

## Mode expansion for $\chi^i(\tau, \sigma)$

Even here we shall make use of the light cone gauge, where we set,

$$\chi^+(\tau) = \tau, \quad \partial_\sigma \gamma_{\sigma\sigma} = 0 \quad \& \quad \det(\gamma_{ab}) = -1$$

Then we get exactly the same results for the Polyakov lagrangian, Hamiltonian, Eq<sup>n</sup> of motions, as that of Open strings.

$$L_P = \frac{-l \gamma_{\sigma\sigma}}{2\pi\alpha'} \partial_\tau x^-(\tau)$$

$$+ \frac{1}{4\pi\alpha'} \int_0^l d\sigma \left[ \gamma_{\sigma\sigma} \dot{\chi}^i \dot{\chi}_i - (\gamma_{\sigma\sigma})^{-1} \dot{\chi}^i \dot{\chi}_i \right]$$

$$\rightarrow p_- = -p^+ = \frac{\partial L_P}{\partial \dot{x}^-} = \frac{-l \gamma_{\sigma\sigma}}{2\pi\alpha'}$$

$$\rightarrow \underline{\gamma_{\sigma\sigma} = \frac{2\pi\alpha'}{l} p^+} \quad \left. \vphantom{\gamma_{\sigma\sigma}} \right\} \text{Constraint on } \gamma_{\sigma\sigma}$$

In closed strings, there is an extra degree of freedom in ' $\sigma$ ' translations.

$$\sigma' = \sigma + s(\tau)_{\text{mod}(l)}$$

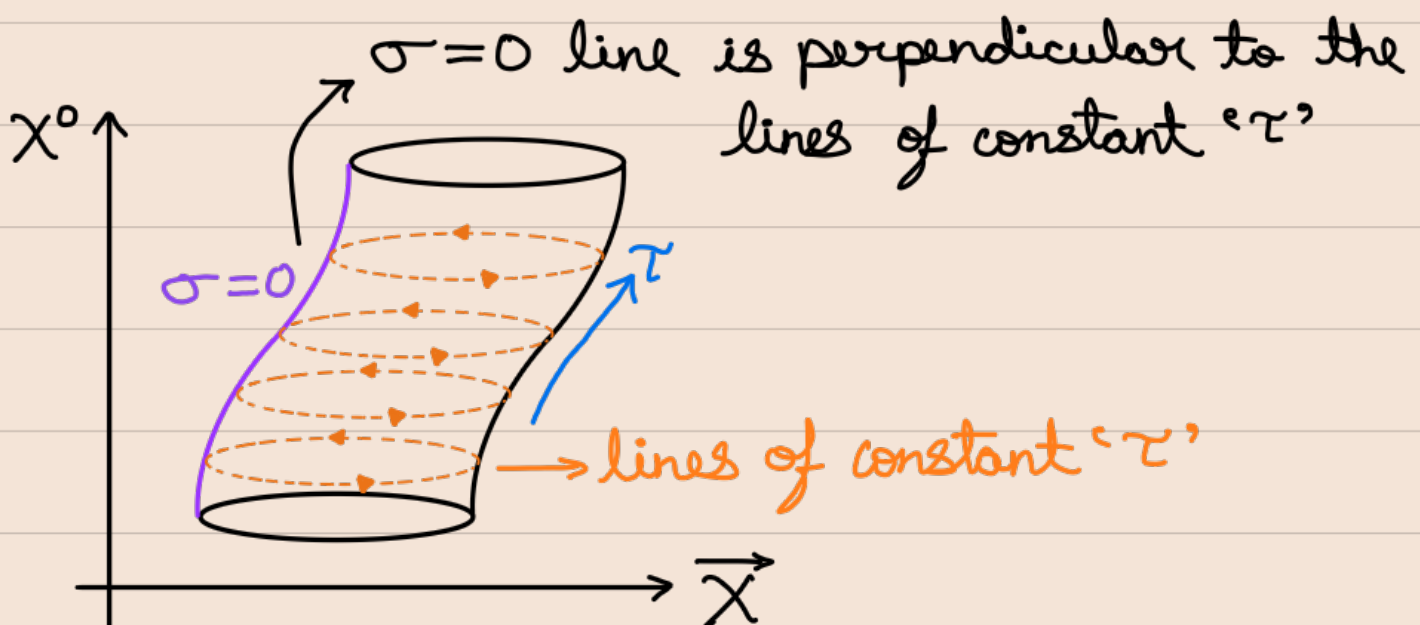
Because the point  $\sigma=0$  can be chosen anywhere in the world sheet, at any time ( $\tau$ ) value.

So we shall choose ' $\sigma$ ' lines such that the line ' $\sigma=0$ ' is orthogonal to all the lines of constant ' $\tau$ '. By

Setting

$$\gamma^{\tau\sigma}(\tau, \sigma=0) = \gamma^{\tau\sigma}(\tau, \sigma=l) = 0$$

This results in the following,



$$\text{Then } \pi_i = \frac{\partial L_P}{\partial \dot{x}^i} = \frac{1}{4\pi\alpha'} (2\gamma_{\sigma\sigma} \dot{x}_i)$$

$$\rightarrow \pi^i = \frac{\dot{x}^i p^+}{l}$$


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Then, the Hamiltonian is

$$H_P = p_- \dot{x}^- + \int_0^l d\sigma \pi_i \dot{x}^i - L_P$$

$$\rightarrow H_P = \frac{l}{2p^+} \int_0^l d\sigma \left[ \pi^i \pi_i + \frac{\dot{x}^i \dot{x}_i}{(2\pi\alpha')^2} \right]$$

This hamiltonian describes the  $D-2$  fields  $x^2, x^3, \dots, x^{D-1}$  as waves in the world sheet with speed  $c = \frac{l}{2\pi\alpha' p^+}$

$$\Rightarrow \boxed{\frac{\partial^2 x^i}{\partial \tau^2} = c^2 \frac{\partial^2 x^i}{\partial \sigma^2}}$$

Now, to describe the closed strings, the periodic boundary conditions should be used.

Periodic boundary conditions are given by

$$\underline{\chi^i(\tau, 0) = \chi^i(\tau, l)}, \quad \gamma_{ab}(\tau, 0) = \gamma_{ab}(\tau, l)$$

$$\& \quad \partial^a \chi^i \Big|_{\sigma=0} = \partial^a \chi^i \Big|_{\sigma=l}$$

$$\rightarrow (\gamma_{\sigma\sigma})^{-1} \partial_{\sigma} \chi^i \Big|_{\sigma=0} = (\gamma_{\sigma\sigma})^{-1} \partial_{\sigma} \chi^i \Big|_{\sigma=l}$$

$$\rightarrow \underline{\dot{\chi}^i(\tau, 0) = \dot{\chi}^i(\tau, l)}$$

A general sol<sup>n</sup> as a fourier expansion is given by,

$$\chi^i(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n^i \phi_n(\sigma) e^{-i\omega_n \tau}$$

where  $c_n^i$  are fourier coefficients  $\phi_n(\sigma)$  are spatial functions which satisfy the boundary conditions.  $\omega_n$  are mode frequencies.

$$\chi^i \text{ satisfies } \frac{\partial^2 \chi^i}{\partial \tau^2} = c^2 \frac{\partial^2 \chi^i}{\partial \sigma^2}$$

$$\rightarrow \sum_{n=-\infty}^{\infty} -\omega_n^2 c_n^i \phi_n(\sigma) e^{-i\omega_n \tau}$$

$$= \sum_{n=-\infty}^{\infty} c^2 c_n^i \phi_n''(\sigma) e^{-i\omega_n \tau}$$

$$\rightarrow \sum_{n=-\infty}^{\infty} \left[ \phi_n''(\sigma) + \frac{\omega_n^2}{c^2} \phi_n(\sigma) \right] c_n^j e^{-i\omega_n \tau} c^2 = 0$$

$\rightarrow \phi_n(\sigma)$  should satisfy,

$$\phi_n''(\sigma) + k_n^2 \phi_n(\sigma) = 0, \text{ where } k_n = \frac{\omega_n}{c}$$

$$\rightarrow \phi_n(\sigma) = A_n e^{-ik_n \sigma} + B_n e^{ik_n \sigma}$$

$$\rightarrow \partial_\sigma \phi_n = -ik_n A_n e^{-ik_n \sigma} + ik_n B_n e^{ik_n \sigma}$$

Boundary conditions give

$$A_n e^{-ik_n l} + B_n e^{ik_n l} = A_n + B_n$$

$$-A_n e^{-ik_n l} + B_n e^{ik_n l} = -A_n + B_n$$

It is satisfied only for  $e^{\pm ik_n l} = 1$

$$\Rightarrow \boxed{k_n = \frac{2n\pi}{l}} \text{ \& } \boxed{\omega_n = \frac{2n\pi c}{l}}$$

$$\rightarrow \chi^j(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n^j A_n e^{-i(k_n \sigma + \omega_n \tau)} + c_n^j B_n e^{i(k_n \sigma - \omega_n \tau)}$$

$$\text{Let } \alpha_n^j = c_n^j A_n \neq \tilde{\alpha}_n^j = c_n^j B_n$$

$$\rightarrow \chi^j(\tau, \sigma) = \sum_{n=-\infty}^{\infty} \alpha_n^j e^{-i(k_n \sigma + \omega_n \tau)} + \tilde{\alpha}_n^j e^{i(k_n \sigma - \omega_n \tau)}$$

$$\text{Since } \chi^j(\tau, \sigma) = [\chi^j(\tau, \sigma)]^*$$

$$\rightarrow \alpha_n^{j*} = \alpha_{-n}^j \neq \tilde{\alpha}_n^{j*} = \tilde{\alpha}_{-n}^j$$

$$\text{For } n=0, \phi_0(\sigma) = 0 \text{ as } \omega_0 = 0$$

$$\rightarrow \frac{d^2 \chi_{(n=0)}^j}{d\tau^2} = 0 \Rightarrow \chi_{n=0}^j = d_0^j + \tilde{d}_0^j \tau$$

The zero mode terms are independent of ' $\sigma$ '. It implies that all string parts move together in the ' $\tau$ ' direction. So it describes the center of mass motion.

$\Rightarrow d_0^j$  should relate to the initial location of the center of mass  $\neq \tilde{d}_0^j$  should relate to the center of mass momentum.

$$\rightarrow \chi^j(\tau, \sigma) = \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \alpha_n^j e^{-i(k_n \sigma + \omega_n \tau)} + \tilde{\alpha}_n^j e^{i(k_n \sigma - \omega_n \tau)} \right] + d_0^j + \tilde{d}_0^j \tau$$

The CoM is  $R^i(\tau) = \frac{1}{l} \int_0^l \chi^i(\tau, \sigma) d\sigma$

$$\int_0^l e^{\pm i k_n \sigma} d\sigma = \frac{e^{\pm i k_n \sigma}}{\pm i k_n} \Big|_{\sigma=0}^{\sigma=l} = \frac{1-1}{\pm i k_n} = 0$$

$$\rightarrow R^i(\tau) = d_0^j + \tilde{d}_0^j \tau \rightarrow R^i(\tau) = \underbrace{R^i(0)}_{\text{Free particle Uniform motion}} + \tilde{d}_0^j \tau$$

Let  $\chi^i = R^i(0)$   
(Initial CoM location)

Free particle  
Uniform motion

Now,  $P^i(\tau) = \int_0^l d\sigma \pi^i(\tau, \sigma)$  is the CoM momenta. But,

$$\partial_\tau \chi^i(\tau, \sigma) = \frac{l \pi^i}{p^+}$$

$$\rightarrow P^i(\tau) = \frac{p^+}{l} \int_0^l d\sigma \partial_\tau X^i(\tau, \sigma) = \frac{p^+}{l} \int_0^l d\sigma \tilde{\alpha}_0^i$$

$$\rightarrow P^i(\tau) = p^+ \tilde{\alpha}_0^i \rightarrow \text{Independent of } \tau$$

Thus even the center of mass momenta is conserved quantity.

For later convenience, we shall change  $\alpha_n^i$  &  $\tilde{\alpha}_n^i$  such that,

$$\alpha_n^i \rightarrow \frac{i}{n} \sqrt{\frac{\alpha'}{2}} \alpha_n^i \quad \& \quad \tilde{\alpha}_n^i \rightarrow \frac{i}{n} \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}_n^i$$

Thus,

$$\begin{aligned} X^j(\tau, \sigma) = & x^j + \frac{p^j}{p^+} \tau \\ & + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \frac{\alpha_n^j}{n} e^{-i(k_n \sigma + \omega_n \tau)} + \frac{\tilde{\alpha}_n^j}{n} e^{i(k_n \sigma - \omega_n \tau)} \right] \end{aligned}$$

$$\text{where } k_n = \frac{2\pi n}{l} \quad \& \quad \omega_n = \frac{2\pi n c}{l}$$

$$\text{Since, } \Pi^i = \frac{p^+}{l} \partial_\tau \chi^i(\tau, \sigma)$$

$$= \frac{p^i}{l} + \frac{p^+}{l} \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \omega_n \frac{\alpha_n^j}{n} e^{-i(k_n \sigma + \omega_n \tau)} + \omega_n \frac{\tilde{\alpha}_n^j}{n} e^{i(k_n \sigma - \omega_n \tau)} \right]$$

$$= \frac{p^i}{l} + \frac{2\pi c p^+}{l^2} \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \alpha_n^j e^{-i(k_n \sigma + \omega_n \tau)} + \tilde{\alpha}_n^j e^{i(k_n \sigma - \omega_n \tau)} \right]$$

$$\text{Since } c = \frac{l}{2\pi \alpha' p^+},$$

$$\Pi^i = \frac{p^i}{l} + \frac{1}{l \sqrt{2\alpha'}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \alpha_n^j e^{-i(k_n \sigma + \omega_n \tau)} + \tilde{\alpha}_n^j e^{i(k_n \sigma - \omega_n \tau)} \right]$$

Thus in Closed strings, there are two independent oscillating components for every 'j' & 'n'. ' $\alpha_n^j$ ' is for left moving waves & ' $\tilde{\alpha}_n^j$ ' is for right moving waves along the string.

## Quantization of Closed Strings

The canonical variables are,  $(x^-, p^+)$   
 $(x^i, p^+)$ ,  $(X^i, \Pi^i)$ ,  $(\alpha_n^j, \alpha_n^{j*})$  &  
 $(\tilde{\alpha}_n^j, \tilde{\alpha}_n^{j*})$ . To quantize the closed strings  
we shall impose equal time canonical  
commutation relations b/w the canonically  
conjugate variables,

$$[\hat{x}^-, \hat{p}^-] = i \rightarrow [\hat{x}^-, \hat{p}^+] = -i$$

$$[\hat{x}^j, \hat{p}^k] = i \delta^{jk}$$

$$[\hat{X}^j(\tau, \sigma), \hat{\Pi}^k(\tau, \sigma')] = i \delta^{jk} \delta(\sigma - \sigma')$$

For the canonical commutation relations  
to hold the fourier components  $\hat{\alpha}_n^j$ 's should  
satisfy  $[\alpha_n^j, \alpha_m^{k*}] = n \delta^{jk} \delta_{nm}$

$$\& [\tilde{\alpha}_n^j, \tilde{\alpha}_m^{k*}] = n \delta^{jk} \delta_{nm}$$

& all other commutators are zero.

$$\rightarrow \left[ \frac{\alpha_n^j}{\sqrt{n}}, \frac{\alpha_m^{k*}}{\sqrt{m}} \right] = \delta^{jk} \delta_{nm}$$

$$\nexists \left[ \frac{\tilde{\alpha}_n^j}{\sqrt{n}}, \frac{\tilde{\alpha}_m^{k*}}{\sqrt{m}} \right] = \delta^{jk} \delta_{nm} \left. \vphantom{\begin{matrix} \rightarrow \\ \nexists \end{matrix}} \right\} \begin{matrix} \text{independent} \\ \text{Harmonic} \\ \text{oscillators} \end{matrix}$$

Thus for any transverse direction 'i'  $\nexists$  for each mode 'n', the modes will satisfy the Harmonic oscillator algebra but with a non standard normalization.

Substituting the mode expansions in the Hamiltonian, we get,

$$\hat{H}_p = \frac{l}{2\hat{p}^+} \int_0^l d\sigma \left[ \hat{\Pi}^i \hat{\Pi}_i + \frac{\hat{\chi}^i \hat{\chi}_i}{(2\pi\alpha')^2} \right]$$

$$\text{where, } \frac{1}{\hat{p}^+} = (\hat{p}^+)^{-1}$$

$$\hat{\Pi}^i = \frac{\hat{p}^i}{l} + \frac{1}{l\sqrt{2\alpha'}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \hat{\alpha}_n^j e^{-i(k_n\sigma + \omega_n\tau)} + \hat{\tilde{\alpha}}_n^j e^{i(k_n\sigma - \omega_n\tau)} \right]$$

$$\hat{\Pi}^j \hat{\Pi}_j = \frac{\hat{p}^j \hat{p}_j}{l^2} + \frac{2\hat{p}_j}{l^2 \sqrt{2\alpha'}} \sum_n \left[ \hat{\alpha}_n^j e^{-i(k_n \sigma + \omega_n \tau)} + \hat{\tilde{\alpha}}_n^j e^{i(k_n \sigma - \omega_n \tau)} \right]$$

$$+ \frac{1}{2\alpha' l^2} \sum_{n,m} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jm} e^{-i(k_n + k_m)\sigma - i(\omega_n + \omega_m)\tau} + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jm} e^{i(k_n + k_m)\sigma - i(\omega_n + \omega_m)\tau} + 2\hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jm} e^{i(k_m - k_n)\sigma - i(\omega_n + \omega_m)\tau} \right]$$

$$\Rightarrow \frac{l}{2\hat{p}^+} \int_0^l d\sigma \hat{\Pi}^i \hat{\Pi}_i \text{ is given by,}$$

$$\frac{\hat{p}^j \hat{p}_j}{2\hat{p}^+} + \frac{\hat{p}_j}{l\sqrt{2\alpha'}\hat{p}^+} \sum_n \left[ \hat{\alpha}_n^j \int_0^l d\sigma e^{-ik_n \sigma} e^{-i\omega_n \tau} + \hat{\beta}_n^j \int_0^l d\sigma e^{ik_n \sigma} e^{-i\omega_n \tau} \right]$$

$$+ \frac{1}{4\hat{p}^+ \alpha' l} \sum_{n,m} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jm} \int_0^l d\sigma e^{-i(k_n + k_m)\sigma} e^{-i(\omega_n + \omega_m)\tau} + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jm} \int_0^l d\sigma e^{i(k_n + k_m)\sigma} e^{-i(\omega_n + \omega_m)\tau} + 2\hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jm} \int_0^l d\sigma e^{i(k_m - k_n)\sigma} e^{-i(\omega_n + \omega_m)\tau} \right]$$

$$\int_0^l d\sigma e^{-i(k_n + k_m)\sigma} = l \delta_{m, -n}$$

$$\int_0^l d\sigma e^{-i(k_n - k_m)\sigma} = l \delta_{mn}$$

$$\int_0^l d\sigma e^{\pm i k_n \sigma} = l \delta_{n0}$$

$$\Rightarrow \frac{l}{2\hat{p}^+} \int_0^l d\sigma \hat{\Pi}^i \hat{\Pi}_i = \frac{\hat{p}^j \hat{p}_j}{2\hat{p}^+}$$

$$+ \frac{1}{4\hat{p}^+ \alpha'} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* + 2\hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jn} e^{-i2\omega_n \tau} \right]$$

Decompose,  $\sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} f_n = \sum_{n=1}^{\infty} f_n + f_{-n}$

$$\Rightarrow \frac{l}{2\hat{p}^+} \int_0^l d\sigma \hat{\Pi}^i \hat{\Pi}_i = \frac{\hat{p}^j \hat{p}_j}{2\hat{p}^+}$$

$$+ \frac{1}{4\hat{p}^+ \alpha'} \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \right. \\ \left. + 2\hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jn} e^{-i2\omega_n \tau} + 2\hat{\alpha}_n^{j*} \hat{\tilde{\alpha}}_{jn}^* e^{i2\omega_n \tau} \right]$$

$$\hat{\chi}^j(\tau, \sigma) = \frac{2\pi}{l} \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \begin{bmatrix} \hat{\alpha}_n^j e^{-i(k_n \sigma + \omega_n \tau)} \\ -\hat{\tilde{\alpha}}_n^j e^{i(k_n \sigma - \omega_n \tau)} \end{bmatrix}$$

$$\hat{\chi}^j \hat{\chi}_j = \frac{2\pi^2 \alpha'}{l^2} \sum_{n,m} \begin{bmatrix} \hat{\alpha}_n^j \hat{\alpha}_{jm} e^{-i(k_n + k_m) \sigma - i(\omega_n + \omega_m) \tau} \\ + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jm} e^{i(k_n + k_m) \sigma - i(\omega_n + \omega_m) \tau} \\ - 2 \hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jm} e^{i(k_m - k_n) \sigma - i(\omega_n + \omega_m) \tau} \end{bmatrix}$$

$$\begin{aligned} \frac{l}{2\hat{p}^+} \int_0^l d\sigma \frac{\hat{\chi}^i \hat{\chi}_i}{(2\pi\alpha')^2} \\ = \frac{1}{4\alpha' \hat{p}^+} \sum_{n,m} \begin{bmatrix} \hat{\alpha}_n^j \hat{\alpha}_{jm} + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jm} \\ - 2 \hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jm} e^{-2i\omega_n \tau} \end{bmatrix} \end{aligned}$$

$$\text{Decompose, } \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} f_n = \sum_{n=1}^{\infty} f_n + f_{-n}$$

$$\Rightarrow \frac{l}{2\hat{p}^+} \int_0^l d\sigma \frac{\hat{\chi}^i \hat{\chi}_i}{(2\pi\alpha')^2}$$

$$= \frac{1}{4\hat{p}^+ \alpha'} \sum_{n=1}^{\infty} \begin{bmatrix} \hat{\alpha}_n^j \hat{\alpha}_{jn}^* + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \\ - 2 \hat{\alpha}_n^j \hat{\tilde{\alpha}}_{jn} e^{-i2\omega_n \tau} - 2 \hat{\alpha}_n^{j*} \hat{\tilde{\alpha}}_{jn}^* e^{i2\omega_n \tau} \end{bmatrix}$$

$$\rightarrow \hat{H}_P = \frac{l}{2\hat{p}^+} \int_0^l d\sigma \left[ \hat{\Pi}^i \hat{\Pi}_i + \frac{\hat{\chi}^i \hat{\chi}_i}{(2\pi\alpha')^2} \right]$$

$$= \frac{\hat{p}^j \hat{p}_j}{2\hat{p}^+} + \frac{1}{2\hat{p}^+ \alpha'} \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \right]$$

use,

$$\hat{\alpha}_n^j \hat{\alpha}_{jn}^* = n \delta_j^j + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} = n(D-2) + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn}$$

$$\hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* = n \delta_j^j + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} = n(D-2) + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn}$$

Thus, we get,

$$\hat{H}_P = \frac{\hat{p}^j \hat{p}_j}{2\hat{p}^+} + \frac{1}{\hat{p}^+ \alpha'} \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} + n(D-2) \right]$$

But there is also ' $\tau$ ' independent translations possible in ' $\sigma$ '. At any ' $\tau$ ' if  $\sigma \rightarrow \sigma \oplus_\ell s$ , the eq<sup>n</sup> of motion remain invariant. Thus, there should be an operator ' $\hat{T}$ ' such that,

$$\hat{T}^\dagger(s) \hat{\chi}^i(\tau, \sigma) \hat{T}(s) = \hat{\chi}^i(\tau, \sigma \oplus_\ell s)$$

Or if  $\hat{T} = e^{-i\epsilon \hat{P}_\sigma}$ , then we need the operator ' $\hat{P}_\sigma$ ' to have the commutation with any  $\hat{\chi}^i(\tau, \sigma)$  as

$$[\hat{\chi}^i(\tau, \sigma), \hat{P}_\sigma] = -i \partial_\sigma \hat{\chi}^i(\tau, \sigma)$$

Then ' $\hat{P}_\sigma$ ' is said to generate the translations along ' $\sigma$ '.

Consider, 
$$\hat{P}_\sigma(\tau) = - \int_0^L d\sigma' \hat{\Pi}^k \partial_{\sigma'} \hat{\chi}_k$$

$$\begin{aligned} \rightarrow [\hat{\chi}^j(\tau, \sigma), \hat{P}_\sigma(\tau)] &= \\ &= - \int_0^L d\sigma' \underbrace{[\hat{\chi}^j(\tau, \sigma), \hat{\Pi}^k(\tau, \sigma')]}_{i \delta^{jk} \delta(\sigma - \sigma')} \partial_{\sigma'} \hat{\chi}_k \\ &= - \int_0^L d\sigma' i \delta^{jk} \delta(\sigma - \sigma') \partial_{\sigma'} \hat{\chi}_k \end{aligned}$$

$$\rightarrow [\hat{\chi}^j(\tau, \sigma), \hat{P}_\sigma(\tau)] = -i \partial_\sigma \hat{\chi}^j(\tau, \sigma)$$

Thus  $\hat{P}_\sigma(\tau) = -\int_0^l d\sigma \hat{\Pi}^j \partial_\sigma \hat{\chi}_j$  is the generator of  $\sigma$ -translations.

$$\hat{P}_\sigma(\tau) = -\int_0^l d\sigma \hat{\Pi}^j \partial_\sigma \hat{\chi}_j$$

$$= -\frac{2\pi \hat{P}^i}{l^2} \sqrt{\frac{\alpha'}{2}} \int_0^l d\sigma \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \hat{\alpha}_n^j e^{-i(k_n \sigma + \omega_n \tau)} - \hat{\tilde{\alpha}}_n^j e^{i(k_n \sigma - \omega_n \tau)} \right]$$

Integral of  $d\sigma = 0$

$$- \frac{\pi}{l^2} \int_0^l d\sigma \sum_{n,m} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jm} e^{-i(k_n + k_m)\sigma} e^{-i(\omega_n + \omega_m)\tau} - \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jm} e^{i(k_n + k_m)\sigma} e^{-i(\omega_n + \omega_m)\tau} \right]$$

$$= -\frac{\pi}{l^2} \sum_{n,m} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jm} l \delta_{n,-m} e^{-i(\omega_n + \omega_m)\tau} - \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jm} l \delta_{n,-m} e^{-i(\omega_n + \omega_m)\tau} \right]$$

$$\rightarrow \hat{P}_\sigma(\tau) = -\frac{\pi}{l} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* - \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* \right]$$

Observe that  $\hat{P}_\sigma(\tau)$  is ' $\tau$ ' independent.

$$\sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* - \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* \right]$$

$$= \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} - \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* - \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \right]$$

use,  $\hat{\alpha}_n^j \hat{\alpha}_{jn}^* = n \delta_j^j + \hat{\alpha}_n^{j*} \hat{\alpha}_{jn}$

$$\hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* = n \delta_j^j + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn}$$

$$\rightarrow \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \hat{\alpha}_n^j \hat{\alpha}_{jn}^* - \hat{\tilde{\alpha}}_n^j \hat{\tilde{\alpha}}_{jn}^* \right] = 2 \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} - \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \right]$$

Thus, the generator of  $\sigma$  translations is

$$\rightarrow \underline{\underline{\hat{P}_{\sigma} = \frac{2\pi}{l} \sum_{n=1}^{\infty} \left[ \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} - \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} \right]}}$$

The possible eigen values of ' $\hat{P}_{\sigma}$ ' is given by the multiples of  $\frac{2\pi}{l}$ . This represents the momentum of a wave travelling along the  $\sigma$ -direction.

' $\hat{P}_\sigma$ ' represents the  $\sigma$ -momenta along ' $\sigma$ ' direction of the world sheet. Then then the  $\hat{\alpha}$  type oscillator has opposite momentum when compared to the  $\hat{\alpha}$  type oscillator. We call ' $\hat{\tilde{\alpha}}$ ' to create +ve or right moving oscillations, the ' $\hat{\alpha}$ ' creates -ve or left moving oscillations.

Thus the excitations have a sense of orientation. Hence, what we have built here is the 'Oriented' String theory.

' $\sigma$ ' translation is like a Gauge degree of freedom which is still present in the light cone gauge after quantization.

But the Spectrum should be got by gauge invariant states, which are physical. So, physical states should have the ' $\sigma$ ' translation Symmetry. So those states should have the expectation of ' $\hat{P}_\sigma$ ' to be zero.

## Eigen States of $\hat{H}_p$

$|k^+, k^i\rangle$  forms a continuous basis sets to represents the CoM momenta. This is the point particle degree of freedom. So, CoM acts as a point particle.

$$\hat{p}^i |k^+, k^i\rangle = k^i |k^+, k^i\rangle$$

$$\hat{p}^+ |k^+, k^i\rangle = k^+ |k^+, k^i\rangle$$

Also for the Simple Harmonic oscillator part of Hamiltonian, we create a Fock space by the states  $|\{N_{\alpha m}^j\}; \{N_{\tilde{\alpha} m}^j\}\rangle$

where  $N_{\alpha m}^j$  is the occupation number of the ' $\alpha$ ' type oscillator in  $j^{\text{th}}$  direction at  $m^{\text{th}}$  mode, similarly  $N_{\tilde{\alpha} m}^j$  is the occupation number of the ' $\tilde{\alpha}$ ' type oscillator in  $j^{\text{th}}$  direction at  $m^{\text{th}}$  mode.

Thus,  $|\{N_{\alpha m}^j\}; \{N_{\tilde{\alpha} m}^j\}; \{k^+, k^i\}\rangle$  gives the complete description of a state.

$\hat{N}_\alpha = \sum_{n=1}^{\infty} \sum_{j=2}^{D-1} \hat{\alpha}_n^{j*} \hat{\alpha}_{jn}$  is the level of the left moving or ' $\alpha$ ' type excitations.

$\hat{N}_{\tilde{\alpha}} = \sum_{n=1}^{\infty} \sum_{j=2}^{D-1} \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn}$  is the level of the right moving or ' $\tilde{\alpha}$ ' type excitations.

Then,  $\hat{N} = \hat{N}_\alpha + \hat{N}_{\tilde{\alpha}}$  is the total no: of excitations or the mass level of the string.

The state  $|0; 0; \{k^+, k^i\}\rangle$  represents the zero excitation of the string's vibration mode.

$$\hat{\alpha}_{jn} |0; 0; \{k^+, k^i\}\rangle = 0 \quad \& \quad \hat{\tilde{\alpha}}_{jn} |0; 0; \{k^+, k^i\}\rangle = 0$$

Then by the usual operator algebra, we can construct any General State

$|\{N_{\alpha m}^j\}; \{\tilde{N}_{\alpha m}^j\}; \{k^+, k^i\}\rangle$  is got by

$$\prod_{j=2}^{D-1} \prod_{m=1}^{\infty} \frac{(\hat{\alpha}_m^{j*})^{N_{\alpha m}^j}}{n^{N_{\alpha m}^j} \sqrt{(N_{\alpha m}^j)!}} \frac{(\hat{\tilde{\alpha}}_m^{j*})^{N_{\tilde{\alpha} m}^j}}{n^{N_{\tilde{\alpha} m}^j} \sqrt{(N_{\tilde{\alpha} m}^j)!}} |0; 0; \{k^+, k^i\}\rangle$$

These oscillators represent an infinite degree of internal freedom of the string. Every choice of these occupation numbers corresponds to a different particle / spin state from the point of view of the space-time.

The state  $|0; 0; 0\rangle$  denotes a string state with zero COM momentum (fixed COM) & zero excitations in the transverse modes.

It is the ground state of the single closed string Hamiltonian given by,

$$\hat{H}_P = \frac{\hat{P}^j \hat{P}_j}{2 \hat{P}^+} + \frac{1}{\hat{P}^+ \alpha'} \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} + n(D-2) \right]$$

The various operators defined previously are operators that act on the single string Fock state. They do not destroy or create strings in the space-time.

Now, the Hamiltonian is given by,

$$\hat{H}_P = \frac{\hat{p}^i \hat{p}_i}{2p^+} + \frac{1}{\alpha' p^+} \left[ \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \right] + 2A \right]$$

Where  $A = \sum_{n=1}^{\infty} \frac{(D-2)n}{2}$ . We previously

saw that in the open string spectrum we were able to replace  $\sum_{n=1}^{\infty} n \rightarrow -\frac{1}{12}$

we shall just do the same here. This gives the zero point energy proportional to  $A = \frac{(2-D)}{24}$ . Thus we get,

$$\hat{H}_P = \frac{\hat{p}^i \hat{p}_i}{2p^+} + \frac{1}{\alpha' p^+} \left[ \sum_{n=1}^{\infty} \left[ \hat{\alpha}_n^{j*} \hat{\alpha}_{jn} + \hat{\tilde{\alpha}}_n^{j*} \hat{\tilde{\alpha}}_{jn} \right] + \frac{(2-D)}{12} \right]$$

Now, due to the transverse oscillator energy of the String, the effective mass of the String depends on the number of excitations in the String.

For the Point particle like CoM, we have  $\hat{p}^- = \hat{H}_p$  in the light Cone gauge. Since  $X^+(\tau) = \tau$  is the time coordinate of the world Sheet.

$$\Rightarrow 2\hat{p}^+\hat{p}^- = \hat{p}^i\hat{p}_i + \frac{2}{\alpha'} \left[ \hat{N} + \frac{(2-D)}{12} \right]$$

Then, for any state  $|\{N_{\alpha m}^j\}; \{N_{\tilde{\alpha} m}^j\}; \{p^+, p^i\}\rangle$  we get the eigen values to relate by,

$$2p^+p^- = p^ip_i + \frac{2}{\alpha'} \left[ N + \frac{(2-D)}{12} \right]$$

$$\underline{N = \sum_{n=1}^{\infty} \sum_{j=2}^{D-1} n [N_{\alpha n}^j + N_{\tilde{\alpha} n}^j]} \text{ is the level of excitations}$$

$$\Rightarrow (p^0 + p^1)(p^0 - p^1) - (p^i)^2 = \frac{2}{\alpha'} \left[ N + \frac{(2-D)}{12} \right]$$

$$\Rightarrow \underbrace{(p^0)^2 - (p^1)^2 - (p^2)^2 - \dots - (p^{D-1})^2}_{-p^\mu p_\mu = M^2} = \frac{2}{\alpha'} \left[ N + \frac{(2-D)}{12} \right]$$

$$\rightarrow \underline{\underline{M^2 = \frac{2}{\alpha'} \left[ N + \frac{(2-D)}{12} \right]}}$$

Thus each state  $|\{N_{\alpha m}^i\}; \{N_{\tilde{\alpha} m}^i\}; \{p^+, p^i\}\rangle$  is associated with some mass related to the level of excitation of the string.

The physical states have to be independent of the  $\sigma$ -translation gauge degree of freedom. Thus expectation value of  $\hat{P}_\sigma$  has to be zero.

$$\text{Since } \underline{\underline{\hat{P}_\sigma = \frac{2\pi}{l} [\hat{N}_{\tilde{\alpha}} - \hat{N}_\alpha]}},$$

The physically relevant states given by  $|\{N_{\alpha m}^j\}; \{N_{\tilde{\alpha} m}^j\}; \{p^t, p^i\}\rangle$  should be such that

$$N_{\tilde{\alpha}} = N_{\alpha} \Rightarrow \sum_{n=1}^{\infty} \sum_{j=2}^{D-1} n N_{\tilde{\alpha} m}^j = \sum_{n=1}^{\infty} \sum_{j=2}^{D-1} n N_{\alpha n}^j$$

## Ground State

The ground state or the lightest closed string state is,  $|0; 0; \{p^t, p^i\}\rangle$

whose  $N_{\tilde{\alpha}} = N_{\alpha} = 0 \rightarrow N = 0$

$$\rightarrow \underline{M^2 = \frac{(2-D)}{6\alpha'}} \quad \left. \vphantom{\frac{(2-D)}{6\alpha'}} \right\} \begin{array}{l} \text{Tachyon for} \\ D > 2 \end{array}$$

## First excited State

They are given by  $\hat{\alpha}_1^{i*} \hat{\tilde{\alpha}}_1^{j*} |0; 0; \{p^t, p^i\}\rangle$

whose  $N_{\beta} = N_{\alpha} = 1 \rightarrow N = 2$

$$\rightarrow \underline{M^2 = \frac{26-D}{6\alpha'}}$$

The states  $M^{ij} = \hat{\alpha}_1^{i*} \hat{\alpha}_1^{j*} |0; 0; \{p^+, p^i\}\rangle$  form an incomplete rep<sup>n</sup> of  $SO(D-1)$

$M^{ij}$  gives a rank-2 tensor representation of  $SO(D-2)$ . Now, any  $M^{ij}$  can be written as,  $M^{ij} = \underbrace{\frac{M^{ij} + M^{ji}}{2}}_{\text{Symmetric}} + \underbrace{\frac{M^{ij} - M^{ji}}{2}}_{\text{Anti Symmetric}}$

and the Symmetric part can be further decomposed to trace (Scalar) & traceless Symmetric tensor part.

$$\frac{M^{ij} + M^{ji}}{2} = U^{ij} + \frac{\delta^{ij}}{(D-2)} \text{Trace}(M)$$

$$\text{where } U^{ij} = \frac{M^{ij} + M^{ji}}{2} - \frac{\delta^{ij}}{(D-2)} \text{Trace}(M)$$

$$\text{Let } \Phi = \frac{\text{Trace}(M)}{(D-2)} \quad \& \quad A^{ij} = \frac{M^{ij} - M^{ji}}{2}$$

$$\Rightarrow M^{ij} = U^{ij} + A^{ij} + \mathcal{Z}^{ij} \Phi$$

$\swarrow$   
 Traceless  
 Symmetric  
 rank 2  
 tensor of  
 $SO(D-2)$

$\downarrow$   
 Anti-Symmetric  
 rank 2 tensor  
 of  $SO(D-2)$

$\searrow$   
 Scalar of  
 $SO(D-2)$

These three separate terms do not mix under rotations. Now, this doesn't add up to any rep<sup>n</sup> of  $SO(D-1)$ , thus it only represents a massless particle which has  $SO(D-2)$  symmetry. which is possible only if  $D=26$ .

Thus the physically allowed first excited state is massless given  $D=26$  in the closed string case. This level gives rise to a massless Symmetric tensor particle similar to Graviton and a massless anti-Symmetric tensor particle called as the Kalb-Ramond particle.

## Second excited State

They are such that  $N_{\tilde{\alpha}} = N_{\alpha} = 2 \rightarrow N = 4$

$$\rightarrow M^2 = \frac{4}{\alpha'} \quad (\text{at } D=26)$$

These states are represented by,

$$M^{ij} = \hat{\alpha}_2^{i*} \hat{\alpha}_2^{j*} |0; 0; \{p^+, p^i\}\rangle$$

$$R^{ijkl} = \hat{\alpha}_1^{i*} \hat{\alpha}_1^{j*} \hat{\alpha}_1^{k*} \hat{\alpha}_1^{l*} |0; 0; \{p^+, p^i\}\rangle$$

Now  $M^{ij}$  is got by direct product of two vectors  $V^i \neq W^j$  of  $SO(D-2)$

$$\rightarrow M^{ij} = V^i \otimes W^j$$

$$\text{Similarly, } R^{ijkl} = S^{ij} \otimes T^{ij}$$

where  $S^{ij} \neq T^{ij}$  are symmetric tensors of  $SO(D-2)$ .

Now  $M^{ij}$  and  $R^{ijkl}$  together gives a direct product rep<sup>n</sup> of two traceless symmetric rank-2 tensors of  $SO(D-1)$

Let  $G^{mn}$  &  $H^{qs}$  be traceless symmetric tensors of  $SO(D-1)$  where the indices  $m, n, q, s = 1, 2, 3, \dots, D-1$ .

We previously saw that in the open string case,  $\{V^i, S^{ij}\}$  together formed a rep<sup>n</sup>  $G^{mn}$  of  $SO(D-1)$  in the level  $M^2 = \frac{1}{\alpha'} (N=2)$ . But here in closed string,  $M^{ij} = V^i \otimes W^j$  &  $R^{ijkl} = S^{ij} \otimes T^{kl}$  of  $SO(D-2)$  represents  $G^{mn} \otimes H^{qs}$  of  $SO(D-1)$ . It is exactly same as open string case, but with two copies.

Thus, the closed string spectra at level  $M^2 = \frac{4}{\alpha'} [N-1]$  ( $N=1, 2, 3, \dots$ ) is the product of two copies of the  $M^2 = \frac{N-1}{\alpha'}$  level of the open string spectra.

# World Sheet Parity

## Oriented & Unoriented Theories

The discrete coordinate transformation given by,  $\sigma \rightarrow l - \sigma$ ,  $\tau \rightarrow \tau$  will change the orientation of the world sheet.

The parity transformation is performed by the world sheet Parity operator ' $\hat{\Omega}$ '.  
where  $\hat{\Omega}^2 = 1 \rightarrow$  the eigen values are either  $+1$  (even parity) or  $-1$  (odd).

For open String we have,

$$\hat{X}^i = \hat{x}^i + \frac{\hat{p}^i}{\hat{p}^+} \tau + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left[ \frac{\hat{a}_n^i e^{-i\omega_n \tau} - \hat{a}_n^{i*} e^{+i\omega_n \tau}}{n} \right] \cos(k_n \sigma)$$

Since  $\hat{X}^i$  doesn't change with world sheet parity, but  $\sigma \rightarrow l - \sigma$ , thus we see that  $\cos(k_n \sigma) \xrightarrow{\Omega} \cos(n\pi - k_n \sigma)$   
Thus  $\cos(k_n \sigma) \xrightarrow{\Omega} (-1)^n \cos(k_n \sigma)$ .  
So,  $\hat{a}_n^i \xrightarrow{\Omega} (-1)^n \hat{a}_n^i$  for open string.

In closed String case,

$$X^j(\tau, \sigma) = x^j + \frac{p^j}{p^+} \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[ \frac{\alpha_n^j}{n} e^{-i(k_n \sigma + \omega_n \tau)} + \frac{\tilde{\alpha}_n^j}{n} e^{i(k_n \sigma - \omega_n \tau)} \right]$$

When  $\sigma \xrightarrow{\Omega} l - \sigma$ ,

$$e^{\pm i k_n \sigma} = e^{\pm i \left( \frac{2n\pi}{l} \right) \sigma} \xrightarrow{\Omega} e^{\pm i \left( \frac{2n\pi}{l} \right) (l - \sigma)}$$

$$e^{\pm i k_n \sigma} \xrightarrow{\Omega} e^{\mp i k_n \sigma}$$

Thus Left moving  $\xleftrightarrow[\Omega]{\Omega}$  Right moving

$$\rightarrow \alpha_n^j \xleftrightarrow[\Omega]{\Omega} \tilde{\alpha}_n^j$$

Thus when Parity (Handedness) of the world Sheet changes, the left & right moving oscillations exchange their roles.

Now we shall fix the ground state to always have even world Sheet parity.

$$\rightarrow \hat{\Omega} |g.s\rangle = |g.s\rangle.$$

Thus we get, for open string states,

$$\hat{\Omega} |\{N_m^i\}; \{p^+, p^i\}\rangle = (-1)^N |\{N_m^i\}; \{p^+, p^i\}\rangle$$

where  $N$  is level of open string.

Then for all those physically possible closed string states,

$$\hat{\Omega} |\{N_{\alpha m}^i\}; \{N_{\tilde{\alpha} m}^i\}; \{k^+, k^i\}\rangle = |\{N_{\alpha m}^i\}; \{N_{\tilde{\alpha} m}^i\}; \{k^+, k^i\}\rangle$$

(Left & right oscillator labels exchange)

In Unoriented theories only those states which give eigen value of  $\hat{\Omega}$  of  $+1$  is kept. Thus in the unoriented open string theory, the massless state ( $N=1$ ) which describes a photon is absent. Also in unoriented closed string theory, the massless symmetric tensor & trace like scalar states  $\Phi$  called Dilaton are possible. But no anti-symmetric tensor particles (massive / massless) don't exist.

We call the massless Symmetric tensor particle as Graviton, represent it as  $G_{\mu\nu}$ .  
The Scalar particle as Dilaton  $\Phi$ , the Photon as  $A_\mu$  & anti-Symmetric tensor particle as  $B_{\mu\nu}$ .

- Closed & Unoriented theory has only  $G_{\mu\nu}$  &  $\Phi$ .
- Closed & Oriented theory has  $G_{\mu\nu}$ ,  $\Phi$  &  $B_{\mu\nu}$ .
- Closed & Open Unoriented theory has only  $G_{\mu\nu}$  &  $\Phi$ .
- Closed & Open Oriented theory has all  $G_{\mu\nu}$ ,  $B_{\mu\nu}$ ,  $\Phi$  &  $A_\mu$ .

Note: 'Only closed' string theory is possible.  
But 'only open' string theory is not possible.

In the above fields we have put  $\mu, \nu = 0, 1, 2, \dots, 25$  instead of  $i, j = 2, 3, \dots, 25$ . Since,  $G_{\mu\nu}$  &  $B_{\mu\nu}$  have Space-time gauge Symmetry that allow us to eliminate appropriate modes.

This gauge Symmetry is essential to remove negative norm states in a covariant quantization approach, as these states are not physically possible.

Also  $G_{\mu\nu}$  field has Spin of '2' & is also massless. The argument of Feynman & Weinberg is that any theory of interacting massless-Spin 2 particle should describe Einstein's general relativity in its quantum form. Thus  $G_{\mu\nu}$  should describe Graviton field.

For more info. look into David Tong's Lecture notes on String theory.

## Excercise 1.5 Polchinski

To find  $\sum_{n=1}^{\infty} (n-\theta)$

$$\sum_{n=1}^{\infty} (n-\theta) = \lim_{\Lambda \rightarrow \infty} \sum_{n=1}^{\Lambda} (n-\theta)$$

$$= \lim_{\Lambda \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \sum_{n=1}^{\Lambda} (n-\theta) e^{-\varepsilon(n-\theta)}$$

$$= - \lim_{\Lambda \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} \sum_{n=1}^{\Lambda} e^{-\varepsilon(n-\theta)}$$

$$\text{use } \sum_{n=1}^{\Lambda} e^{-\varepsilon(n-\theta)} = \left[ \frac{1 - e^{-\Lambda \varepsilon}}{(e^{\varepsilon} - 1)} \right] e^{\varepsilon \theta}$$

$\Rightarrow$

$$\lim_{\Lambda \rightarrow \infty} \sum_{n=1}^{\Lambda} (n-\theta) = \lim_{\substack{\Lambda \rightarrow \infty \\ \varepsilon \rightarrow 0}} \frac{\partial}{\partial \varepsilon} \left[ \frac{1 - e^{-\Lambda \varepsilon}}{(e^{\varepsilon} - 1)} \right] e^{\varepsilon \theta}$$

$$\Rightarrow \sum_{n=1}^{\infty} (n - \theta) = \lim_{\varepsilon \rightarrow 0} \frac{\partial}{\partial \varepsilon} \left[ \frac{e^{\varepsilon \theta}}{(e^{\varepsilon} - 1)} \right]$$

$$= \lim_{\varepsilon \rightarrow 0} \left[ \frac{\theta e^{\varepsilon \theta}}{(e^{\varepsilon} - 1)} - \frac{e^{\varepsilon \theta} e^{\varepsilon}}{(e^{\varepsilon} - 1)^2} \right]$$

$$= \lim_{\varepsilon \rightarrow 0} \underbrace{\frac{1}{\varepsilon^2}} - \frac{1}{2} \left( \frac{1}{6} - \theta + \theta^2 \right) + \underbrace{\theta(\varepsilon^2)}_0$$

remove  
by counter term

$$\Rightarrow \sum_{n=1}^{\infty} (n - \theta) = -\frac{1}{12} + \theta - \theta^2$$