

Light-Cone Coordinates

The light cone coordinates for space-time is given by,

$$x^{\pm} = \frac{x^0 \pm x^1}{\sqrt{2}} \quad \& \quad x^i = 2, 3, \dots, D-1$$

Note: Lower case x^{μ} represents the space-time coordinate. Upper case $X^{\mu}(\tau, \sigma)$ define the fields on the world sheet.

$$\begin{aligned} \rightarrow x^{\mu} x_{\mu} &= -(x^0)^2 + (x^1)^2 + \sum_{i=2}^{D-1} (x^i)^2 \\ &= \frac{-(x^+ + x^-)^2 + (x^+ - x^-)^2}{2} + \sum_{i=2}^{D-1} (x^i)^2 \end{aligned}$$

$$\rightarrow x^{\mu} x_{\mu} = -x^- x^+ - x^+ x^- + \sum_{i=2}^{D-1} (x^i)^2$$

Then the metric in light cone coordinates is not diagonal.

$$\text{Light cone metric} = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\rightarrow x_- = -x^+, \quad x_+ = -x^-, \quad x_i = x^i$$

We shall reparameterize τ to be equal to the space time coordinate x^+ . Thus x^+ will play the role of time & its corresponding conjugate p_+ or p^- will play the role of energy.

x^- & $(p_- \text{ or } p^+)$ are the longitudinal variables, x^i & p^i are the transverse space-time coordinate & momenta.

We can check how the relativistic particle action in the presence of the Einbein fields behave in the light - Cone coordinates.

Consider the action given by

$$S = \frac{1}{2} \int d\tau [\varepsilon^{-1} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \varepsilon m^2]$$

where $\varepsilon \rightarrow \varepsilon(\tau)$ is an auxiliary 'Einbein' field & $\dot{X}^\mu = \frac{dX^\mu}{d\tau}$

$$\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} = -\dot{X}^- \dot{X}^+ - \dot{X}^+ \dot{X}^- + \sum_{i=2}^{D-1} (\dot{X}^i)^2$$

Setting $X^+(\tau) = \tau$, $\dot{X}^+ = 1$

$$\Rightarrow S = \frac{1}{2} \int d\tau [-2\varepsilon^{-1} \dot{X}^- + \dot{X}^i \dot{X}_i - \varepsilon m^2]$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} [-2\varepsilon^{-1} \dot{X}^- + \varepsilon^{-1} \dot{X}^i \dot{X}_i - \varepsilon m^2]$$

$$p_- = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = -\varepsilon^{-1}, \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{X}^i} = \varepsilon^{-1} \dot{X}_i$$

$$p^+ = -p_- = \varepsilon^{-1}, \quad p^i = p_i = \varepsilon^{-1} \dot{X}^i$$

Note: $p_+ = p^- = 0$, as $\dot{X}^+$ is not present in the lagrangian.

Since $\dot{\varepsilon}(\tau)$ is not present in action, $p_{\varepsilon} = 0$, there is no conjugate momenta associated with ' ε ' $\neq$ Since $\varepsilon^{-1} = p^+$, this becomes the constraint for $\varepsilon(\tau)$.

Then the Hamiltonian is,

$$H = p_- \dot{\chi}^- + p_i \dot{\chi}^i - L$$

$$\Rightarrow H = p_- \dot{\chi}^- + p_i \dot{\chi}^i + \varepsilon^{-1} \dot{\chi}^- - \frac{1}{2} \varepsilon^{-1} \dot{\chi}^i \dot{\chi}_i + \frac{1}{2} \varepsilon m^2$$

$$\rightarrow H = \frac{1}{2} [\varepsilon^{-1} \dot{\chi}_i \dot{\chi}^i + \varepsilon m^2]$$

$$= \frac{[(\varepsilon^{-1} \dot{\chi}_i)(\varepsilon^{-1} \dot{\chi}_i) + m^2]}{2\varepsilon^{-1}}$$

$$\rightarrow H = \left[\frac{p_i p^i + m^2}{2 p^+} \right] = \left[\frac{p_i p^i + m^2}{(-2 p_-)} \right]$$

Then, to quantize this, we shall promote the p^i 's & p^+ as operators & impose the following commutations

$$[\hat{x}^j, \hat{p}_k] = i\delta_{jk}$$

$$\& [\hat{x}^-, \hat{p}_-] = i$$

$$\Rightarrow \hat{H} = \frac{-1}{2} (\hat{p}_-)^{-1} \left[\sum_i (\hat{p}^i)^2 + m^2 \right]$$

Since all the momentum operators commute, they form a complete set. And the momentum Eigenstates are $|p_-, p^i\rangle$

$$\begin{aligned} \Rightarrow \hat{p}_- |p_-, p^i\rangle &= p_- |p_-, p^i\rangle \\ p^i |p_-, p^i\rangle &= p^i |p_-, p^i\rangle \end{aligned}$$

$$\Rightarrow \hat{H} |p_-, p^i\rangle = E_p |p_-, p^i\rangle$$

$$\text{where } E_p = \frac{-1}{2p_-} \left[\sum_i (p^i)^2 + m^2 \right]$$

$$p_- = -p^+ \rightarrow E_p = \frac{1}{2p^+} \left[\sum_i (p^i)^2 + m^2 \right]$$

Since we have chosen $X^+(\tau) = \tau$, it implies that its conjugate momenta p_+ is energy.

$$\rightarrow -p_+ = p^- = E_p$$

$$\rightarrow 2 p^- p^+ = \left[\sum_i (p^i)^2 + m^2 \right]$$

$$\Rightarrow (p^0 - p') (p^0 + p') = \left[\sum_i (p^i)^2 + m^2 \right]$$

$$\Rightarrow -(p^0)^2 + (p')^2 + (p_i p^i)^2 = -m^2$$

$$\Rightarrow \underline{p^\mu p_\mu = -m^2}$$

Thus, $H = \left[\frac{p_i p^i + m^2}{2 p^+} \right]$ gives

the mass Shell Condition for the relativistic particle in the light cone coordinates.

Polyakov Action in Light Cone Coordinates

In light cone gauge, we set the parameter $X^+(\tau) = \tau$ & to fix the redundancies in World Sheet, we will set two conditions on the metric

$$\partial_\sigma \gamma_{\sigma\sigma} = 0$$

$$\& \quad -\det(\gamma_{ab}) = 1$$

Thus we have exhausted the freedom of two diffeomorphism symmetry & one Weyl symmetry.

The procedure to reach this gauge is first setting $\tau = X^+(\tau)$. Then define a function
$$f(\tau, \sigma) = \frac{\gamma_{\sigma\sigma}}{\sqrt{-\det(\gamma_{ab})}}$$

Under reparameterization of $\sigma \rightarrow \tilde{\sigma}(\tau, \sigma)$ but keeping τ fixed i.e. $\tau = \bar{\tau}$, we get the transformation rule for 'f'.

$$\tilde{f}(\tilde{z}, \tilde{q}) = \frac{\tilde{z}_{\tilde{a}\tilde{b}}}{\sqrt{-\det(\tilde{\gamma}_{ab})}}$$

$$= \frac{1}{\sqrt{-\det(\tilde{\gamma}_{ab})}} \left(\frac{\partial \sigma^a}{\partial \tilde{z}} \right) \left(\frac{\partial \sigma^b}{\partial \tilde{z}} \right) \gamma_{ab}$$

$$= \frac{1}{\sqrt{-\det(\tilde{\gamma}_{ab})}} \left[\left(\frac{\partial \tau}{\partial \tilde{z}} \right)^2 \gamma_{\tau\tau} + \left(\frac{\partial \sigma}{\partial \tilde{z}} \right)^2 \gamma_{\sigma\sigma} + 2 \left(\frac{\partial \tau}{\partial \tilde{z}} \right) \left(\frac{\partial \sigma}{\partial \tilde{z}} \right) \gamma_{\sigma\tau} \right]$$

Since $\tau = \tilde{z} \rightarrow \frac{\partial \tau}{\partial \tilde{z}} = 1$

$\rightarrow \sigma = \sigma(\tilde{z}) \rightarrow \frac{\partial \sigma}{\partial \tilde{z}} = 0$

$$\rightarrow \tilde{f}(\tilde{z}, \tilde{q}) = \frac{1}{\sqrt{-\det(\tilde{\gamma}_{ab})}} \left(\frac{d\sigma}{d\tilde{z}} \right)^2 \gamma_{\sigma\sigma}$$

Now $\left[\frac{\partial \sigma^a}{\partial \tilde{z}^b} \right] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{d\sigma}{d\tilde{z}} \end{bmatrix}$

$$\Rightarrow \tilde{\gamma}_{cd} = \left(\frac{\partial \sigma^a}{\partial \tilde{\sigma}^c} \right) \left(\frac{\partial \sigma^b}{\partial \tilde{\sigma}^d} \right) \gamma_{ab}$$

$$\Rightarrow \sqrt{\det(\tilde{\gamma})} = \left[\det \left(\frac{\partial \sigma^a}{\partial \tilde{\sigma}^c} \right) \right] \sqrt{\det(\gamma)}$$

$$\Rightarrow \sqrt{\det(\tilde{\gamma})} = \left(\frac{d\sigma}{d\tilde{\sigma}} \right) \sqrt{\det(\gamma)}$$

$$\rightarrow \tilde{f}(\tilde{\tau}, \tilde{\sigma}) = \frac{1}{\sqrt{-\det(\tilde{\gamma}_{ab})}} \left(\frac{d\sigma}{d\tilde{\sigma}} \right)^2 \gamma_{\sigma\sigma}$$

$$= \frac{1}{\sqrt{-\det(\gamma_{ab})} \left(\frac{d\sigma}{d\tilde{\sigma}} \right)} \left(\frac{d\sigma}{d\tilde{\sigma}} \right)^2 \gamma_{\sigma\sigma}$$

$$\rightarrow \tilde{f}(\tau, \tilde{\sigma}) = f(\tau, \sigma) \left(\frac{d\sigma}{d\tilde{\sigma}} \right)$$

$$\Rightarrow \underline{\underline{\tilde{f} d\tilde{\sigma} = f d\sigma}}$$

Thus the quantity 'f dσ' is invariant under 'σ' parameterization.

Let $dl = f d\sigma$ be an invariant length. And then we can interpret ' σ ' value of a point on the world sheet as a quantity proportional to distance dl from $\sigma = 0$ to that point. Then the proportionality constant is got by needing $\sigma = l$ at the right hand end point.

Since $f d\sigma = dl$ Constant under σ reparameterization, $f = \frac{dl}{d\sigma}$ is

independent of ' σ ', but depends on ' τ '.

$\rightarrow \partial_\sigma f = 0$. Even under Weyl transformation $f(\tau, \sigma)$ is invariant as the factor $e^{2\omega}$ simply cancels out.

$$f' = \frac{\gamma'_{\sigma\sigma}}{\sqrt{-\det(\gamma'_{ab})}} = \frac{e^{2\omega} \gamma_{\sigma\sigma}}{e^{2\omega} \sqrt{-\det(\gamma_{ab})}} = f$$

$\rightarrow \partial_\sigma f = 0$ holds even under Weyl.

Now we need to set $\det(\gamma_{ab}) = -1$

$\Rightarrow \gamma_{\sigma\sigma}$ is a function of ' τ ' alone.

$$\text{So, } \gamma_{ab} = \begin{bmatrix} \gamma_{\tau\tau} & \gamma_{\tau\sigma} \\ \gamma_{\tau\sigma} & \gamma_{\sigma\sigma} \end{bmatrix}$$

$$\det(\gamma_{ab}) = \gamma_{\tau\tau}\gamma_{\sigma\sigma} - (\gamma_{\tau\sigma})^2 = -1$$

$$\Rightarrow \gamma_{\tau\tau} = -(\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2]$$

Thus only $\gamma_{\sigma\sigma}$ & $\gamma_{\tau\sigma}$ are the remaining independent degrees of freedom of ' γ_{ab} '.

$$\Rightarrow \underline{\partial_\sigma \gamma_{\sigma\sigma} = 0}$$

$$\& \gamma^{ab} = \begin{bmatrix} -\gamma_{\sigma\sigma} & \gamma_{\tau\sigma} \\ \gamma_{\tau\sigma} & (\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] \end{bmatrix}$$

$$\& \quad \gamma_{ab} = \begin{bmatrix} -(\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] & \gamma_{\tau\sigma} \\ \gamma_{\tau\sigma} & \gamma_{\sigma\sigma} \end{bmatrix}$$

Now, we shall write the Polyakov action in this gauge.

$$S_P = \frac{-1}{4\pi\alpha'} \int_W d^2\sigma \sqrt{-\det(\gamma)} \gamma^{ab} (\partial_a x^\mu) (\partial_b x^\nu) \eta_{\mu\nu}$$

$$\sqrt{-\det(\gamma)} = 1$$

$$\begin{aligned} (\partial_a x^\mu) (\partial_b x^\nu) \eta_{\mu\nu} &= -(\partial_a x^0) (\partial_b x^0) + (\partial_a x^1) (\partial_b x^1) \\ &\quad + (\partial_a x^i) (\partial_b x_i) \\ &= -(\partial_a x^-) (\partial_b x^+) - (\partial_a x^+) (\partial_b x^-) + (\partial_a x^i) (\partial_b x_i) \end{aligned}$$

$$(\text{Since } x^+ = \tau)$$

$$\begin{aligned} (\partial_a x^\mu) (\partial_b x^\nu) \eta_{\mu\nu} &= -(\partial_a x^-) \partial_b^\tau - \partial_a^\tau (\partial_b x^-) \\ &\quad + (\partial_a x^i) (\partial_b x_i) \end{aligned}$$

$$\rightarrow \mathcal{L}_p = \frac{-1}{4\pi\alpha'} \gamma^{ab} \left[-(\partial_a x^-) \partial_b^\tau - \partial_a^\tau (\partial_b x^-) + (\partial_a x^i) (\partial_b x_i) \right]$$

$$\mathcal{L}_p = \frac{-1}{4\pi\alpha'} \left[-2(\partial_a x^-) \gamma^{a\tau} + (\partial_a x^i) (\partial_b x_i) \gamma^{ab} \right]$$

$$= \frac{-1}{4\pi\alpha'} \left[-2(\partial_\tau x^-) \gamma^{\tau\tau} - 2(\partial_\sigma x^-) \gamma^{\sigma\tau} + (\partial_\tau x^i) (\partial_\tau x_i) \gamma^{\tau\tau} + (\partial_\sigma x^i) (\partial_\sigma x_i) \gamma^{\sigma\sigma} + 2(\partial_\sigma x^i) (\partial_\tau x_i) \gamma^{\sigma\tau} \right]$$

$$\mathcal{L}_p = \frac{-1}{4\pi\alpha'} \left[\gamma_{\sigma\sigma} \left[2(\partial_\tau x^-) - (\partial_\tau x^i) (\partial_\tau x_i) \right] - 2\gamma_{\tau\sigma} \left[(\partial_\sigma x^-) - (\partial_\sigma x^i) (\partial_\tau x_i) \right] + (\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] (\partial_\sigma x^i) (\partial_\sigma x_i) \right]$$

Then the lagrangian is $L_p = \int_0^l d\sigma \mathcal{L}_p$

$$\neq S_p = \int d\tau L$$

$$L_p = \frac{-1}{4\pi\alpha'} \int_0^l d\sigma \left[\gamma_{\sigma\sigma} [2(\partial_\tau x^-) - (\partial_\tau x^i)(\partial_\tau x_i)] \right. \\ \left. - 2\gamma_{\tau\sigma} [(\partial_\sigma x^-) - (\partial_\sigma x^i)(\partial_\tau x_i)] \right. \\ \left. + (\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] (\partial_\sigma x^i)(\partial_\sigma x_i) \right]$$

Since $\partial_\sigma \gamma_{\sigma\sigma} = 0$

$$L_p = \frac{-\gamma_{\sigma\sigma}}{2\pi\alpha'} \partial_\tau \int_0^l d\sigma x^- \\ + \frac{1}{4\pi\alpha'} \int_0^l d\sigma \left[\gamma_{\sigma\sigma} (\partial_\tau x^i)(\partial_\tau x_i) \right. \\ \left. + 2\gamma_{\tau\sigma} [(\partial_\sigma x^-) - (\partial_\sigma x^i)(\partial_\tau x_i)] \right. \\ \left. - (\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] (\partial_\sigma x^i)(\partial_\sigma x_i) \right]$$

We shall define $x^-(\tau) = \frac{1}{l} \int_0^l d\sigma x^-$

which is the mean value of ' x^- ' at a given ' τ '. $\nabla^-(\tau, \sigma) = x^-(\tau, \sigma) - x^-(\tau)$ is the centralized version of x^- .

$\rightarrow \partial_\sigma X^- = \partial_\sigma Y^-$, so we get,

$$L_p = \frac{-l \gamma_{\sigma\sigma}}{2\pi\alpha'} \partial_\tau X^-(\tau)$$

$$+ \frac{1}{4\pi\alpha'} \int_0^l d\sigma \left[\gamma_{\sigma\sigma} (\partial_\tau X^i)(\partial_\tau X_i) \right. \\ \left. + 2\gamma_{\tau\sigma} [\partial_\sigma Y^- - (\partial_\sigma X^i)(\partial_\tau X_i)] \right. \\ \left. - (\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] (\partial_\sigma X^i)(\partial_\sigma X_i) \right]$$

Now Y^- doesn't appear in any terms with ' τ ' derivatives, so it is not a dynamical field. It acts as a lagrange multiplier which constrains its coefficients.

$\rightarrow \partial_\sigma \gamma_{\tau\sigma} = \text{fixed constant.}$

$$\Rightarrow \underline{\partial_\sigma^2 \gamma_{\tau\sigma} = 0}$$

The open string boundary conditions is given by,

$$\gamma^{\sigma\sigma} \partial_\sigma X^\mu + \gamma^{\sigma\tau} \partial_\tau X^\mu = 0, \text{ at } \sigma = 0, l$$

Then in this Gauge, at $\sigma = 0, 1$

$$(\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] \partial_{\sigma} X^{\mu} + \gamma_{\tau\sigma} \partial_{\tau} X^{\mu} = 0$$

For $\mu = '+'$

$$(\gamma_{\sigma\sigma})^{-1} [1 - (\gamma_{\tau\sigma})^2] \partial_{\sigma} X^{+} + \gamma_{\tau\sigma} \partial_{\tau} X^{+} = 0$$

$$\partial_{\sigma} X^{\mu} = 0 \Rightarrow \underline{\gamma_{\tau\sigma}(\tau, \sigma = 0 \text{ or } 1) = 0}$$

But $\partial_{\sigma}^2 \gamma_{\tau\sigma} = 0$ everywhere
& $\gamma_{\tau\sigma} = 0$ at σ 's boundary.
for any ' τ '

$\Rightarrow \gamma_{\tau\sigma}$ should be zero everywhere

$$\& \text{ for } \mu = i, (\gamma_{\sigma\sigma})^{-1} \partial_{\sigma} X^i = 0$$

$$\Rightarrow \partial_{\sigma} X^i = 0 \text{ at } \sigma\text{'s boundary.}$$

Thus finally after Gauge fixing & then using the lagrange multiplier, we get the system with only $\gamma_{\sigma\sigma}(\tau)$, $x^-(\tau)$ & $X^i(\tau, \sigma)$

Then, the lagrangian is,

$$L_p = \frac{-l \gamma_{\sigma\sigma}}{2\pi\alpha'} \partial_\tau x^-(\tau) + \frac{1}{4\pi\alpha'} \int_0^l d\sigma \left[\gamma_{\sigma\sigma} \dot{X}^i \dot{X}_i - (\gamma_{\sigma\sigma})^{-1} \dot{X}^i \dot{X}_i \right]$$

$$\rightarrow p_- = -p^+ = \frac{\partial L_p}{\partial \dot{x}^-} = \frac{-l \gamma_{\sigma\sigma}}{2\pi\alpha'}$$

$$\rightarrow \underline{\gamma_{\sigma\sigma} = \frac{2\pi\alpha'}{l} p^+} \quad \left. \vphantom{\gamma_{\sigma\sigma}} \right\} \text{Constraint on } \gamma_{\sigma\sigma}$$

$$\text{Then } \pi_i = \frac{\partial L_p}{\partial \dot{X}^i} = \frac{1}{4\pi\alpha'} (2\gamma_{\sigma\sigma} \dot{X}_i)$$

$$\rightarrow \underline{\pi^i = \frac{\dot{X}^i p^+}{l}}$$

Then, the Hamiltonian is

$$H_p = p_- \dot{x}^- + \int_0^l d\sigma \pi_i \dot{X}^i - L_p$$

$$\rightarrow H_p = \underline{p_-} \dot{x}^- + \int_0^l d\sigma \pi_i \left(\frac{l \pi^i}{p^+} \right) + p^+ \dot{x}^-$$

$$- \frac{1}{4\pi\alpha'} \int_0^l d\sigma \left[2\pi\alpha' \frac{l \pi^i \pi_i}{p^+} - (\gamma_{\sigma\sigma})^{-1} \dot{x}^i \dot{x}_i \right]$$

$$\rightarrow H_p = \frac{l}{2p^+} \int_0^l d\sigma \left[\pi^i \pi_i + \frac{\dot{x}^i \dot{x}_i}{(2\pi\alpha')^2} \right]$$

This hamiltonian describes just the $D-2$ fields $x^2, x^3, \dots, x^{D-1}$, where $p^+ = -p_-$ is conserved. since, $\frac{\partial H}{\partial x^-} = 0$

Then, the eqⁿ for motions are,

$$\frac{\partial H}{\partial p_-} = \dot{x}^- \rightarrow \dot{x}^- = \frac{H}{p^+}$$

$$\nabla \frac{\partial H}{\partial x^-} = -\dot{p}_- = 0 \Rightarrow \dot{p}_- = 0 \Rightarrow \dot{p}^+ = 0$$

$$\text{Then, } \frac{\partial H}{\partial \pi_i} = \dot{x}^i = \frac{l \pi^i}{p^+}$$

$$4 \quad \dot{\pi}_i = -\frac{\partial H}{\partial \dot{\chi}^i} = -l \frac{\partial}{\partial \dot{\chi}^i} \int_0^l d\sigma \underbrace{\frac{\partial_\sigma \chi^i \partial_\sigma \chi_i}{2p^+ (2\pi\alpha')^2}}_{\text{Int. by parts}}$$

$$= -l \frac{\partial}{\partial \dot{\chi}^i} \left(\underbrace{\frac{[\partial_\sigma \chi^i] \chi_i}{2p^+ (2\pi\alpha')^2}}_{\text{boundary } 0} - \int_0^l d\sigma \frac{(\chi^i \partial_\sigma^2 \chi_i)}{2p^+ (2\pi\alpha')^2} \right)$$

$$= l \frac{\partial}{\partial \dot{\chi}^i} \int_0^l d\sigma \frac{(\chi^i \partial_\sigma^2 \chi_i)}{2p^+ (2\pi\alpha')^2} = \frac{l \partial_\sigma^2 \chi_i}{p^+ (2\pi\alpha')^2}$$

$$\Rightarrow \dot{\pi}^i = \frac{l \ddot{\chi}^i}{p^+ (2\pi\alpha')^2}$$

$$\Rightarrow \ddot{\chi}^i = \frac{l \dot{\pi}^i}{p^+} = \frac{(l)^2 \ddot{\chi}^i}{(2\pi\alpha' p^+)^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \chi^i}{\partial \tau^2} = c^2 \frac{\partial^2 \chi^i}{\partial \sigma^2}}$$

Thus χ^i act as waves in the world sheet with speed $c = l/(2\pi\alpha' p^+)$.

Mode expansion of Transverse Coordinates

Here we need to find the general solⁿ for the wave eqⁿ,

$$\frac{\partial^2 \chi^i}{\partial \tau^2} = c^2 \frac{\partial^2 \chi^i}{\partial \sigma^2}$$

with the boundary conditions

$$\left. \partial_\sigma \chi^i \right|_{\sigma=0} = \left. \partial_\sigma \chi^i \right|_{\sigma=l} = 0$$

(at any τ) (at any τ)

This conditions mean that the spatial endpoints of strings are not fixed & are free to move in space but no force is exerted at σ 's boundary at any ' τ '.

A general solⁿ as a fourier expansion is given by,

$$\chi^i(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n^i \phi_n(\sigma) e^{-i\omega_n \tau}$$

where C_n^i are fourier coefficients $\phi_n(\sigma)$ are spatial functions which satisfy the boundary conditions. ω_n are mode frequencies.

$$\chi^i \text{ satisfies } \frac{\partial^2 \chi^i}{\partial \tau^2} = c^2 \frac{\partial^2 \chi^i}{\partial \sigma^2}$$

$$\begin{aligned} \rightarrow \sum_{n=-\infty}^{\infty} -\omega_n^2 C_n^i \phi_n(\sigma) e^{-i\omega_n \tau} \\ = \sum_{n=-\infty}^{\infty} c^2 C_n^i \phi_n''(\sigma) e^{-i\omega_n \tau} \end{aligned}$$

$$\rightarrow \sum_{n=-\infty}^{\infty} \left[\phi_n''(\sigma) + \frac{\omega_n^2}{c^2} \phi_n(\sigma) \right] C_n^i e^{-i\omega_n \tau} c^2 = 0$$

$\rightarrow \phi_n(\sigma)$ should satisfy,

$$\phi_n''(\sigma) + k_n^2 \phi_n(\sigma) = 0, \text{ where } k_n = \frac{\omega_n}{c}$$

with $\partial_\sigma \phi_n = 0$ at $\sigma = 0, l$

$$\rightarrow \phi_n(\sigma) = A_n \cos(k_n \sigma) + B_n \sin(k_n \sigma)$$

$$\rightarrow \partial_\sigma \phi_n = -A_n k_n \sin(k_n \sigma) + B_n k_n \cos(k_n \sigma)$$

$$\text{at } \underline{\sigma=0}, -A_n k_n \sin(0) + B_n k_n \cos(0) = 0 \\ \Rightarrow B_n = 0$$

$$\text{at } \underline{\sigma=l}, -A_n k_n \sin(k_n l) = 0 \\ \Rightarrow k_n l = n\pi \Rightarrow \underline{k_n = \frac{n\pi}{l}}$$

$$\rightarrow \phi_n(\sigma) = A_n \cos\left(\frac{n\pi\sigma}{l}\right) \quad \& \quad \underline{\omega_n = \frac{n\pi c}{l}}$$

$$\rightarrow \chi^i(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n^i \cos(k_n \sigma) e^{-i\omega_n \tau}$$

We have absorbed A_n in the definition of ' c_n^i '. Since $[\chi^i(\tau, \sigma)]^* = \chi^i(\tau, \sigma)$ is real, we need,

$$c_n^{i*} e^{i\omega_n \tau} = c_n^i e^{-i\omega_n \tau}$$

$$\Rightarrow (c_n^i)^* e^{\frac{i n \pi c \tau}{l}} = c_n^i e^{\frac{-i n \pi c \tau}{l}} = c_{-n}^i e^{\frac{i n \pi c \tau}{l}}$$

Thus c_n^i should satisfy $(c_n^i)^* = c_{-n}^i$

Hence,

$$\underline{\chi^i(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n^i \cos\left(\frac{n\pi\sigma}{l}\right) e^{\frac{-i n \pi c \tau}{l}}}$$

For $n=0$, $\phi_0(\sigma)=0$ as $\omega_0=0$

$$\Rightarrow \frac{d^2 \chi_{(n=0)}^i}{d\tau^2} = 0$$

$$\Rightarrow \chi_{n=0}^i = A_0^i + B_0^i \tau$$

$$\Rightarrow \chi^i(\tau, \sigma) = A_0^i + B_0^i \tau + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} C_n^i \cos\left(\frac{n\pi\sigma}{l}\right) e^{\frac{-in\pi\tau}{l}}$$

The zero mode terms are independent of ' σ '. It implies that all string parts move together in the ' τ ' direction. So it describes the center of mass motion.

$\Rightarrow A_0^i$ should relate to the initial location of the center of mass & B_0^i should relate to the center of mass momentum.

The CoM is
$$R^i(\tau) = \frac{1}{l} \int_0^l \chi^i(\tau, \sigma) d\sigma$$

$$R^i(\tau) = \frac{1}{l} \int_0^l \chi^i(\tau, \sigma) d\sigma$$

$$= \frac{1}{l} \int_0^l d\sigma (A_0^i + B_0^i \tau)$$

$$+ \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} c_n^i e^{\frac{-in\pi\tau}{l}} \frac{1}{l} \int_0^l d\sigma \underbrace{\cos\left(\frac{n\pi\sigma}{l}\right)}_0$$

$$\rightarrow R^i(\tau) = A_0^i + B_0^i \tau \rightarrow R^i(\tau) = \underbrace{R^i(0)}_{\text{Free particle Uniform motion}} + B_0^i \tau$$

Let $\chi^i = R^i(0)$
(Initial CoM location)

Free particle
Uniform motion

$$\text{Now, } P^i(\tau) = \int_0^l d\sigma \pi^i(\tau, \sigma) \text{ is the}$$

Center of mass momenta. But, by hamilton eqⁿ of motion, $\partial_\tau \chi^i(\tau, \sigma) = \frac{l \pi^i}{p^+}$

$$\rightarrow P^i(\tau) = \frac{p^+}{l} \int_0^l d\sigma \partial_\tau \chi^i(\tau, \sigma)$$

$$\rightarrow P^i(\tau) = \frac{p^+}{l} \int_0^l d\sigma \left[B_0^i - i \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} C_n^i \omega_n \underbrace{\cos\left(\frac{n\pi\sigma}{l}\right)}_{\text{Vanishes}} e^{-i\omega_n \tau} \right]$$

$$\rightarrow P^i(\tau) = \frac{p^+}{l} \int_0^l d\sigma B_0^i = \underbrace{p^+ B_0^i}_{\text{Independent of } \tau'}$$

Thus even the center of mass momenta is conserved quantity.

$$\rightarrow B_0^i = \frac{P^i}{p^+} \quad \& \quad A_0^i = \underbrace{x^i}_{\text{Initial CoM position}}$$

$$\rightarrow X^i(\tau, \sigma) = x^i + \frac{P^i}{p^+} \tau + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} C_n^i \cos\left(\frac{n\pi\sigma}{l}\right) e^{\frac{-in\pi\tau}{l}}$$

Since, $\pi^i = \frac{p^+}{l} \partial_\tau X^i(\tau, \sigma)$

$$\Rightarrow \pi^i = \frac{p^+}{l} \left[\frac{P^i}{p^+} - i \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} C_n^i \omega_n \cos\left(\frac{n\pi\sigma}{l}\right) e^{-i\omega_n \tau} \right]$$

$$\rightarrow \Pi^i = \frac{p^i}{l} - \frac{i p^+}{l} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} c_n^i \frac{n \pi c}{l} \cos\left(\frac{n \pi \sigma}{l}\right) e^{-i \omega_n \tau}$$

$$\rightarrow \Pi^i = \frac{p^i}{l} + \frac{(-i)}{2\alpha' l} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} c_n^i n \cos\left(\frac{n \pi \sigma}{l}\right) e^{-i \omega_n \tau}$$

For later convenience, we shall define a_n^i such that,

$$c_n^i = \frac{i \sqrt{2\alpha'}}{n} a_n^i \rightarrow \frac{-i n c_n^i}{2\alpha' l} = \frac{a_n^i}{\sqrt{2\alpha'} l}$$

Thus we finally get $\chi^i(\tau, \sigma)$ & $\Pi^i(\tau, \sigma)$ as,

$$\chi^i = x^i + \frac{p^i}{p^+} \tau + i \sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{a_n^i}{n} \cos\left(\frac{n \pi \sigma}{l}\right) e^{\frac{-i n \pi c \tau}{l}}$$

$$\Pi^i = \frac{p^i}{l} + \frac{1}{l \sqrt{2\alpha'}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} a_n^i \cos\left(\frac{n \pi \sigma}{l}\right) e^{\frac{-i n \pi c \tau}{l}}$$

Quantization of Open Strings

The canonical coordinates are $x^-(\tau)$ & $\chi^i(\tau, \sigma)$. The canonical momenta conjugate to x^- is $p_- = -p^+$ & for χ^i is π_i .

To quantize the open string's we shall impose equal time canonical commutation relations b/w the canonically conjugate variables,

$$[\hat{x}^-, \hat{p}_-] = i$$

$$\& [\hat{\chi}^j(\tau, \sigma), \hat{\pi}_k(\tau, \sigma')] = i \delta^j_k \delta(\sigma - \sigma')$$
$$(\hat{p}_- = -\hat{p}^+, \quad \hat{\pi}^k = \hat{\pi}_k)$$

$$\rightarrow [\hat{x}^-, \hat{p}^+] = -i \quad \text{and}$$

$$[\hat{\chi}^j(\tau, \sigma), \hat{\pi}^k(\tau, \sigma')] = i \delta^{jk} \delta(\sigma - \sigma')$$

& all other commutators b/w the canonical variables are Zero.

The center of mass coordinates have

$$[\hat{x}^j, \hat{p}^k] = i\delta^{jk}$$

These result in corresponding commutation relations b/w the fourier components.

For the canonical commutation relations to hold the fourier components $\hat{a}_n^j$'s should satisfy

$$[\hat{a}_n^j, \hat{a}_m^k] = n\delta^{jk}\delta_{n,-m}$$

This can be written as,

$$[\hat{a}_n^j, \hat{a}_{-m}^k] = n\delta^{jk}\delta_{nm}$$

$$\rightarrow [\hat{a}_n^j, (\hat{a}_m^k)^*] = n\delta^{jk}\delta_{nm}$$

$$\text{or } \left[\frac{\hat{a}_n^j}{\sqrt{n}}, \frac{(\hat{a}_m^k)^*}{\sqrt{m}} \right] = \delta^{jk}\delta_{nm}$$

So, if $\hat{b}_n^j = \frac{\hat{a}_n^j}{\sqrt{n}} \rightarrow [\hat{b}_n^j, (\hat{b}_m^k)^*] = \delta^{jk}\delta_{nm}$
Harmonic oscillator algebra

Thus for any transverse direction 'i' & for each mode 'n', the modes will satisfy the Harmonic oscillator algebra but with a non standard normalization.

Substituting the mode expansions in the Hamiltonian, we get,

$$\hat{H}_P = \frac{l}{2\hat{p}^+} \int_0^l d\sigma \left[\hat{\Pi}^i \hat{\Pi}_i + \frac{\hat{\chi}^i \hat{\chi}_i}{(2\pi\alpha')^2} \right]$$

where, $\frac{1}{\hat{p}^+} = (\hat{p}^+)^{-1}$,

$$\hat{\Pi}^i = \frac{\hat{p}^i}{l} + \frac{1}{l\sqrt{2\alpha'}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \hat{a}_n^i \cos(k_n \sigma) e^{-i\omega_n \tau}$$

&

$$\hat{\chi}^i = \frac{-i\sqrt{2\alpha'}\pi}{l} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \hat{a}_n^i \sin(k_n \sigma) e^{-i\omega_n \tau}$$

Decompose, $\sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} f_n = \sum_{n=1}^{\infty} f_n + f_{-n}$

Thus,

$$\hat{\chi}^i = \hat{x}^i + \frac{\hat{p}^i}{\hat{p}^+} + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[\frac{\hat{a}_n^i e^{-i\omega_n \tau} - \hat{a}_n^{i*} e^{+i\omega_n \tau}}{n} \right] \cos(k_n \sigma)$$

$$\hat{\Pi}^i = \frac{\hat{p}^i}{l} + \frac{1}{l\sqrt{2\alpha'}} \sum_{n=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} + \hat{a}_n^{i*} e^{+i\omega_n \tau} \right] \cos(k_n \sigma)$$

$$\hat{\dot{\chi}}^i = -\frac{i\sqrt{2\alpha'}\pi}{l} \sum_{n=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} - \hat{a}_n^{i*} e^{+i\omega_n \tau} \right] \sin(k_n \sigma)$$

Let,

$$I_1 = \frac{l}{2\hat{p}_0^+} \int_0^l d\sigma (\hat{\Pi}^i)^2 \quad \& \quad I_2 = \frac{l}{2\hat{p}_0^+} \int_0^l d\sigma \frac{(\hat{\dot{\chi}}^i)^2}{(2\pi\alpha')^2}$$

$$(\hat{\Pi}^i)^2 = \left(\frac{\hat{p}^i}{l} \right)^2 + \frac{2\hat{p}^i}{l^2\sqrt{2\alpha'}} \sum_{n=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} + \hat{a}_n^{i*} e^{+i\omega_n \tau} \right] \cos(k_n \sigma)$$

$$+ \frac{1}{l^2(2\alpha')} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} + \hat{a}_n^{i*} e^{+i\omega_n \tau} \right] \cos(k_n \sigma) \\ \times \left[\hat{a}_m^i e^{-i\omega_m \tau} + \hat{a}_m^{i*} e^{+i\omega_m \tau} \right] \cos(k_m \sigma)$$

$\int_0^l d\sigma (\hat{T}^i)^2$ is given by.

$$= (\hat{p}^i)^2 + \frac{1}{4\alpha'} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} + \hat{a}_n^{i*} e^{+i\omega_n \tau} \right] \delta_{nm} \\ \times \left[\hat{a}_m^i e^{-i\omega_m \tau} + \hat{a}_m^{i*} e^{+i\omega_m \tau} \right]$$

Since $\int_0^l d\sigma \cos(k_n \sigma) = 0$

$$\neq \int_0^l d\sigma \cos(k_n \sigma) \cos(k_m \sigma) = \frac{\delta_{nm} l}{2}$$

$$\rightarrow \hat{I}_1 = \frac{\hat{p}^i \hat{p}_i}{2 \hat{p}^+} + \frac{1}{8\alpha' \hat{p}^+} \sum_{n=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} + \hat{a}_n^{i*} e^{+i\omega_n \tau} \right]^2$$

$$\rightarrow \hat{I}_1 = \frac{\hat{p}^i \hat{p}_i}{2 \hat{p}^+} + \frac{1}{8\alpha' \hat{p}^+} \sum_{n=1}^{\infty} \left[\begin{aligned} &\hat{a}_n^i \hat{a}_n^i e^{-2i\omega_n \tau} \\ &+ \hat{a}_n^{i*} \hat{a}_n^{i*} e^{+2i\omega_n \tau} \\ &+ \hat{a}_n^{i*} \hat{a}_n^i + \hat{a}_n^i \hat{a}_n^{i*} \end{aligned} \right]$$

Similarly,

$$(\hat{X}^i)^2 = -\frac{2\alpha' \pi^2}{l^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n \tau} - \hat{a}_n^{i*} e^{+i\omega_n \tau} \right] \sin(k_n \sigma) \\ \times \left[\hat{a}_m^i e^{-i\omega_m \tau} - \hat{a}_m^{i*} e^{+i\omega_m \tau} \right] \sin(k_m \sigma)$$

$$\hat{I}_2 = \frac{l}{2\hat{p}_0^+} \int_0^l d\sigma \frac{(\hat{\chi}^i)^2}{(2\pi\alpha')^2}$$

$$= \frac{-\alpha'\pi^2}{l(2\pi\alpha')^2\hat{p}^+} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n\tau} - \hat{a}_n^{i*} e^{+i\omega_n\tau} \right] \frac{\delta_{nm} l}{2} \\ \times \left[\hat{a}_m^i e^{-i\omega_m\tau} - \hat{a}_m^{i*} e^{+i\omega_m\tau} \right]$$

$$= \frac{-\alpha'\pi^2}{2(2\pi\alpha')^2\hat{p}^+} \sum_{n=1}^{\infty} \left[\hat{a}_n^i e^{-i\omega_n\tau} - \hat{a}_n^{i*} e^{+i\omega_n\tau} \right]^2$$

$$\rightarrow \hat{I}_2 = \frac{-1}{8\alpha'\hat{p}^+} \sum_{n=1}^{\infty} \left[\begin{aligned} &\hat{a}_n^i \hat{a}_n^i e^{-2i\omega_n\tau} \\ &+ \hat{a}_n^{i*} \hat{a}_n^{i*} e^{+2i\omega_n\tau} \\ &- \hat{a}_n^{i*} \hat{a}_n^i - \hat{a}_n^i \hat{a}_n^{i*} \end{aligned} \right]$$

$$\hat{H}_p = \hat{I}_1 + \hat{I}_2$$

$$\rightarrow \hat{H}_p = \frac{\hat{p}^i \hat{p}_i}{2\hat{p}^+} + \frac{1}{8\alpha'\hat{p}^+} \sum_{n=1}^{\infty} \left[\begin{aligned} &\hat{a}_n^i \hat{a}_n^i e^{-2i\omega_n\tau} \\ &+ \hat{a}_n^{i*} \hat{a}_n^{i*} e^{+2i\omega_n\tau} \\ &+ \hat{a}_n^{i*} \hat{a}_n^i + \hat{a}_n^i \hat{a}_n^{i*} \end{aligned} \right]$$

$$- \frac{1}{8\alpha'\hat{p}^+} \sum_{n=1}^{\infty} \left[\begin{aligned} &\hat{a}_n^i \hat{a}_n^i e^{-2i\omega_n\tau} \\ &+ \hat{a}_n^{i*} \hat{a}_n^{i*} e^{+2i\omega_n\tau} \\ &- \hat{a}_n^{i*} \hat{a}_n^i - \hat{a}_n^i \hat{a}_n^{i*} \end{aligned} \right]$$

$$\Rightarrow \hat{H}_p = \frac{\hat{p}^i \hat{p}_i}{2\hat{p}^+} + \frac{1}{4\alpha'\hat{p}^+} \sum_{n=1}^{\infty} \left[\hat{a}_n^{i*} \hat{a}_n^i + \hat{a}_n^i \hat{a}_n^{i*} \right]$$

$$\text{Now, } [\hat{a}_n^i, \hat{a}_n^{i*}] = n \delta^{ii} = (D-2)n$$

$$\rightarrow \hat{a}_n^i \hat{a}_n^{i*} = \hat{a}_n^{i*} \hat{a}_n^i + (D-2)n$$

$$\Rightarrow \hat{H}_p = \frac{\hat{p}^i \hat{p}_i}{2 \hat{p}^+} + \frac{1}{4\alpha' \hat{p}^+} \sum_{n=1}^{\infty} [2 \hat{a}_n^{i*} \hat{a}_n^i + (D-2)n]$$

$$\Rightarrow \hat{H}_p = \frac{\hat{p}^i \hat{p}_i}{2 \hat{p}^+} + \frac{1}{2\alpha' \hat{p}^+} \sum_{n=1}^{\infty} \left[\hat{a}_n^{i*} \hat{a}_n^i + \frac{(D-2)n}{2} \right]$$

$$\begin{aligned} [\hat{H}_p, \hat{a}_m^j] &= \frac{1}{2\alpha' \hat{p}^+} \sum_{n=1}^{\infty} [\hat{a}_n^{i*} \hat{a}_n^i, \hat{a}_m^j] \\ &= \frac{-1}{2\alpha' \hat{p}^+} \sum_{n=1}^{\infty} n \delta^{ij} \delta_{nm} \hat{a}_n^i \end{aligned}$$

Thus we get,

$$[\hat{H}_p, \hat{a}_m^j] = \frac{-m \hat{a}_m^j}{2\alpha' \hat{p}^+}, \quad [\hat{H}_p, \hat{a}_m^{j*}] = \frac{m \hat{a}_m^{j*}}{2\alpha' \hat{p}^+}$$

If we define $\hat{b}_n^j = \frac{\hat{a}_n^j}{\sqrt{n}}$, then we can simplify the expressions

$$\hat{H}_P = \frac{\hat{p}^i \hat{p}_i}{2\hat{p}^+} + \frac{1}{2\alpha' \hat{p}^+} \sum_{n=1}^{\infty} \left[\frac{1}{n} \hat{b}_n^{i*} \hat{b}_n^i + \frac{(D-2)n}{2} \right]$$

where, $[\hat{b}_n^i, \hat{b}_m^{j*}] = \delta^{ij} \delta_{nm}$

$$\rightarrow [\hat{H}_P, \hat{b}_m^j] = \frac{-\hat{b}_m^j}{2\alpha' \hat{p}^+ m}$$

$$\& [\hat{H}_P, \hat{b}_m^{j*}] = \frac{\hat{b}_m^{j*}}{2\alpha' \hat{p}^+ m}$$

Now, $\hat{p}^i$ & $\hat{p}^+$ operators commute with each other & also with $\hat{b}_n^i$ & $\hat{b}_n^{i*}$.

$\hat{p}^i$ represents the transverse momenta of CoM & $\hat{p}^+$ represents the longitudinal momenta of CoM.

Thus $|k^+, k^i\rangle$ forms a continuous basis sets to represents the CoM momenta. This is the point particle degree of freedom. So, CoM acts as a point particle.

$$\hat{p}^i |k^+, k^i\rangle = k^i |k^+, k^i\rangle$$

$$\hat{p}^+ |k^+, k^i\rangle = k^+ |k^+, k^i\rangle$$

Now to account the oscillators,
Consider,

$$\hat{H}_0 = \frac{1}{2\alpha' \hat{p}^+} \sum_{n=1}^{\infty} \left[\frac{1}{n} \hat{b}_n^{j*} \hat{b}_n^j + \frac{(D-2)n}{2} \right]$$

$$\rightarrow [\hat{H}_0, \hat{b}_m^j] = \frac{-\hat{b}_m^j}{2\alpha' \hat{p}^+ m}$$

$$\& [\hat{H}_0, \hat{b}_m^{j*}] = \frac{\hat{b}_m^{j*}}{2\alpha' \hat{p}^+ m}$$

$$\rightarrow \text{If } \hat{H}_0 |N_m^j; k^+, k^i\rangle = E_{N_m^j}^j |N_m^j; k^+, k^i\rangle$$

$$\text{Then, } \hat{H}_0 \hat{b}_m^j |N_m^j; k^+, k^i\rangle$$

$$= \left[E_{N_m^j}^j - \frac{1}{2\alpha' k^+ m} \right] |N_m^j; k^+, k^i\rangle$$

$$\& \hat{H}_0 \hat{b}_m^{j*} |N_m^j; k^+, k^i\rangle$$

$$= \left[E_{N_m^j}^j + \frac{1}{2\alpha' k^+ m} \right] |N_m^j; k^+, k^i\rangle$$

Thus $\hat{b}_m^{j*}$ increases the value of energy of the m^{th} mode along j^{th} direction by $(2\alpha' k^+ m)^{-1}$
And $\hat{b}_m^j$ decreases the value of energy of the m^{th} mode along j^{th} direction by $(2\alpha' k^+ m)^{-1}$

Thus we can create a Fock Space basis for each mode & each direction as $|\{N_m^j\}\rangle$ whose expanded version is,

$$\left| N_{m=1}^{j=2}, N_{m=1}^{j=3}, \dots, N_{m=1}^{j=D-1}; N_{m=2}^{j=2}, N_{m=2}^{j=3}, \dots, N_{m=2}^{j=D-1}; \right. \\ \left. N_{m=n}^{j=2}, N_{m=n}^{j=3}, \dots, N_{m=n}^{j=D-1}; \dots \right\rangle$$

where N_m^j is the occupation number of the oscillator in j^{th} direction at m^{th} mode.

But to describe $\hat{H}_p = \hat{H}_{\text{com}} + \hat{H}_0$ we need to use the complete basis given by,

$$|\{N_m^j\}; \{k^+, k^i\}\rangle$$

The state $|0; \{k^+, k^i\}\rangle$ represents the zero excitation of the string's vibration mode.

$$\rightarrow \underline{\hat{b}_m^j |0; \{k^+, k^i\}\rangle = 0}$$

Then by the usual operator algebra, we can construct any General State $|\{N_m^j\}; \{k^+, k^i\}\rangle$ by

$$\prod_{j=2}^{D-1} \prod_{n=1}^{\infty} \frac{(\hat{\alpha}_m^j)^{N_m^j}}{\sqrt{(N_m^j)!}} |0; \{k^+, k^i\}\rangle$$

These oscillators represent an infinite degree of internal freedom of the string. Every choice of these occupation numbers corresponds to a different particle / spin state from the point of view of the space-time.

The state $|0; 0\rangle$ denotes a string state with zero CoM momentum (fixed CoM) & zero excitations in the transverse modes. It is the ground state of the single string Hamiltonian given by,

$$\hat{H}_p = \frac{\hat{p}^i \hat{p}_i}{2p^+} + \frac{1}{2\alpha' p^+} \sum_{n=1}^{\infty} \left[\hat{a}_n^{i*} \hat{a}_n^i + \frac{(D-2)n}{2} \right]$$

The various operators defined previously are operators that act on the single string Fock state. They do not destroy or create strings in the space-time.

A single string Hilbert space is got by the span of $\{|\{N_m^i\}; \{k^+, k^i\}\rangle\}$, where $|0;0\rangle$ represent the ground state of the single string Hamiltonian. It is not the vacuum state (zero string state).

Then the n -string Hilbert space is got by making ' n ' copies of the single string Hilbert space & the vacuum space $\{|\Omega\rangle\}$ by direct sum.
$$\mathbb{H}^{(n)} = \{|\Omega\rangle\} \oplus \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \mathbb{H}_3 \cdots \oplus \mathbb{H}_n$$

Here we described the strings by operators that have fundamental 'commutation' relations. So they are Bosonic strings and have integer spins. Thus the multistring wave function has to be symmetrized.

Now, the Hamiltonian is given by,

$$\rightarrow \hat{H}_p = \frac{\hat{p}^i \hat{p}_i}{2p^+} + \frac{1}{2\alpha' p^+} \left[\sum_{n=1}^{\infty} \hat{a}_n^{i*} \hat{a}_n^i + A \right]$$

Where $A = \sum_{n=1}^{\infty} \frac{(D-2)n}{2}$. Now, let us try to find,

$$A = \sum_{n=1}^{\infty} \frac{(D-2)n}{2} = \frac{(D-2)}{2} \sum_{n=1}^{\infty} n$$

Logic says that $\sum_{n=1}^{\infty} n = \infty$. But Ramanujan proved it to equal to $-1/12$.

$$\begin{aligned} \text{Consider } \lim_{\Lambda \rightarrow \infty} \sum_{n=1}^{\Lambda} n &= \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \sum_{n=1}^{\Lambda} n e^{-\epsilon n} \\ &= - \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{\partial}{\partial \epsilon} \sum_{n=1}^{\Lambda} e^{-\epsilon n} \end{aligned}$$

$$\text{use } \sum_{n=1}^{\Lambda} e^{-\epsilon n} = \frac{1 - e^{-\Lambda \epsilon}}{(e^{\epsilon} - 1)}$$

$$\Rightarrow \lim_{\Lambda \rightarrow \infty} \sum_{n=1}^{\Lambda} n = \lim_{\Lambda \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} \frac{e^{-\Lambda \varepsilon} - 1}{(e^{\varepsilon} - 1)}$$

$$\Rightarrow \sum_{n=1}^{\infty} n = \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} \frac{-1}{(e^{\varepsilon} - 1)}$$

$$\Rightarrow \sum_{n=1}^{\infty} n = \lim_{\varepsilon \rightarrow 0} \underbrace{\frac{e^{\varepsilon}}{(e^{\varepsilon} - 1)^2}}_{\text{Taylor Series}}$$

$$\Rightarrow \sum_{n=1}^{\infty} n = \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\varepsilon^2} - \frac{1}{12} + \frac{\varepsilon^2}{240} + \mathcal{O}(\varepsilon^4) \right]$$

Thus we see that as we take the lim of $\varepsilon \rightarrow 0$, the $\frac{1}{\varepsilon^2}$ term diverges.

But we can actually introduce a Counter term in the Hamiltonian to remove this divergence. This is a necessity to maintain Weyl Invariance.

Thus we shall just replace $\sum_{n=1}^{\infty} n \rightarrow -\frac{1}{12}$

This gives the Zero point energy to be proportional to $A = \frac{(2-D)}{24}$

$$\rightarrow \hat{H}_P = \frac{\hat{p}^i \hat{p}_i}{2p^+} + \frac{1}{2\alpha' p^+} \left[\sum_{n=1}^{\infty} \hat{a}_n^{i*} \hat{a}_n^i + \frac{(2-D)}{24} \right]$$

This remainder even after regularization is the Casimir Energy coming from the fact that the String has finite length.

Now, due to the transverse oscillator energy of the String, the effective mass of the String depends on the number of excitations in the String.

For the Point particle like CoM, we have $\hat{p}^- = \hat{H}_P$ in the light Cone gauge. Since $X^+(\tau) = \tau$ is the time coordinate of the world Sheet.

$$\Rightarrow 2\hat{p}^+\hat{p}^- = \hat{p}^i\hat{p}_i + \frac{1}{\alpha'} \left[\sum_{n=1}^{\infty} a_n^{i*} a_n^i + \frac{(2-D)}{24} \right]$$

Then, for any state $|\{N_m^j\}; \{p^+, p^i\}\rangle$,
we get the eigen values to relate by,

$$2p^+p^- = p^i p_i + \frac{1}{\alpha'} \left[N + \frac{(2-D)}{24} \right]$$

where N is given by,

$$\begin{aligned} & \langle \{N_m^j\}; \{p^+, p^i\} | \sum_{n=1}^{\infty} a_n^{i*} a_n^i | \{N_m^j\}; \{p^+, p^i\} \rangle \\ &= \sum_{n=1}^{\infty} \langle \{N_m^j\}; \{p^+, p^i\} | n \hat{b}_n^{i*} \hat{b}_n^i | \{N_m^j\}; \{p^+, p^i\} \rangle \end{aligned}$$

Since $a_n^{i*} a_n^i = n \hat{b}_n^{i*} \hat{b}_n^i$

&

$$\langle \{N_m^j\}; \{p^+, p^i\} | \hat{b}_n^{i*} \hat{b}_n^i | \{N_m^j\}; \{p^+, p^i\} \rangle = \sum_{j=2}^{D-1} N_n^j$$

$$\Rightarrow \underline{N = \sum_{n=1}^{\infty} \sum_{j=2}^{D-1} n N_n^j} \quad \text{is the level of excitations}$$

$$\text{So, } 2p^+p^- = p^i p_i + \frac{1}{\alpha'} \left[N + \frac{(2-D)}{24} \right]$$

$$\Rightarrow (p^0 + p^1)(p^0 - p^1) - (p^i)^2 = \frac{1}{\alpha'} \left[N + \frac{(2-D)}{24} \right]$$

Thus,

$$\underbrace{(p^0)^2 - (p^1)^2 - (p^2)^2 - \dots - (p^{D-1})^2}_{-p^\mu p_\mu = M^2} = \frac{1}{\alpha'} \left[N + \frac{(2-D)}{24} \right]$$

$$\Rightarrow \underline{\underline{M^2 = \frac{1}{\alpha'} \left[N + \frac{(2-D)}{24} \right]}} \rightarrow \text{Mass Squared}$$

Thus each state $|\{N_m^i\}; \{p^+, p^i\}\rangle$ is associated with some mass related to the level of excitation of the string.

The Ground state $|0; \{p^+, p^i\}\rangle$ has $N=0$.

$$\Rightarrow \underline{\underline{M^2 = \frac{(2-D)}{24\alpha'}}$$

If $D > 2$, then M^2 is -ve. Which implies the state has imaginary mass which corresponds to tachyons, which always travels at speed greater than that of light.

The lowest excited state has $N = 1$. But there are $D - 2$ states which can give $N = 1$, which have only one excitation only along one direction.

$$\rightarrow M^2 = \frac{1}{\alpha'} \left[1 + \frac{(2-D)}{24} \right]$$

So 1st excited state has,

$$\underline{M^2 = \frac{26-D}{24\alpha'}}$$

Thus the mass of the string depends on the dimensionality of Space-time.

The choice of light-cone gauge has removed the Lorentz invariance of our theory, as two directions $x^0 \neq x^1$ are treated differently. But still the theory has the symmetries of sub-group of Lorentz group.

For a massive particle, we can go to its rest frame, $P^\mu = (m, 0, 0, \dots, 0)$ & then the internal states have $SO(D-1)$ symmetry (rot. in ' $D-1$ ' dimensional space).

But for a massless particle, rest frame doesn't exist. Since $P_\mu P^\mu = 0$, we can choose a frame where $P_\mu = (E, E, 0, \dots, 0)$. Thus, internal states have $SO(D-2)$ symmetry of rot. in ' $D-2$ ' dimensional space. Here $SO(D-2)$ acting on transverse directions leaves p^μ invariant.

These results are familiar in $D=4$, where photon has only 2-helicity degree of freedom.

But massive particles have $SO(3)$ symmetry & have spin labelled by 'j' & has $2j+1$ spin states.

Thus generalizing this to D -dimensions, a massive vector particle has $SO(D-1)$ symmetry & massless vector particle has only $SO(D-2)$ symmetry.

In our quantized bosonic string, at the 1st level ($N=1$), we got only $D-2$ states got by $\hat{\alpha}_m^j |0; \{k^+, k^i\}\rangle$

($j = 2, 3, 4, \dots, D-1$). $\Rightarrow D-2$ transverse directions

Thus the internal space of states has only $SO(D-2)$ symmetry. Hence these have to describe a massless particle to have the complete $SO(1, D-1)$ Lorentz symmetry. $\Rightarrow M=0$ is needed at 1st level.

$$\Rightarrow M^2 = \frac{26-D}{24\alpha'} = 0 \Rightarrow \underline{\underline{D=26}}$$

Hence for 1st level to have the full Lorentz Symmetry, we must have the dimensions of Space-time to be, $D=26$. Only then the Open String Spectrum has Lorentz invariance at first level $N=1$.

The Spin generators for the $SO(D-2)$ transformation are given by,

$$\hat{S}^{ij} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left[\hat{b}_n^{*i} \hat{b}_n^j - \hat{b}_n^{*j} \hat{b}_n^i \right]$$

For massless particle, the $SO(D-2)$ Spin Symmetry is got by choosing the momentum of string to be along the 1-direction singled out during light-cone quantization.

But in excited states where $N=2$ or higher, massive particles are allowed.

For $D=26$

$$M^2 = \frac{1}{\alpha'} \left[N + \frac{(2-D)}{24} \right] = \frac{1}{\alpha'} [N-1]$$

$$\rightarrow M^2 = \frac{N-1}{\alpha'}$$

Thus $M^2 = 0, \frac{1}{\alpha'}, \frac{2}{\alpha'}, \dots$
massless $\swarrow$
Excited State Massive Excited States

Consider the States of the open string at $N=2$, whose $m^2 = \frac{1}{\alpha'}$, are of two types given by

$$\hat{a}_1^{i*} \hat{a}_1^{j*} |0; \{p^+, p^k\}\rangle \neq \hat{a}_2^{j*} |0\rangle$$

where $i, j, k, l = \{2, 3, 4, \dots, D-1\}$, $D=26$

The 1st type of States have 1 excitation in two different directions each.

The 2nd type of states have 2 excitations in one direction.

Due to bosonic symmetry

$$\hat{a}_1^{i*} \hat{a}_1^{j*} |0; \{p^+, p^k\}\rangle = \hat{a}_1^{j*} \hat{a}_1^{i*} |0; \{p^+, p^k\}\rangle$$

Thus these states form a symmetric rank two tensor repⁿ of $SO(D-2)$.

$$\text{Let } T^{ij} = \hat{a}_1^{i*} \hat{a}_1^{j*} |0; \{p^+, p^k\}\rangle$$

$$\text{now } T^{ij} = \left(T^{ij} - \frac{\delta^{ij} T^k_k}{(D-2)} \right) + \frac{\delta^{ij} T^k_k}{(D-2)}$$

$$\text{Let } U^{ij} = T^{ij} - \frac{\delta^{ij} T^k_k}{(D-2)}$$

U^{ij} is a symmetric traceless rank two tensor.

$$\Rightarrow T^{ij} = U^{ij} + \frac{\delta^{ij} \text{Trace}(T)}{D-2}$$

Now T^{ij} has $\frac{(D-2)(D-2+1)}{2}$

independent States. & Trace(T) part has just one independent State.

→ U^{ij} has $\frac{(D-2)(D-1)-2}{2} = \frac{D^2-3D}{2}$ states.

→ No: of States in T^{ij} is given by $\frac{D(D-3)}{2} + 1$

The second type of State $\hat{a}_2^{j*}|0\rangle$ for a Vector in $SO(D-2)$ which has $D-2$ States in it.

⇒ Total no: of States are given by
rank 2 traceless repⁿ of $SO(D-2)$
+ rank 1 repⁿ of $SO(D-2)$
+ rank 0 scalar repⁿ of $SO(D-2)$
 $= \left[\frac{D(D-3)}{2} \right] + (D-2) + 1$

Now, consider the traceless Symmetric repⁿ of $SO(D-1)$ given by G^{mn} , where $m, n = 1, 2, 3, \dots, D-1$.

$$\text{States of } G^{mn} = \frac{(D-1)D}{2} - 1$$

$\underbrace{\hspace{10em}}_{SO(N): \frac{N(N+1)}{2}} \quad \downarrow \text{tracelessness}$

$$\rightarrow G^{mn} \text{ has } \frac{D^2 - D - 2}{2} \text{ states.}$$

Since G^{mn} is traceless, we can write $G^{11} = -\delta_{ij} G^{ij}$. This will transform as a scalar under $SO(D-2)$.

$$\text{Consider } G^{ij} - \frac{1}{D-2} \delta^{ij} \delta_{kl} G^{kl}$$

This transforms as a traceless Symmetric tensor of $SO(D-2)$.

G^{1j} transform as a vector under $SO(D-2)$.

$$\begin{bmatrix} G_1^{11} & G_1^{12} & G_1^{13} & \dots & G_1^{1,D-1} \\ G_1^{12} & G_1^{22} & G_1^{23} & \dots & G_1^{2,D-1} \\ G_1^{13} & G_1^{23} & G_1^{33} & \dots & G_1^{3,D-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_1^{1,D-1} & G_1^{2,D-1} & G_1^{3,D-1} & \dots & G_1^{D-1,D-1} \end{bmatrix}$$

$G_1^{11} \rightarrow \text{Scalar in } SO(D-2)$

$G_1^{j1} = G_1^{1j} \Rightarrow \text{Vector in } SO(D-2)$

$G_1^{ij} - \frac{1}{D-2} \delta^{ij} \delta_{kl} G_1^{kl} \Rightarrow \text{Symmetric traceless tensor of } SO(D-2)$

Thus $T^{ij} = \hat{a}_1^{i*} \hat{a}_1^{j*} |0; \{p^+, p^k\}\rangle$

encloses a scalar of $SO(D-2)$: Trace,
 $\neq$ Symmetric traceless tensor of $SO(D-2)$
 $\hat{a}_2^{j*} |0\rangle$ is a vector of $SO(D-2)$.

Thus all possible 2nd level states will give a traceless symmetric rank two representation of $SO(D-1)$. Hence this works out in our theory & can exist.

Consider the states of the open string at $N=3$, whose $M^2 = \frac{2}{\alpha'}$, are of two types given by

$$T^{ijk} = \hat{a}_1^{i*} \hat{a}_1^{j*} \hat{a}_1^{k*} |0; \{p^+, p^k\}\rangle$$

$$M^{ij} = \hat{a}_2^{i*} \hat{a}_1^{j*} |0; \{p^+, p^k\}\rangle \neq V^j = \hat{a}_3^{j*} |0\rangle$$

T^{ijk} is a completely symmetric rank 3 tensor. M^{ij} is neither symmetric nor anti-symmetric. V^j is a rank-1 (vector) of $SO(D-2)$.

$$T^{ijk} = \underbrace{\left[T^{ijk} - \frac{\partial^{ij} \partial_{lh} T^{lhk}}{D-2} \right]}_{S^{ijk}} + \underbrace{\frac{\partial^{ij} \partial_{lh} T^{lhk}}{D-2}}_{\partial^{ij} W^k}$$

$$\rightarrow T^{ijk} = \underbrace{S^{ijk}}_{\substack{\text{traceless} \\ \text{Symmetric rank-3} \\ \text{tensor of } SO(D-2)}} + \underbrace{\partial^{ij} W^k}_{\text{Vector of } SO(D-2)}$$

$$M^{ij} = \underbrace{\frac{M^{ij} + M^{ji}}{2}}_{\text{Symmetric}} + \underbrace{\frac{M^{ij} - M^{ji}}{2}}_{\text{Anti Symmetric}}$$

$$\frac{M^{ij} + M^{ji}}{2} = \frac{M^{ij} + M^{ji}}{2} - \frac{\delta^{ij} \text{Trace}(M)}{(D-2)} + \frac{\delta^{ij} \text{Trace}(M)}{(D-2)}$$

$$\text{Let } \Phi = \frac{\text{Trace}(M)}{(D-2)}$$

$$U^{ij} = \frac{M^{ij} + M^{ji}}{2} - \frac{\delta^{ij} \text{Trace}(M)}{(D-2)}$$

$$A^{ij} = \frac{M^{ij} - M^{ji}}{2}$$

$$\Rightarrow M^{ij} = U^{ij} + A^{ij} + \delta^{ij} \Phi$$

Traceless
Symmetric
rank 2
tensor of
 $SO(D-2)$

↓
Anti-Symmetric
rank 2 tensor
of $SO(D-2)$

↘ Scalar of
 $SO(D-2)$

Thus $N=3$ level consists of

$$\{S^{ijk}, W^k, U^{ij}, A^{ij}, \Phi, V^j\}$$

It has rank-3 symmetric traceless tensor + two Vectors + rank 2 symmetric traceless tensor + rank 2 anti-symmetric tensor + a scalar.

$\{A^{ij}, V^j\}$ forms a Anti-symmetric rank-2 tensor ' B^{mn} ' of $SO(D-1)$

$$B^{mn} = \{B^{ij} \approx A^{ij}, B^{1i} = -B^{i1} = V^i\}$$

$\nexists \{U^{ij}, W^k, \Phi\}$ forms a Symmetric traceless 3-tensor Z^{mnq} of $SO(D-1)$

$$Z^{mnq} = \left\{ \begin{array}{l} Z^{ij1} = Z^{1ij} = Z^{i1j} \approx U^{ij}, \\ Z^{11j} = Z^{1j1} = Z^{j11} \approx W^j, \\ Z^{111} \approx \Phi \end{array} \right\}$$