Neglect the higher powers}$$

$$\rightarrow S_{NG} \approx \frac{-1}{2\pi\alpha'} \int_W dt d\sigma |\vec{u}| \left[1 - \frac{|\vec{v}|^2}{2} + \frac{(\vec{v} \cdot \vec{u})^2}{2|\vec{u}|^2} \right]$$

Thus in non-relativistic limit,

$$S_{NG} \approx \int_W dt d\sigma \frac{|\vec{u}|}{2\pi\alpha'} \left(\frac{|\vec{v}|^2}{2} - 1 - \frac{(\vec{v} \cdot \vec{u})^2}{2|\vec{u}|^2} \right)$$

$S_{NG} = \int dt L$, where L is the Lagrangian.

$$\Rightarrow L = \int d\sigma \frac{|\vec{u}|}{2\pi\alpha'} \left(\frac{|\vec{v}|^2}{2} - 1 - \frac{(\vec{v} \cdot \vec{u})^2}{2|\vec{u}|^2} \right)$$

Since $\vec{u}$ is the longitudinal velocity,

$$\vec{v}_T = \vec{v} - \frac{(\vec{u} \cdot \vec{v})}{|\vec{u}|^2} \vec{u} \rightarrow \text{transverse velocity}$$

$$\& v_T^2 = v^2 - \frac{(\vec{u} \cdot \vec{v})^2}{|\vec{u}|^2}$$

$$\text{Thus, } \mathcal{L} = \int d\sigma \frac{|\vec{u}|}{2\pi\alpha'} \left(\frac{|\vec{v}_T|^2}{2} - 1 \right)$$

$$= \int d\sigma \frac{|\vec{u}|}{2\pi\alpha'} \frac{|\vec{v}_T|^2}{2} - \frac{1}{2\pi\alpha'} \int d\sigma |\vec{u}|$$

$|\vec{u}| d\sigma$ gives the length element ' dl_s ' of the string. Thus $\int d\sigma |\vec{u}| = l_s$

$\Rightarrow l_s = \text{Physical length of string}$

Let $T = \frac{1}{2\pi\alpha'}$ $\rightarrow$ String Tension

which is equal to mass per length (ρ) in natural units. $\rightarrow T = \rho = \frac{1}{2\pi\alpha'}$

$$\Rightarrow \mathcal{L} = \int dl_s \underbrace{\left[\frac{1}{2} \rho |\vec{v}_T|^2 - T l_s \right]}_{\text{Lagrangian density}}$$

$$K.E = \frac{1}{2} \rho |\vec{v}_T|^2 \quad \& \quad P.E = T l_s$$

Symmetries of the Nambu-Goto Action

a) S_{NG} has isometry of flat space-time given by the D -dim Poincare group, under the transform,

$$X^\mu \longrightarrow \bar{X}^\mu = \Lambda^\mu_\nu X^\nu + a^\mu$$

Λ^μ_ν is the Lorentz transform ' $SO(1, D-1)$ ' and a^μ is the D -dim translation.

b) S_{NG} has reparameterization invariance (also called as diffeomorphism). Thus,

$$X^\mu(\tau, \sigma) = \bar{X}^\mu(\bar{\tau}, \bar{\sigma})$$

where
$$\left. \begin{aligned} \bar{\tau} &= \bar{\tau}(\tau, \sigma) \\ \bar{\sigma} &= \bar{\sigma}(\tau, \sigma) \end{aligned} \right\} \text{reparameterization}$$

This reflects the fact that we have a redundancy in our description because the world sheet coordinates (τ, σ) have no any physical meaning.

Eqⁿ of motion for a string using the Nambu Goto action

The Nambu-Goto lagrangian density is

$$\mathcal{L}_{NG} = \frac{-1}{2\pi\alpha'} \sqrt{-\det(h_{ab})}$$

The canonical conjugate momenta for X^μ are given by,

$$\pi_\mu^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \quad \& \quad \pi_\mu^\sigma = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$$

$$\rightarrow \pi_\mu^\tau = \frac{[(\dot{X}^\alpha \dot{X}^\beta \eta_{\alpha\beta}) \dot{X}^\nu - (\dot{X}^\alpha \dot{X}^\beta \eta_{\alpha\beta}) \dot{X}^\nu] \eta_{\mu\nu}}{(2\pi\alpha')^2 \mathcal{L}_{NG}}$$

$$\& \pi_\mu^\sigma = \frac{[(\dot{X}^\alpha \dot{X}^\beta \eta_{\alpha\beta}) \dot{X}^\nu - (\dot{X}^\alpha \dot{X}^\beta \eta_{\alpha\beta}) \dot{X}^\nu] \eta_{\mu\nu}}{(2\pi\alpha')^2 \mathcal{L}_{NG}}$$

$$\text{Then the EoM is, } \underline{\underline{\frac{\partial \pi_\mu^\tau}{\partial \tau} + \frac{\partial \pi_\mu^\sigma}{\partial \sigma} = 0}}$$

But these eqⁿ are very tough to work with & are very complicated. Thus we shall directly consider the variation of S_{NG} & use that variation of determinant of pull back metric.

$$\delta \sqrt{-\det(h)} = \frac{1}{2} \sqrt{-\det(h)} h^{ab} \delta h_{ab}$$

This finally results in Eqⁿ of motion,

$$\frac{\partial}{\partial \sigma^a} \left[\sqrt{-\det(h)} h^{ab} \frac{\partial x^u}{\partial \sigma^b} \right] = 0$$

where h^{ab} is the inverse metric of the induced metric ' h_{ab} '.

$$\rightarrow \underline{h^{ab} h_{bc} = \delta_c^a}$$

Only then $\delta \sqrt{-\det(h)} = \frac{\sqrt{-\det(h)}}{2} h^{ab} \delta h_{ab}$

The Polyakov Action

The Nambu Goto action is very complicated to quantize because of the square root factor. So we introduce a new dynamical field to make the action simple. Now, the world sheet of the string itself defines a 2-dim manifold with coordinates $\sigma^a = (\tau, \sigma)$ which has its own metric ' γ_{ab} '. τ is time-like & σ is space-like. So, we take γ_{ab} to have the metric signature of $(-, +)$.

Using this World Sheet metric we can actually construct a Simplified Action, whose eqⁿ of motion are that of N-G action. It is given by the Polyakov Action.

$$S_P = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} \eta_{\mu\nu}$$

$$\rightarrow S_P = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} h_{ab}$$

Remember that:

- Greek indices $\mu, \nu, \dots = (0, 1, 2, \dots, D-1) \rightarrow \text{Space-time}$
- Latin indices $i, j, k, \dots = (1, 2, 3, \dots, D-1) \Rightarrow \text{Space}$
- Latin indices $a, b, c, d = (0, 1) \rightarrow \text{World sheet}$

$$\rightarrow \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_i = \frac{\partial}{\partial x^i} \quad \& \quad \partial_a = \frac{\partial}{\partial \sigma^a}$$

Then the raising & lowering of indices also depend on the type of coordinate.

x^μ is Space-time coordinate $\rightarrow x_\mu = x^\nu \eta_{\mu\nu}$
where $\eta_{\mu\nu}$ is the metric of Space-time.

σ^a is World sheet coordinate $\Rightarrow \sigma_a = \sigma^b \gamma_{ab}$
where γ_{ab} is the world sheet metric.

$$\rightarrow \underline{\gamma^{ab} \gamma_{bc} = \delta_c^a}$$

Moreover $\gamma^{ab} \partial_b$ need not be equal to ' ∂^a '.
Because γ^{ab} also depends on the world sheet coordinates. $\Rightarrow \partial_a \gamma^{ab} \neq 0$.

But if $\phi(\tau, \sigma)$ is some scalar function of ' σ^a ', i.e. $\tilde{\phi}(\tilde{\tau}, \tilde{\sigma}) = \phi(\tau, \sigma)$ under the transformation $\sigma^a \rightarrow \tilde{\sigma}^a$, only then

$$\gamma^a \partial_a \phi = \partial^a \phi$$

This is because the covariant derivative $\nabla_a \phi = \partial_a \phi$ (Partial derivative) only for scalars. And also $\nabla_a \gamma_{bc} = 0$, thus,

$$\gamma^a \partial_a \phi = \gamma^a \nabla_a \phi = \nabla^a \phi = \partial^a \phi$$

Since $\chi^\mu(\tau, \sigma) = \tilde{\chi}^\mu(\tilde{\tau}, \tilde{\sigma})$ is a scalar with respect to σ^a , $\gamma^a \partial_a \chi^\mu = \partial^a \chi^\mu$

Thus even $h_{ab} = (\partial_a \chi^\mu)(\partial_b \chi_\mu)$, which is the induced metric, is also raised & lowered by γ^{ab} itself.

$$\rightarrow \underline{\gamma^{ac} \gamma^{bd} h_{ab} = h^{cd}}$$

$$\rightarrow h^{ab} h_{bc} \neq \delta^a_c$$

This is different from what we saw in the N.G. action.

Symmetries of the Polyakov Action

a) Poincare Invariance

$$\text{Under } X^\mu \rightarrow \tilde{X}^\mu = \Lambda^\mu_\nu X^\nu + C^\mu$$

the action is invariant, because,

$$\partial_a C^\mu = 0 \quad \& \quad \Lambda^\mu_\alpha \Lambda^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$$

$$\rightarrow (\partial_a X^\mu)(\partial_b X^\nu) \eta_{\mu\nu} \xrightarrow{\text{Poincare}} (\partial_a \tilde{X}^\alpha)(\partial_b \tilde{X}^\beta) \eta_{\alpha\beta}$$

Poincare Invariance is an internal symmetry of $X^\mu(\tau, \sigma)$ & doesn't have anything to do with the world sheets.

Thus the induced metric and the World Sheet metric are invariant under the Poincare transformation.

$$\Rightarrow h_{ab}(\tau, \sigma) \xrightarrow{\text{Poincare}} h_{ab}(\tau, \sigma)$$

$$\& \gamma_{ab}(\tau, \sigma) \xrightarrow{\text{Poincare}} \gamma_{ab}(\tau, \sigma)$$

b) Diffeomorphism Invariance

This is the symmetry of the lagrangian under the reparameterization of the World sheet Coordinates. It is a Gauge Symmetry on the World sheet.

$$\text{Let } \overset{\text{old}}{\sigma^a} \rightarrow \overset{\text{new}}{\tilde{\sigma}^a} = \tilde{\sigma}^a(\tau, \sigma)$$

$$\rightarrow \frac{\partial}{\partial \sigma^a} \rightarrow \frac{\partial}{\partial \tilde{\sigma}^a} = \underbrace{\left(\frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \right)}_{\text{chain rule}} \frac{\partial}{\partial \sigma^c}$$

$$\& \quad \chi^\mu(\tau, \sigma) = \tilde{\chi}^\mu(\tilde{\tau}, \tilde{\sigma})$$

(χ^μ is a scalar to world sheet)

$$\Rightarrow \frac{\partial \chi^\mu}{\partial \sigma^a} \frac{\partial \chi^\nu}{\partial \sigma^b} \eta_{\mu\nu} \rightarrow \frac{\partial \tilde{\chi}^\mu}{\partial \tilde{\sigma}^a} \frac{\partial \tilde{\chi}^\nu}{\partial \tilde{\sigma}^b} \eta_{\mu\nu}$$

where,

$$\frac{\partial \tilde{\chi}^\mu}{\partial \tilde{\sigma}^a} \frac{\partial \tilde{\chi}^\nu}{\partial \tilde{\sigma}^b} \eta_{\mu\nu} = \left(\frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \right) \left(\frac{\partial \sigma^d}{\partial \tilde{\sigma}^b} \right) \frac{\partial \chi^\mu}{\partial \sigma^c} \frac{\partial \chi^\nu}{\partial \sigma^d} \eta_{\mu\nu}$$

$$\rightarrow \tilde{h}_{ab} = \left(\frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \right) \left(\frac{\partial \sigma^d}{\partial \tilde{\sigma}^b} \right) h_{cd}$$

$$\text{or } h_{ab} = \left(\frac{\partial \tilde{\sigma}^c}{\partial \sigma^a} \right) \left(\frac{\partial \tilde{\sigma}^d}{\partial \sigma^b} \right) \tilde{h}_{cd}$$

In the next section, we will be showing that $\gamma_{ab} = 2 \underbrace{f(\tau, \sigma)}_{f \text{ is some scalar in } (\tau, \sigma)} h_{ab}$

f is some scalar in (τ, σ)

$$\rightarrow \gamma_{ab}(\tau, \sigma) \rightarrow \tilde{\gamma}_{ab}(\tilde{\tau}, \tilde{\sigma})$$

$$\text{where } \tilde{\gamma}_{ab} = \left(\frac{\partial \sigma^c}{\partial \tilde{\sigma}^a} \right) \left(\frac{\partial \sigma^d}{\partial \tilde{\sigma}^b} \right) \gamma_{cd}$$

$$\text{or } \gamma_{ab} = \left(\frac{\partial \tilde{\sigma}^c}{\partial \sigma^a} \right) \left(\frac{\partial \tilde{\sigma}^d}{\partial \sigma^b} \right) \tilde{\gamma}_{cd}$$

Since $\gamma^{ab} \gamma_{bc} = \delta_c^a$ we can get

$$\tilde{\gamma}^{ab} = \left(\frac{\partial \tilde{\sigma}^a}{\partial \sigma^c} \right) \left(\frac{\partial \tilde{\sigma}^b}{\partial \sigma^d} \right) \gamma^{cd}$$

$$\text{or } \gamma^{ab} = \left(\frac{\partial \sigma^a}{\partial \tilde{\sigma}^c} \right) \left(\frac{\partial \sigma^b}{\partial \tilde{\sigma}^d} \right) \tilde{\gamma}^{cd}$$

The matrix got by $\left(\frac{\partial \tilde{\sigma}^a}{\partial \sigma^c}\right) = J^a_c$ is

the jacobian matrix of reparameterization

$\rightarrow \left(\frac{\partial \sigma^a}{\partial \tilde{\sigma}^c}\right) = (J^{-1})^a_c$ forms the inverse

jacobian matrix. Since,

$$J^a_c (J^{-1})^c_b = \left(\frac{\partial \tilde{\sigma}^a}{\partial \sigma^c}\right) \left(\frac{\partial \sigma^c}{\partial \tilde{\sigma}^b}\right) = \frac{\partial \tilde{\sigma}^a}{\partial \tilde{\sigma}^b} = \delta^a_b$$

$$\text{Now since } \gamma_{ab} = \left(\frac{\partial \tilde{\sigma}^c}{\partial \sigma^a}\right) \left(\frac{\partial \tilde{\sigma}^d}{\partial \sigma^b}\right) \tilde{\gamma}_{cd}$$

$$\rightarrow \gamma_{ab} = J^c_a \tilde{\gamma}_{cd} J^d_b$$

$$\rightarrow [\gamma] = [J]^T [\tilde{\gamma}] [J]$$

$\downarrow$ Transpose of $[J]$

$$\rightarrow \det(\gamma) = [\det(J)]^2 \det(\tilde{\gamma})$$

$$\Rightarrow \sqrt{-\det(\gamma)} = [\det(J)] \sqrt{-\det(\tilde{\gamma})}$$

Now the integration measure of the action will change under reparameterization, as,

$$d\tau d\sigma = \underbrace{\left| \frac{\partial(\tau, \sigma)}{\partial(\tilde{\tau}, \tilde{\sigma})} \right|}_{\det(J^{-1})} d\tilde{\tau} d\tilde{\sigma}$$

$$\rightarrow d\tau d\sigma = [\det(J)]^{-1} d\tilde{\tau} d\tilde{\sigma}$$

$$\rightarrow d\tau d\sigma \sqrt{-\det(\gamma)} = [\det(J)] \sqrt{-\det(\tilde{\gamma})} \times [\det(J)]^{-1} d\tilde{\tau} d\tilde{\sigma}$$

$$\rightarrow d\tau d\sigma \sqrt{-\det(\gamma)} = d\tilde{\tau} d\tilde{\sigma} \sqrt{-\det(\tilde{\gamma})}$$

$$\text{Also, } \gamma^{ab} = (J^{-1})^a_c (J^{-1})^b_d \gamma^{cd}$$

$$\& h_{ab} = J^e_a J^f_b h_{ef}$$

$$\rightarrow \gamma^{ab} h_{ab} = (J^{-1})^a_c J^e_a (J^{-1})^b_d J^f_b \times \tilde{\gamma}^{cd} \tilde{h}_{ef}$$

$$\rightarrow \gamma^{ab} h_{ab} = \partial_c^e \partial_d^f \tilde{\gamma}^{cd} \tilde{h}_{ef}$$

$$\rightarrow \gamma^{ab} h_{ab} = \tilde{\gamma}^{cd} \tilde{h}_{cd}$$

The Polyakov action is

$$S_P = \frac{-1}{4\pi\alpha'} \int \underbrace{d\tau d\sigma \sqrt{-\det(\gamma)}}_{d\tilde{\tau} d\tilde{\sigma} \sqrt{-\det(\tilde{\gamma})}} \underbrace{\gamma^{ab} h_{ab}}_{\tilde{\gamma}^{cd} \tilde{h}_{cd}}$$

$$= \frac{-1}{4\pi\alpha'} \int d\tilde{\tau} d\tilde{\sigma} \sqrt{-\det(\tilde{\gamma})} \tilde{\gamma}^{ab} \tilde{h}_{ab}$$

$$\rightarrow S_P = \tilde{S}_P$$

Hence Polyakov action has Reparameterization invariance or Diffeomorphism Symmetry.

Thus the action doesn't depend on the World Sheet Coordinates.

c) Weyl Invariance

This is a Symmetry in Polyakov action, that doesn't have an analogue in the Nambu Goto action. Now, consider

$$S_P = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} h_{ab}$$

If $\gamma_{ab} \xrightarrow{\text{Weyl transform}} e^{2\omega(\tau, \sigma)} \gamma_{ab}$, then

Since γ^{ab} is a 2 dim. rank-2 tensor,

$$\det(\gamma) \xrightarrow{\text{Weyl transform}} \left[e^{2\omega(\tau, \sigma)} \right]^2 \det(\gamma)$$

$$\rightarrow \sqrt{-\det(\gamma)} \rightarrow e^{2\omega(\tau, \sigma)} \sqrt{-\det(\gamma)}$$

But γ^{ab} is inverse metric tensor.
i.e. $\gamma^{ab} \gamma_{bc} = \delta^a_c$. Thus,

$$\gamma^{ab} \xrightarrow{\text{Weyl transform}} e^{-2\omega(\tau, \sigma)} \gamma^{ab}$$

Thus, under $\gamma_{ab} \xrightarrow{\text{Weyl transform}} \underbrace{e^{2\omega(\tau, \sigma)}}_{\text{local rescale}} \gamma_{ab}$,

$$\sqrt{-\det(\gamma)} \gamma^{ab} \longrightarrow \sqrt{-\det(\gamma)} \gamma^{ab}$$

$$\& \quad \chi^\mu(\tau, \sigma) \longrightarrow \chi^\mu(\tau, \sigma)$$

(Space-time coordinates don't depend on γ_{ab})

Thus S_p remains invariant. This implies that under the local rescaling of the metric (as ω depends on τ & σ), the action remains invariant.

Weyl Symmetry is an extra Gauge Symmetry of Polyakov action due to the presence of the two-dimensional and a dynamical World Sheet metric which is not present in the N.G. action.

Thus two metrics related by Weyl transformation are considered to be the same physical state of World Sheet embedded in the Space-time.

Finding the Constraint eqⁿ for the World Sheet Metric

Varying the action with respect to γ_{ab} gives the eqⁿ of motion of γ_{ab} which sets the constraint on γ_{ab} itself.

$$\delta_\gamma S_P = \frac{-1}{4\pi\alpha'} \int d^2\sigma \delta_\gamma \left[\sqrt{-\det(\gamma)} \gamma^{ab} h_{ab} \right]$$

$$= \frac{-1}{4\pi\alpha'} \int d^2\sigma \delta_\gamma \left[\sqrt{-\det(\gamma)} \gamma^{ab} h_{ab} \right]$$

$$= - \int \frac{d^2\sigma}{4\pi\alpha'} \left[\left(\delta \sqrt{-\det(\gamma)} \right) \gamma^{ab} + \sqrt{-\det(\gamma)} \left(\delta \gamma^{ab} \right) \right] h_{ab}$$

$$\text{use, } \delta_\gamma \det(\gamma) = \det(\gamma) \gamma^{cd} \delta \gamma_{cd}$$

$$\rightarrow \delta \sqrt{-\det(\gamma)} = -\frac{1}{2} \sqrt{-\det(\gamma)} \gamma^{cd} \delta \gamma_{cd}$$

$$\rightarrow \delta_\gamma S_P = - \int \frac{d^2\sigma}{4\pi\alpha'} \sqrt{-\det(\gamma)} \left[-\frac{1}{2} \gamma^{cd} \gamma^{ab} \delta \gamma_{cd} + \delta \gamma^{ab} \right] h_{ab}$$

Thus the variation in action w.r.t ' γ^{ab} ' is,

$$\delta_\gamma S_P = - \int \frac{d\sigma}{4\pi\alpha'} \sqrt{-\det(\gamma)} \left[h_{cd} - \frac{1}{2} \gamma_{cd} \gamma^{ab} h_{ab} \right] \delta \gamma^{cd}$$

The eqⁿ of motion for γ^{ab} is got when $\delta_\gamma S_P$ is zero.

$$\Rightarrow \underline{h_{ab} = \frac{1}{2} \gamma_{ab} \gamma^{cd} h_{cd}}$$

This is the constraint for γ_{ab} metric.

$$\Rightarrow \underline{\gamma_{ab} = 2 f(\tau, \sigma) h_{ab}}$$

$$\text{where } [f(\tau, \sigma)]^{-1} = \gamma^{ab} h_{ab} = h^a_a$$

Thus $\gamma_{ab} \propto h_{ab}$, but they are not equal.
So, we get,

$$\det(\gamma) = \det[2 f(\tau, \sigma) h_{ab}]$$

$$\Rightarrow \det(\gamma) = [2 f(\tau, \sigma)]^2 \det(h)$$

$$\rightarrow \sqrt{-\det(\gamma)} = 2f(\tau, \sigma) \sqrt{-\det(h)}$$

$$\rightarrow \frac{\gamma_{ab}}{\sqrt{-\det(\gamma)}} = \frac{h_{ab}}{\sqrt{-\det(h)}}$$

Substituting this in the polyakov action

$$S_P = \frac{-1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} h_{ab}$$

$$\Rightarrow S_P \rightarrow \frac{-1}{4\pi\alpha'} \int d^2\sigma \gamma^{ab} \underbrace{\sqrt{-\det(\gamma)} h_{ab}}_{\gamma_{ab} \sqrt{-\det(h)}}$$

$$\& \text{ using } \gamma^{ab} \gamma_{ab} = \delta_a^a = 2$$

$$\rightarrow S_P \rightarrow \frac{-1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det(h)} = S_{NG}$$

Thus the eqⁿ of motion of $X^\mu(\tau, \sigma)$ got by S_P & S_{NG} should be the same.

So after we find EoM of X^μ using S_P , we have to eliminate the γ^{ab} from EoM by using the constraint ' $h_{cd} = \frac{1}{2} \gamma_{cd} \gamma^{ab} h_{ab}$ '

Energy - Momentum tensor for String

Here we use the Hilbert's definition of Energy-momentum tensor given by,

$$T^{ab} = \frac{-4\pi}{\sqrt{-\det(\gamma)}} \frac{\partial S}{\partial \gamma_{ab}}$$

The action given here describes a Matter field & not gravity, for such non-Gravitational but Matter-full action, we usually define the Hilbert Energy - Momentum tensor. And in String theory convention, there a factor of 2π extra in the numerator, compared to the usual definition in General relativity.

Given,

$$S_p = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_a X^\mu) (\partial_b X_\mu)$$

$$\text{use, } \partial \det(\gamma) = \det(\gamma) \gamma^{cd} \partial \gamma_{cd}$$

$$\rightarrow \delta \sqrt{-\det(\gamma)} = -\frac{1}{2} \sqrt{-\det(\gamma)} \gamma^{cd} \delta \gamma_{cd}$$

$$\rightarrow \delta S_p = - \int \frac{d\sigma}{4\pi\alpha'} \sqrt{-\det(\gamma)} \left[-\frac{1}{2} \gamma^{cd} \gamma^{ab} \delta \gamma_{cd} + \delta \gamma^{ab} \right] h_{ab}$$

(w.r.t γ)

Thus the variation in action w.r.t ' γ^{ab} ' is,

$$\delta S_p = - \int \frac{d\sigma}{4\pi\alpha'} \sqrt{-\det(\gamma)} \left[h_{cd} - \frac{1}{2} \gamma_{cd} \gamma^{ab} h_{ab} \right] \delta \gamma^{cd}$$

$$\rightarrow \frac{\delta S_p}{\delta \gamma_{cd}} = \frac{\sqrt{-\det(\gamma)}}{4\pi\alpha'} \left[h^{cd} - \frac{1}{2} \gamma^{cd} \gamma^{ab} h_{ab} \right]$$

$$\rightarrow T^{ab} = -\frac{1}{\alpha'} \left[h^{ab} - \frac{1}{2} \gamma^{ab} h^c_c \right]$$

Thus we get,

$$\boxed{T^{ab} = -\frac{1}{\alpha'} \left[(\partial^a x^\mu)(\partial^b x_\mu) - \frac{1}{2} \gamma^{ab} (\partial^c x^\mu)(\partial_c x_\mu) \right]}$$

Since T^{ab} is completely constructed by metric tensors, we can show that T^{ab} is conserved, i.e. $\nabla_a T^{ab} = 0$.

This property of conservation of T^{ab} i.e. $\nabla_a T^{ab} = 0$, is a result of the Diffeomorphism invariance of S_P .

$$\text{Now, } T_{ab} = -\frac{1}{\alpha'} \left[h_{ab} - \frac{1}{2} \gamma_{ab} h^c_c \right]$$

$$\gamma^{ab} T_{ab} = -\frac{1}{\alpha'} \left[h^a_a - \frac{1}{2} \gamma^a_a h^c_c \right]$$

$(\gamma^a_a = 2)$

$$\rightarrow T^a_a = -\frac{1}{\alpha'} (h^a_a - h^c_c) = 0$$

$$\rightarrow \underline{\text{Trace}(T) = 0}$$

This property of tracelessness of the Polyakov energy momentum tensor is because of the Weyl Symmetry.

Also the constraint on γ_{ab} is got by the γ 's eqⁿ of motion $T_{ab} = 0$.

Equation of Motion for χ^μ

The Polyakov action is

$$S_P = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_a \chi^\mu) (\partial_b \chi^\nu) \eta_{\mu\nu}$$

$$\rightarrow \mathcal{L}_P = \frac{-1}{4\pi\alpha'} \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_a \chi^\mu) (\partial_b \chi^\nu) \eta_{\mu\nu}$$

$$\text{EoM is, } \frac{\partial \mathcal{L}}{\partial \chi^\mu} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial (\partial_a \chi^\mu)} \right) = 0$$

$$\pi_\mu^a = \frac{\partial \mathcal{L}}{\partial (\partial_a \chi^\mu)} = \frac{-1}{4\pi\alpha'} \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_b \chi^\nu) \eta_{\mu\nu}$$

$$\rightarrow \underline{\partial_a \left[\sqrt{-\det(\gamma)} \gamma^{ab} (\partial_b \chi^\nu) \eta_{\mu\nu} \right] = 0}$$

But the laplacian operator on a curved manifold is given by $\nabla^2 = \nabla_a \nabla^a$, which is the Laplace-Beltrami operator.

Since $X^\mu(\tau, \sigma)$ is a world sheet scalar,

$$\nabla^2 X^\mu = \nabla_a (\nabla^a X^\mu) = \nabla_a (\partial^a X^\mu)$$

now, $\nabla_a (\partial^a X^\mu)$ is the divergence of the vector field ' $\partial^a X^\mu$ ' in curved manifold.

$$\rightarrow \nabla_a (\partial^a X^\mu) = \frac{1}{\sqrt{-\det(\gamma)}} \partial_a (\sqrt{-\det(\gamma)} \partial^a X^\mu)$$

$$\rightarrow \nabla^2 X^\mu = \frac{1}{\sqrt{-\det(\gamma)}} \partial_a (\sqrt{-\det(\gamma)} \gamma^{ab} \partial_b X^\mu)$$

Thus, the eqⁿ of motion for X^μ is given by,

$$\partial_a [\sqrt{-\det(\gamma)} \gamma^{ab} (\partial_b X^\mu) \eta_{\mu\nu}] = 0$$

$$\rightarrow \sqrt{-\det(\gamma)} \nabla^2 (X^\mu \eta_{\mu\nu}) = 0$$

$$\Rightarrow \boxed{\nabla^2 X^\mu = 0}$$

Thus the Polyakov Action describes ' X^μ ' as the massless Klein-Gordon Scalars w.r.t the world sheet coordinates (τ, σ) which is covariantly coupled to the world sheet metric γ^{ab} .

Hence the Polyakov & N.G. action describes 2-dimensional scalar field theory of the space-time coordinates $X^\mu(\tau, \sigma)$ on the string's world sheet as the manifold. Also from this point of view, the D-dim Poincare Invariance is an internal symmetry of X 's, which acts on X 's by fixing the world sheet coordinates (τ, σ) .

Thus in String Theory, the amplitudes for space-time scattering processes are given by matrix elements in the 2-dim Quantum field theories on the world sheet.

Boundary Conditions for Strings

Let the coordinate region for world sheet be $-\infty < \tau < \infty$ & $0 < \sigma < l$.

Then the Polyakov action is,

$$S_P = \frac{-1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_a x^\mu) (\partial_b x_\mu)$$

Varying with respect to x 's, we get,

$$\delta S_P = \frac{-1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma \sqrt{-\det(\gamma)} \gamma^{ab} \times \left[\frac{\partial (\partial_a x^\mu) (\partial_b x_\mu)}{\partial (\partial_c x^\nu)} \right] \delta (\partial_c x^\nu)$$

$$= \frac{-1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma \sqrt{-\det(\gamma)} \gamma^{ac} 2 (\partial_a x_\nu) \partial_c (\delta x^\nu)$$

$$= \frac{-1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma \sqrt{-\det(\gamma)} (\partial^c x_\nu) \partial_c (\delta x^\nu)$$

(doing integral by parts term by term)

$$= \frac{-1}{2\pi\alpha'} \int_0^l d\sigma \int_{-\infty}^{\infty} d\tau \sqrt{-\det(\gamma)} (\partial^\tau \chi_\nu) \partial_\tau (\partial \chi^\nu) \\ - \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma \sqrt{-\det(\gamma)} (\partial^\sigma \chi_\nu) \partial_\sigma (\partial \chi^\nu)$$

$$= \frac{-1}{2\pi\alpha'} \int_0^l d\sigma \left[\sqrt{-\det(\gamma)} (\partial^\tau \chi_\nu) (\partial \chi^\nu) \right]_{\tau=-\infty}^{\tau=\infty} \\ - \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \left[\sqrt{-\det(\gamma)} (\partial^\sigma \chi_\nu) (\partial \chi^\nu) \right]_{\sigma=0}^{\sigma=l}$$

$$+ \frac{1}{2\pi\alpha'} \int_0^l d\sigma \int_{-\infty}^{\infty} d\tau (\partial \chi^\nu) \partial_\tau \left[\sqrt{-\det(\gamma)} (\partial^\tau \chi_\nu) \right]$$

$$+ \frac{1}{2\pi\alpha'} \int_0^l d\sigma \int_{-\infty}^{\infty} d\tau (\partial \chi^\nu) \partial_\sigma \left[\sqrt{-\det(\gamma)} (\partial^\sigma \chi_\nu) \right]$$

$$\begin{aligned}
&= \frac{-1}{2\pi\alpha'} \int_0^l d\sigma \left[\sqrt{-\det(\gamma)} (\partial^\tau \chi_\nu) (\partial \chi^\nu) \right]_{\tau=-\infty}^{\tau=\infty} \\
&\quad - \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \left[\sqrt{-\det(\gamma)} (\partial^\sigma \chi_\nu) (\partial \chi^\nu) \right]_{\sigma=0}^{\sigma=l} \\
&\quad + \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma (\partial \chi^\nu) \partial_c \left(\sqrt{-\det(\gamma)} (\partial^c \chi_\nu) \right)
\end{aligned}$$

$$\left[\text{Use, } \nabla^2 \chi_\nu = \frac{1}{\sqrt{-\det(\gamma)}} \partial_c \left(\sqrt{-\det(\gamma)} \partial^c \chi_\nu \right) \right]$$

The variations are done such that at 'τ' end points, $\partial \chi^\nu = 0$.

$$\begin{aligned}
\rightarrow \delta S_p &= \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^l d\sigma (\partial \chi^\nu) \sqrt{-\det(\gamma)} (\nabla^2 \chi_\nu) \\
&\quad - \frac{1}{2\pi\alpha'} \left[\int_{-\infty}^{\infty} d\tau \sqrt{-\det(\gamma)} \gamma^{\tau c} (\partial_c \chi_\nu) (\partial \chi^\nu) \right]_{\sigma=0}^{\sigma=l}
\end{aligned}$$

Surface term

Neumann Boundary cond. for Open Strings

The eqⁿ of motion results in the known wave eqⁿ only if the surface term is 0.

One possibility is when, for any ' τ '

$$\gamma^{2c}(\partial_c X_\nu) \Big|_{\sigma=l} = \gamma^{2c}(\partial_c X_\nu) \Big|_{\sigma=0} = 0$$

This is the Neumann boundary condition on X_ν is given by,

$$\Rightarrow \left[\gamma^{22} \frac{\partial X_\nu}{\partial \sigma} + \gamma^{21} \frac{\partial X_\nu}{\partial \tau} \right]_{\substack{\text{any } \tau \\ \sigma=l,0}} = 0$$

Similarly, if we had interchanged the order of integration, then we need,

$$\left[\gamma^{11} \frac{\partial X_\nu}{\partial \tau} + \gamma^{12} \frac{\partial X_\nu}{\partial \sigma} \right]_{\substack{\text{any } \sigma \\ \tau=\pm\infty}} = 0$$

$\tau = \text{any}$, $\sigma = 0, l$ is a Constant ' σ ' Surface. Similarly, $\sigma = \text{any}$, $\tau = \pm \infty$ is the Constant ' τ ' Surface.

Normal to Constant ' σ ': $n_a = \partial_a \sigma$

$$\rightarrow n_a = \partial_a^2 \rightarrow n_a \gamma^{ab} = \partial_a^2 \gamma^{ab} \rightarrow n^b = \gamma^{2b}$$

Hence, at the ' σ ' boundary the normal n^a is given by, $n^a = \gamma^{2a}$ ($\tau = \text{any}$, $\sigma = \text{boundary}$)

Then Normal to constant ' τ ': $n_a = \partial_a \tau$
 $\rightarrow n_a = \partial_a^1 \rightarrow n^b = \gamma^{1b}$

Hence, at the ' τ ' boundary the normal n^a is given by, $n^a = \gamma^{1a}$ ($\tau = \text{boundary}$, $\sigma = \text{any}$)

Thus we can write the Neumann boundary Conditions covariantly as,

$$\boxed{n^a \partial_a \chi^\mu = 0, \text{ on the boundary}}$$

They tell that at the boundary the strings move freely.

Periodic Boundary cond. for Closed Strings

The surface term also goes to zero if,

$$\chi^\mu(\tau, 0) = \chi^\mu(\tau, l)$$

$$\gamma^{ab}(\tau, 0) = \gamma^{ab}(\tau, l)$$

$$\& [\gamma^{2a} \partial_a \chi^\mu]_{(\tau, 0)} = [\gamma^{2a} \partial_a \chi^\mu]_{(\tau, l)}$$

i.e. The χ^μ 's, their derivative & the metric are same at both the space-like ends of the string at all the τ values. This is the periodic boundary condition where the string's are always closed.

Thus the Neumann boundary conditions that describe Open strings and the Periodic boundary conditions that describe Closed strings are the only two allowed possibilities consistent with D -dim Poincare invariance and eqⁿ of motion.

Conformal Gauge & Classical Open String

The Polyakov eqⁿ of motion for X^μ is,

$$\partial_a (\sqrt{-\det(\gamma)} \gamma^{ab} \partial_b X^\mu) = 0$$

For an open string whose space-like ends are at $\sigma = 0$ and $\sigma = l$, the surface term is removed by the Neumann Boundary Condition given by

$$\left[\gamma^{2a} \partial_a X^\mu \right]_{(\tau, 0)} = \left[\gamma^{2a} \partial_a X^\mu \right]_{(\tau, l)} = 0$$

$$\text{or } (\partial^2 X^\mu)_{(\tau, 0)} = (\partial^2 X^\mu)_{(\tau, l)} = 0$$

i.e. Contravariant ' σ ' component of the gradient of X^μ vanishes at σ 's boundary.

The action is symmetric under diffeomorphism. This gives rise to two reparameterization invariance of action when we change $\tau \leftrightarrow \sigma$.

Thus, we have freedom to set two out of the three independent components of γ_{ab} by our choice by choosing appropriate coordinates & scale. So, we can locally set γ_{ab} to be proportional upto a scalar function to $\eta_{ab} = \text{diag}(-1, 1)$.

Setting $\gamma_{ab} = e^{2\phi(\tau, \sigma)} \eta_{ab}$ is called as the Conformal Gauge.

$$\Rightarrow \gamma^{ab} = e^{-2\phi(\tau, \sigma)} \eta^{ab}$$

$$\& \sqrt{-\det(\gamma_{ab})} = e^{2\phi(\tau, \sigma)} \sqrt{-\det(\eta_{ab})}$$

$$\Rightarrow \sqrt{-\det(\gamma_{ab})} \gamma^{ab} = \sqrt{-\det(\eta_{ab})} \eta^{ab}$$

Then the eqⁿ of motions for X^μ is,

$$\partial_a (\sqrt{-\det(\eta_{ab})} \eta^{ab} \partial_b X^\mu) = 0$$

$$\Rightarrow \sqrt{-\det(\eta_{ab})} \nabla_a \nabla^a X^\mu = 0$$

$$\Rightarrow -\partial_\tau^2 \chi^\mu + \partial_\sigma^2 \chi^\mu = 0$$

Then the boundary term becomes,

$$\left[\eta^{2a} \partial_a \chi^\mu \right]_{(\tau,0)} = 0 = \left[\eta^{2a} \partial_a \chi^\mu \right]_{(\tau,l)}$$

$$\Rightarrow \left[\eta^{22} (\partial_2 \chi^\mu) \right]_{(\tau,0)} = 0 = \left[\eta^{22} \partial_2 \chi^\mu \right]_{(\tau,l)}$$

$$\Rightarrow \underline{(\partial_\sigma \chi^\mu)_{\text{at boundary}} = 0}$$

This is the boundary condition for a space-like open string.

Then the energy momentum tensor is,

$$T_{ab} = \frac{-1}{\alpha'} \left[(\partial_a \chi^\mu) (\partial_b \chi_\mu) - \frac{\gamma_{ab} \gamma^{cd}}{2} (\partial_c \chi^\mu) (\partial_d \chi_\mu) \right]$$

≠ In Conformal Gauge,

$$T_{ab} = \frac{-1}{\alpha'} \left[(\partial_a \chi^\mu) (\partial_b \chi_\mu) - \frac{\eta_{ab} \eta^{cd}}{2} (\partial_c \chi^\mu) (\partial_d \chi_\mu) \right]$$

$$\text{So, } T_{11} = \frac{-1}{\alpha'} \left[\dot{\chi}^\mu \dot{\chi}_\mu - \frac{1}{2} (\dot{\chi}^\mu \dot{\chi}_\mu - \dot{\bar{\chi}}^\mu \dot{\bar{\chi}}_\mu) \right]$$

$$\rightarrow T_{11} = \frac{-1}{2\alpha'} (\dot{\chi}^\mu \dot{\chi}_\mu + \dot{\bar{\chi}}^\mu \dot{\bar{\chi}}_\mu)$$

$$T_{22} = \frac{-1}{\alpha'} \left[\dot{\bar{\chi}}^\mu \dot{\bar{\chi}}_\mu + \frac{1}{2} (\dot{\chi}^\mu \dot{\chi}_\mu - \dot{\bar{\chi}}^\mu \dot{\bar{\chi}}_\mu) \right]$$

$$\rightarrow T_{22} = \frac{-1}{2\alpha'} (\dot{\chi}^\mu \dot{\chi}_\mu + \dot{\bar{\chi}}^\mu \dot{\bar{\chi}}_\mu)$$

$$\& T_{12} = T_{21} = \frac{-1}{\alpha'} \dot{\chi}^\mu \dot{\bar{\chi}}_\mu$$

Now, the constraint on γ_{ab} fixes

$$T_{ab} = 0. \rightarrow \dot{\chi}^\mu \dot{\chi}_\mu + \dot{\bar{\chi}}^\mu \dot{\bar{\chi}}_\mu = 0$$

$$\& \dot{\chi}^\mu \dot{\bar{\chi}}_\mu = 0$$

The condition $\dot{\chi}^\mu \dot{\bar{\chi}}_\mu = 0$ implies that the induced metric components h_{12} & h_{21} are zero. Since $\gamma \propto h$, it implies the World Sheet metric is diagonal in Conformal gauge.

So the parameters $\tau \neq \sigma$ should be chosen such that their coordinate lines are always perpendicular to each other.

$$\text{Now, } (\partial_\sigma \chi^\mu)_{\text{boundary}} = (\dot{\chi}^\mu)_{\text{boundary}} = 0$$

Since $\dot{\chi}^\mu \dot{\chi}_\mu + \ddot{\chi}^\mu \ddot{\chi}_\mu = 0$ should also be satisfied at boundary, it implies that
$$[(\dot{\chi}^\mu)(\dot{\chi}_\mu)]_{\text{boundary}} = 0.$$

Thus the tangent vector to the world sheet at the open ends of the string is light like. It points along the null/light like direction.

If we set $\tau = \chi^0 = t$ (Static gauge) allowed by residual Gauge symmetry, then,
$$\dot{\chi}^\mu \dot{\chi}_\mu = -1 + |\vec{\dot{\chi}}|^2 = 0 \Rightarrow |\vec{\dot{\chi}}| = 1.$$

Thus the open ends travel at the speed of light.

In static gauge, $\dot{X}^i = v^i$, $\dot{\bar{X}}^i = u^i$
 v^i is the velocity of string & u^i is the longitudinal velocity of string.

$$\rightarrow \frac{dX^0}{d\sigma} = 0, \quad (\dot{X}^\mu \dot{\bar{X}}^\nu \eta_{\mu\nu}) = (\vec{v} \cdot \vec{u}),$$

$$(\dot{X}^\mu \dot{\bar{X}}^\nu \eta_{\mu\nu}) = -1 + |\vec{v}|^2,$$

$$\& (\dot{X}^\alpha \dot{\bar{X}}^\beta \eta_{\alpha\beta}) = |\vec{u}|^2$$

$$\dot{X}^\mu \dot{X}_\mu + \dot{\bar{X}}^\mu \dot{\bar{X}}_\mu = 0 \Rightarrow |\vec{v}|^2 + |\vec{u}|^2 = 1$$

$$\& \dot{X}^\mu \dot{\bar{X}}_\mu = 0 \Rightarrow (\vec{v} \cdot \vec{u}) = 0$$

$(\vec{v} \cdot \vec{u}) = 0$ tells that the motion of string must be perpendicular to the string itself. There are no longitudinal modes available. All physical modes are always transverse oscillations.

Any metric γ_{ab} is equivalent to flat one

Consider γ_{ab} as a world sheet metric. If we Weyl transform the metric to $\tilde{\gamma}_{ab}$, then they both will represent the same state.

$$\text{Let } \tilde{\gamma}_{ab}(\tau, \sigma) = e^{2\phi(\tau, \sigma)} \gamma_{ab}(\tau, \sigma)$$

Then the Ricci Scalars (R) of these metrics are related by,

$$\sqrt{-\det(\tilde{\gamma})} \tilde{R} = \sqrt{-\det(\gamma)} (R - 2 \nabla^2 \phi)$$

This can be derived using Bianchi Identities.

So if we need the 2-dim metric $\tilde{\gamma}_{ab}$ to describe a flat manifold, then the new $\tilde{R}$ should be 0. Thus the function ϕ

Should satisfy $(R - 2 \nabla^2 \phi) = 0$

$$\text{or } \underline{\underline{\nabla^2 \phi = \frac{1}{2} R}}$$

This is enough to ensure that the new metric describes a flat manifold because in 2 dimensions, the Riemann tensor is given by,

$$\tilde{R}_{abcd} = \frac{\tilde{R}}{2} [\tilde{\gamma}_{ac} \tilde{\gamma}_{bd} - \tilde{\gamma}_{ad} \tilde{\gamma}_{bc}]$$

So if $\tilde{R} = 0$, then all the components of the Riemann tensor is zero. Then, the manifold described by $\tilde{\gamma}_{ab}$ is flat & there is no curvature.

Thus given a curved 2d manifold γ_{ab} whose Ricci scalar is 'R', then the Weyl transformation to

$$\tilde{\gamma}_{ab}(\tau, \sigma) = e^{2\phi(\tau, \sigma)} \gamma_{ab}(\tau, \sigma)$$

where $\nabla^2 \phi = \frac{1}{2} R$ will make $\tilde{\gamma}_{ab}(\tau, \sigma)$ to represent a flat 2 dimensional manifold.

Additional allowed terms to Polyakov action

Let us generalize the Polyakov action by introducing new terms to it but requiring all the symmetries to still be maintained and these terms should be polynomials in derivatives.

One such allowed term is given by the Euler characteristic of the world sheet. which is a topological invariant & does not depend on the metric.

For a world sheet without a boundary i.e., for a closed string, it is given by,

$$\chi = \frac{1}{4\pi} \int_W d\tau d\sigma \sqrt{-\det(\gamma)} R$$

where R is the Ricci scalar.

This has the diffeomorphism invariance as all factors are world sheet scalars.

Under a Weyl transform,

$$\tilde{\gamma}_{ab}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \gamma_{ab}(\tau, \sigma)$$

$$\text{Then } \sqrt{-\det(\tilde{\gamma})} = e^{2\omega(\tau, \sigma)} \sqrt{-\det(\gamma)}$$

& also the Ricci scalar will transform as,

$$\underline{\sqrt{-\det(\tilde{\gamma})} \tilde{R} = \sqrt{-\det(\gamma)} (R - 2\nabla^2 \omega)}$$

$$\rightarrow \tilde{R} = e^{-2\omega(\tau, \sigma)} (R - 2\nabla^2 \omega)$$

$$\rightarrow \chi = \frac{1}{4\pi} \int_W d\tau d\sigma \sqrt{-\det(\gamma)} R$$

$\Downarrow$ Weyl transform

$$\tilde{\chi} = \frac{1}{4\pi} \int_W d\tau d\sigma \sqrt{-\det(\tilde{\gamma})} \tilde{R}$$

$$= \frac{1}{4\pi} \int_W d\tau d\sigma \left[\sqrt{-\det(\gamma)} R - 2\sqrt{-\det(\gamma)} \nabla^2 \omega \right]$$

$$\rightarrow \tilde{\chi} = \chi - \frac{1}{2\pi} \int_W dz d\sigma \sqrt{-\det(\gamma)} \nabla^2 \omega$$

Now, for any vector $\mathcal{V}^a$

$$\nabla_a \mathcal{V}^a = \frac{1}{\sqrt{-\det(\gamma)}} \partial_a \left[\sqrt{-\det(\gamma)} \mathcal{V}^a \right]$$

$$\rightarrow \nabla^2 \omega = \nabla_a (\nabla^a \omega) = \frac{\partial_a \left[\sqrt{-\det(\gamma)} \nabla^a \omega \right]}{\sqrt{-\det(\gamma)}}$$

$$\rightarrow \tilde{\chi} = \chi - \frac{1}{2\pi} \int_W dz d\sigma \partial_a \left[\sqrt{-\det(\gamma)} \partial^a \omega \right]$$

$$\text{Let } \mathcal{U}^a = \sqrt{-\det(\gamma)} \partial^a \omega$$

$$\rightarrow \tilde{\chi} = \chi - \frac{1}{2\pi} \int_W dz d\sigma (\partial_a \mathcal{U}^a)$$

Since we are dealing with closed string, the boundary conditions are periodic.

$$\Rightarrow \begin{aligned} \mathcal{U}^a(\tau, 0) &= \mathcal{U}^a(\tau, l) \\ \partial_a \mathcal{U}^a(\tau, 0) &= \partial_a \mathcal{U}^a(\tau, l) \end{aligned}$$

Thus, there is no boundary term, we can use 2-dim Gauss divergence theorem in arbitrary coordinates given by

$$\int_W dz d\sigma \partial_a u^a = \int_{\partial W} ds n_a u^a$$

No boundary for closed string

Thus, $\tilde{\chi} = \chi$. It implies that for closed strings

$$\chi = \frac{1}{4\pi} \int_W dz d\sigma \sqrt{-\det(\gamma)} R$$

is Weyl invariant & as well as invariant under reparameterization. Hence this term can be added to Polyakov action & still have same symmetries.

But in the presence of a boundary, then we need to add an additional term in ' χ ' to remove boundary effect.

For Open String whose World Sheet boundary exists, the appropriate ' χ ' which is Weyl Invariant is,

$$\chi = \frac{1}{4\pi} \int_W dz d\sigma \sqrt{-\det(\gamma)} R + \frac{1}{2\pi} \int_{\partial W} ds k$$

where $ds = \sqrt{(-\gamma_{\tau\tau})} d\tau$ is the proper time along the space like boundary & similarly $ds = \sqrt{(\gamma_{\sigma\sigma})} d\sigma$ for the time like boundary. And k is the geodesic curvature of the boundary given by,

$$k = \pm t^a n_b \nabla_a t^b$$

where t^a is the unit vector tangent to the boundary, n^a is a unit outward normal vector orthogonal to t^a . The '+' sign is for time like boundary & '-' sign is for space like boundary.

Under a Weyl transform,

$$\tilde{\gamma}_{ab}(\tau, \sigma) = e^{2\omega(\tau, \sigma)} \gamma_{ab}(\tau, \sigma)$$

The metric connection coefficients are,

$$\Gamma_{bc}^a = \frac{\gamma^{ad}}{2} [\partial_b \gamma_{cd} + \partial_c \gamma_{bd} - \partial_d \gamma_{bc}]$$

Under Weyl transformation,

$$\tilde{\Gamma}_{bc}^a = \frac{\tilde{\gamma}^{ad}}{2} [\partial_b \tilde{\gamma}_{cd} + \partial_c \tilde{\gamma}_{bd} - \partial_d \tilde{\gamma}_{bc}]$$

$$= \frac{\gamma^{ad}}{2} \left[\partial_b \gamma_{cd} + 2\gamma_{cd} \partial_b \omega + \partial_c \gamma_{bd} \right. \\ \left. + 2\gamma_{cd} \partial_c \omega - \partial_d \gamma_{bc} - 2\gamma_{bc} \partial_d \omega \right]$$

$$\rightarrow \tilde{\Gamma}_{bc}^a = \Gamma_{bc}^a + \delta_c^a \partial_b \omega + \delta_b^a \partial_c \omega - \gamma_{bc} \partial^a \omega$$

$$\& \tilde{R} = e^{-2\omega} [R - 2\nabla^2 \omega]$$

The normal vector (n) & tangent vector (t) at boundary are normalized.

$$\rightarrow n^a n_a = 1 \quad (\omega \cdot \gamma \cdot t \quad \gamma_{ab})$$

$$\& \underbrace{t^a t_a}_{\omega \cdot \gamma \cdot t \quad \gamma_{ab}} = \begin{cases} -1, & \text{time like boundary} \\ +1, & \text{space like boundary} \end{cases}$$

Then under Weyl transformation,

$$\tilde{t}^a \tilde{t}_a = \mp 1 \text{ is needed}$$

$$\Rightarrow \tilde{t}^a \tilde{t}^b \tilde{\gamma}_{ab} = \mp 1$$

$$\Rightarrow \tilde{t}^a \tilde{t}^b e^{2\omega} \gamma_{ab} = \mp 1$$

This is consistent only if

$$\underline{\underline{\tilde{t}^a = e^{-\omega} t^a}}$$

Similarly Since $n^a n_a = 1$
& $\tilde{n}^a \tilde{n}_a = 1$

$$\Rightarrow \underline{\tilde{n}^a = e^{-\omega} n^a} \text{ is needed.}$$
$$\text{or } \underline{\underline{\tilde{n}_a = e^{\omega}}}$$

Now, in the original metric

$$k = \pm t^a n_b \nabla_a t^b$$

$$\Rightarrow k = \pm t^a n_b (\partial_a t^b + \Gamma_{ac}^b t^c)$$

Then the transformed one is

$$\tilde{k} = \pm \tilde{t}^a \tilde{n}_b (\partial_a \tilde{t}^b + \tilde{\Gamma}_{ac}^b \tilde{t}^c)$$

$$= \pm t^a n_b \left[\partial_a (e^{-\omega} t^b) + \Gamma_{ac}^b e^{-\omega} t^c + e^{-\omega} t^c \left(\partial_c^b \partial_a \omega + \partial_a^b \partial_c \omega - \gamma_{ac} \partial^b \omega \right) \right]$$

$$\begin{aligned}
 \rightarrow \tilde{k} &= \pm t^a n_b (\bar{e}^{-\omega} \partial_a t^b - t^b \partial_a \omega) \\
 &\quad \pm t^a n_b \Gamma_{ac}^b t^c \bar{e}^{-\omega} \\
 &\quad \pm \bar{e}^{-\omega} \left[t^a n_c t^c \partial_a \omega \right. \\
 &\quad \quad \left. + t^a n_a t^c \partial_c \omega \right. \\
 &\quad \quad \left. - t_c n_b t^c \partial^b \omega \right]
 \end{aligned}$$

Since t & n are orthogonal,
 $t^a n_a = 0$. & $t_c t^c = \mp 1$

$$\begin{aligned}
 \Rightarrow \tilde{k} &= \pm t^a n_b [\partial_a t^b + \Gamma_{ac}^b t^c] \bar{e}^{-\omega} \\
 &\quad \pm \bar{e}^{-\omega} (\mp n_b \partial^b \omega)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \tilde{k} &= \pm [t^a n_b (\nabla_a t^b)] \bar{e}^{-\omega} \\
 &\quad + \bar{e}^{-\omega} n_b \partial^b \omega
 \end{aligned}$$

$$\Rightarrow \underline{\underline{\tilde{k} = [k + n^a \partial_a \omega] \bar{e}^{-\omega}}}$$

$$\text{Then } ds = \sqrt{\gamma_{ab} d\sigma^a d\sigma^b}$$

$$\rightarrow \tilde{ds} = \sqrt{e^{+2\omega} \gamma_{ab} d\sigma^a d\sigma^b}$$

$$\rightarrow \underline{\tilde{ds} = e^{\omega} ds}$$

$$\begin{aligned} \rightarrow \frac{1}{2\pi} \int_{\partial M} \tilde{ds} \tilde{K} &= \frac{1}{2\pi} \int_{\partial M} ds e^{\omega} [K + n^a \partial_a \omega] e^{-\omega} \end{aligned}$$

Thus finally we get,

$$\frac{1}{2\pi} \int_{\partial M} \tilde{ds} \tilde{K} = \frac{1}{2\pi} \int_{\partial M} ds K + \overbrace{\frac{1}{2\pi} \int_{\partial M} ds n^a \partial_a \omega}^{\text{Extra term}}$$

But previously we found

$$\frac{1}{4\pi} \int_W dx d\sigma \sqrt{-\det(\tilde{\gamma})} \tilde{R}$$

$$= \frac{1}{4\pi} \int_W dx d\sigma \sqrt{-\det(\gamma)} (R - 2 \nabla_a \partial^a \omega)$$

$$\Rightarrow \frac{1}{4\pi} \int_W d^2\sigma \sqrt{-\det(\tilde{\gamma})} \tilde{R} = \frac{1}{4\pi} \int_W d^2\sigma \sqrt{-\det(\gamma)} R$$

$$- \frac{1}{2\pi} \int_W d^2\sigma \sqrt{-\det(\gamma)} \nabla_a (\partial^a \omega)$$

Using divergence theorem,

$$\frac{1}{2\pi} \int_W d^2\sigma \sqrt{-\det(\gamma)} \nabla_a (\partial^a \omega) = \frac{1}{2\pi} \int_{\partial W} ds n_a (\partial^a \omega)$$

$$\Rightarrow \frac{1}{4\pi} \int_W d^2\sigma \sqrt{-\det(\tilde{\gamma})} \tilde{R} = \frac{1}{4\pi} \int_W d^2\sigma \sqrt{-\det(\gamma)} R$$

$$- \frac{1}{2\pi} \int_{\partial W} ds n_a (\partial^a \omega)$$

Thus the Surface term of this integral exactly cancels with the extra term produced by the boundary correction. Thus χ is Weyl invariant.

$$\chi = \frac{1}{4\pi} \int_W dz d\sigma \sqrt{-\det(\gamma)} R + \frac{1}{2\pi} \int_{\partial W} ds k$$

Minimal Coupling of World Sheet's Hilbert Action to D Scalar Fields

If we add the Euler characteristic ' χ ' to the Polyakov action ' S_P ' where,

$$S_P = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_a x^\mu) (\partial_b x^\nu) \eta_{\mu\nu}$$

$$\chi = \frac{1}{4\pi} \int_W d^2\sigma \sqrt{-\det(\gamma)} R \quad \left(\text{for closed string} \right)$$

Then $S_M = S_P - \lambda \chi$, where λ is some coupling constant.

$$\rightarrow S_M = \int_W d^2\sigma \mathcal{L}_M$$

where,

$$\mathcal{L}_M = \frac{-1}{4\pi} \sqrt{-\det(\gamma)} \left[\frac{\gamma^{ab}}{\alpha'} (\partial_a x^\mu) (\partial_b x_\mu) + \lambda R \right]$$

This is the Most general structure of an action which has diffeomorphism, Weyl & Poincare invariance with only the fields $X^\mu(\tau, \sigma)$ & $\gamma_{ab}(\tau, \sigma)$.

Note that X^μ is a scalar field on the 2 dim world sheet.

This action looks like the Hilbert action of the 2dim world sheet given by

$$\frac{1}{4\pi} \int_W d^2\sigma \sqrt{-\det(\gamma)} R \quad \left. \vphantom{\int} \right\} \begin{array}{l} \text{Einstein} \\ \text{Hilbert} \\ \text{action of} \\ \text{2dim world.} \end{array}$$

& the Polyakov part is like a Minimal coupling of D scalar fields X^μ , living on the 2dim world sheet coupled by the world Sheet Metric ' γ '.

$$\frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \underbrace{\gamma^{ab} (\partial_a X^\mu) (\partial_b X^\nu) \eta_{\mu\nu}}_{\text{Minimal coupling}}$$