

## World Sheet Symmetry transformation & Noether's Theorem

Consider a general field theory with action  $S[\phi]$  in a  $d$ -dimensional world sheet, where  $\phi_A(\sigma)$  denoting a general set of fields. The indices  $\{A, B, C, \dots\}$  denote the fields. Then  $\sigma \equiv \sigma^a$  denote the usual world sheet coordinates. Let there be a transformation,

$$\phi_A(\sigma) \longrightarrow \phi'_A(\sigma) = \phi_A(\sigma) + \delta\phi_A(\sigma)$$

where  $\delta\phi_A(\sigma)$  is proportional to infinitesimal parameter ' $\epsilon$ ',  $\rightarrow \delta\phi_A(\sigma) = \epsilon \Delta\phi_A$

The euclidean path integral is given by

$$Z = \int \mathcal{D}[\phi] e^{-S[\phi]}$$

The presence of Symmetry implies that the above path integral is invariant.

Under this transformation,  $\mathcal{D}[\phi'] = \mathcal{J} \mathcal{D}[\phi]$

$$\mathcal{J} = \det \left[ \frac{\partial \phi'_B(\sigma')}{\partial \phi_A(\sigma)} \right] = \exp \left( \text{Tr} \left( \ln \left[ \frac{\partial \phi'_B(\sigma')}{\partial \phi_A(\sigma)} \right] \right) \right)$$

$$\left[ \frac{\partial \phi'_B(\sigma')}{\partial \phi_A(\sigma)} \right] = \left[ \frac{\partial (\phi_B(\sigma') + \delta \phi_B(\sigma'))}{\partial \phi_A(\sigma)} \right]$$

$$= \left[ \delta_B^A \delta(\sigma - \sigma') + \frac{\partial (\delta \phi_B(\sigma'))}{\partial \phi_A(\sigma)} \right]$$

Thus upto 1<sup>st</sup> order in ' $\epsilon$ '

$$\mathcal{J} = 1 + \text{Tr} \left[ \frac{\partial (\delta \phi_B(\sigma'))}{\partial \phi_A(\sigma)} \right] + \mathcal{O}(\epsilon^2)$$

$$\rightarrow \mathcal{J} = 1 + \int d^d \sigma \sqrt{g} \frac{\partial (\delta \phi_A(\sigma))}{\partial \phi_A(\sigma)} + \mathcal{O}(\epsilon^2)$$

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$\sqrt{g} \rightarrow$  Square root of determinant of world sheet metric tensor.

Then the action,

$$S[\phi'] = S[\phi + \delta\phi] = S[\phi] + \int d^d\sigma \sqrt{g} \frac{\delta S[\phi]}{\delta \phi_A(\sigma)} \delta\phi_A + \mathcal{O}(\epsilon^2)$$

where  $g_{ab}$  is a fixed World Sheet metric.  
→

$$e^{-S[\phi']} = e^{-S[\phi]} \left( 1 - \int d^d\sigma \sqrt{g} \frac{\delta S[\phi]}{\delta \phi_A(\sigma)} \delta\phi_A + \mathcal{O}(\epsilon^2) \right)$$

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Thus the path integral transforms as,

$$\rightarrow \int \mathcal{D}[\phi'] e^{-S[\phi']} \quad \xrightarrow{\phi'_A(\sigma) = \phi_A(\sigma) + \delta\phi_A(\sigma)}$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \left( 1 - \int d^d\sigma \sqrt{g} \frac{\delta S[\phi]}{\delta \phi_A(\sigma)} \delta\phi_A + \mathcal{O}(\epsilon^2) \right)$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \left( 1 + \int d^d\sigma \sqrt{g} \frac{\delta(\delta\phi_A(\sigma))}{\delta \phi_A(\sigma)} + \mathcal{O}(\epsilon^2) \right) \\ \times \left( 1 - \int d^d\sigma \sqrt{g} \frac{\delta S[\phi]}{\delta \phi_A(\sigma)} \delta\phi_A + \mathcal{O}(\epsilon^2) \right)$$

$$\rightarrow \int \mathcal{D}[\phi'] e^{-S[\phi']} \quad \longrightarrow \quad \phi'_A(\sigma) = \phi_A(\sigma) + \delta\phi_A(\sigma)$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \left[ 1 + \int d^d\sigma \sqrt{g} \left( \frac{\partial(\delta\phi_A(\sigma))}{\partial\phi_A(\sigma)} - \frac{\partial S[\phi]}{\partial\phi_A(\sigma)} \delta\phi_A \right) \right]$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} + \int \mathcal{D}[\phi] e^{-S[\phi]} \int d^d\sigma \sqrt{g} \frac{\partial(\delta\phi_A(\sigma))}{\partial\phi_A(\sigma)} - \int \mathcal{D}[\phi] e^{-S[\phi]} \left( \int d^d\sigma \sqrt{g} \frac{\partial S[\phi]}{\partial\phi_A(\sigma)} \delta\phi_A \right)$$

$$\text{Since } \int \mathcal{D}[\phi'] e^{-S[\phi']} = \int \mathcal{D}[\phi] e^{-S[\phi]}$$

just a change of variable

$$\rightarrow \int \mathcal{D}[\phi] e^{-S[\phi]} \left( \int d^d\sigma \sqrt{g} \frac{\partial(\delta\phi_A(\sigma))}{\partial\phi_A(\sigma)} \right)$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \left( \int d^d\sigma \sqrt{g} \frac{\partial S[\phi]}{\partial\phi_A(\sigma)} \delta\phi_A \right)$$

$$\rightarrow \left\langle \int d^d\sigma \sqrt{g} \frac{\partial(\delta\phi_A(\sigma))}{\partial\phi_A(\sigma)} \right\rangle = \left\langle \int d^d\sigma \sqrt{g} \frac{\partial S[\phi]}{\partial\phi_A(\sigma)} \delta\phi_A \right\rangle$$

If  $\partial\phi_A(\sigma)$  is a constant w.r.t the coordinates ' $\sigma$ ', then  $\frac{\partial(\partial\phi_A(\sigma))}{\partial\phi_A(\sigma)} = 0$

$\rightarrow \frac{\partial S[\phi]}{\partial\phi_A(\sigma)} = 0 \rightarrow$  There is a symmetry

Let  $\phi'_A(\sigma) = \phi_A(\sigma) + \epsilon P(\sigma) \Delta\phi_A$ , which makes the transformation local. Then, in general  $\frac{\partial S[\phi]}{\partial\phi_A(\sigma)}$  need not be zero.

Then the change in the path integral should vary proportional to  $\partial_\alpha P(\sigma)$ .

Thus we can write  $\int \mathcal{D}[\phi'] e^{-S[\phi']}$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \left( 1 + \frac{i\epsilon}{2\pi} \int d\sigma \sqrt{g} J^a(\sigma) \partial_\alpha P(\sigma) \right)$$

where  $J^a(\sigma)$  is some unknown constant coming from the variation of action & path integral measure. Since the change is local,  $J^a(\sigma)$  is also local, & depends on the fields & its derivatives.

Let  $P(\sigma)$  be non-zero in some region 'R' & zero outside 'R'. Then we can include insertions '.....' that depend on points which are outside 'R'. Thus these insertions are invariant under transformation. Then,

$$\int \mathcal{D}[\phi'] e^{-S[\phi']} \dots = \int \mathcal{D}[\phi] e^{-S[\phi]} \dots + \int \mathcal{D}[\phi] e^{-S[\phi]} \left( \frac{i\epsilon}{2\pi} \int_R d^d \sigma \sqrt{g} J^a(\sigma) \partial_a P(\sigma) \dots \right)$$

$$\rightarrow \int \mathcal{D}[\phi] e^{-S[\phi]} \left( \frac{i\epsilon}{2\pi} \int_R d^d \sigma \sqrt{g} J^a(\sigma) \partial_a P(\sigma) \dots \right) = 0$$

Integrate by parts

$$\int_R d^d \sigma \sqrt{g} J^a(\sigma) \partial_a P(\sigma) = \sqrt{g} J^a(\sigma) \partial_a P(\sigma) \Big|_{\text{boundary}} - \int_R d^d \sigma P(\sigma) \partial_a [\sqrt{g} J^a(\sigma)]$$

$P(\sigma) = 0$  at boundary.

$$\begin{aligned} \rightarrow \int_{\mathcal{R}} d^d \sigma \sqrt{g} J^a(\sigma) \partial_a \rho(\sigma) \\ = - \int_{\mathcal{R}} d^d \sigma \sqrt{g} \rho(\sigma) \left( \frac{1}{\sqrt{g}} \partial_a [\sqrt{g} J^a(\sigma)] \right) \end{aligned}$$

$$\nabla_a J^a(\sigma) = \frac{1}{\sqrt{g}} \partial_a [\sqrt{g} J^a(\sigma)] \rightarrow \text{Covariant divergence}$$

$$\rightarrow \int_{\mathcal{R}} d^d \sigma \sqrt{g} J^a(\sigma) \partial_a \rho(\sigma) = - \int_{\mathcal{R}} d^d \sigma \sqrt{g} \rho(\sigma) \nabla_a J^a(\sigma)$$

$$\rightarrow \int \mathcal{D}[\phi] \bar{e}^{-S[\phi]} \left( \frac{-i\epsilon}{2\pi} \int_{\mathcal{R}} d^d \sigma \sqrt{g} \rho(\sigma) \nabla_a J^a(\sigma) \dots \right) = 0$$

This can be written as,

$$\frac{-i\epsilon}{2\pi} \int_{\mathcal{R}} d^d \sigma \sqrt{g} \rho(\sigma) \left( \int \mathcal{D}[\phi] \bar{e}^{-S[\phi]} \nabla_a J^a(\sigma) \dots \right) = 0$$

The above is an operator equation.

$$\Rightarrow \frac{\epsilon}{2\pi i} \int_{\mathcal{R}} d^d \sigma \sqrt{g} \rho(\sigma) \langle \nabla_a J^a(\sigma) \dots \rangle = 0$$

Since  $\epsilon \neq P(\sigma)$  is arbitrary, we get,

$$\Rightarrow \boxed{\frac{1}{2\pi i} \langle \nabla_a J^a(\sigma) \dots \rangle = 0}$$

This is the Noether's theorem.

Now we shall derive the form of  $J^a(\sigma)$ .  
Let the action be written as,

$$S[\phi] = \int d^d \sigma \sqrt{g} \mathcal{L}(\phi_A(\sigma), \partial_a \phi_A(\sigma))$$

Now, under  $\phi_A(\sigma) \rightarrow \phi'_A(\sigma) = \phi_A(\sigma) + \delta \phi_A(\sigma)$

$$\delta S[\phi] = \int d^d \sigma \sqrt{g} \delta \mathcal{L}(\phi_A(\sigma), \partial_a \phi_A(\sigma))$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} \delta \phi_A(\sigma) + \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \delta (\partial_a \phi_A(\sigma))$$

Now,  $\delta (\partial_a \phi_A(\sigma)) = \partial_a (\delta \phi_A(\sigma))$ , & then, we can do integration by parts in the 2nd term.

$$\rightarrow \int d^d \sigma \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \delta (\partial_a \phi_A(\sigma))$$

$$= \delta \phi_A(\sigma) \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \Big|_{\text{boundary}}$$

$$- \int d^d \sigma \delta \phi_A(\sigma) \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right)$$

Keeping the variations in the boundary fixed, we discard the surface term.

$$\rightarrow \delta S[\phi] = \int d^d \sigma \sqrt{g} \frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} \delta \phi_A(\sigma)$$

$$- \int d^d \sigma \delta \phi_A(\sigma) \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right)$$

$$= \int d^d \sigma \sqrt{g} \delta \phi_A(\sigma) \left[ \frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} - \frac{1}{\sqrt{g}} \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) \right]$$

$$\text{Let } \frac{1}{\sqrt{g}} \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) = \nabla_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right)$$

be the covariant derivative.

$$\rightarrow \delta S[\phi] = \int d^d \sigma \sqrt{g} \delta \phi_A(\sigma) \left[ \frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} - \nabla_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) \right]$$

Thus for eq<sup>n</sup> of motion, we need  $\delta S[\phi]$ , by the variational principle, for any ' $\delta \phi_A(\sigma)$ '

$$\rightarrow \text{EoM is, } \left[ \frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} - \nabla_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) \right] = 0$$

Now, consider a different setting. Let ' $\phi_A(\sigma)$ ' be the solution to the EoM, but we shall consider a symmetry transform,  $\phi_A(\sigma) \rightarrow \phi'_A(\sigma) = \phi_A(\sigma) + \delta \phi_A(\sigma)$ . Then we need the action to be invariant under this transformation for the symmetry to exist. Thus, for this to hold, the variation of ' $\mathcal{L}$ ' utmost can be a four divergence.

$$\delta S[\phi] = \int d^d \sigma \sqrt{g} \delta \mathcal{L}(\phi_A(\sigma), \partial_a \phi_A(\sigma))$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} \delta \phi_A(\sigma) + \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \delta (\partial_a \phi_A(\sigma))$$

Now,  $\delta (\partial_a \phi_A(\sigma)) = \partial_a (\delta \phi_A(\sigma))$ , and then, by Lagrange eq<sup>n</sup> of motion

$$\frac{\partial \mathcal{L}}{\partial \phi_A(\sigma)} = \nabla_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right)$$

$$\rightarrow \delta S[\phi] = \int d^d \sigma \sqrt{g} \left[ \nabla_a \left( \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) \delta \phi_A(\sigma) + \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \partial_a (\delta \phi_A(\sigma)) \right]$$

$$= \int d^d \sigma \left[ \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) \delta \phi_A(\sigma) + \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \partial_a (\delta \phi_A(\sigma)) \right]$$

$$\rightarrow \delta S[\phi] = \int d^d \sigma \sqrt{g} \left[ \frac{1}{\sqrt{g}} \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \delta \phi_A(\sigma) \right) \right]$$

Let  $\delta \phi_A(\sigma) = \epsilon \Delta \phi_A(\sigma)$ . Then, if

$\delta S[\phi] = \int d^d \sigma \partial_a (\sqrt{g} \epsilon \mathcal{K}^a)$  for some  $\mathcal{K}^a$ ,  
then we get  $\delta S[\phi] = 0$ . Thus we need

$$\delta \mathcal{L}(\phi_A(\sigma), \partial_a \phi_A(\sigma)) = \partial_a (\sqrt{g} \epsilon \mathcal{K}^a)$$

$$\rightarrow \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \epsilon \Delta \phi_A(\sigma) \right) = \partial_a (\sqrt{g} \epsilon \mathcal{K}^a)$$

$$\rightarrow \frac{\epsilon}{2\pi i} \partial_a \left( 2\pi i \sqrt{g} \left[ \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \Delta \phi_A(\sigma) - \mathcal{K}^a \right] \right) = 0$$

$$\text{Let } \mathcal{J}^a = 2\pi i \left[ \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \Delta \phi_A(\sigma) - \mathcal{K}^a \right]$$

$$\Rightarrow \frac{\epsilon}{2\pi i} \nabla_a \mathcal{J}^a(\sigma) = 0$$

But, if  $\delta\phi_A(\sigma) = \epsilon P(\sigma) \Delta\phi_A(\sigma)$

$$\Rightarrow \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_A(\sigma)} \delta\phi_A(\sigma) + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \delta(\partial_a\phi_A(\sigma))$$

$$= \frac{\partial\mathcal{L}}{\partial\phi_A(\sigma)} (\epsilon P(\sigma) \Delta\phi_A(\sigma)) + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \partial_a(\epsilon P(\sigma) \Delta\phi_A(\sigma))$$

$$= \left[ \frac{\partial\mathcal{L}}{\partial\phi_A(\sigma)} (\epsilon \Delta\phi_A(\sigma)) + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \partial_a(\epsilon \Delta\phi_A(\sigma)) \right] P(\sigma)$$

$$+ \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \epsilon \Delta\phi_A(\sigma) (\partial_a P(\sigma))$$

Use EoM in 1<sup>st</sup> term

$$\rightarrow \delta S[\phi] = \int d^d\sigma \sqrt{g} \left[ \nabla_a \left( \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \right) (\epsilon \Delta\phi_A(\sigma)) P(\sigma) \right. \\ \left. + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \partial_a(\epsilon \Delta\phi_A(\sigma)) P(\sigma) \right. \\ \left. + \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_A(\sigma))} \epsilon \Delta\phi_A(\sigma) (\partial_a P(\sigma)) \right]$$

$$\rightarrow \delta S[\phi] = \int d^d \sigma \left[ \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \right) (\epsilon \Delta \phi_A(\sigma)) P(\sigma) \right. \\ \left. + \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \partial_a (\epsilon \Delta \phi_A(\sigma)) P(\sigma) \right. \\ \left. + \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \epsilon \Delta \phi_A(\sigma) (\partial_a P(\sigma)) \right]$$

$$\rightarrow \delta S[\phi] = \int d^d \sigma \left[ \partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} (\epsilon \Delta \phi_A(\sigma)) \right) P(\sigma) \right. \\ \left. + \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \epsilon \Delta \phi_A(\sigma) (\partial_a P(\sigma)) \right]$$

Use,

$$\partial_a \left( \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \epsilon \Delta \phi_A(\sigma) \right) = \partial_a (\sqrt{g} \epsilon \mathcal{K}^a)$$

from the previous result.

$$\rightarrow \delta S[\phi] = \int d^d \sigma \left[ \partial_a (\sqrt{g} \epsilon \mathcal{K}^a) P(\sigma) \right. \\ \left. + \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \epsilon \Delta \phi_A(\sigma) (\partial_a P(\sigma)) \right]$$

Perform integration by parts on 1<sup>st</sup> term.  
 & take  $P(\sigma)=0$  at boundary.

$$\rightarrow \delta S[\phi] = \int d^d \sigma \left[ -(\sqrt{g} \epsilon \kappa^a) \partial_a P(\sigma) + \sqrt{g} \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \epsilon \Delta \phi_A(\sigma) (\partial_a P(\sigma)) \right]$$

$$= \int d^d \sigma \sqrt{g} \epsilon \left[ -\kappa^a + \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \Delta \phi_A(\sigma) \right] (\partial_a P(\sigma))$$

$$\Rightarrow \delta S[\phi] = \frac{\epsilon}{2\pi i} \int d^d \sigma \sqrt{g} \mathcal{T}^a (\partial_a P(\sigma))$$


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Thus,

$$\mathcal{T}^a = 2\pi i \left[ \frac{\partial \mathcal{L}}{\partial (\partial_a \phi_A(\sigma))} \Delta \phi_A(\sigma) - \kappa^a \right]$$

where  $\partial_a (\sqrt{g} \epsilon \kappa^a) = \partial \mathcal{L}(\phi_A(\sigma), \partial_a \phi_A(\sigma))$

## Ward Identity

Consider the usual path integral in a region 'R' but with an insertion of some general local operator  $\hat{O}(\sigma_0)$ , where  $\sigma_0 \in R$ .

$$\langle \hat{O}(\sigma_0) \dots \rangle = \int \mathcal{D}[\phi] \hat{O}(\sigma_0) e^{-S[\phi]} \dots$$

Now, we wish to see the consequence of  $\phi_A(\sigma) \rightarrow \phi'_A(\sigma) = \phi_A(\sigma) + \epsilon P(\sigma) \Delta \phi_A$ , where

$$P(\sigma) = \begin{cases} 0, & \text{outside } R \\ 1, & \text{inside } R \end{cases}$$

The insertions '.....' are outside R. Now,

$$\langle \hat{O}'(\sigma_0) \dots \rangle = \int \mathcal{D}[\phi'] \hat{O}'(\sigma_0) e^{-S[\phi']} \dots$$

The operator  $\hat{O}(\sigma_0)$  also changes under the symmetry transform as

$$\hat{O}(\sigma_0) \rightarrow \hat{O}'(\sigma_0) = \hat{O}(\sigma_0) + \epsilon P(\sigma_0) \Delta \hat{O}(\sigma_0)$$

Now,

$$\int \mathcal{D}[\phi'] \hat{\Theta}'(\sigma_0) e^{-S[\phi']} \dots = \int \mathcal{D}[\phi] e^{-S[\phi]} \hat{\Theta}(\sigma_0) \dots$$

just a change of variable

In the same procedure as before,

$$\rightarrow \langle \hat{\Theta}(\sigma_0) \dots \rangle = \langle \hat{\Theta}'(\sigma_0) \dots \rangle$$

$$= \int \mathcal{D}[\phi'] \hat{\Theta}'(\sigma_0) e^{-S[\phi']} \dots$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \left( 1 + \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d\sigma \sqrt{g} J^a(\sigma) \partial_a \rho(\sigma) \right) \\ \times [\hat{\Theta}(\sigma_0) + \epsilon \rho(\sigma) \Delta \hat{\Theta}(\sigma_0)] \dots$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \hat{\Theta}(\sigma_0) \dots$$

$$+ \int \mathcal{D}[\phi] e^{-S[\phi]} \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d\sigma \sqrt{g} J^a(\sigma) \partial_a \rho(\sigma) \hat{\Theta}(\sigma_0) \dots$$

$$+ \int \mathcal{D}[\phi] e^{-S[\phi]} \epsilon \rho(\sigma) \Delta \hat{\Theta}(\sigma_0) \dots$$

$$\rightarrow \langle \hat{\Theta}(\sigma_0) \dots \rangle = \langle \hat{\Theta}(\sigma_0) \dots \rangle$$

$$+ \int \mathcal{D}[\phi] e^{-S[\phi]} \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d\sigma \sqrt{g} T^a(\sigma) \partial_a \rho(\sigma) \hat{\Theta}(\sigma_0) \dots$$

$$+ \int \mathcal{D}[\phi] e^{-S[\phi]} \epsilon \rho(\sigma) \Delta \hat{\Theta}(\sigma_0) \dots$$

$$\rightarrow \int \mathcal{D}[\phi] e^{-S[\phi]} \epsilon \rho(\sigma) \Delta \hat{\Theta}(\sigma_0) \dots$$

$$= - \int \mathcal{D}[\phi] e^{-S[\phi]} \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d\sigma \sqrt{g} T^a(\sigma) \partial_a \rho(\sigma) \hat{\Theta}(\sigma_0) \dots$$

Integrate by parts

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d\sigma \rho(\sigma) \partial_a [\sqrt{g} T^a(\sigma) \hat{\Theta}(\sigma_0) \dots]$$

Using def<sup>n</sup> of covariant divergence &  $\rho(\sigma) = 1$  inside the region, we get,

$$\int \mathcal{D}[\phi] e^{-S[\phi]} \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d\sigma \sqrt{g} \nabla_a [T^a(\sigma) \hat{\Theta}(\sigma_0) \dots]$$

$$= \int \mathcal{D}[\phi] e^{-S[\phi]} \epsilon \Delta \hat{\Theta}(\sigma_0) \dots$$

$$\text{Let } \delta \hat{\Theta}(\sigma_0) = \epsilon \Delta \hat{\Theta}(\sigma_0),$$

$$\Rightarrow \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d\sigma \sqrt{g} \left\langle \nabla_a [T^a(\sigma) \hat{\Theta}(\sigma_0) \dots] \right\rangle = \langle \delta \hat{\Theta}(\sigma_0) \dots \rangle$$

This is the Ward Identity. It can also be written as an operator relation as,

$$\frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d\sigma \sqrt{g} \nabla_a [T^a(\sigma) \hat{\Theta}(\sigma_0)] = \delta \hat{\Theta}(\sigma_0)$$

The above relation can be inverted as,

$$\rightarrow \nabla_a [T^a(\sigma) \hat{\Theta}(\sigma_0)] = \frac{2\pi}{i\epsilon \sqrt{g}} \delta(\sigma - \sigma_0) \delta \hat{\Theta}(\sigma) + \text{total } \sigma\text{-derivative}$$

where the total  $\sigma$ -derivative is of the form  $\nabla_a f^a = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} f^a)$ , for some ' $f^a$ ', that vanish at boundary.

$$\int_{\mathcal{R}} d\sigma \sqrt{g} \nabla_a f^a = \int_{\mathcal{R}} d\sigma \partial_a (\sqrt{g} f^a) = \int_{\partial \mathcal{R}} da \underbrace{n_a \sqrt{g} f^a}_{=0} = 0$$

$$\rightarrow \int_{\mathcal{R}} d^d \sigma \sqrt{g} \nabla_a [T^a(\sigma) \hat{\Theta}(\sigma_0)] = \frac{2\pi}{i\epsilon} \int_{\mathcal{R}} d^d \sigma \partial \hat{\Theta}(\sigma) \partial(\sigma - \sigma_0) + \underbrace{\int_{\mathcal{R}} d^d \sigma \sqrt{g} \nabla_a f^a}_0$$

$$\rightarrow \underline{\frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d \sigma \sqrt{g} \nabla_a [T^a(\sigma) \hat{\Theta}(\sigma_0)] = \partial \hat{\Theta}(\sigma_0)}$$

Now, we shall use the divergence theorem

$$\rightarrow \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d \sigma \partial_a [\sqrt{g} T^a(\sigma) \hat{\Theta}(\sigma_0)] = \partial \hat{\Theta}(\sigma_0)$$

$$\rightarrow \frac{i\epsilon}{2\pi} \int_{\partial \mathcal{R}} d^{d-1} \sigma \sqrt{g} n_a (T^a(\sigma) \hat{\Theta}(\sigma_0)) = \partial \hat{\Theta}(\sigma_0)$$

( $n_a \rightarrow$  outward normal)

$$\rightarrow \frac{i\epsilon}{2\pi} \int_{\partial \mathcal{R}} da n_a (T^a(\sigma) \hat{\Theta}(\sigma_0)) = \partial \hat{\Theta}(\sigma_0)$$

$$\rightarrow \boxed{\int_{\partial \mathcal{R}} da n_a T^a(\sigma) \hat{\Theta}(\sigma_0) = \frac{2\pi}{i\epsilon} \partial \hat{\Theta}(\sigma_0)}$$

In 2D, if the curve is parameterized by 'u'

$$\rightarrow \oint_{\partial \mathcal{R}} n_a T^a(\sigma) \hat{\mathcal{O}}(\sigma_0) = \frac{2\pi}{i\epsilon} \delta \hat{\mathcal{O}}(\sigma_0)$$

but  $n^a du = (d\sigma^2, -d\sigma^1)$  seen previously.

$$\rightarrow \oint_{\partial \mathcal{R}} (T_1 d\sigma^2 - T_2 d\sigma^1) \hat{\mathcal{O}}(\sigma_0) = \frac{2\pi}{i\epsilon} \delta \hat{\mathcal{O}}(\sigma_0)$$

Converting to complex coordinates,

$$d\sigma^1 = \frac{1}{2}(d\bar{z} + dz), \quad d\sigma^2 = \frac{i}{2}(d\bar{z} - dz)$$

$$T_1 = T_{\bar{z}} + T_z, \quad T_2 = i(T_z - T_{\bar{z}})$$

$$\rightarrow T_1 d\sigma^2 - T_2 d\sigma^1 = i(T_{\bar{z}} d\bar{z} - T_z dz)$$

$$\& \quad \sigma_0 = (z_0, \bar{z}_0)$$

$$\rightarrow \oint_{\partial \mathcal{R}} i(T_{\bar{z}} d\bar{z} - T_z dz) \hat{\mathcal{O}}(z_0, \bar{z}_0) = \frac{2\pi}{i\epsilon} \delta \hat{\mathcal{O}}(z_0, \bar{z}_0)$$

$$\rightarrow \boxed{\oint_{\partial \mathcal{R}} (T_z dz - T_{\bar{z}} d\bar{z}) \hat{\mathcal{O}}(z_0, \bar{z}_0) = \frac{2\pi}{\epsilon} \delta \hat{\mathcal{O}}(z_0, \bar{z}_0)}$$

Note that the in lorentzian world sheet  $J^a = (J^0, J^1)$ , they are physical observables, so  $J^0 \neq J^1$  are Hermitian. So  $J^2 = iJ^0$  is actually anti-Hermitian. Thus,

$$J_z = \frac{1}{2} (J_1 - iJ_2)$$

$$J_{\bar{z}} = \frac{1}{2} (J_1 + iJ_2)$$

$\left( \begin{array}{l} J_2 \text{ is} \\ \text{anti-hermitian} \end{array} \right)$

$$\rightarrow \underline{J_z^* = J_z \quad \neq \quad J_{\bar{z}}^* = J_{\bar{z}}}$$

If  $\hat{O}(\sigma_0)$  was not inserted in the path integral, then we would get Noether's theorem in complex coordinates as,

$$\boxed{\frac{1}{2\pi i} \langle \partial_z J^{\bar{z}} + \partial_{\bar{z}} J^z \rangle = 0}$$

The above is a operator eq<sup>n</sup>.

If  $T^z$  is holomorphic and  $T^{\bar{z}}$  is anti-holomorphic, then the currents given by,  $(T^z, 0) \neq (0, T^{\bar{z}})$  are separately conserved. Also, then

$$\underline{\partial_z T^{\bar{z}} = \partial_{\bar{z}} T^z = 0}$$

So,  $T^z$  is infinitely differentiable in  $z$   
 $\neq T^{\bar{z}}$  is infinitely differentiable in  $\bar{z}$ .

If there are no operator insertions in region of interest, then there are no poles for  $T^z \neq T^{\bar{z}}$ .

But, if there are any operator insertions in the region of interest, then

$\oint_{\partial R} dz T_z \hat{O}(z_0, \bar{z}_0)$  picks out the

residue of OPE ' $T_z \hat{O}(z_0, \bar{z}_0)$ ' as  $z \rightarrow z_0$ .

Similarly,  $\oint_{\partial\mathcal{R}} d\bar{z} J_{\bar{z}} \hat{\mathcal{O}}(z_0, \bar{z}_0)$  picks out the residue of OPE  $J_{\bar{z}} \hat{\mathcal{O}}(z_0, \bar{z}_0)$  as  $\bar{z} \rightarrow \bar{z}_0$ .

$$\rightarrow \oint_{\partial\mathcal{R}} J_z \hat{\mathcal{O}}(z_0, \bar{z}_0) dz = 2\pi i \operatorname{Res}_{z \rightarrow z_0} [J_z \hat{\mathcal{O}}(z_0, \bar{z}_0)]$$

$$\& \oint_{\partial\mathcal{R}} J_{\bar{z}} \hat{\mathcal{O}}(z_0, \bar{z}_0) d\bar{z} = -2\pi i \overline{\operatorname{Res}_{\bar{z} \rightarrow \bar{z}_0} [J_{\bar{z}} \hat{\mathcal{O}}(z_0, \bar{z}_0)]}$$

$$\text{So, } \operatorname{Res}_{z \rightarrow z_0} [J_z \hat{\mathcal{O}}(z_0, \bar{z}_0)] \& \overline{\operatorname{Res}_{\bar{z} \rightarrow \bar{z}_0} [J_{\bar{z}} \hat{\mathcal{O}}(z_0, \bar{z}_0)]} \text{ are}$$

the coefficients of  $(z - z_0)^{-1}$  &  $(\bar{z} - \bar{z}_0)^{-1}$  in OPE of  $J_z \hat{\mathcal{O}}(z_0, \bar{z}_0)$ .

Then the Ward identity in 2D complex coordinates become,

$$\boxed{\operatorname{Res}_{z \rightarrow z_0} [J_z \hat{\mathcal{O}}(z_0, \bar{z}_0)] + \overline{\operatorname{Res}_{\bar{z} \rightarrow \bar{z}_0} [J_{\bar{z}} \hat{\mathcal{O}}(z_0, \bar{z}_0)]} = \frac{\partial \hat{\mathcal{O}}(z_0, \bar{z}_0)}{\partial \epsilon}}$$

## Application to Massless Scalar $\chi^\mu$ fields

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu) (\partial_b \chi_\mu) \delta^{ab}$$

### Space-time translations

Under space-time translations,

$$\chi_\mu \longrightarrow \chi'_\mu = \chi_\mu + \epsilon c_\mu, \quad \delta S_E = 0.$$

But if,  $\chi_\mu \longrightarrow \chi'_\mu = \chi_\mu + \epsilon \rho(\sigma) c_\mu$

$$\rightarrow \delta \chi_\mu = \epsilon \rho(\sigma) c_\mu \quad \& \quad \Delta \chi_\mu = \rho(\sigma) c_\mu$$

$$\rightarrow \delta (\partial_a \chi^\mu) (\partial_a \chi_\mu) = 2 (\partial^a \chi^\mu) \partial_a (\delta \chi_\mu)$$

$$\text{Then } \delta S_E = \frac{\epsilon c_\mu}{2\pi\alpha'} \int_{\mathcal{R}} d^2\sigma (\partial^a \chi^\mu) \partial_a \rho(\sigma)$$

$$\text{But, } \delta S = \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d\sigma \underbrace{\sqrt{g}}_1 T^a(\sigma) \partial_a \rho(\sigma)$$

$$\rightarrow T_a = \frac{i}{\alpha'} c_\mu (\partial_a \chi^\mu)$$

We define field Specific Noether current as,

$$\underline{J_a^\mu = \frac{i}{\alpha'} \partial_a X^\mu}$$

$J_a^\mu$  is the noether current when only  $X^\mu$  is translated.

$$J_z^\mu = \frac{1}{2} (J_1^\mu - i J_2^\mu) = \frac{i}{2\alpha'} (\partial_1 X^\mu - i \partial_2 X^\mu)$$

$$\Rightarrow \underline{J_z^\mu = \frac{i}{\alpha'} \partial_z X^\mu(z, \bar{z})}$$

$$\text{Similarly, } J_{\bar{z}}^\mu = \frac{1}{2} (J_1^\mu + i J_2^\mu)$$

$$\Rightarrow \underline{J_{\bar{z}}^\mu = \frac{i}{\alpha'} \partial_{\bar{z}} X^\mu(z, \bar{z})}$$

The eq<sup>n</sup> of motion for  $X^\mu(z, \bar{z})$  is

$$\partial_{\bar{z}} \partial_z X^\mu(z, \bar{z}) = 0 = \partial_z \partial_{\bar{z}} X^\mu(z, \bar{z})$$

$\rightarrow \partial_z X^\mu(z, \bar{z})$  is Holomorphic &  
 $\partial_{\bar{z}} X^\mu(z, \bar{z})$  is anti-Holomorphic.

Thus for space-time translations of Polyakov (Massless  $X^\mu$  field theory), the Noether currents are such that,

$$J_z^\mu = \frac{i}{\alpha'} \partial_z X^\mu(z, \bar{z}) \text{ is Holomorphic}$$

$$\& J_{\bar{z}}^\mu = \frac{i}{\alpha'} \partial_{\bar{z}} X^\mu(z, \bar{z}) \text{ is anti-Holomorphic}$$

Consider an operator  $e^{ik^\mu X_\mu(z_0, \bar{z}_0)}$ , now we shall try to find OPE of

$$J_z^\mu : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} : = \frac{i}{\alpha'} \partial_z X^\mu(z, \bar{z}) : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} :$$

$$= \frac{i}{\alpha'} : \partial_z X^\mu(z, \bar{z}) : \sum_{n=0}^{\infty} \frac{(ik_\nu)^n}{n!} : [X^\nu(z_0, \bar{z}_0)]^n :$$

$$= \frac{i}{\alpha'} \sum_{n=0}^{\infty} \frac{(ik_\nu)^n}{n!} : \partial_z X^\mu(z, \bar{z}) : : [X^\nu(z_0, \bar{z}_0)]^n :$$

Now,  $\underbrace{X^\mu(z, \bar{z}) X^\nu(z_0, \bar{z}_0)}_{\text{Contraction}} = -\frac{\alpha' \eta^{\mu\nu}}{2} \ln |z - z_0|^2$

$$\rightarrow : \partial_{\bar{z}} X^{\mu}(z, \bar{z}) :: X^{\nu}(z_0, \bar{z}_0) : = - \frac{\alpha' \eta^{\mu\nu}}{2} \partial_{\bar{z}} \ln |z - z_0|^2$$

$$\rightarrow : \partial_{\bar{z}} X^{\mu}(z, \bar{z}) :: X^{\nu}(z_0, \bar{z}_0) : = - \frac{\alpha' \eta^{\mu\nu}}{2(z - z_0)}$$

$$\rightarrow k_{\nu} : \partial_{\bar{z}} X^{\mu}(z, \bar{z}) :: X^{\nu}(z_0, \bar{z}_0) : = - \frac{\alpha' k^{\mu}}{2(z - z_0)}$$

$$\rightarrow k_{\nu_1} k_{\nu_2} \dots k_{\nu_n} : \partial_{\bar{z}} X^{\mu}(z, \bar{z}) :: X^{\nu_1} X^{\nu_2} \dots X^{\nu_n}(z_0, \bar{z}_0) :$$

$$= - \frac{\alpha' k^{\mu}}{2(z - z_0)} k_{\nu_2} k_{\nu_3} \dots k_{\nu_n} : X^{\nu_2} X^{\nu_3} \dots X^{\nu_n}(z_0, \bar{z}_0) :$$

$$- \frac{\alpha' k^{\mu}}{2(z - z_0)} k_{\nu_1} k_{\nu_3} \dots k_{\nu_n} : X^{\nu_1} X^{\nu_3} \dots X^{\nu_n}(z_0, \bar{z}_0) :$$

⋮

$$- \frac{\alpha' k^{\mu}}{2(z - z_0)} k_{\nu_1} k_{\nu_2} \dots k_{\nu_{n-1}} : X^{\nu_1} X^{\nu_3} \dots X^{\nu_{n-1}}(z_0, \bar{z}_0) :$$

$$\rightarrow : \partial_{\bar{z}} X^{\mu}(z, \bar{z}) :: \left[ k_{\nu} X^{\nu}(z, \bar{z}_0) \right]^n :$$

$$= - \frac{\alpha' k^{\mu}}{2(z - z_0)} (n-1) : \left[ k_{\nu} X^{\nu}(z, \bar{z}_0) \right]^{n-1} :$$

$$\rightarrow J_z^\mu : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} : =$$

$$= \frac{i}{\alpha'} \sum_{n=0}^{\infty} \frac{i^n}{n!} : \partial_z X^\mu(z, \bar{z}) : : [k_\nu X^\nu(z_0, \bar{z}_0)]^n :$$

$$= \frac{i}{\alpha'} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[ \frac{-\alpha' k^\mu (n-1)}{2(z-z_0)} \right] : [k_\nu X^\nu(z_0, \bar{z}_0)]^{n-1} :$$

$$= \frac{k^\mu}{2(z-z_0)} \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} : [k_\nu X^\nu(z_0, \bar{z}_0)]^{n-1} :$$

$$\rightarrow J_z^\mu : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} : = \frac{k^\mu}{2(z-z_0)} : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} :$$


---

$$\rightarrow \text{Res}_{z \rightarrow z_0} [J_z^\mu e^{ik_\nu X^\nu(z_0, \bar{z}_0)}] = \frac{k^\mu e^{ik_\nu X^\nu(z_0, \bar{z}_0)}}{2}$$

Similarly,

$$J_{\bar{z}}^\mu : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} : = \frac{k^\mu}{2(\bar{z}-\bar{z}_0)} : e^{ik_\nu X^\nu(z_0, \bar{z}_0)} :$$


---

$$\rightarrow \overline{\text{Res}}_{\bar{z} \rightarrow \bar{z}_0} [J_{\bar{z}}^\mu e^{ik_\nu X^\nu(z_0, \bar{z}_0)}] = \frac{k^\mu e^{ik_\nu X^\nu(z_0, \bar{z}_0)}}{2}$$

Now, we shall try to show this using Ward identity.

Under Space-time translations,

$$\chi_\mu \longrightarrow \chi'_\mu = \chi_\mu + \epsilon C_\mu$$

$$e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)} \longrightarrow e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)} (1 + i\epsilon k^\nu C_\nu)$$

$$\Rightarrow \partial e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)} = i\epsilon k^\nu C_\nu e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)}$$

$$\begin{aligned} \text{Res}_{z \rightarrow z_0} [J_z e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)}] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{z}_0} [J_{\bar{z}} e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)}]} \\ = \frac{\partial e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)}}{i\epsilon} = k^\nu C_\nu e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)} \end{aligned}$$

$$\text{Since } J_z = J_z^\nu C_\nu \text{ \& } J_{\bar{z}} = J_{\bar{z}}^\nu C_\nu$$

$$\begin{aligned} \rightarrow \text{Res}_{z \rightarrow z_0} [J_z e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)}] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{z}_0} [J_{\bar{z}} e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)}]} \\ = k^\mu e^{ik^\nu \chi_\nu(z_0, \bar{z}_0)} \end{aligned}$$

---

Thus we got the same results.

## World Sheet translations

Consider the action in arbitrary world sheet coordinates.

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} (\partial_a \chi^\mu) (\partial_b \chi_\mu)$$

Under  $\sigma^a \rightarrow \sigma^a + \epsilon C^a$ , the action remains invariant since  $\partial \sigma^a = \text{constant}$ . But if  $\sigma^a \rightarrow \sigma^a + \epsilon P(\sigma) C^a$ , then we should think of this as diffeomorphism transform. And for the theory to remain invariant, we need all tensors to change in a certain way.

$$\bar{g}^{ab} = \frac{\partial \bar{\sigma}^a}{\partial \sigma^c} \frac{\partial \bar{\sigma}^b}{\partial \sigma^d} g^{cd}$$

$$\begin{aligned} \bar{g}^{ab} &= [\partial_c^a + \epsilon C^a \partial_c P(\sigma)] [\partial_d^b + \epsilon C^b \partial_d P(\sigma)] g^{cd} \\ &= g^{ab} + \epsilon C^a g^{cb} \partial_c P(\sigma) + \epsilon C^b g^{ad} \partial_d P(\sigma) \end{aligned}$$

$$\Rightarrow \underline{\delta g^{ab} = \epsilon [c^a \partial^b \rho(\sigma) + c^b \partial^a \rho(\sigma)]}$$

$$\text{Similarly, } \delta \sqrt{g} = -\frac{1}{2} \sqrt{g} g_{ab} \delta g^{ab}$$

$$\begin{aligned} S'_E &= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left( 1 - \frac{1}{2} g_{cd} \delta g^{cd} \right) \\ &\quad \times (g^{ab} + \delta g^{ab}) (\partial_a \chi^\mu) (\partial_b \chi_\mu) \\ &= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left( g^{ab} + \delta g^{ab} - \frac{1}{2} g^{ab} g^{cd} \delta g_{cd} \right) \\ &\quad \times (\partial_a \chi^\mu) (\partial_b \chi_\mu) \end{aligned}$$

$$= \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} (\partial_a \chi^\mu) (\partial_b \chi_\mu) \quad \underbrace{S_E}$$

$$\begin{aligned} &+ \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left( \partial_c^a \partial_d^b - \frac{1}{2} g^{ab} g_{cd} \right) \delta g^{cd} \\ &\quad \times (\partial_a \chi^\mu) (\partial_b \chi_\mu) \end{aligned}$$

$$\begin{aligned} \Rightarrow \delta S_E &= \int_{\mathcal{R}} d^2\sigma \sqrt{g} \left( \partial_c^a \partial_d^b - \frac{1}{2} g^{ab} g_{cd} \right) \delta g^{cd} \\ &\quad \times \frac{1}{4\pi\alpha'} (\partial_a \chi^\mu) (\partial_b \chi_\mu) \end{aligned}$$

Thus we get ' $\delta S_E$ ' to be proportional to the variation in the metric tensor.

We define the variation of action under arbitrary variation of metric tensor, keeping  $X^\mu$  fixed as follows,

$$\rightarrow \delta S_E = \int_{\mathcal{R}} d^2\sigma \left( \frac{\delta S^{ab}}{\delta g} \right) \delta g^{ab}$$

$$= \frac{-1}{4\pi} \int d^2\sigma \sqrt{g} \left[ -\frac{4\pi}{\sqrt{g}} \left( \frac{\delta S}{\delta g^{ab}} \right) \right] \delta g^{ab}$$

$$\rightarrow \boxed{\delta S_E = \frac{-1}{4\pi} \int_{\mathcal{R}} d^2\sigma \sqrt{g} T_{ab} \delta g^{ab}}$$

$$\text{where } T_{ab} = -\frac{4\pi}{\sqrt{g}} \left( \frac{\delta S}{\delta g^{ab}} \right)$$

Thus the Coefficient of  $\delta g^{ab}$  defines the Energy momentum tensor.

Let us find  $T_{ab}$  for  $X^\mu$  fields.

$$X^\mu(\sigma^a) \rightarrow X^\mu(\sigma^a + \epsilon C^a)$$

$$\Rightarrow X^\mu(\sigma^a) \rightarrow X^\mu(\sigma^a) + \epsilon C^a \partial_a X^\mu$$

$$\rightarrow \delta X^\mu(\sigma^a) = \epsilon C^a \partial_a X^\mu$$

$$\delta g_{ab} = \delta (g^{-1})^{ab} = -g_{ac} g_{bd} \delta g^{cd}$$

$$\delta \sqrt{g} = \frac{1}{2} \sqrt{g} g^{cd} \delta g_{cd} = -\frac{1}{2} \sqrt{g} g_{cd} \delta g^{cd}$$

$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{ab} (\partial_a X^\mu) (\partial_b X_\mu)$$

$$\delta S_E = \frac{1}{4\pi\alpha'} \int_{\mathcal{R}} d^2\sigma \delta [\sqrt{g} g^{ab} (\partial_a X^\mu) (\partial_b X_\mu)]$$

$$= \frac{1}{4\pi\alpha'} \int_{\mathcal{R}} d^2\sigma \left[ \sqrt{g} \delta g^{ab} (\partial_a X^\mu) (\partial_b X_\mu) - \frac{1}{2} \sqrt{g} g_{cd} \delta g^{cd} (\partial_a X^\mu) (\partial^a X_\mu) \right]$$

$$\delta S_E = \frac{1}{4\pi\alpha'} \int_{\mathcal{R}} d^2\sigma \sqrt{g} \left[ \delta g^{cd} (\partial_c \chi^\mu) (\partial_d \chi_\mu) \right. \\ \left. - \frac{1}{2} g_{cd} \delta g^{cd} (\partial_a \chi^\mu) (\partial^a \chi_\mu) \right]$$

$$\delta S_E = \frac{-1}{4\pi} \int_{\mathcal{R}} d^2\sigma \frac{\sqrt{g}}{\alpha'} \left[ -(\partial_c \chi^\mu) (\partial_d \chi_\mu) \right. \\ \left. + \frac{1}{2} g_{cd} (\partial_a \chi^\mu) (\partial^a \chi_\mu) \right] \delta g^{cd}$$

Thus we get,

$$T_{ab} = \frac{-1}{\alpha'} \left[ (\partial_a \chi^\mu) (\partial_b \chi_\mu) - \frac{1}{2} g_{ab} (\partial_c \chi^\mu) (\partial^c \chi_\mu) \right]$$

Observe that  $T_{ab} = T_{ba}$  (Symmetric)

The Noether current is given by,

$$\delta S_E = \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^d\sigma \sqrt{g} J^a(\sigma) \partial_a \rho(\sigma) \\ = \frac{-1}{4\pi} \int_{\mathcal{R}} d^2\sigma \sqrt{g} T_{ab} \delta g^{ab}$$

For world sheet translations, if  $\sigma^a \rightarrow \sigma^a + \epsilon P(\sigma) c^a$ , we previously got  $\delta g^{ab} = \epsilon [c^a \partial^b P(\sigma) + c^b \partial^a P(\sigma)] = \delta g^{ba}$ . So,

$$\delta S_E = -\frac{\epsilon}{4\pi} \int_{\mathcal{R}} d^2\sigma \sqrt{g} T_{ab} [c^a \partial^b P(\sigma) + c^b \partial^a P(\sigma)]$$

$$= -\frac{\epsilon}{4\pi} \int_{\mathcal{R}} d^2\sigma \sqrt{g} T_{ab} 2c^b \partial^a P(\sigma)$$

$$\rightarrow \delta S_E = \frac{i\epsilon}{2\pi} \int_{\mathcal{R}} d^2\sigma \sqrt{g} \underbrace{(ic^b T_{ab})}_{\mathcal{J}_a} \partial^a P(\sigma)$$

Thus, for world sheet translations, the Noether current is given by

$$\boxed{\mathcal{J}_a = ic^b T_{ab}}$$

Where the Energy momentum tensor is,

$$T_{ab} = \frac{-1}{\alpha'} \left[ (\partial_a \chi^\mu)(\partial_b \chi_\mu) - \frac{1}{2} g_{ab} (\partial_c \chi^\mu)(\partial^c \chi_\mu) \right]$$

For a flat world sheet  $g_{ab} = \delta_{ab}$ , the world sheet energy momentum tensor is traceless in the scalar  $X^\mu$ -theory.

$$T_{11} = \frac{-1}{2\alpha'} \left[ (\partial_1 X^\mu)(\partial_1 X_\mu) - (\partial_2 X^\mu)(\partial_2 X_\mu) \right]$$

$$T_{22} = \frac{1}{2\alpha'} \left[ (\partial_1 X^\mu)(\partial_1 X_\mu) - (\partial_2 X^\mu)(\partial_2 X_\mu) \right]$$

$$T_{12} = T_{21} = \frac{-1}{\alpha'} (\partial_1 X^\mu)(\partial_2 X_\mu)$$

$$\rightarrow T_{ab} = \frac{-1}{\alpha'} \begin{bmatrix} \frac{(\partial_1 X^\mu)^2 - (\partial_2 X^\mu)^2}{2} & (\partial_1 X^\mu)(\partial_2 X_\mu) \\ (\partial_1 X^\mu)(\partial_2 X_\mu) & \frac{(\partial_2 X^\mu)^2 - (\partial_1 X^\mu)^2}{2} \end{bmatrix}$$

$$\rightarrow \underline{T_{\text{trace}}(T_{ab}) = \delta^{ab} T_{ab} = 0}$$

The Conservation of Noether Current in presence of a symmetry is given by

$$\partial^a J_a = 0$$

$$\rightarrow \partial^a c^b T_{ab} = 0$$

$$\rightarrow c^b \partial^a T_{ab} = 0$$

If all  $c^b = (\varepsilon, \varepsilon, \dots)$ ,  $\rightarrow \boxed{\partial^a T_{ab} = 0}$

This is the Conservation of Energy momentum tensor for World Sheet.

Thus  $X^\mu$  theory has traceless and a conserved Energy-momentum tensor.

Given such a tensor in Euclidean coordinates, we shall see its form in Complex coordinates.

In complex coordinates,

$$T_{\text{Complex}} = \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}}_{(J^{-1})^T} T_{\text{euclidean}} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{bmatrix}}_{(J^{-1})}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} T_{11} - iT_{12} & T_{11} + iT_{12} \\ T_{21} - iT_{22} & T_{21} + iT_{22} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} T_{11} - iT_{12} & T_{11} + iT_{12} \\ T_{21} - iT_{22} & T_{21} + iT_{22} \end{bmatrix}$$

$$\rightarrow T_{z\bar{z}} = \frac{1}{4} (T_{11} - iT_{12} - iT_{21} - T_{22})$$

$$T_{\bar{z}z} = \frac{1}{4} (T_{11} + iT_{12} + iT_{21} - T_{22})$$

$$T_{z\bar{z}} = \frac{1}{4} (T_{11} + iT_{12} - iT_{21} + T_{22})$$

$$T_{\bar{z}z} = \frac{1}{4} (T_{11} - iT_{12} + iT_{21} + T_{22})$$

If  $T_a^a = 0 \rightarrow T_1^1 + T_2^2 = 0$   
 or  $T_{11} + T_{22} = 0$  in euclidean,

$$T_{zz} = \frac{1}{4} (2T_{11} - iT_{12} - iT_{21})$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4} (2T_{11} + iT_{12} + iT_{21})$$

$$T_{z\bar{z}} = \frac{1}{4} (iT_{12} - iT_{21}) = -T_{\bar{z}z}$$

So, if  $T_{ab} = T_{ba}$  &  $T_a^a = 0$ , in the Euclidean coordinates, then,

$$T_{zz} = \frac{1}{2} (T_{11} - iT_{12})$$

$$T_{\bar{z}\bar{z}} = \frac{1}{2} (T_{11} + iT_{12})$$

$$\underline{T_{z\bar{z}} = T_{\bar{z}z} = 0}$$

Now, use the conservation eq<sup>n</sup>,

$$\partial^a T_{ab} = 0 \rightarrow \partial^z T_{zz} + \underbrace{\partial^{\bar{z}} T_{\bar{z}z}}_0 = 0$$

$$\rightarrow \partial^z T_{zz} = 0. \text{ Since, } \partial^z = 2\partial_{\bar{z}}$$

$$\Rightarrow \boxed{\partial_{\bar{z}} T_{zz} = 0}$$

$$\text{Similarly, } \partial^z \underbrace{T_{z\bar{z}}}_0 + \partial^{\bar{z}} T_{\bar{z}\bar{z}} = 0$$

$$\rightarrow \partial^{\bar{z}} T_{\bar{z}\bar{z}} = 0. \text{ Since, } \partial^{\bar{z}} = 2\partial_z$$

$$\rightarrow \boxed{\partial_z T_{\bar{z}\bar{z}} = 0}$$

Thus in any theory where the energy momentum tensor is symmetric & traceless in Euclidean coordinates, then in Complex coordinates,

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{\text{Trace}(T_{ab})}{2} = 0$$

$$\rightarrow T_{zz} \text{ is Holomorphic.}$$

$$\& T_{\bar{z}\bar{z}} \text{ is anti-Holomorphic.}$$

For  $X^\mu$  fields, the Energy-momentum tensor in Complex coordinates is,

$$T_{ab} = \frac{-1}{\alpha'} \left[ (\partial_a X^\mu)(\partial_b X_\mu) - \frac{1}{2} g_{ab} (\partial_c X^\mu)(\partial^c X_\mu) \right]$$

$$\text{where } g_{ab} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\rightarrow T_{zz} = \frac{-1}{\alpha'} (\partial_z X^\mu)(\partial_z X_\mu)$$

$$T_{\bar{z}\bar{z}} = \frac{-1}{\alpha'} (\partial_{\bar{z}} X^\mu)(\partial_{\bar{z}} X_\mu)$$

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$

$$\text{Let } T(z) = T_{zz} = \frac{-1}{\alpha'} (\partial_z X^\mu)(\partial_z X_\mu)$$

$$\widetilde{T}(\bar{z}) = T_{\bar{z}\bar{z}} = \frac{-1}{\alpha'} (\partial_{\bar{z}} X^\mu)(\partial_{\bar{z}} X_\mu)$$

$\partial_z X^\mu$  is holomorphic &  $\partial_{\bar{z}} X^\mu$  is anti-holomorphic as a consequence of EoM.

This is seen even in  $T(z)$  &  $\widetilde{T}(\bar{z})$