

Complex World Sheet Coordinates

We define complex coordinates as,

$$Z = \sigma^1 + i\sigma^2$$

$$\bar{Z} = \sigma^1 - i\sigma^2$$

$$\rightarrow dZ = d\sigma_1 + i d\sigma_2 \quad \& \quad d\bar{Z} = d\sigma_1 - i d\sigma_2$$

$$\Rightarrow \text{Jacobian matrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

$$\text{Jacobian} = |-2i| = 2$$

$$\rightarrow d^2 Z = 2 d^2 \sigma$$

Now,

$$\sigma^1 = \frac{1}{2}(\bar{Z} + Z), \quad \sigma^2 = \frac{i}{2}(\bar{Z} - Z)$$

$$\& \quad \frac{\partial}{\partial Z} = \frac{\partial \sigma^1}{\partial Z} \partial_1 + \frac{\partial \sigma^2}{\partial Z} \partial_2$$

$$\frac{\partial}{\partial \bar{Z}} = \frac{\partial \sigma^1}{\partial \bar{Z}} \partial_1 + \frac{\partial \sigma^2}{\partial \bar{Z}} \partial_2$$

$$\Rightarrow \partial \equiv \partial_z = \frac{1}{2} (\partial_1 - i\partial_2)$$

$$\& \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2} (\partial_1 + i\partial_2)$$

Since, σ_1 & σ_2 are independent variables i.e. $\partial_i \sigma_j = \delta_{ij}$, we get z & $\bar{z}$ also to be independent of each other.

$$\text{Thus, } \partial_z z = \partial_{\bar{z}} \bar{z} = 1$$

$$\& \partial_z \bar{z} = \partial_{\bar{z}} z = 0$$

For any world sheet vector $v^a = (v^1, v^2)$ we define the complex transformation

$$[v^1(\sigma^1, \sigma^2), v^2(\sigma^1, \sigma^2)] \longrightarrow [v^z(z, \bar{z}), v^{\bar{z}}(z, \bar{z})]$$

$$\left. \begin{aligned} v^z &= \frac{\partial z}{\partial \sigma^1} v^1 + \frac{\partial z}{\partial \sigma^2} v^2 \\ v^{\bar{z}} &= \frac{\partial \bar{z}}{\partial \sigma^1} v^1 + \frac{\partial \bar{z}}{\partial \sigma^2} v^2 \end{aligned} \right\} \begin{array}{l} \text{Contravariant} \\ \text{Components} \\ \text{transformation} \end{array}$$

$$\rightarrow \varphi^{\bar{z}} = \varphi^1 + i\varphi^2$$

$$\varphi^{\bar{\bar{z}}} = \varphi^1 - i\varphi^2$$

Similarly,

$$\left. \begin{aligned} \varphi_{\bar{z}} &= \frac{\partial \sigma^1}{\partial \bar{z}} \varphi_1 + \frac{\partial \sigma^2}{\partial \bar{z}} \varphi_2 \\ \varphi_{\bar{\bar{z}}} &= \frac{\partial \sigma^1}{\partial \bar{\bar{z}}} \varphi_1 + \frac{\partial \sigma^2}{\partial \bar{\bar{z}}} \varphi_2 \end{aligned} \right\} \begin{array}{l} \text{Covariant} \\ \text{Components} \\ \text{transformation} \end{array}$$

$$\Rightarrow \varphi_{\bar{z}} = \frac{1}{2} (\varphi_1 - i\varphi_2)$$

$$\varphi_{\bar{\bar{z}}} = \frac{1}{2} (\varphi_1 + i\varphi_2)$$

The metric tensor in complex is 'g'

$$g = \begin{bmatrix} g_{\bar{z}\bar{z}} & g_{\bar{z}\bar{\bar{z}}} \\ g_{\bar{\bar{z}}\bar{z}} & g_{\bar{\bar{z}}\bar{\bar{z}}} \end{bmatrix}$$

This is related to the euclidian Metric δ_{ab} by a rank-2 covariant tensor transformation relation.

$$\rightarrow g_{zz} = \frac{\partial \sigma^a}{\partial z} \frac{\partial \sigma^b}{\partial z} \delta_{ab}$$

$$= \left(\frac{\partial \sigma^1}{\partial z} \right)^2 + \left(\frac{\partial \sigma^2}{\partial z} \right)^2 = \left(\frac{1}{2} \right)^2 + \left(\frac{-i}{2} \right)^2$$

$$\rightarrow \underline{g_{zz} = 0}$$

Similarly, $g_{\bar{z}\bar{z}} = \frac{\partial \sigma^a}{\partial \bar{z}} \frac{\partial \sigma^b}{\partial \bar{z}} \delta_{ab}$

$$= \left(\frac{\partial \sigma^1}{\partial \bar{z}} \right)^2 + \left(\frac{\partial \sigma^2}{\partial \bar{z}} \right)^2 = \left(\frac{1}{2} \right)^2 + \left(\frac{i}{2} \right)^2$$

$$\rightarrow \underline{g_{\bar{z}\bar{z}} = 0}$$

Now,

$$g_{\bar{z}z} = g_{z\bar{z}} = \frac{\partial \sigma^a}{\partial z} \frac{\partial \sigma^b}{\partial \bar{z}} \delta_{ab}$$

$$= \left(\frac{\partial \sigma^1}{\partial z} \right) \left(\frac{\partial \sigma^1}{\partial \bar{z}} \right) + \left(\frac{\partial \sigma^2}{\partial z} \right) \left(\frac{\partial \sigma^2}{\partial \bar{z}} \right)$$

$$= \frac{1}{4} + \frac{-i^2}{4} = \frac{1}{2}$$

$$\underline{g_{\bar{z}z} = g_{z\bar{z}} = \frac{1}{2}}$$

$$\rightarrow \underline{g = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}}$$

$$\rightarrow \underline{g^{-1} = \begin{bmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}}$$

Gegeben $(v^z, v^{\bar{z}})$

$$v_z = g_{z\bar{z}} v^{\bar{z}} = \frac{v^{\bar{z}}}{2}$$

$$v_{\bar{z}} = g_{\bar{z}z} v^z = \frac{v^z}{2}$$

$$\rightarrow v^z = 2 v_{\bar{z}} \quad \& \quad v^{\bar{z}} = 2 v_z$$

$$|\vec{v}|^2 = g_{z\bar{z}} v^z v^{\bar{z}} + g_{\bar{z}z} v^{\bar{z}} v^z$$

$$\Rightarrow \underline{|\vec{v}|^2 = v^z v^{\bar{z}}}$$

In euclidean, $|\vec{v}|^2 = (v^1)^2 + (v^2)^2$

The dirac delta functions are,

$$\delta^{(2)}(\sigma^1, \sigma^2) = \delta(\sigma^1) \delta(\sigma^2)$$

$$\text{where } \int d^2\sigma \delta^{(2)}(\sigma^1, \sigma^2) = 1$$

$$\text{But we need, } \int d^2z \delta^{(2)}(z, \bar{z}) = 1$$

$$\text{Since } d^2\sigma = \frac{d^2z}{2}$$

$$\Rightarrow \frac{\delta^{(2)}(\sigma^1, \sigma^2)}{2} = \delta^{(2)}(z, \bar{z})$$

$$\underline{\rightarrow \delta^{(2)}(z, \bar{z}) = \frac{1}{2} \delta(\sigma^1) \delta(\sigma^2)}$$

Divergence theorem in 2D

Let $[v^1(\sigma^a), v^2(\sigma^a)]$ be a vector in Euclidean coordinates, then

$$\iint_R d^2\sigma (\partial_a v^a) = \oint_{\partial R} (v^a n_a) dl$$

where $\hat{n}_a$ gives the unit vector normal to the boundary of the area.
'dl' = line element of boundary.

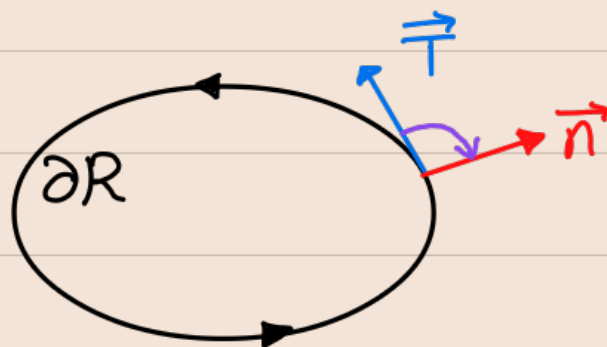
Parameterize the boundary curve ∂R by $[\sigma^1(u), \sigma^2(u)]$, ($u \rightarrow$ Parameter), where the curve is oriented in the Counter-clockwise direction.

Tangent vector is $T^a(u) = \left(\frac{d\sigma^1}{du}, \frac{d\sigma^2}{du} \right)$

$$\text{and } dl^2 = (d\sigma^1)^2 + (d\sigma^2)^2$$

$$\Rightarrow dl = \sqrt{T^a(u) T_a(u)} du = |\vec{T}| du$$

Then the normal vector is got by rotating the tangent vector by 90° clockwise.



$$\rightarrow \vec{n} = \begin{bmatrix} \cos(-90^\circ) & -\sin(-90^\circ) \\ \sin(-90^\circ) & \cos(-90^\circ) \end{bmatrix} \vec{T}$$

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|}, \quad \vec{n} = \left(\frac{d\sigma^2}{du}, -\frac{d\sigma^1}{du} \right)$$

$$\rightarrow \underline{\vec{n} du = (d\sigma^2, -d\sigma^1)}$$

$$\text{Now } |\vec{n}| du = |\vec{T}| du = dl$$

$$\Rightarrow \hat{n} dl = \frac{\vec{n}}{|\vec{n}|} dl = \frac{\vec{n}}{|\vec{n}|} |\vec{n}| du$$

$$\rightarrow \underline{\hat{n} dl = \vec{n} du}$$

$$\begin{aligned} \rightarrow \iint_R d^2\sigma (\partial_a v^a) &= \oint_{\partial R} (v^a n_a) |\vec{n}| du \\ &= \oint_{\partial R} \vec{v} \cdot \vec{n} du \end{aligned}$$

$$\rightarrow \iint_R d^2\sigma (\partial_a v^a) = \oint_{\partial R} (v^1 d\sigma^2 - v^2 d\sigma^1)$$

In complex coordinates,

$$d^2\sigma = \frac{d^2z}{2}$$

$$\partial_a v^a = \partial_z v^z + \partial_{\bar{z}} v^{\bar{z}}$$

$$v^1 = \frac{1}{2}(v^{\bar{z}} + v^z), \quad v^2 = \frac{i}{2}(v^{\bar{z}} - v^z)$$

$$d\sigma^1 = \frac{1}{2}(d\bar{z} + dz), \quad d\sigma^2 = \frac{i}{2}(d\bar{z} - dz)$$

$$\begin{aligned} (v^1 d\sigma^2 - v^2 d\sigma^1) &= \frac{1}{2}(v^{\bar{z}} + v^z) \frac{i}{2}(d\bar{z} - dz) \\ &\quad - \frac{i}{2}(v^{\bar{z}} - v^z) \frac{1}{2}(d\bar{z} + dz) \end{aligned}$$

$$= \frac{i}{4} \begin{bmatrix} v^{\bar{z}} d\bar{z} + v^z d\bar{z} - v^{\bar{z}} dz - v^z dz \\ -v^{\bar{z}} d\bar{z} - v^{\bar{z}} dz + v^z d\bar{z} + v^z dz \end{bmatrix}$$

$$\Rightarrow (v^1 d\sigma^2 - v^2 d\sigma^1) = \frac{i}{2} (v^z d\bar{z} - v^{\bar{z}} dz)$$

Thus the divergence theorem,

$$\rightarrow \iint_R d^2\sigma (\partial_a v^a) = \oint_{\partial R} (v^1 d\sigma^2 - v^2 d\sigma^1)$$

in complex coordinates becomes

$$\iint_R d^2z (\partial_z v^{\bar{z}} + \partial_{\bar{z}} v^z) = i \oint_{\partial R} (v^{\bar{z}} d\bar{z} - v^z dz)$$

$$\rightarrow \iint_R d^2z (\partial_z v_{\bar{z}} + \partial_{\bar{z}} v_z) = i \oint_{\partial R} (v_{\bar{z}} d\bar{z} - v_z dz)$$

The inner products of two vectors in complex coordinates is,

$$(v^z, v^{\bar{z}}) \cdot (u^z, u^{\bar{z}}) = \frac{1}{2} (v^z u^{\bar{z}} + v^{\bar{z}} u^z)$$

$$\rightarrow |(v^z, v^{\bar{z}})|^2 = v^z v^{\bar{z}} = v^z v^{\bar{z}}$$

Path integral Formalism

Appendix A: Hilbert Space Formalism

Here we will show how the Path integral formalism relates to the usual Quantum mechanics in Hilbert Space. We know that, in the usual quantum mechanics,

$$\langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle = \langle \vec{x}_f | e^{-i\hat{H}(t_f - t_i)} | \vec{x}_i \rangle$$

But in path integral formalism,

$$\langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle = \int_{\vec{x}(t_i)}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]}$$

where $S[\vec{x}(t)] = \int_{t_i}^{t_f} dt \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)]$ is the

classical action of the System.

We can introduce a complete states of $\vec{x}'(t')$ in above eqⁿ.

$$\langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle =$$

$$\int d[\vec{x}'(t')] \langle \vec{x}_f, t_f | \vec{x}', t' \rangle \langle \vec{x}', t' | \vec{x}_i, t_i \rangle$$

$\Rightarrow \langle \vec{x}_f, t_f | \vec{x}_i, t_i \rangle$ is given by,

$$= \int d[\vec{x}'(t')] \int_{\vec{x}'(t')}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] \exp \left[i \int_{t'}^{t_f} dt \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] \right] \\ \times \int_{\vec{x}(t_i)}^{\vec{x}'(t')} \mathcal{D}[\vec{x}(t)] \exp \left[i \int_{t_i}^{t'} dt \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] \right]$$

But there is a restriction $t_f > t' > t_i$. This procedure is called cutting & Sewing technique. Now, consider the matrix element $\langle \vec{x}_f, t_f | \hat{\vec{x}}(t') | \vec{x}_i, t_i \rangle$, where ' $\hat{\vec{x}}(t')$ ' is the position operator. It can be written as,

$$= \int d[\vec{x}'(t')] \langle \vec{x}_f, t_f | \hat{\vec{x}}(t') | \vec{x}', t' \rangle \langle \vec{x}', t' | \vec{x}_i, t_i \rangle$$

$$= \int d[\vec{x}'(t')] \langle \vec{x}_f, t_f | \vec{x}', t' \rangle \langle \vec{x}', t' | \vec{x}_i, t_i \rangle \vec{x}'(t')$$

$$\rightarrow \langle \vec{x}_f, t_f | \hat{\vec{x}}(t') | \vec{x}_i, t_i \rangle$$

$$= \int d[\vec{x}'(t')] \int_{\vec{x}'(t')}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] \exp \left[i \int_{t'}^{t_f} dt \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] \right] \vec{x}'(t')$$

$$\times \int_{\vec{x}(t_i)}^{\vec{x}'(t')} \mathcal{D}[\vec{x}(t)] \exp \left[i \int_{t_i}^{t'} dt \mathcal{L}[\vec{x}(t), \dot{\vec{x}}(t)] \right]$$

$$= \int_{\vec{x}(t_i)}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \vec{x}(t)$$

Thus we get,

$$\langle \vec{x}_f, t_f | \hat{\vec{x}}(t) | \vec{x}_i, t_i \rangle = \int_{\vec{x}(t_i)}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \vec{x}(t)$$

Observe that $\vec{x}(t)$ in the functional integral translates to $\hat{\vec{x}}(t)$ operator's matrix element provided $t_f > t > t_i$.

Similarly, we get $\int_{\vec{x}(t_i)}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \vec{x}(t_1) \vec{x}(t_2)$

as the time ordered product's matrix element

$$\rightarrow \langle \vec{x}_f, t_f | T [\hat{\vec{x}}(t_1) \hat{\vec{x}}(t_2) \hat{\vec{x}}(t_3) \dots] | \vec{x}_i, t_i \rangle$$

$$= \int_{\vec{x}(t_i)}^{\vec{x}(t_f)} \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \vec{x}(t_1) \vec{x}(t_2) \hat{\vec{x}}(t_3) \dots$$

This is because the path integrals are built out of successive infinitesimal time slices. So any insertions of position variables in path integrals will always result in the matrix element of the time ordered product of the position operators.

Now, we shall try to get the operator eqⁿ by using the classical eqⁿ of motion.

Classical eqⁿ of motion is got by the variational principle by setting $\frac{\delta S}{\delta x(t)} = 0$. Consider this

inside the path integral, but with other insertions $\mathcal{F}[x(t_1), x(t_2), \dots]$, but none of the times $t_1, t_2, \dots \neq t'$. Thus we get,

$$\int \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \frac{\delta S}{\delta x(t')} \mathcal{F}$$

$$= \int \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \frac{\delta S}{\delta x(t')} \mathcal{F}$$

$$\left(\text{Since, } \frac{\partial e^{iS[\vec{x}(t)]}}{\partial x(t')} = i e^{iS[\vec{x}(t)]} \frac{\delta S}{\delta x(t')} \right)$$

$$= -i \int \mathcal{D}[\vec{x}(t)] \mathcal{F} \frac{\partial e^{iS[\vec{x}(t)]}}{\partial x(t')}$$

(Int. by parts)

$$= i \int \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \frac{\delta \mathcal{F}}{\delta x(t')}$$

Now F doesn't depend on any operator at time ' t '. Thus $\frac{\partial F}{\partial x(t)} = 0$.

$$\rightarrow \int \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \nabla \frac{\partial S}{\partial x(t)} = 0$$

This also holds as an operator eqⁿ as seen in the previous results.

$$\Rightarrow \langle x_f, t_f | \nabla \frac{\partial S}{\partial x(t)} | x_i, t_i \rangle = 0$$

But if $F = x(t'')$, which can come arbitrarily close to $x(t')$, then,

$$\begin{aligned} & \int \mathcal{D}[\vec{x}(t)] \frac{\partial S}{\partial x(t)} x(t'') e^{iS[\vec{x}(t)]} \\ &= i \int \mathcal{D}[\vec{x}(t)] e^{iS[\vec{x}(t)]} \frac{\partial x(t'')}{\partial x(t)} \end{aligned}$$

But $\frac{\partial x(t'')}{\partial x(t')} = \delta(t' - t'')$

$$\rightarrow \left\langle \frac{\partial S}{\partial x(t')} x(t'') \right\rangle = i \left\langle \delta(t' - t'') \right\rangle$$

This also holds if there are other insertions $F[x(t_1), x(t_2), \dots]$, where none of the times $t_1, t_2, \dots \neq t', t''$. Thus we get

$$\left\langle \frac{\partial S}{\partial x(t')} x(t'') F \right\rangle = i \left\langle \delta(t' - t'') F \right\rangle$$

This is known as Schwinger - Dyson eqⁿ.

This can also be written as operator equation as,

$$\widehat{\left(\frac{\partial S}{\partial x(t')} \right)} \hat{x}(t'') = i \delta(t' - t'')$$

Eg: For harmonic oscillator, we have

$$\frac{\partial S}{\partial x(t')} = -m\ddot{x}(t') - \omega^2 x(t')$$

$$\rightarrow - \left[m \frac{d^2}{dt'^2} + \omega^2 \right] \hat{x}(t') = 0 \quad \rightarrow \text{Heisenberg eq}^n \text{ of motion}$$

$$\& \left[m \frac{d^2}{dt'^2} + \omega^2 \right] T [\hat{x}(t) \hat{x}(t')] = -i \delta(t - t')$$

D free Scalar fields in Euclidean 2 dim. World Sheet

Consider a 2 dim Euclidean world sheet Embedded in a 'D' dimensional Minkowski Space time. Then the Polyakov action for the world sheet is given by,

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu)(\partial_b \chi_\mu) \bar{g}^{ab}$$

where $\sigma^a = (\sigma^1, \sigma^2)$ are world sheet coordinates. This is got by replacing the γ^{ab} of arbitrary world sheet to $\bar{g}^{ab}$ & by analytic continuation $i\tau \rightarrow \sigma^2$, where $\sigma^2 \rightarrow \sigma$ is the space-like coordinate. This is called as the Euclidean Signature.

The index ' $\mu, \nu, \dots$ ' are Minkowski indices.

There is an overall sign difference in ' S_E ' as compared to S_p of Lorentzian Signature.

Polyakov Action in Complex Coordinates for Closed Strings

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu) (\partial_b \chi_\mu) \eta^{ab}$$

$$d^2\sigma = \frac{d^2z}{2}$$

$(\partial_a \chi^\mu) (\partial_b \chi_\mu) \eta^{ab}$ becomes

$$= (\partial_z \chi^\mu) (\partial_z \chi_\mu) \underbrace{g^{zz}}_0 + (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu) g^{z\bar{z}} \\ + (\partial_{\bar{z}} \chi^\mu) (\partial_z \chi_\mu) g^{\bar{z}z} + (\partial_{\bar{z}} \chi^\mu) (\partial_{\bar{z}} \chi_\mu) \underbrace{g^{\bar{z}\bar{z}}}_0$$

$$= 2(\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu) + 2(\partial_{\bar{z}} \chi^\mu) (\partial_z \chi_\mu)$$

$$= 4(\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

$$S_E = \frac{1}{4\pi\alpha'} \int \frac{d^2z}{2} 4(\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

$$\Rightarrow \boxed{S_E = \frac{1}{2\pi\alpha'} \int d^2z (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)}$$

Eqⁿ of motion is given by varying the above action.

$$\delta S_E = \frac{1}{2\pi\alpha'} \int d^2\bar{z} \, \delta \left[(\partial_{\bar{z}} \chi^\mu) (\partial_{\bar{z}} \chi_\mu) \right]$$

$$= \frac{1}{2\pi\alpha'} \int d^2\bar{z} \left[(\partial_{\bar{z}} \chi^\mu) \delta (\partial_{\bar{z}} \chi_\mu) + (\partial_{\bar{z}} \chi_\mu) \delta (\partial_{\bar{z}} \chi^\mu) \right]$$

$$= \frac{1}{2\pi\alpha'} \int d^2\bar{z} \left[(\partial_{\bar{z}} \chi^\mu) \partial_{\bar{z}} (\delta \chi_\mu) + (\partial_{\bar{z}} \chi_\mu) \partial_{\bar{z}} (\delta \chi^\mu) \right]$$

$$= \frac{1}{2\pi\alpha'} \left[\int d\bar{z} \int d\bar{z} (\partial_{\bar{z}} \chi^\mu) \partial_{\bar{z}} (\delta \chi_\mu) + \int d\bar{z} \int d\bar{z} (\partial_{\bar{z}} \chi_\mu) \partial_{\bar{z}} (\delta \chi^\mu) \right]$$

(Integrate by parts)

$$\delta S_E = \frac{1}{2\pi\alpha'} \left[\int d\bar{z} \left[(\partial_{\bar{z}} \chi^\mu) (\delta \chi_\mu) \right]_{\text{boundary}} + \int d\bar{z} \left[(\partial_{\bar{z}} \chi_\mu) (\delta \chi^\mu) \right]_{\text{boundary}} - \int d^2 z (\delta \chi_\mu) \left[\partial_{\bar{z}} (\partial_z \chi^\mu) + \partial_z (\partial_{\bar{z}} \chi^\mu) \right] \right]$$

Under appropriate closed string boundary conditions, the surface term vanishes.

$$\rightarrow \delta S_E = \frac{-1}{2\pi\alpha'} \int d^2 z (\delta \chi_\mu) \left[\partial_{\bar{z}} (\partial_z \chi^\mu) + \partial_z (\partial_{\bar{z}} \chi^\mu) \right]$$

Thus the path taken is when $\delta S_E = 0$.
for any $\delta \chi_\mu \neq 0$.

$$\rightarrow \frac{-1}{2\pi\alpha'} \left[\partial_{\bar{z}} (\partial_z \chi^\mu) + \partial_z (\partial_{\bar{z}} \chi^\mu) \right] = 0$$

$$\text{If } \partial_{\bar{z}} (\partial_z \chi^\mu) = \partial_z (\partial_{\bar{z}} \chi^\mu)$$

$$\text{Eq}^n \text{ of motion is } \underline{\partial_z \partial_{\bar{z}} \chi^\mu(z, \bar{z}) = 0}$$

$$\text{Writing } \partial_{\bar{z}}(\partial_z \chi^\mu) = 0$$

$\rightarrow \partial_z \chi^\mu$ is Holomorphic, which means it is infinitely differentiable w.r.t z .
So it is a function of $z = \sigma_1 + i\sigma_2$ & not of $\bar{z} = \sigma_1 - i\sigma_2$.

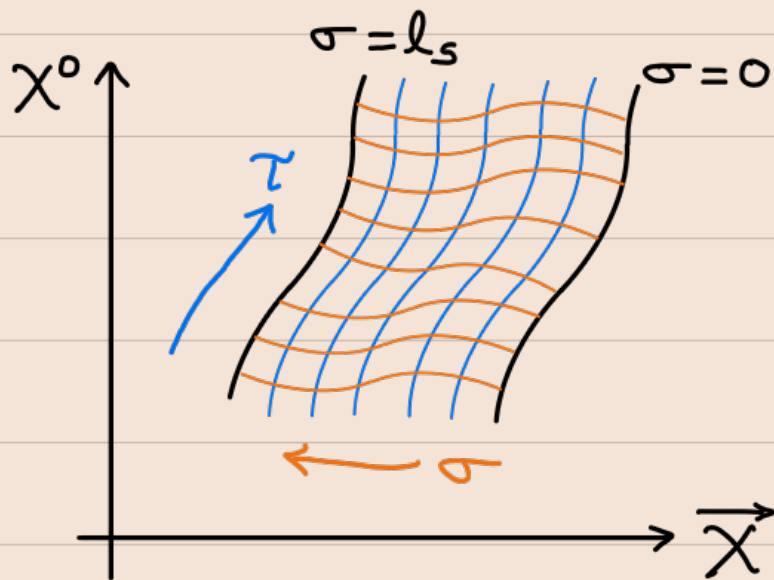
$$\text{Similarly, since } \partial_z(\partial_{\bar{z}} \chi^\mu) = 0$$

$\rightarrow \partial_{\bar{z}} \chi^\mu$ is anti-Holomorphic, which means that it is infinitely differentiable w.r.t $\bar{z}$.
So it is a function of $\bar{z} = \sigma_1 - i\sigma_2$ & not of $z = \sigma_1 + i\sigma_2$.

But by analytic continuation, $\sigma^2 = i\tau$
 $\sigma^1 = \sigma$, So $z = \sigma - \tau$ & $\bar{z} = \sigma + \tau$.

To match with the existing terminology
Holomorphic $\sim$ left-moving and
Anti-Holomorphic $\sim$ right-moving.

Then we should actually consider that ' σ^1 ' value increases from right to left.



$$\sigma_{\text{current}} = -|\sigma_{\text{Previous}}|$$

$$\rightarrow \sigma - \tau \Rightarrow -(|\sigma_{\text{Previous}}| + \tau)$$

Physically left going wave

$$\sigma + \tau \Rightarrow -|\sigma_{\text{Previous}}| + \tau$$

Physically right going wave

Path integral for Scalar χ^μ fields

The un-normalized expectation value of $F[\chi^\mu]$, a functional is given by the Path integral as

$$\langle F[\chi^\mu] \rangle = \int \mathcal{D}[\chi^\mu] F[\chi^\mu] e^{-S_E}$$

This is a path integral over all possible ' χ^μ '. And the action is written in Euclidean Signature $(+, +)$. But the usual action of the world sheet in Minkowski (Lorentzian) signature is, $S_M = \int d\tau L$ where L is the Lagrangian. Then the path integral carries a factor of ' e^{iS_M} '. But to go to Euclidean Signature, by Analytic continuation, $\tau \rightarrow -i\sigma^2$ & $\sigma \rightarrow \sigma^1$. This results in the action, given by $-i \int d\sigma^2 L$, and a factor of $e^{\int d\sigma^2 L}$. But we introduce an overall -ve sign to the action as it doesn't change the physics & eqⁿ of motion.

$$\oint S_E = -\int d\sigma^2 \mathcal{L} \Rightarrow S_E = -\int (i d\tau) \mathcal{L}$$

$$S_E = -i S_M$$

That is reason why the path integral has e^{-S_E} factor in it.

$$\mathcal{L} = \int d\sigma^1 \mathcal{L} = \frac{1}{4\pi\alpha'} \int d\sigma^1 (\partial_a \chi^\mu) (\partial_b \chi_\mu) \delta^{ab}$$

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu) (\partial_b \chi_\mu) \delta^{ab}$$

$$e^{-S_E} = \exp \left[\frac{-1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu) (\partial_b \chi_\mu) \delta^{ab} \right]$$

$$(\partial_a \chi^\mu) (\partial_b \chi_\mu) \delta^{ab} = (\partial_1 \chi^\mu) (\partial_1 \chi_\mu) + (\partial_2 \chi^\mu) (\partial_2 \chi_\mu)$$

$$= (\partial_1 \chi^\mu) (\partial_1 \chi_\mu) + (\partial_2 \chi^\mu) (\partial_2 \chi_\mu)$$

$$- (\partial_1 \chi^0)^2 + \sum_{j=1}^{D-1} (\partial_1 \chi^j)^2 - (\partial_2 \chi^0)^2 + \sum_{j=1}^{D-1} (\partial_2 \chi^j)^2$$

$$\rightarrow (\partial_a \chi^\mu)(\partial_b \chi_\mu) \delta^{ab} = \sum_{j=1}^{D-1} [(\partial_1 \chi^j)^2 + (\partial_2 \chi^j)^2] - [(\partial_1 \chi^0)^2 + (\partial_2 \chi^0)^2]$$

$$\Rightarrow e^{-S_E} = \exp \left[\frac{-1}{4\pi\alpha'} \int d^2\sigma \sum_{j=1}^{D-1} [(\partial_1 \chi^j)^2 + (\partial_2 \chi^j)^2] \right] \times \exp \left[\frac{1}{4\pi\alpha'} \int d^2\sigma [(\partial_1 \chi^0)^2 + (\partial_2 \chi^0)^2] \right]$$

Thus we see that the time part gives rise to a wrong-signed Gaussian integral. Hence, we shall do analytic continuation of $\chi^0 \rightarrow -i\chi^D$ while evaluating the path integral. So,

$$(\partial_1 \chi^\mu)(\partial_1 \chi_\mu) + (\partial_2 \chi^\mu)(\partial_2 \chi_\mu) = \sum_{j=1}^D [(\partial_1 \chi^j)^2 + (\partial_2 \chi^j)^2]$$

$$\Rightarrow e^{-S_E} = \exp \left[\frac{-1}{4\pi\alpha'} \int d^2\sigma \sum_{j=1}^D [(\partial_1 \chi^j)^2 + (\partial_2 \chi^j)^2] \right]$$

After analytic continuation of χ^0

Now, the total derivative of path-integral is zero. i.e.

$\int \mathcal{D}[\chi^\mu] e^{-S_E}$ is independent of χ^μ 's.

In complex coordinates,

$$\rightarrow \int \mathcal{D}[\chi^\mu] \frac{\partial}{\partial \chi_\mu(\bar{z}, \bar{z})} e^{-S_E} = 0$$

$$\rightarrow - \int \mathcal{D}[\chi^\mu] \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] e^{-S_E} = 0$$

$$\rightarrow \left\langle - \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] \right\rangle = 0$$

$$\rightarrow \left\langle \frac{1}{2\pi\alpha'} \left[\partial_{\bar{z}} \partial_z \chi^\mu(\bar{z}, \bar{z}) + \partial_z \partial_{\bar{z}} \chi^\mu(\bar{z}, \bar{z}) \right] \right\rangle = 0$$

Equal

$$\rightarrow \frac{1}{\pi\alpha'} \langle \partial_z \partial_{\bar{z}} \chi^\mu(\bar{z}, \bar{z}) \rangle = 0$$

Let $F[\chi^\nu(\bar{z}', \bar{z}')]$ be some function of χ^ν but at a different point $(\bar{z}', \bar{z}')$.

$$\text{Consider } \left\langle F[\chi^\nu(\bar{z}', \bar{z}')] \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] \right\rangle$$

$$= \int \mathcal{D}[\chi^\mu] \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] e^{-S_E} F[\chi^\nu(\bar{z}', \bar{z}')]$$

$$= - \int \mathcal{D}[\chi^\mu] \left[\frac{\partial e^{-S_E}}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] F[\chi^\nu(\bar{z}', \bar{z}')]$$

Int. by parts

$$= \int \mathcal{D}[\chi^\mu] e^{-S_E} \left[\frac{\partial F[\chi^\nu(\bar{z}', \bar{z}')] }{\partial \chi_\mu(\bar{z}, \bar{z})} \right]$$

If all the χ 's in F are at different points $(\bar{z}', \bar{z}')$, then this results in '0'.

$$\Rightarrow \left\langle F[\chi^\nu(\bar{z}', \bar{z}')] \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] \right\rangle = 0$$

Since, $\left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] = \frac{-1}{\pi \alpha'} \partial_{\bar{z}} \partial_{\bar{z}} \chi^\mu(\bar{z}, \bar{z})$

Thus if $z \neq z' \nmid z \neq \bar{z}'$,

$$\left\langle \mathcal{F}[\chi^\nu(\bar{z}', \bar{z}')] \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] \right\rangle = 0$$

$$\rightarrow \frac{-1}{\pi \alpha'} \left\langle \mathcal{F}[\chi^\nu(\bar{z}', \bar{z}')] \partial_{\bar{z}} \partial_{\bar{z}} \chi^\mu(\bar{z}, \bar{z}) \right\rangle = 0$$

Consider, $\langle \chi^\nu(\bar{z}', \bar{z}') \rangle = \int \mathcal{D}[\chi^\mu] \chi^\nu(\bar{z}', \bar{z}') e^{-S_E}$

$$\Rightarrow \frac{\partial \langle \chi^\nu(\bar{z}', \bar{z}') \rangle}{\partial \chi_\mu(\bar{z}, \bar{z})} = \int \mathcal{D}[\chi^\mu] \frac{\partial}{\partial \chi_\mu(\bar{z}, \bar{z})} [\chi^\nu(\bar{z}', \bar{z}') e^{-S_E}]$$

$$= \int \mathcal{D}[\chi^\mu] \left[\frac{\partial \chi^\nu(\bar{z}', \bar{z}')}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] e^{-S_E}$$

$$- \int \mathcal{D}[\chi^\mu] \chi^\nu(\bar{z}', \bar{z}') \left[\frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} \right] e^{-S_E}$$

Use, $\frac{\partial \chi^\nu(\bar{z}', \bar{z}')}{\partial \chi_\mu(\bar{z}, \bar{z})} = \eta^{\mu\nu} \partial^{(2)}(\bar{z} - \bar{z}', \bar{z} - \bar{z}')$

$$\neq \frac{\partial S_E}{\partial \chi_\mu(\bar{z}, \bar{z})} = -\frac{1}{\pi\alpha'} \partial_{\bar{z}} \partial_{\bar{z}'} \chi^\mu(\bar{z}, \bar{z})$$

Also, $\frac{\partial \langle \chi^\nu(\bar{z}', \bar{z}') \rangle}{\partial \chi_\mu(\bar{z}, \bar{z})} = 0$ (Using Previous Proof)

$$\begin{aligned} &\Rightarrow \eta^{\mu\nu} \int \mathcal{D}[\chi^\mu] \partial^{(2)}(\bar{z} - \bar{z}', \bar{z} - \bar{z}') e^{-S_E} \\ &\quad + \frac{1}{\pi\alpha'} \int \mathcal{D}[\chi^\mu] \chi^\nu(\bar{z}', \bar{z}') \partial_{\bar{z}} \partial_{\bar{z}'} \chi^\mu(\bar{z}, \bar{z}) e^{-S_E} \\ &= \eta^{\mu\nu} \int \mathcal{D}[\chi^\mu] \partial^{(2)}(\bar{z} - \bar{z}', \bar{z} - \bar{z}') e^{-S_E} \\ &\quad + \frac{1}{\pi\alpha'} \partial_{\bar{z}} \partial_{\bar{z}'} \int \mathcal{D}[\chi^\mu] \chi^\nu(\bar{z}', \bar{z}') \chi^\mu(\bar{z}, \bar{z}) e^{-S_E} \\ &= \eta^{\mu\nu} \langle \partial^{(2)}(\bar{z} - \bar{z}', \bar{z} - \bar{z}') \rangle \\ &\quad + \frac{1}{\pi\alpha'} \partial_{\bar{z}} \partial_{\bar{z}'} \langle \chi^\nu(\bar{z}', \bar{z}') \chi^\mu(\bar{z}, \bar{z}) \rangle = 0 \end{aligned}$$

$$\begin{aligned} &\frac{1}{\pi\alpha'} \partial_{\bar{z}} \partial_{\bar{z}'} \langle \chi^\nu(\bar{z}', \bar{z}') \chi^\mu(\bar{z}, \bar{z}) \rangle \\ &= -\eta^{\mu\nu} \langle \partial^{(2)}(\bar{z} - \bar{z}', \bar{z} - \bar{z}') \rangle \end{aligned}$$

Thus by Hilbert Space formalism, we can write the expectation value relations as operator equations as,

$$\partial_z \partial_{\bar{z}} \hat{X}^\mu(z, \bar{z}) = 0$$

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$$\frac{1}{\pi \alpha'} \partial_z \partial_{\bar{z}} \underbrace{[\hat{X}^\nu(z', \bar{z}') \hat{X}^\mu(z, \bar{z})]}_{\text{Should be Time ordered}} = -\eta^{\mu\nu} \partial^{(2)}(z - z', \bar{z} - \bar{z}')$$

Should be Time ordered

Normal ordering of Operators

Normal ordering in complex coordinates for X^μ fields is defined by,

$$:\hat{X}^\mu(z, \bar{z}): = \hat{X}^\mu(z, \bar{z})$$

$$:\hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') : = \hat{X}^\mu(z, \bar{z}) \hat{X}^\nu(z', \bar{z}') + \alpha' \eta^{\mu\nu} \ln|z - z'|$$

We shall see the connection of the above defⁿ of normal ordering w.r.t the usual ladder operators defⁿ later.

Ex: 2.1 : Prove $\partial_z \partial_{\bar{z}} \ln|z|^2 = 2\pi \delta^{(2)}(z, \bar{z})$

Consider ' $\partial_z \partial_{\bar{z}} \ln|z|^2$ ', going to σ^1, σ^2 ,
we get,

$$\partial_z \partial_{\bar{z}} = \frac{1}{4} (\partial_1^2 + \partial_2^2) = \frac{\nabla^2}{4}$$

& $|z|^2 = r^2$, where $r^2 = (\sigma_1)^2 + (\sigma_2)^2$

$$\rightarrow \partial_z \partial_{\bar{z}} \ln|z|^2 = \frac{\nabla^2}{4} \ln(r^2) = \lim_{\varepsilon \rightarrow 0} \frac{\nabla^2 \ln(r^2 + \varepsilon^2)}{4}$$

(Polar coordinates)

$$= \frac{1}{4} \lim_{\varepsilon \rightarrow 0} \frac{1}{r} \partial_r [r \partial_r \ln(r^2 + \varepsilon^2)]$$

$$= \lim_{\varepsilon \rightarrow 0} \frac{1}{4r} \partial_r \left[\frac{2r^2}{(r^2 + \varepsilon^2)} \right] = \lim_{\varepsilon \rightarrow 0} \underbrace{\frac{\varepsilon^2}{(r^2 + \varepsilon^2)^2}}_{\pi \delta(\sigma_1, \sigma_2)}$$

$$\rightarrow \partial_z \partial_{\bar{z}} \ln|z|^2 = \pi \delta(\sigma_1, \sigma_2) = 2\pi \delta^{(2)}(z, \bar{z})$$

$$\rightarrow \underline{\partial_z \partial_{\bar{z}} \ln|z - z'|^2 = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')} \underline{\hspace{1cm}}$$

We can also show this using the divergence theorem.

$$\partial_{\bar{z}} \partial_z \ln|z|^2 = \partial_{\bar{z}} \partial_z [\ln(z) + \ln(\bar{z})] = \partial_{\bar{z}} \left(\frac{1}{z} \right)$$

Consider an anti-holomorphic test function $\phi(\bar{z})$ & integrate it with ' $\partial_z \partial_{\bar{z}} \ln|z|^2$ '

$$\rightarrow \int_R d^2z (\partial_z \partial_{\bar{z}} \ln|z|^2) \phi(\bar{z}) = \int_R d^2z \partial_z \left(\frac{1}{z} \right) \phi(\bar{z})$$

$$= \int_R d^2z \partial_z \left(\frac{1}{z} \phi(\bar{z}) \right) \underset{\text{(divergence theorem)}}{\uparrow} = i \oint_{\partial R} d\bar{z} \frac{\phi(\bar{z})}{\bar{z}} = i [-2\pi i \phi(0)]$$

$$\rightarrow \int_R d^2z (\partial_z \partial_{\bar{z}} \ln|z|^2) \phi(\bar{z}) = 2\pi \phi(0)$$

$$\begin{aligned} \text{Similarly, } \partial_z \partial_{\bar{z}} \ln|z|^2 &= \partial_{\bar{z}} \partial_z \ln|z|^2 \\ &= \partial_{\bar{z}} \partial_z [\ln(z) + \ln(\bar{z})] = \partial_{\bar{z}} \left(\frac{1}{\bar{z}} \right) \end{aligned}$$

Then, for a Holomorphic $\phi(z)$ test function,

$$\rightarrow \int_R d^2 \bar{z} (\partial_z \partial_{\bar{z}} \ln |z|^2) \phi(z) = \int_R d^2 \bar{z} \partial_{\bar{z}} \left(\frac{1}{z} \right) \phi(\bar{z})$$

$$= \int_R d^2 \bar{z} \partial_{\bar{z}} \left(\frac{1}{z} \phi(\bar{z}) \right) = -i \oint_{\partial R} dz \frac{\phi(z)}{z}$$

(By divergence theorem) $= -i [2\pi i \phi(0)]$

$$\rightarrow \int_R d^2 \bar{z} (\partial_z \partial_{\bar{z}} \ln |z|^2) \phi(z) = 2\pi \phi(0)$$

$$\Rightarrow (\partial_z \partial_{\bar{z}} \ln |z|^2) = 2\pi \delta(z, z)$$

$$\rightarrow \underline{\partial_z \partial_{\bar{z}} \ln |z - z'|^2 = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')}$$

Then, $\partial_z \partial_{\bar{z}}$ acting on $:\hat{\chi}^\mu(z, \bar{z}) \hat{\chi}^\nu(z', \bar{z}'):$ gives,

$$\partial_z \partial_{\bar{z}} : \hat{\chi}^\mu(z, \bar{z}) \hat{\chi}^\nu(z', \bar{z}') :$$

$$= \underbrace{\partial_z \partial_{\bar{z}} \hat{\chi}^\mu(z, \bar{z}) \hat{\chi}^\nu(z', \bar{z}')} + \alpha' \eta^{\mu\nu} \partial_z \partial_{\bar{z}} \ln|z - z'|$$

$$= -\pi \alpha' \eta^{\mu\nu} \partial^{(2)}(z - z', \bar{z} - \bar{z}') + \alpha' \eta^{\mu\nu} \partial_z \partial_{\bar{z}} \ln|z - z'|$$

$$\left(\begin{array}{l} \text{By previous result,} \\ \partial_z \partial_{\bar{z}} \ln|z - z'| = \pi \partial^{(2)}(z - z', \bar{z} - \bar{z}') \end{array} \right)$$

$$= -\pi \alpha' \eta^{\mu\nu} \partial^{(2)}(z - z', \bar{z} - \bar{z}') + \underbrace{\alpha' \eta^{\mu\nu} \pi \partial^{(2)}(z - z', \bar{z} - \bar{z}')}_0$$

$$\Rightarrow \underline{\underline{\partial_z \partial_{\bar{z}} : \hat{\chi}^\mu(z, \bar{z}) \hat{\chi}^\nu(z', \bar{z}') : = 0}}$$