

Conformal Transformation

Tracelessness of Energy Momentum Tensor

Consider T_{ab} in Euclidean coordinates, then in complex coordinates,

$$T_{\text{Complex}} = \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}}_{(J^{-1})^T} \underbrace{\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}}_{T_{\text{euclidean}}} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{i}{2} \end{bmatrix}}_{(J^{-1}) \text{ (Inverse Jacobian)}}$$

$$\rightarrow T_{zz} = \frac{1}{4} (T_{11} - iT_{12} - iT_{21} - T_{22})$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4} (T_{11} + iT_{12} + iT_{21} - T_{22})$$

$$T_{z\bar{z}} = \frac{1}{4} (T_{11} + iT_{12} - iT_{21} + T_{22})$$

$$T_{\bar{z}z} = \frac{1}{4} (T_{11} - iT_{12} + iT_{21} + T_{22})$$

So, if $T_{ab} = T_{ba} \neq T_a^a = 0$, in the Euclidean coordinates, then,

$$T_{zz} = \frac{1}{2} (T_{11} - iT_{12})$$

$$T_{\bar{z}\bar{z}} = \frac{1}{2} (T_{11} + iT_{12})$$

$$\underline{T_{z\bar{z}} = T_{\bar{z}z} = 0}$$

Since the Energy-momentum tensor is Conserved in the presence of World Sheet translation, $\rightarrow \partial^a T_{ab} = 0$

$$\rightarrow \partial^z T_{zz} = 0. \text{ Since, } \partial^z = 2\partial_{\bar{z}}$$

$$\Rightarrow \boxed{\partial_{\bar{z}} T_{zz} = 0}$$

Similarly, $\partial^{\bar{z}} T_{\bar{z}\bar{z}} = 0$. Since, $\partial^{\bar{z}} = 2\partial_z$

$$\rightarrow \boxed{\partial_z T_{\bar{z}\bar{z}} = 0}$$

Thus in any theory where the energy momentum tensor is symmetric & traceless in Euclidean coordinates, then in Complex coordinates,

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{\text{Trace}(T_{ab})}{4} = 0$$

$\rightarrow T(z) = T_{zz}$ is Holomorphic.

& $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$ is anti-Holomorphic.

All this was possible because the trace of energy-momentum tensor is Zero. This becomes the key feature for conformal field theories in any dimensions at the classical level. But in the quantum level, it will become a problem, where only 2 dim theories succeed in preserving the Conformal Symmetry in the quantum level, also it possible only in flat theories where $T_a^a = 0$.

Let $\mathcal{V}(z)$ be any holomorphic function.
Then, consider the current $(j_z, j_{\bar{z}})$, where

$$j_z = i\mathcal{V}(z)T(z) \equiv j(z) \quad (\text{Holomorphic})$$

&

$$j_{\bar{z}} = i\bar{\mathcal{V}}(\bar{z})\bar{T}(\bar{z}) \equiv \bar{j}(\bar{z}) \quad (\text{anti-Holomorphic})$$

(where $\bar{\mathcal{V}}(\bar{z}) = [\mathcal{V}(z)]^*$)

Now, the divergence of this current is,

$$\partial^z j_z + \partial^{\bar{z}} j_{\bar{z}} = 2(\partial_{\bar{z}} j_z + \partial_z j_{\bar{z}})$$

Since $\partial_{\bar{z}} j_z = 0$ & $\partial_z j_{\bar{z}} = 0$, we get,

$$\underline{\partial^z j_z + \partial^{\bar{z}} j_{\bar{z}} = 0}$$

$\Rightarrow (j_z, j_{\bar{z}})$ is a conserved current.

Moreover, the currents $(j_z, 0)$ & $(0, j_{\bar{z}})$ are separately conserved. These are also known as Chiral currents.

Conformal Symmetry in Scalar X^μ theory

Now, let us find what transformations lead to $j_z = i\vartheta(z)T(z)$ & $j_{\bar{z}} = i\bar{\vartheta}(\bar{z})\widetilde{T}(\bar{z})$ as the conserved Noether currents.

$$T(z) = -\frac{1}{\alpha'} (\partial_z X^\mu)(\partial_z X_\mu)$$

$$\widetilde{T}(\bar{z}) = -\frac{1}{\alpha'} (\partial_{\bar{z}} X^\mu)(\partial_{\bar{z}} X_\mu)$$

Consider the OPE,

$$:T(z)::X^\mu(z_0, \bar{z}_0):$$

$$= -\frac{1}{\alpha'} :(\partial_z X^\nu)(\partial_z X_\nu)::X^\mu(z_0, \bar{z}_0):$$

$$\sim -\frac{1}{\alpha'} \left[\overline{(\partial_z X^\nu) X^\mu(z_0, \bar{z}_0)} :(\partial_z X_\nu): + \overline{(\partial_z X_\nu) X^\mu(z_0, \bar{z}_0)} :(\partial_z X^\nu): \right]$$

$$= -\frac{2}{\alpha'} \overline{(\partial_z X^\nu) X^\mu(z_0, \bar{z}_0)} :(\partial_z X_\nu):$$

$$\overline{(\partial_z \chi^\nu) \chi^\mu(z_0, \bar{z}_0)} = \partial_{\bar{z}} \left(\overline{\chi^\nu(z, \bar{z}) \chi^\mu(z_0, \bar{z}_0)} \right)$$

$$= \partial_{\bar{z}} \left(-\frac{\alpha'}{2} \eta^{\nu\mu} \ln |z - z_0|^2 \right)$$

$$\rightarrow \overline{(\partial_z \chi^\nu) \chi^\mu(z_0, \bar{z}_0)} = -\frac{\alpha'}{2} \eta^{\mu\nu} \left(\frac{1}{z - z_0} \right)$$

$$\Rightarrow :T(z)::\chi^\mu(z_0, \bar{z}_0): \sim \frac{\partial_z \chi^\mu}{z - z_0}$$

Similarly, $:\tilde{T}(\bar{z})::\chi^\mu(z_0, \bar{z}_0): \sim \frac{\partial_{\bar{z}} \chi^\mu}{\bar{z} - \bar{z}_0}$

$$\Rightarrow :j_z::\chi^\mu(z_0, \bar{z}_0): \sim \frac{i\mathcal{V}(z)\partial_z \chi^\mu}{z - z_0}$$

$$\neq :j_{\bar{z}}::\chi^\mu(z_0, \bar{z}_0): \sim \frac{i\bar{\mathcal{V}}(\bar{z})\partial_{\bar{z}} \chi^\mu}{\bar{z} - \bar{z}_0}$$

$$\Rightarrow \text{Res}_{z \rightarrow z_0} [j_z \chi^\mu(z_0, \bar{z}_0)] = i\mathcal{V}(z)\partial_z \chi^\mu$$

$$\neq \overline{\text{Res}_{\bar{z} \rightarrow \bar{z}_0} [j_{\bar{z}} \chi^\mu(z_0, \bar{z}_0)]} = i\bar{\mathcal{V}}(\bar{z})\partial_{\bar{z}} \chi^\mu$$

Using Ward Identity,

$$\text{Res}_{\substack{z \rightarrow z_0}} [J_z \hat{O}(z_0, \bar{z}_0)] + \overline{\text{Res}_{\substack{\bar{z} \rightarrow \bar{z}_0}}} [J_{\bar{z}} \hat{O}(z_0, \bar{z}_0)] = \frac{\partial \hat{O}(z_0, \bar{z}_0)}{i\epsilon}$$

$$\text{Res}_{\substack{z \rightarrow z_0}} [j_z \chi^\mu(z_0, \bar{z}_0)] + \overline{\text{Res}_{\substack{\bar{z} \rightarrow \bar{z}_0}}} [j_{\bar{z}} \chi^\mu(z_0, \bar{z}_0)] = \frac{\partial \chi^\mu}{i\epsilon}$$

$$\rightarrow i\mathcal{V}(z)\partial_z \chi^\mu + i\bar{\mathcal{V}}(\bar{z})\partial_{\bar{z}} \chi^\mu = \frac{\partial \chi^\mu}{i\epsilon}$$

$$\rightarrow \underline{\underline{\partial \chi^\mu = -\epsilon \mathcal{V}(z)\partial_z \chi^\mu - \epsilon \bar{\mathcal{V}}(\bar{z})\partial_{\bar{z}} \chi^\mu}}$$

This change in $\chi^\mu(z, \bar{z})$ is possible only when there is a transformation in both $z \neq \bar{z}$ as,

$$z \longrightarrow z + \epsilon \mathcal{V}(z)$$
$$\bar{z} \longrightarrow \bar{z} + \epsilon \bar{\mathcal{V}}(\bar{z})$$

Changing z is active transform and result of $\partial \chi^\mu$ because of change in z is passive. That is why there is a -ve sign change in both.

Then the finite transformation is got by changing

$$z \longrightarrow z' = f(z)$$

$$\& \bar{z} \longrightarrow \bar{z}' = \bar{f}(\bar{z})$$

where $f(z)$ is holomorphic & $\bar{f}(\bar{z})$ is anti-holomorphic. This is called as the Conformal transformation.

So we can state that the massless string action $S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a X^\mu)(\partial_b X_\mu) \delta^{ab}$

has Conformal Symmetry. Because, under the conformal transformation $z \rightarrow z' = f(z)$ & $\bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z})$, we get the resulting Noether current $(j_z, j_{\bar{z}})$, where

$$j_z = \frac{\mathcal{V}(z)}{i\alpha'} (\partial_z X^\mu)(\partial_z X_\mu) \& j_{\bar{z}} = \frac{\bar{\mathcal{V}}(\bar{z})}{i\alpha'} (\partial_{\bar{z}} X^\mu)(\partial_{\bar{z}} X_\mu)$$

to be conserved where $\mathcal{V}(z)$ & $\bar{\mathcal{V}}(\bar{z})$ are infinitesimal conformal transform of $f(z)$ & $\bar{f}(\bar{z})$ respectively.

This conformal invariance is an actual physical symmetry. We are in flat space fixed metric world sheet, with no any dynamical independent metric field. So the transformation $z \rightarrow f(z)$ actually changes the distances b/w two points.

This symmetry should be thought of as a Global Symmetry with corresponding Conserved Currents.

More formally, a Conformal transformation is a change of coordinates $\sigma^a \rightarrow \sigma^{a'}$, such that the metric tensor is rescaled as $g_{ab}(\sigma) \rightarrow g'_{ab}(\sigma') = \Omega^2(\sigma) g_{ab}(\sigma)$.

In 2D complex coordinates, the line element is given by

$$ds^2 = \frac{1}{2} dz d\bar{z} + \frac{1}{2} d\bar{z} dz$$

$$\rightarrow \underline{ds^2 = dz d\bar{z}}$$

Under conformal transformation given by,
 $z \rightarrow z' = f(z) \quad \& \quad \bar{z} \rightarrow \bar{z}' = \bar{f}(\bar{z}),$

$$ds^2 \rightarrow ds'^2 = dz' d\bar{z}' = f'(z) dz \bar{f}'(\bar{z}) d\bar{z} \\ \Rightarrow ds'^2 = \left| \frac{df}{dz} \right|^2 dz d\bar{z}$$

Thus under conformal transformation the line element changes as

$$ds^2 \longrightarrow \left| \frac{df}{dz} \right|^2 ds^2$$

$$\Rightarrow g_{ab} \longrightarrow |f'(z)|^2 g_{ab} \Rightarrow \Omega^2(z) = |f'(z)|^2$$

Thus conformal transformation rescales the distance elements by a position dependent factor $\Omega(z)$.

Notice that we have an infinite number of possible conformal transformations in 2 dimensions for all possible $f(z)$

For the Scalar χ^μ theory in complex coordinates the action is,

$$S_E = \frac{1}{2\pi\alpha'} \int d^2z (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

under conformal transformation $z \rightarrow z' = f(z)$,

$$dz' = \left(\frac{df}{dz} \right) dz \Rightarrow d^2z' = \left| \frac{df}{dz} \right|^2 d^2z$$

$$\frac{\partial}{\partial z'} = \underbrace{\left(\frac{\partial z}{\partial z'} \right)}_{\left(\frac{df}{dz} \right)^{-1}} \frac{\partial}{\partial z} \quad \quad \frac{\partial}{\partial \bar{z}'} = \underbrace{\left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right)}_{\left(\frac{d\bar{f}}{d\bar{z}} \right)^{-1}} \frac{\partial}{\partial \bar{z}}$$

$$\Rightarrow (\partial_{z'} \chi^\mu) (\partial_{\bar{z}'} \chi_\mu) = \left| \frac{df}{dz} \right|^{-2} (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

Thus we get that,

$$d^2z' (\partial_{z'} \chi^\mu) (\partial_{\bar{z}'} \chi_\mu) = \left| \frac{df}{dz} \right|^2 d^2z \left| \frac{df}{dz} \right|^{-2} (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

$$\Rightarrow d^2z' (\partial_{z'} \chi^\mu) (\partial_{\bar{z}'} \chi_\mu) = d^2z (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

Thus the massless string action is invariant under Conformal transformations. But this won't be the case when mass term is added.

Conformal Transformations & OPE's

Translations

Translations are given by the conformal transformations of the form,

$$z \rightarrow z' = z + \epsilon z_0 \quad \& \quad \bar{z} \rightarrow \bar{z}' = \bar{z} + \epsilon \bar{z}_0 \\ \rightarrow \partial z = \epsilon z_0 \quad \& \quad \partial \bar{z} = \epsilon \bar{z}_0$$

here $v(z) = z_0$ is a constant.

Any general local operator $\hat{\mathcal{O}}(z, \bar{z})$ will transform passively under translations as,

$$\hat{\mathcal{O}}(z, \bar{z}) \longrightarrow \hat{\mathcal{O}}(z', \bar{z}') = \hat{\mathcal{O}}(z - \epsilon z_0, \bar{z} - \epsilon \bar{z}_0)$$

$$\Rightarrow \partial \hat{\mathcal{O}} = -\epsilon [z_0 \partial_z \hat{\mathcal{O}}(z, \bar{z}) + \bar{z}_0 \partial_{\bar{z}} \hat{\mathcal{O}}(z, \bar{z})]$$

The Noether current is given by the Energy momentum tensor

$$J_z = i z_0 T(z)$$

$$J_{\bar{z}} = i \bar{z}_0 \widetilde{T}(\bar{z})$$

By Ward Identity we can relate the OPE to the residues as

$$\text{Res}_{z \rightarrow \omega} [T_z \hat{\Theta}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\bar{T}_{\bar{z}} \hat{\Theta}(\omega, \bar{\omega})] = \frac{\partial \hat{\Theta}(\omega, \bar{\omega})}{i\epsilon}$$

$$\begin{aligned} \Rightarrow z_0 \text{Res}_{z \rightarrow \omega} [T(z) \hat{\Theta}(\omega, \bar{\omega})] + \bar{z}_0 \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})] \\ = [z_0 \partial_\omega \hat{\Theta}(\omega, \bar{\omega}) + \bar{z}_0 \partial_{\bar{\omega}} \hat{\Theta}(\omega, \bar{\omega})] \end{aligned}$$

Since $T(z)$ is holomorphic & $\bar{T}(\bar{z})$ is anti-holomorphic and since they are separately conserved, the OPE's of $\hat{\Theta}(\omega, \bar{\omega})$ with T & $\bar{T}$ are of the form,

$$T(z) \hat{\Theta}(\omega, \bar{\omega}) = \dots + \left(\frac{\partial_\omega \hat{\Theta}(\omega, \bar{\omega})}{z - \omega} \right) + \dots$$

$$\bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega}) = \dots + \left(\frac{\partial_{\bar{\omega}} \hat{\Theta}(\omega, \bar{\omega})}{\bar{z} - \bar{\omega}} \right) + \dots$$

Thus $v(z) = \text{constant}$ determines the coefficient of 1st order pole.

Rotations & Scaling

Consider the conformal transformation given by, $z' = \zeta z$ & $\bar{z}' = \zeta^* \bar{z}$, where ζ is a complex number, $\zeta = m e^{i\theta}$. But the complex coordinates in polar form are given by,

$$z = r e^{i\varphi} \quad \& \quad \bar{z} = r e^{-i\varphi}$$

$$\rightarrow z' = m r e^{i(\varphi + \theta)} \quad \& \quad \bar{z}' = m r e^{-i(\varphi + \theta)}$$

Thus the radial coordinate changed as $r \rightarrow r' = m r$ & polar angle changed as $\varphi \rightarrow \varphi' = \varphi + \theta$. Hence the phase ' θ ' of ζ induces a rotation in the system and the magnitude ' m ' of ζ rescales the size of the system.

The presence of this conformal invariance gives rise to scale invariant and rotation independent theories.

Consider the infinitesimal form of the 'rescaling + rotation' conformal transformation given by

$$\mathcal{C}_3 = (1 + \delta m) e^{i\delta\theta}$$

where δm & $\delta\theta$ are infinitesimal.

$$\Rightarrow \mathcal{C}_3 = (1 + \delta m)(1 + i\delta\theta) = 1 + \delta m + i\delta\theta$$

Let $\epsilon = \delta m + i\delta\theta$, whose real part induces a rescale & imaginary part will induce a rotation.

$$\Rightarrow z \longrightarrow z + \epsilon z \quad \Rightarrow \delta z = \epsilon z$$

$$\& \bar{z} \longrightarrow \bar{z} + \epsilon^* \bar{z} \quad \Rightarrow \delta \bar{z} = \epsilon^* \bar{z}$$

Then any general local operator $\mathcal{O}(z, \bar{z})$ will transform passively under this as,

$$\mathcal{O}(z, \bar{z}) \longrightarrow \mathcal{O}(z - \epsilon z, \bar{z} - \epsilon^* \bar{z})$$

$$\rightarrow \delta \mathcal{O}(z, \bar{z}) = -\epsilon z \partial_z \mathcal{O}(z, \bar{z}) - \epsilon^* \bar{z} \partial_{\bar{z}} \mathcal{O}(z, \bar{z})$$

Thus under this transformation,

$$\begin{aligned}\rightarrow \delta \Theta(z, \bar{z}) &= -(\delta m + i\delta\theta)z \partial_z \Theta(z, \bar{z}) \\ &\quad - (\delta m - i\delta\theta)\bar{z} \partial_{\bar{z}} \Theta(z, \bar{z}) \\ &= -\delta m [z \partial_z + \bar{z} \partial_{\bar{z}}] \Theta(z, \bar{z}) - i\delta\theta [z \partial_z - \bar{z} \partial_{\bar{z}}] \Theta(z, \bar{z})\end{aligned}$$

Thus under pure rescaling with real ' ϵ '

$$\xrightarrow[\text{(rescale)}]{} \delta \Theta(z, \bar{z}) = -\delta m [z \partial_z + \bar{z} \partial_{\bar{z}}] \Theta(z, \bar{z})$$

$$\rightarrow \frac{-\delta \Theta(z, \bar{z})}{\delta m} = [z \partial_z + \bar{z} \partial_{\bar{z}}] \Theta(z, \bar{z})$$

$$\rightarrow \underline{D = z \partial_z + \bar{z} \partial_{\bar{z}}} \text{ :- Dilation/Scaling operator}$$

Similarly, under pure rotation with purely imaginary ' ϵ ' we get,

$$\delta \Theta(z, \bar{z}) = -\delta\theta (i[z \partial_z - \bar{z} \partial_{\bar{z}}]) \Theta(z, \bar{z})$$

(pure rot.)

$$\rightarrow \frac{i\delta \Theta(z, \bar{z})}{\delta\theta} = [z \partial_z - \bar{z} \partial_{\bar{z}}] \Theta(z, \bar{z})$$

$$\rightarrow \underline{R = z \partial_z - \bar{z} \partial_{\bar{z}}} \text{ :- Rotation operator}$$

Consider Pure rescaling. The Noether current is

$$J_z = i z T(z)$$

$$J_{\bar{z}} = i \bar{z} \bar{T}(\bar{z})$$

By Ward Identity we can relate the OPE to the residues as

$$\text{Res}_{z \rightarrow \omega} [J_z \hat{\Theta}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [J_{\bar{z}} \Theta(\omega, \bar{\omega})] = \frac{\partial \hat{\Theta}(\omega, \bar{\omega})}{i \partial m}$$

$$\begin{aligned} \rightarrow \text{Res}_{z \rightarrow \omega} [z T(z) \hat{\Theta}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})] \\ = [\omega \partial_\omega + \bar{\omega} \partial_{\bar{\omega}}] \Theta(\omega, \bar{\omega}) \end{aligned}$$

Consider Pure rotation. The Noether current is $J_z = -z T(z) \neq J_{\bar{z}} = \bar{z} \bar{T}(\bar{z})$

By Ward Identity we can relate the OPE to the residues as

$$\text{Res}_{z \rightarrow \omega} [J_z \hat{\Theta}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [J_{\bar{z}} \Theta(\omega, \bar{\omega})] = \frac{\partial \hat{\Theta}(\omega, \bar{\omega})}{i \partial \theta}$$

$$\begin{aligned} \rightarrow -\operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})] + \overline{\operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})]} \\ = -[\omega \partial_{\omega} - \bar{\omega} \partial_{\bar{\omega}}] \Theta(\omega, \bar{\omega}) \end{aligned}$$

$$\begin{aligned} \rightarrow \operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})] - \overline{\operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})]} \\ = [\omega \partial_{\omega} - \bar{\omega} \partial_{\bar{\omega}}] \Theta(\omega, \bar{\omega}) \end{aligned}$$

Thus by these two results we get,

$$\operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})] = \omega \partial_{\omega} \Theta(\omega, \bar{\omega})$$

$$\nabla \quad \overline{\operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})]} = \bar{\omega} \partial_{\bar{\omega}} \Theta(\omega, \bar{\omega})$$

Consider the first relation,

$$\operatorname{Res}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega})] = \omega \partial_{\omega} \Theta(\omega, \bar{\omega})$$

$$\text{Let } \bar{z} \bar{T}(\bar{z}) \hat{\Theta}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{C_n(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})^{n+1}} + \text{Regular terms}$$

$C_0 \rightarrow$ Coefficient of simple pole

$C_n \rightarrow$ Coefficient of $(n+1)^{\text{th}}$ order pole.

$$\rightarrow \mathbb{Z} T(\mathbb{Z}) \hat{\Theta}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\mathbb{Z} C_n(\omega, \bar{\omega})}{(\mathbb{Z} - \omega)^{n+1}} + \mathbb{Z} (\text{Regular terms})$$

$$= \sum_{n=0}^{\infty} \frac{\omega + (\mathbb{Z} - \omega) C_n(\omega, \bar{\omega})}{(\mathbb{Z} - \omega)^{n+1}} + \mathbb{Z} (\text{Regular terms})$$

$$= \sum_{n=0}^{\infty} \frac{\omega C_n(\omega, \bar{\omega})}{(\mathbb{Z} - \omega)^{n+1}} + \sum_{n=0}^{\infty} \frac{C_n(\omega, \bar{\omega})}{(\mathbb{Z} - \omega)^n} + \mathbb{Z} (\text{Regular terms})$$

$$\rightarrow \text{Res}_{\mathbb{Z} \rightarrow \omega} [\mathbb{Z} T(\mathbb{Z}) \hat{\Theta}(\omega, \bar{\omega})] = \omega C_0(\omega, \bar{\omega}) + C_1(\omega, \bar{\omega})$$

$$\rightarrow \underline{\omega \partial_{\omega} \Theta(\omega, \bar{\omega}) = \omega C_0(\omega, \bar{\omega}) + C_1(\omega, \bar{\omega})}$$

$\rightarrow \mathcal{V}(\mathbb{Z}) = \mathbb{Z}$ determines the coefficient of the 2nd order pole in the OPE.

Thus, all singular terms of the OPE

$$T(\mathbb{Z}) \hat{\Theta}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{C_n(\omega, \bar{\omega})}{(\mathbb{Z} - \omega)^{n+1}} + \text{Regular terms}$$

can be derived by various possible conformal transformations of $\hat{\Theta}(\omega, \bar{\omega})$.

Since $T(z)$ & $\bar{T}(\bar{z})$ are holomorphic & anti-holomorphic respectively, the OPE of any general operator $\hat{O}(\omega, \bar{\omega})$ with $T(z)$ & $\bar{T}(\bar{z})$ will be Laurent expansion of ' z ' & ' $\bar{z}$ ' respectively about ω & $\bar{\omega}$.

$$\rightarrow T(z) \hat{O}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{C_n(\omega, \bar{\omega})}{(z-\omega)^{n+1}} + \text{Regular terms}$$

$$\& \bar{T}(\bar{z}) \hat{O}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\bar{C}_n(\omega, \bar{\omega})}{(\bar{z}-\bar{\omega})^{n+1}} + \text{Regular terms}$$

Under infinitesimal conformal transformation $z \rightarrow z' = z + \epsilon \mathcal{V}(z)$ & $\bar{z} \rightarrow \bar{z}' = \bar{z} + \epsilon \mathcal{V}^*(\bar{z})$, we get the noether current as

$$(J_z, J_{\bar{z}}) = (i \mathcal{V}(z) T(z), i \mathcal{V}^*(\bar{z}) \bar{T}(\bar{z})).$$

Let $\delta \hat{O}(z, \bar{z})$ be the change of $\hat{O}(z, \bar{z})$ by this infinitesimal transform. Then, by the ward Identity,

$$\text{Res}_{z \rightarrow \omega} [J_z \hat{O}(\omega, \bar{\omega})] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{\omega}}} [J_{\bar{z}} \hat{O}(\omega, \bar{\omega})] = \frac{\delta \hat{O}(\omega, \bar{\omega})}{i\epsilon}$$

Thus we can get the change in some general operator as

$$\delta \hat{O}(\omega, \bar{\omega}) = -\epsilon \left(\text{Res}_{z \rightarrow \omega} [\mathcal{V}(z) T(z) \hat{O}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\mathcal{V}^*(\bar{z}) \bar{T}(\bar{z}) \hat{O}(\omega, \bar{\omega})] \right)$$

Now since $\mathcal{V}(z)$ is holomorphic & if it has no poles. Then, by Taylor expansion of $\mathcal{V}(z)$ about ' ω ', we get,

$$\begin{aligned} \mathcal{V}(z) &= \mathcal{V}(\omega) + \mathcal{V}'(\omega)(z-\omega) + \frac{\mathcal{V}''(\omega)}{2!}(z-\omega)^2 + \dots \\ \Rightarrow \mathcal{V}(z) &= \sum_{n=0}^{\infty} \frac{\mathcal{V}^{(n)}(\omega)}{n!} (z-\omega)^n \end{aligned}$$

Now,

$$T(z) \hat{O}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{C_n(\omega, \bar{\omega})}{(z-\omega)^{n+1}} + \text{Regular terms}$$

$\rightarrow$

$$\begin{aligned} \mathcal{V}(z) T(z) \hat{O}(\omega, \bar{\omega}) &= \\ &= \sum_{n=0}^{\infty} \frac{\mathcal{V}^{(n)}(\omega)}{n!} (z-\omega)^n \sum_{m=0}^{\infty} \frac{C_m(\omega, \bar{\omega})}{(z-\omega)^{m+1}} + \text{Regular terms} \end{aligned}$$

$$\Rightarrow \mathcal{V}(z) T(z) \hat{\Theta}(\omega, \bar{\omega}) = \sum_{n,m} \frac{\mathcal{V}^{(n)}(\omega) C_m(\omega, \bar{\omega})}{n! (z-\omega)^{m-n+1}}$$

$\Rightarrow \text{Res}_{z \rightarrow \omega} [\mathcal{V}(z) T(z) \hat{\Theta}(\omega, \bar{\omega})]$ picks out the coefficient of $(z-\omega)^{-1}$ only.

Thus, we get,

$$\text{Res}_{z \rightarrow \omega} [\mathcal{V}(z) T(z) \hat{\Theta}(\omega, \bar{\omega})] = \sum_{n=0}^{\infty} \frac{\mathcal{V}^{(n)}(\omega) C_n(\omega, \bar{\omega})}{n!}$$

In the exact same manner,

$$\overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\mathcal{V}^*(\bar{z}) \bar{T}(\bar{z}) \Theta(\omega, \bar{\omega})] = \sum_{n=0}^{\infty} \frac{\mathcal{V}^{*(n)}(\bar{\omega}) \tilde{C}_n(\omega, \bar{\omega})}{n!}$$

$$\partial \hat{\Theta}(\omega, \bar{\omega}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} [\mathcal{V}^{(n)}(\omega) C_n(\omega, \bar{\omega}) + \mathcal{V}^{*(n)}(\bar{\omega}) \tilde{C}_n(\omega, \bar{\omega})]$$

where $\mathcal{V}^{(n)}(\omega) = \frac{\partial^n \mathcal{V}(\omega)}{\partial \omega^n}$ & $\mathcal{V}^{*(n)}(\bar{\omega}) = \frac{\partial^n \mathcal{V}^*(\bar{\omega})}{\partial \bar{\omega}^n}$

Thus we can get all $C_n(\omega, \bar{\omega})$ by all possible conformal transformations on $\hat{\Theta}(\omega, \bar{\omega})$.

Eigen States of Rotation + Dilation ($\bar{z}' = \bar{c}_3 \bar{z}$)

We shall consider a basis of local operators $\hat{A}(z, \bar{z})$ that are eigen states of the transformation $z \rightarrow z' = c_3 z$. So they transform as following.

$$\hat{A}'(z', \bar{z}') = c_3^{-h} c_3^{*\bar{h}} \hat{A}(z, \bar{z})$$

where $(h, \bar{h})$ are known as weights of the local operator $\hat{A}(z, \bar{z})$. For $c_3 = m e^{i\theta}$ we get

$$\hat{A}'(c_3 z, c_3^* \bar{z}) = m^{-h} e^{-ih\theta} m^{-\bar{h}} e^{i\bar{h}\theta} \hat{A}(z, \bar{z})$$

$$\Rightarrow \hat{A}'(c_3 z, c_3^* \bar{z}) = m^{-(h+\bar{h})} e^{-i(h-\bar{h})\theta} \hat{A}(z, \bar{z})$$

Thus $h+\bar{h}$ determines the behaviour under scaling/dilation which is called as dimension (scaling dimension) of $\hat{A}$. If $h+\bar{h} = 0$, then the operator doesn't scale at all.

Similarly, $h - \tilde{h}$ determines the behaviour under rotations & is called as the Spin of $\hat{A}$. If $h - \tilde{h} = 0$, then the operator doesn't change under rotations.

Now consider ' $\partial_z \hat{A}(z, \bar{z})$ ', under this transformation, $\partial_z \rightarrow \mathcal{C}^{-1} \partial_z$

$$\rightarrow \partial_z \hat{A}(z, \bar{z}) \rightarrow \mathcal{C}^{-1} \mathcal{C}^{-h} \mathcal{C}^{*\tilde{h}} \partial_z \hat{A}(z, \bar{z})$$

Similarly,

$$\partial_{\bar{z}} \hat{A}(z, \bar{z}) \rightarrow \mathcal{C}^{*-1} \mathcal{C}^{-h} \mathcal{C}^{*\tilde{h}} \partial_{\bar{z}} \hat{A}(z, \bar{z})$$

Since $\partial_{\bar{z}} \rightarrow \mathcal{C}^{*-1} \partial_{\bar{z}}$. Thus ' ∂_z ' increases ' h ' value by '1'. And ' $\partial_{\bar{z}}$ ' increases ' $\tilde{h}$ ' value by '1'.

For infinitesimal transformations, $\mathcal{C} = 1 + \delta m + i\delta\theta$

$$\hat{A}(z, \bar{z}) \rightarrow (1 + \delta m + i\delta\theta)^{-h} (1 + \delta m - i\delta\theta)^{-\tilde{h}} \hat{A}(z, \bar{z})$$

$$\hat{A}(z, \bar{z}) \rightarrow [1 - h(\delta m + i\delta\theta)] [1 - \tilde{h}(\delta m - i\delta\theta)] \hat{A}(z, \bar{z})$$

$$\rightarrow \hat{A}'(z', \bar{z}') = \hat{A}(z, \bar{z}) - (\hbar\epsilon + \tilde{\hbar}\epsilon^*)\hat{A}(z, \bar{z})$$

where $\epsilon = \partial m + i\partial\theta$.

Now, we have only changed $\partial z = \epsilon z$ & got the corresponding transformation in $\hat{A}(z, \bar{z})$. But to find $\partial\hat{A}(z, \bar{z})$ we need to see the change in the same physical point. i.e. we need to find

$$\partial\hat{A}(z, \bar{z}) = \hat{A}'(z, \bar{z}) - \hat{A}(z, \bar{z})$$

Now $z' = z + \epsilon z$. So to bring back $\hat{A}'(z', \bar{z}')$ to z , we need, the value of $\hat{A}'$ at z .

$$\Rightarrow \hat{A}'(z, \bar{z}) = \hat{A}'(z', \bar{z}') \Big|_{\substack{\text{'Pull back of } \hat{A}' \\ z \rightarrow z - \epsilon z}}$$

$$\begin{aligned} \Rightarrow \hat{A}'(z, \bar{z}) &= \hat{A}'(z - \epsilon z, \bar{z} - \epsilon^* \bar{z}) \\ &= \hat{A}(z - \epsilon z, \bar{z} - \epsilon^* \bar{z}) [1 - (\hbar\epsilon + \tilde{\hbar}\epsilon^*)] \end{aligned}$$

$$\begin{aligned} &= \hat{A}(z, \bar{z}) - \epsilon z \partial \hat{A}(z, \bar{z}) - \epsilon^* \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) \\ &\quad - (\hbar\epsilon + \tilde{\hbar}\epsilon^*) \hat{A}(z, \bar{z}) \end{aligned}$$

$$\rightarrow \partial \hat{A}(z, \bar{z}) = -\epsilon z \partial \hat{A}(z, \bar{z}) - \epsilon^* \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) - (\hbar \epsilon + \tilde{\hbar} \epsilon^*) \hat{A}(z, \bar{z})$$

For pure scaling,

$$\partial \hat{A}(z, \bar{z}) = -\partial m (\hbar + \tilde{\hbar}) \hat{A}(z, \bar{z}) - \partial m [z \partial \hat{A}(z, \bar{z}) + \bar{z} \bar{\partial} \hat{A}(z, \bar{z})]$$

The Noether current is $J_z = i z T(z)$ and $J_{\bar{z}} = i \bar{z} \tilde{T}(\bar{z})$. By Ward Identity, we get

$$\text{Res}_{z \rightarrow \omega} [J_z \hat{A}(\omega, \bar{\omega})] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{\omega}}} [J_{\bar{z}} \hat{A}(\omega, \bar{\omega})] = \frac{\partial \hat{A}(\omega, \bar{\omega})}{i \partial m}$$

$$\begin{aligned} \rightarrow \text{Res}_{z \rightarrow \omega} [z T(z) \hat{A}(\omega, \bar{\omega})] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{\omega}}} [\bar{z} \tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] \\ = (\hbar + \tilde{\hbar}) \hat{A}(\omega, \bar{\omega}) + \omega \partial \hat{A}(\omega, \bar{\omega}) + \bar{\omega} \bar{\partial} \hat{A}(\omega, \bar{\omega}) \end{aligned}$$

For pure rotⁿ,

$$\partial \hat{A}(z, \bar{z}) = -i \partial \theta z \partial \hat{A}(z, \bar{z}) + i \partial \theta \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) - i \partial \theta (\hbar - \tilde{\hbar}) \hat{A}(z, \bar{z})$$

The Noether current is $J_z = -zT(z) \neq J_{\bar{z}} = \bar{z}\tilde{T}(\bar{z})$. By Ward Identity we get

$$\text{Res}_{z \rightarrow \omega} [J_z \hat{A}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [J_{\bar{z}} \hat{A}(\omega, \bar{\omega})] = \frac{\partial \hat{A}(\omega, \bar{\omega})}{i\partial\theta}$$

$$\begin{aligned} \rightarrow & \text{Res}_{z \rightarrow \omega} [zT(z) \hat{A}(\omega, \bar{\omega})] - \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z}\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] \\ & = \omega \partial \hat{A}(\omega, \bar{\omega}) - \bar{\omega} \bar{\partial} \hat{A}(\omega, \bar{\omega}) + (h - \tilde{h}) \hat{A}(\omega, \bar{\omega}) \end{aligned}$$

By using the residue relations of pure rescaling & pure rotations

$$\text{Res}_{z \rightarrow \omega} [zT(z) \hat{A}(\omega, \bar{\omega})] = \omega \partial \hat{A}(\omega, \bar{\omega}) + h \hat{A}(\omega, \bar{\omega})$$

$$\overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z}\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \bar{\omega} \bar{\partial} \hat{A}(\omega, \bar{\omega}) + \tilde{h} \hat{A}(\omega, \bar{\omega})$$

As shown previously, for $\partial z = \mathcal{V}(z)$,

$$\text{Res}_{z \rightarrow \omega} [\mathcal{V}(z) T(z) \hat{A}(\omega, \bar{\omega})] = \sum_{n=0}^{\infty} \frac{\mathcal{V}^{(n)}(\omega) \hat{A}^{(n)}(\omega, \bar{\omega})}{n!}$$

where $\mathcal{V}^{(n)}(\omega) = \partial_{\bar{z}}^n \mathcal{V}(\bar{z}) \Big|_{\bar{z}=\omega}$. with

$$T(\bar{z}) \hat{A}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{A}^{(n)}(\omega, \bar{\omega})}{(\bar{z} - \omega)^{n+1}} + \text{Regular terms}$$

is the OPE of $\hat{A}$ with $T(\bar{z})$.

$$\rightarrow \text{Res}_{\bar{z} \rightarrow \omega} [\bar{z} T(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \omega \hat{A}^{(0)}(\omega, \bar{\omega}) + \hat{A}^{(1)}(\omega, \bar{\omega})$$

$$\rightarrow \omega \partial \hat{A}(\omega, \bar{\omega}) + \hbar \hat{A}(\omega, \bar{\omega}) = \omega \hat{A}^{(0)}(\omega, \bar{\omega}) + \hat{A}^{(1)}(\omega, \bar{\omega})$$

But under translations, $\rightarrow \partial \bar{z} = \epsilon \bar{z}_0$
with ' ϵ ' as real, we previously got,
 $\partial \hat{A}(\omega, \bar{\omega}) = -\epsilon \bar{z}_0 \partial \hat{A}(\omega, \bar{\omega})$. And by ward

identity, $\text{Res}_{\bar{z} \rightarrow \omega} [T(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \partial \hat{A}(\omega, \bar{\omega})$

$$\rightarrow \underline{\hat{A}^{(0)}(\omega, \bar{\omega}) = \partial \hat{A}(\omega, \bar{\omega})}$$

$$\rightarrow \omega \partial \hat{A}(\omega, \bar{\omega}) + \hbar \hat{A}(\omega, \bar{\omega}) = \omega \partial \hat{A}(\omega, \bar{\omega}) + \hat{A}^{(1)}(\omega, \bar{\omega})$$

$$\text{Thus } \underline{\hat{A}^{(1)}(\omega, \bar{\omega}) = \hbar \hat{A}(\omega, \bar{\omega})}$$

So, the Ward identities for translations, rotations, & rescaling determines the part of OPE of ' $T(z)\hat{A}(\omega, \bar{\omega})$ ' giving the coefficient of simple pole & the 2nd order pole.

$$\rightarrow T(z)\hat{A}(\omega, \bar{\omega}) = \underset{\text{terms}}{\text{Regular}} + \frac{\partial \hat{A}(\omega, \bar{\omega})}{(z - \omega)} + \frac{h \hat{A}(\omega, \bar{\omega})}{(z - \omega)^2} + \dots$$

(Here ' $\dots$ ' represents possible higher order poles)

Now, by similar calculations for the OPE with $\tilde{T}(\bar{z})$, we can use,

$$\overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\mathcal{V}^*(\bar{z}) \tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \sum_{n=0}^{\infty} \frac{\mathcal{V}^{*(n)}(\bar{\omega})}{n!} \hat{A}^{(n)}(\omega, \bar{\omega})$$

where $\mathcal{V}^{*(n)}(\bar{\omega}) = \partial_{\bar{z}}^n \mathcal{V}^*(\bar{z})|_{\bar{z}=\bar{\omega}}$. and the conformal transform is given by $\partial \bar{z} = \mathcal{V}^*(\bar{z})$
 Here $\hat{A}^{(n)}(\omega, \bar{\omega})$ are the coefficients of the OPE with $\tilde{T}(\bar{z})$

$$\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{A}^{(n)}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})^{n+1}} + \text{Regular terms}$$

$$\rightarrow \overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\bar{z} \tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \bar{\omega} \hat{A}^{(0)}(\omega, \bar{\omega}) + \hat{A}^{(1)}(\omega, \bar{\omega})$$

$$\rightarrow \bar{\omega} \bar{\partial} \hat{A}(\omega, \bar{\omega}) + \hbar \hat{A}(\omega, \bar{\omega}) = \bar{\omega} \hat{A}^{(0)}(\omega, \bar{\omega}) + \hat{A}^{(1)}(\omega, \bar{\omega})$$

But under translations, $\rightarrow \partial \bar{z} = \epsilon \bar{z}_0$
with ' ϵ ' as real, we previously got,

$$\Rightarrow \partial \hat{A}(\omega, \bar{\omega}) = -\epsilon \bar{z}_0 \bar{\partial} \hat{A}(\omega, \bar{\omega})$$

And by ward identity,

$$\overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \underline{\hat{A}^{(0)}(\omega, \bar{\omega}) = \bar{\partial} \hat{A}(\omega, \bar{\omega})}$$

$$\Rightarrow \underline{\hat{A}^{(1)}(\omega, \bar{\omega}) = \hbar \hat{A}(\omega, \bar{\omega})}$$

$$\rightarrow \boxed{\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega}) = \underset{\text{terms}}{\text{Regular}} + \frac{\bar{\partial} \hat{A}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})} + \frac{\hbar \hat{A}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})^2} + \dots}$$

(Here ' $\dots$ ' represents possible higher order poles)

Primary/Tensor Local Operator

A local operator is said to be Primary or Tensor operator ' $\hat{A}(z, \bar{z})$ ' of weight $(h, \tilde{h})$ if under a general conformal transformation $z \rightarrow f(z) \neq \bar{z} \rightarrow \bar{f}(\bar{z})$, the primary operator transforms as,

$$\hat{A}(z, \bar{z}) \longrightarrow [f'(z)]^{-h} [\bar{f}'(\bar{z})]^{-\tilde{h}} \hat{A}(z, \bar{z})$$

For infinitesimal transformation given by $f(z) = z + \epsilon \mathcal{V}(z)$, where ' ϵ ' is real, we get $\delta z = \epsilon \mathcal{V}(z)$ and $f'(z) = 1 + \epsilon \mathcal{V}'(z)$.

Thus the Noether currents are given by,
 $J_z = i \mathcal{V}(z) T(z) \neq J_{\bar{z}} = i \mathcal{V}^*(\bar{z}) \tilde{T}(\bar{z})$.

Then through ward identity, we get,

$$\delta \hat{A}(\omega, \bar{\omega}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} [\mathcal{V}^{(n)}(\omega) \hat{A}^{(n)}(\omega, \bar{\omega}) + \mathcal{V}^{*(n)}(\bar{\omega}) \hat{A}^{(n)}(\omega, \bar{\omega})]$$

$$\mathcal{V}^{(n)}(\omega) = \left. \partial_{\bar{z}}^n \mathcal{V}(z) \right|_{z=\omega} \neq \mathcal{V}^{*(n)}(\bar{\omega}) = \left. \partial_{\bar{z}}^n \mathcal{V}^*(\bar{z}) \right|_{\bar{z}=\bar{\omega}}$$

$\hat{A}^{(n)}(\omega, \bar{\omega})$ & $\hat{\bar{A}}^{(n)}(\omega, \bar{\omega})$ are coefficients of $(n+1)^{\text{th}}$ order pole in Laurent series expansion of the OPE $T(z)\hat{A}(\omega, \bar{\omega})$ and $\bar{T}(\bar{z})\hat{A}(\omega, \bar{\omega})$ respectively. Thus they are given by,

$$T(z)\hat{A}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{A}^{(n)}(\omega, \bar{\omega})}{(z-\omega)^{n+1}} + \text{Regular terms}$$

&

$$\bar{T}(\bar{z})\hat{A}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{\bar{A}}^{(n)}(\omega, \bar{\omega})}{(\bar{z}-\bar{\omega})^{n+1}} + \text{Regular terms}$$

Now, let us find $\partial\hat{A}(\omega, \bar{\omega})$ for primary operator under ' $\partial z = \epsilon \mathcal{V}(z)$ ', which implies $f(z) = z + \epsilon \mathcal{V}(z)$ & $f'(z) = 1 + \epsilon \mathcal{V}'(z)$.

$$\begin{aligned} \rightarrow \hat{A}(z, \bar{z}) &\rightarrow [f'(z)]^{-h} [\bar{f}'(\bar{z})]^{-\bar{h}} \hat{A}(z, \bar{z}) \\ \rightarrow \hat{A}(z, \bar{z}) &\rightarrow [1 + \epsilon \mathcal{V}'(z)]^{-h} [1 + \epsilon \mathcal{V}'^*(\bar{z})]^{-\bar{h}} \hat{A}(z, \bar{z}) \\ \rightarrow \hat{A}(z, \bar{z}) &\rightarrow [1 - \epsilon h \mathcal{V}'(z)] [1 - \epsilon \bar{h} \mathcal{V}'^*(\bar{z})] \hat{A}(z, \bar{z}) \\ \rightarrow \hat{A}'(z', \bar{z}') &= \hat{A}(z, \bar{z}) [1 - \epsilon (h \mathcal{V}'(z) + \bar{h} \mathcal{V}'^*(\bar{z}))] \end{aligned}$$

We need $\partial \hat{A}(z, \bar{z}) = \hat{A}'(z, \bar{z}) - \hat{A}(z, \bar{z})$

$$\text{But, } \hat{A}'(z, \bar{z}) = \hat{A}'(z', \bar{z}') \Big|_{\bar{z} \rightarrow \bar{z} - \epsilon \bar{\nu}(\bar{z})}$$

$$= \hat{A}(z - \epsilon \nu(z), \bar{z} - \epsilon \bar{\nu}^*(\bar{z})) \left[1 - \epsilon (h \nu'(z) + \bar{h} \bar{\nu}'^*(\bar{z})) \right]$$

$$= \hat{A}(z, \bar{z}) - \epsilon \nu(z) \partial \hat{A}(z, \bar{z}) - \epsilon \bar{\nu}^*(\bar{z}) \bar{\partial} \hat{A}(z, \bar{z}) - \epsilon (h \nu'(z) + \bar{h} \bar{\nu}'^*(\bar{z})) \hat{A}(z, \bar{z})$$

$$\rightarrow \partial \hat{A}(z, \bar{z}) = -\epsilon \left[\nu(z) \partial \hat{A}(z, \bar{z}) + \bar{\nu}^*(\bar{z}) \bar{\partial} \hat{A}(z, \bar{z}) + (h \nu'(z) + \bar{h} \bar{\nu}'^*(\bar{z})) \hat{A}(z, \bar{z}) \right]$$

Thus we get using ward identity,

$$\text{Res}_{z \rightarrow \omega} [\nu(z) T(z) \hat{A}(\omega, \bar{\omega})] = \nu(\omega) \partial \hat{A}(\omega, \bar{\omega}) + h \nu'(\omega) \hat{A}(\omega, \bar{\omega})$$

But,

$$\text{Res}_{z \rightarrow \omega} [\nu(z) T(z) \hat{A}(\omega, \bar{\omega})] = \sum_{n=0}^{\infty} \frac{\nu^{(n)}(\omega) \hat{A}^{(n)}(\omega, \bar{\omega})}{n!}$$

$$\rightarrow \underline{\underline{\hat{A}^{(0)}(\omega, \bar{\omega}) = \partial \hat{A}(\omega, \bar{\omega})}} \quad \& \quad \underline{\underline{\hat{A}^{(1)}(\omega, \bar{\omega}) = h \hat{A}(\omega, \bar{\omega})}}$$

and all other $\hat{A}^{(n)}(\omega, \bar{\omega})$ with $n > 1$ is Zero.

Similarly, for the anti-holomorphic part,

$$\overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\mathcal{V}^*(\bar{z}) \tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \mathcal{V}^*(\bar{\omega}) \bar{\partial} \hat{A}(\omega, \bar{\omega}) + \tilde{h} \mathcal{V}^*(\bar{\omega}) \hat{A}(\omega, \bar{\omega})$$

But,

$$\overline{\text{Res}}_{\bar{z} \rightarrow \bar{\omega}} [\mathcal{V}^*(\bar{z}) \tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega})] = \sum_{n=0}^{\infty} \frac{\mathcal{V}^{*(n)}(\bar{\omega}) \hat{A}^{(n)}(\omega, \bar{\omega})}{n!}$$

$$\rightarrow \underline{\hat{A}^{(0)}(\omega, \bar{\omega}) = \bar{\partial} \hat{A}(\omega, \bar{\omega})} \quad \& \quad \underline{\hat{A}^{(1)}(\omega, \bar{\omega}) = \tilde{h} \hat{A}(\omega, \bar{\omega})}$$

Thus for primary operators, their OPE with the components of Energy momentum tensor is,

$$T(z) \hat{A}(\omega, \bar{\omega}) = \frac{h \hat{A}(\omega, \bar{\omega})}{(z - \omega)^2} + \frac{\partial \hat{A}(\omega, \bar{\omega})}{(z - \omega)} + \text{Regular terms}$$

$$\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega}) = \frac{\tilde{h} \hat{A}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})^2} + \frac{\bar{\partial} \hat{A}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})} + \text{Regular terms}$$

Thus the primary/tensor operators have singular terms only upto 2nd order.

For eigen states of rotation & dilation, the transformation is $z \rightarrow z' = \zeta z$ & $\bar{z} \rightarrow \bar{z}' = \zeta^* \bar{z}$, & they transform in the following manner, $\hat{A}'(z', \bar{z}') = \zeta^{-h} \zeta^{*\bar{h}} \hat{A}(z, \bar{z})$

But $\zeta = \frac{\partial z'}{\partial z}$ & $\zeta^* = \frac{\partial \bar{z}'}{\partial \bar{z}}$.

Thus the Primary/Tensor operators are also the eigen states of rot^n & dilation.

The converse of the above statement need not be true. Not all eigen states of rot^n & dilation are primary operators.

Only the Eigen states of rot^n & dilation have good transformation property under $z' = \zeta z$. This is analogous to stating that not all quantum states transform nicely under the action of Angular momentum ($\hat{L}$) and the Hamiltonian ($\hat{H}$). But we can always choose their eigen states as basis.

OPEs between Eigen basis operators of Rotations and Dilations

Let $\hat{A}_i(z_1, \bar{z}_1) \neq \hat{A}_j(z_2, \bar{z}_2)$ be any two local operators that have weights $(h_i, \tilde{h}_i) \neq (h_j, \tilde{h}_j)$ respectively, that are also the eigen states of the transformation $z \rightarrow z' = \zeta z$. So they transform as,

$$\hat{A}'_i(z', \bar{z}') = \zeta^{-h_i} \zeta^{*\tilde{h}_i} \hat{A}_i(z, \bar{z})$$

$$\text{and } \hat{A}'_j(z', \bar{z}') = \zeta^{-h_j} \zeta^{*\tilde{h}_j} \hat{A}_j(z, \bar{z})$$

Then, the product of these two eigen states can be expanded by the set of all possible eigen states of rotⁿ & dilation by OPE as,

$$\hat{A}_i(z_1, \bar{z}_1) \hat{A}_j(z_2, \bar{z}_2) = \sum_k C_{ij}^k(z_{12}, \bar{z}_{12}) \hat{A}_k(z_2, \bar{z}_2)$$

where $(z_{12}, \bar{z}_{12}) = (z_1 - z_2, \bar{z}_1 - \bar{z}_2) \neq C_{ij}^k$ are a set of functions that depend only on the separation between the operators.

The conformal invariance under translations is what restricts the coefficient functions to depend only on the separation $(z_{12}, \bar{z}_{12})$.

$\hat{A}_k(z_2, \bar{z}_2)$ are also eigen states of rot^n & dilations with weights $(h_k, \tilde{h}_k)$.

Now, under $z \rightarrow z' = \mathcal{C}_3 z$ & $\bar{z} \rightarrow \bar{z}' = \mathcal{C}_3^* \bar{z}$ OPE transforms as,

$$\hat{A}'_i(z'_1, \bar{z}'_1) \hat{A}'_j(z'_2, \bar{z}'_2) = \sum_k C_{ij}^k(z'_{12}, \bar{z}'_{12}) \hat{A}'_k(z'_2, \bar{z}'_2)$$

$$\Rightarrow \mathcal{C}_3^{-h_i} \mathcal{C}_3^{*\tilde{h}_i} \hat{A}_i(z, \bar{z}) \mathcal{C}_3^{-h_j} \mathcal{C}_3^{*\tilde{h}_j} \hat{A}_j(z, \bar{z})$$

$$= \sum_k C_{ij}^k(z'_{12}, \bar{z}'_{12}) \mathcal{C}_3^{-h_k} \mathcal{C}_3^{*\tilde{h}_k} \hat{A}_k(z_2, \bar{z}_2)$$

$$\Rightarrow \hat{A}_i(z, \bar{z}) \hat{A}_j(z, \bar{z}) =$$

$$\sum_k C_{ij}^k(z'_{12}, \bar{z}'_{12}) \mathcal{C}_3^{(h_i+h_j-h_k)} \mathcal{C}_3^{*(\tilde{h}_i+\tilde{h}_j-\tilde{h}_k)} \hat{A}_k(z_2, \bar{z}_2)$$

Thus the coefficients C_{ij}^k should satisfy,

$$C_{ij}^k(z'_{12}, \bar{z}'_{12}) \ell_g^{(h_i+h_j-h_k)} \ell_g^*(\tilde{h}_i+\tilde{h}_j-\tilde{h}_k) = C_{ij}^k(z_{12}, \bar{z}_{12})$$

or,

$$C_{ij}^k(\ell_g z_{12}, \ell_g^* \bar{z}_{12}) = \ell_g^{(h_k-h_i-h_j)} \ell_g^*(\tilde{h}_k-\tilde{h}_i-\tilde{h}_j) C_{ij}^k(z_{12}, \bar{z}_{12})$$

Thus C_{ij}^k is homogenous function of $(z_{12}, \bar{z}_{12})$ with degree $(h_k-h_i-h_j, \tilde{h}_k-\tilde{h}_i-\tilde{h}_j)$ respectively.

Let $\ell_g = m e^{i\theta}$, then

$$C_{ij}^k(m e^{i\theta} z_{12}, m e^{-i\theta} \bar{z}_{12}) = m^{[(h_k+\tilde{h}_k)-(h_i+\tilde{h}_i)-(h_j+\tilde{h}_j)]} e^{i[(h_k-\tilde{h}_k)-(h_i-\tilde{h}_i)-(h_j-\tilde{h}_j)]\theta} C_{ij}^k(z_{12}, \bar{z}_{12})$$

Let $z_{12} = \gamma e^{i\varphi}$ & $\bar{z}_{12} = \gamma e^{-i\varphi}$. So, any $C_{ij}^k(z_{12}, \bar{z}_{12}) \equiv f(\gamma, \varphi)$. Also, let, $p = h_k - h_i - h_j$ & $q = \tilde{h}_k - \tilde{h}_i - \tilde{h}_j$. Then, under $z \rightarrow \ell_g z$, we see that $\gamma \rightarrow m\gamma$ & $\varphi \rightarrow \varphi + \theta$.

$$\rightarrow f[mx, \varphi] = m^{p+q} f(x, \varphi)$$

$$\& f(x, \varphi + \theta) = e^{i(p-q)\theta} f(x, \varphi)$$

Thus ' φ ' dependence of ' f ' has to be such that

$$f(x, \varphi) = g(x) e^{i(p-q)\varphi}$$

$$\text{where } g(mx) = m^{p+q} g(x)$$

$\Rightarrow g(x)$ has to be a monomial proportional to x^{p+q}

$$\Rightarrow g(x) = c x^{p+q}$$

$$\begin{aligned} \Rightarrow f(x, \varphi) &= c x^{p+q} e^{i(p-q)\varphi} \\ &= c [x e^{i\varphi}]^p [x e^{-i\varphi}]^q \end{aligned}$$

$$\Rightarrow \underline{f(x, \varphi) = c z_{12}^p \bar{z}_{12}^q}$$

Thus all coefficients

$$C_{ij}^k(z_{12}, \bar{z}_{12}) = C_{ij}^k z_{12}^{(h_k - h_i - h_j)} \bar{z}_{12}^{(\tilde{h}_k - \tilde{h}_i - \tilde{h}_j)}$$

where C_{ij}^k are constant numbers that are independent of z_{12} or $\bar{z}_{12}$.

Thus for eigen states of rot^n & dilation we can write,

$$\hat{A}_i(\bar{z}_1, \bar{\bar{z}}_1) \hat{A}_j(\bar{z}_2, \bar{\bar{z}}_2)$$

$$= \sum_k \bar{z}_{12}^{(h_k - h_i - h_j)} \bar{\bar{z}}_{12}^{(\tilde{h}_k - \tilde{h}_i - \tilde{h}_j)} C_{ij}^k \hat{A}_k(\bar{z}_2, \bar{\bar{z}}_2)$$

Thus translations, rotations & dilations put the above constraints on 'z' dependence of the OPE coefficients completely.

' C_{ij}^k ' $\rightarrow$ structure coefficients of the theory.

The weights of all $\hat{A}_k(\bar{z}_2, \bar{\bar{z}}_2)$ cannot be arbitrarily -ve. So all weights are bounded below for the theories which we will study. So none of the OPE expansions have singularity of infinite order. There won't be any essential singularity in the theories of our interest.

OPE relations, Weights & Tensors in the Scalar χ^μ CFT

For fields $\chi^\mu(\omega, \bar{\omega})$

For the scalar χ^μ massless theory,

$$T(z) = -\frac{1}{\alpha'} (\partial_z \chi^\mu)(\partial_z \chi_\mu)$$

$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} (\partial_{\bar{z}} \chi^\mu)(\partial_{\bar{z}} \chi_\mu)$$

And previously we got,

$$:T(z)::\chi^\mu(\omega, \bar{\omega}): \sim \frac{\partial_z \chi^\mu(z, \bar{z})}{z - \omega}$$

$$:\tilde{T}(\bar{z})::\chi^\mu(\omega, \bar{\omega}): \sim \frac{\partial_{\bar{z}} \chi^\mu(z, \bar{z})}{\bar{z} - \bar{\omega}}$$

Now, expand ' $\partial_z \chi^\mu(z, \bar{z})$ ' around ' ω '.

$$\begin{aligned} \partial_z \chi^\mu(z, \bar{z}) &= \partial_\omega \chi^\mu(\omega, \bar{\omega}) + (z - \omega) \partial_\omega^2 \chi^\mu(\omega, \bar{\omega}) \\ &\quad + (\bar{z} - \bar{\omega}) \underbrace{\partial_{\bar{\omega}} \partial_\omega \chi^\mu(\omega, \bar{\omega})}_{\text{higher order}} \end{aligned}$$

Vanishes by Eqⁿ of motion

Thus,

$$\begin{aligned} :T(z)::X^\mu(\omega, \bar{\omega}): &\sim \frac{\partial_\omega X^\mu(\omega, \bar{\omega})}{z - \omega} \\ &+ \sum_{n=0}^{\infty} \frac{(z - \omega)^n}{n!} \partial_\omega^{n+2} X^\mu(\omega, \bar{\omega}) \end{aligned}$$

All cross derivative terms vanish since $\partial_{\bar{\omega}} \partial_\omega X^\mu(\omega, \bar{\omega}) = 0$ by eqⁿ of motion.

Similarly we get for conjugate terms,

$$\begin{aligned} :\widetilde{T}(\bar{z})::X^\mu(\omega, \bar{\omega}): &\sim \frac{\partial_{\bar{\omega}} X^\mu(\omega, \bar{\omega})}{\bar{z} - \bar{\omega}} \\ &+ \sum_{n=0}^{\infty} \frac{(\bar{z} - \bar{\omega})^n}{n!} \partial_{\bar{\omega}}^{n+2} X^\mu(\omega, \bar{\omega}) \end{aligned}$$

Thus ' $X^\mu(\omega, \bar{\omega})$ ' are tensor operators with weights $(0, 0)$. Their pole's maximum order is '1'. (No 2nd order pole)

For $\partial_\omega \chi^\mu(\omega, \bar{\omega})$

$$:T(z)::\partial_\omega \chi^\mu(\omega, \bar{\omega}):$$

$$= \frac{-1}{\alpha'} :(\partial_z \chi^\nu)(\partial_z \chi_\nu(z, \bar{z}))::\partial_\omega \chi^\mu(\omega, \bar{\omega}):$$

$$\sim \frac{-1}{\alpha'} \left[\overline{\partial_z \chi^\nu(z, \bar{z}) \partial_\omega \chi^\mu(\omega, \bar{\omega})} (\partial_z \chi_\nu(z, \bar{z})) \right. \\ \left. + \overline{\partial_z \chi_\nu(z, \bar{z}) \partial_\omega \chi^\mu(\omega, \bar{\omega})} (\partial_z \chi^\nu(z, \bar{z})) \right]$$

$$= \frac{-2}{\alpha'} \overline{\partial_z \chi^\nu(z, \bar{z}) \partial_\omega \chi^\mu(\omega, \bar{\omega})} (\partial_z \chi_\nu(z, \bar{z}))$$

We can find,

$$\overline{\partial_z \chi^\nu(z, \bar{z}) \partial_\omega \chi^\mu(\omega, \bar{\omega})} = \partial_z \partial_\omega \overline{\chi^\nu(z, \bar{z}) \chi^\mu(\omega, \bar{\omega})}$$

$$= \partial_z \partial_\omega \left[-\frac{\alpha' \eta^{\nu\mu}}{2} \ln |z - \omega|^2 \right]$$

$$= \partial_z \left(\frac{\alpha' \eta^{\nu\mu}}{2(z - \omega)} \right)$$

$$\rightarrow \overline{\partial_z \chi^\nu(z, \bar{z}) \partial_\omega \chi^\mu(\omega, \bar{\omega})} = -\frac{\alpha' \eta^{\mu\nu}}{2} \frac{1}{(z - \omega)^2}$$

$$\rightarrow :T(z)::\partial_\omega X^\mu(\omega, \bar{\omega}): \sim \frac{\partial_z X^\mu(z, \bar{z})}{(z-\omega)^2}$$

$$\partial_z X^\mu(z, \bar{z}) = \partial_\omega X^\mu(\omega, \bar{\omega}) + (z-\omega) \partial_\omega^2 X^\mu(\omega, \bar{\omega}) + \text{higher order}$$

$$\begin{aligned} \rightarrow :T(z)::\partial_\omega X^\mu(\omega, \bar{\omega}): &\sim \frac{\partial_\omega X^\mu(\omega, \bar{\omega})}{(z-\omega)^2} \\ &+ \frac{\partial_\omega^2 X^\mu(\omega, \bar{\omega})}{(z-\omega)} + \text{Non-singular terms} \end{aligned}$$

$\Rightarrow$ 'h' of $\partial_\omega X^\mu$ is 1

Similarly,

$$:\tilde{T}(\bar{z})::\partial_{\bar{\omega}} X^\mu(\omega, \bar{\omega}): \sim \frac{\partial_{\bar{z}} X^\mu(\omega, \bar{\omega})}{(\bar{z}-\bar{\omega})^2}$$

$$+ \frac{\partial_{\bar{\omega}}^2 X^\mu(\omega, \bar{\omega})}{(\bar{z}-\bar{\omega})} + \text{Non-singular terms}$$

So $\tilde{h}$ of $\partial_{\bar{\omega}} X^\mu$ is 1.

But Consider $:\widetilde{T}(\bar{z})::\partial_\omega X^\mu(\omega, \bar{\omega}):$

$$:\widetilde{T}(\bar{z})::\partial_\omega X^\mu(\omega, \bar{\omega}):$$

$$= \frac{-1}{\alpha'} :(\partial_{\bar{z}} X^\nu)(\partial_{\bar{z}} X_\nu(z, \bar{z}))::\partial_\omega X^\mu(\omega, \bar{\omega}):$$

$$\sim \frac{-1}{\alpha'} \left[\overline{\partial_{\bar{z}} X^\nu(z, \bar{z}) \partial_\omega X^\mu(\omega, \bar{\omega})} (\partial_{\bar{z}} X_\nu(z, \bar{z})) \right. \\ \left. + \overline{\partial_{\bar{z}} X_\nu(z, \bar{z}) \partial_\omega X^\mu(\omega, \bar{\omega})} (\partial_{\bar{z}} X^\nu(z, \bar{z})) \right]$$

$$= \frac{-2}{\alpha'} \overline{\partial_{\bar{z}} X^\nu(z, \bar{z}) \partial_\omega X^\mu(\omega, \bar{\omega})} (\partial_{\bar{z}} X_\nu(z, \bar{z}))$$

$$\overline{\partial_{\bar{z}} X^\nu(z, \bar{z}) \partial_\omega X^\mu(\omega, \bar{\omega})} = \partial_{\bar{z}} \partial_\omega \overline{X^\nu(z, \bar{z}) X^\mu(\omega, \bar{\omega})}$$

$$= \partial_{\bar{z}} \partial_\omega \left[-\frac{\alpha'}{2} \eta^{\nu\mu} \ln|z-\omega|^2 \right]$$

$$= \partial_{\bar{z}} \left(\frac{\alpha' \eta^{\nu\mu}}{2(z-\omega)} \right) = 0, \text{ as } \partial_{\bar{z}} z = 0$$

$$\Rightarrow \underline{:\widetilde{T}(\bar{z})::\partial_\omega X^\mu(\omega, \bar{\omega}): \sim 0}$$

$$\Rightarrow \underline{\widetilde{h} \text{ of } \partial_\omega X^\mu \text{ is } 0.}$$

This tells that ∂X^μ has no $\bar{z}$ -singularity.

Similarly, for $:T(z)::\partial_{\bar{\omega}}\chi^{\mu}(\omega,\bar{\omega}):$

$$:T(z)::\partial_{\bar{\omega}}\chi^{\mu}(\omega,\bar{\omega}):$$

$$\sim \frac{-2}{\alpha'} \overline{\partial_z \chi^{\nu}(z, \bar{z}) \partial_{\bar{\omega}} \chi^{\mu}(\omega, \bar{\omega})} (\partial_z \chi_{\nu}(z, \bar{z}))$$

$$\overline{\partial_z \chi^{\nu}(z, \bar{z}) \partial_{\bar{\omega}} \chi^{\mu}(\omega, \bar{\omega})} = \partial_z \partial_{\bar{\omega}} \overline{\chi^{\nu}(z, \bar{z}) \chi^{\mu}(\omega, \bar{\omega})}$$

$$= \partial_z \partial_{\bar{\omega}} \left[-\frac{\alpha' \eta^{\nu\mu}}{2} \ln |z - \omega|^2 \right]$$

$$= \partial_z \left(\frac{\alpha' \eta^{\nu\mu}}{2(\bar{z} - \bar{\omega})} \right) = 0, \text{ as } \partial_z \bar{z} = 0$$

$$\rightarrow :T(z)::\partial_{\bar{\omega}}\chi^{\mu}(\omega,\bar{\omega}): \sim 0$$

$$\rightarrow h \text{ of } \partial_{\bar{\omega}}\chi^{\mu} \text{ is } 0.$$

This tells that $\bar{\partial}\chi^{\mu}$ has no z -singularity.

Thus, $(h, \tilde{h})$ of $\partial\chi^{\mu}$ is $(1, 0) \neq$ for $\bar{\partial}\chi^{\mu}$ is $(0, 1)$. These are tensor operators as their OPE with $T(z)$ has singularity of utmost order '2'.

Consider $\partial^n \chi^\mu$. Its OPE with $T(z)$ will be,

$$:T(z)::\partial_\omega^n \chi^\mu(\omega, \bar{\omega}):$$

$$\sim \frac{-2}{\alpha'} \overline{\partial_{\bar{z}} \chi^\nu(z, \bar{z}) \partial_\omega^n \chi^\mu(\omega, \bar{\omega})} (\partial_{\bar{z}} \chi_\nu(z, \bar{z}))$$

$$\overline{\partial_{\bar{z}} \chi^\nu(z, \bar{z}) \partial_\omega^n \chi^\mu(\omega, \bar{\omega})} = \partial_\omega^n \left[\frac{-\alpha' \eta^{\mu\nu}}{2(z-\omega)} \right]$$

$$= \frac{\alpha' \eta^{\mu\nu}}{2} \partial_\omega^n \left(\frac{1}{(\omega - z)} \right) = \frac{\alpha' \eta^{\mu\nu}}{2} \frac{(-1)^n n!}{(\omega - z)^{n+1}}$$

$$\rightarrow \overline{\partial_{\bar{z}} \chi^\nu(z, \bar{z}) \partial_\omega^n \chi^\mu(\omega, \bar{\omega})} = \frac{-\alpha' \eta^{\mu\nu} n!}{2(z-\omega)^{n+1}}$$

$$\rightarrow :T(z)::\partial_\omega^n \chi^\mu(\omega, \bar{\omega}): \sim \frac{n! (\partial_{\bar{z}} \chi^\mu(z, \bar{z}))}{(z-\omega)^{n+1}}$$

$$\rightarrow :T(z)::\partial_\omega^n \chi^\mu(\omega, \bar{\omega}): \sim \sum_{m=0}^{\infty} \frac{n! \partial_\omega^{m+1} \chi^\mu(\omega, \bar{\omega})}{m! (z-\omega)^{n+1-m}}$$

Thus $(h, \bar{h}) = (n, 0)$. And $\partial^n \chi^\mu$ for $n \geq 2$ is not a tensor operator.

For Powers of χ^μ

$$:T(z)::[\chi^\mu(\omega, \bar{\omega})]^n:$$

$$= \frac{-1}{\alpha'} :(\partial_{\bar{z}} \chi^\nu)(\partial_{\bar{z}} \chi_\nu(z, \bar{z}))::\chi^{\mu_1} \chi^{\mu_2} \dots \chi^{\mu_n}(\omega, \bar{\omega}):$$

(we need only cross contractions)

$$\sim \left(\frac{1}{z - \omega} \right) \sum_{m=1}^n :(\partial_{\bar{z}} \chi^{\mu_m}(z, \bar{z})) \prod_{\substack{i=1 \\ i \neq m}}^n \chi^{\mu_i}(\omega, \bar{\omega}):$$

$$- \frac{\alpha'}{2(z - \omega)^2} \sum_{m=1}^n \sum_{k=1}^{m-1} \eta^{\mu_m \mu_k} : \prod_{\substack{i=1 \\ i \neq m, k}}^n \chi^{\mu_i}(\omega, \bar{\omega}):$$

1st term is got by all possible single cross-contractions & 2nd term is got by all possible double cross-contractions.

$$\text{Eg: } :T(z)::\chi^{\mu_1} \chi^{\mu_2}(\omega, \bar{\omega}):$$

$$\left(\frac{-1}{z - \omega} \right) \left[:(\partial_{\bar{z}} \chi^{\mu_1}(z, \bar{z})) \chi^{\mu_2}(\omega, \bar{\omega}): + :(\partial_{\bar{z}} \chi^{\mu_2}(z, \bar{z})) \chi^{\mu_1}(\omega, \bar{\omega}): \right]$$

$$- \frac{\alpha'}{2(z - \omega)^2} \eta^{\mu_1 \mu_2}$$

For $n \geq 2$, this operator is not eigen state of $\text{rot}^n \neq$ dilations. Since the 2nd order pole is not proportional to the operator itself.

Since it is not even an eigen state of $\text{rot}^n \neq$ dilations, assigning weight should not be done to such operators.

For $(k_\mu X^\mu)^n$

$$:T(z)::[k_\mu X^\mu(\omega, \bar{\omega})]^n:$$

$$= \frac{-k_{\mu_1} k_{\mu_2} \dots k_{\mu_n}}{\alpha'}$$

$$x : (\partial_z X^\nu)(\partial_{\bar{z}} X_\nu(z, \bar{z})) :: X^{\mu_1} X^{\mu_2} \dots X^{\mu_n}(\omega, \bar{\omega}) :$$

$$\sim \left(\frac{1}{z - \omega} \right) \sum_{m=1}^n k_{\mu_m} : (\partial_z X^{\mu_m}(z, \bar{z})) \prod_{\substack{i=1 \\ i \neq m}}^n k_{\mu_i} X^{\mu_i}(\omega, \bar{\omega}) :$$

$$- \frac{\alpha'}{2(z - \omega)^2} \sum_{\substack{m, l \\ l < m}} k_{\mu_m} k_{\mu_l} \eta^{\mu_m \mu_l} : \prod_{\substack{i=1 \\ i \neq m, l}}^n k_{\mu_i} X^{\mu_i}(\omega, \bar{\omega}) :$$

(1st term has 'n' identical copies of it &
2nd term has $\frac{n(n-1)}{2}$ copies of it.)

$$\text{Thus : } T(z) :: [k_\mu X^\mu(\omega, \bar{\omega})]^n :$$

$$\sim \left(\frac{n}{z - \omega} \right) : \partial_z (k_\mu X^\mu(z, \bar{z})) [k_\mu X^\mu(\omega, \bar{\omega})]^{n-1} : \\ - \frac{\alpha' n(n-1)}{4(z - \omega)^2} (k_\mu k^\mu) : [k_\mu X^\mu(\omega, \bar{\omega})]^{n-2} :$$

In exact same steps,

$$:\tilde{T}(\bar{z}) :: [k_\mu X^\mu(\omega, \bar{\omega})]^n :$$

$$\sim \left(\frac{n}{\bar{z} - \bar{\omega}} \right) : \partial_{\bar{z}} (k_\mu X^\mu(z, \bar{z})) [k_\mu X^\mu(\omega, \bar{\omega})]^{n-1} : \\ - \frac{\alpha' n(n-1)}{4(\bar{z} - \bar{\omega})^2} (k_\mu k^\mu) : [k_\mu X^\mu(\omega, \bar{\omega})]^{n-2} :$$

This operator has no weights as it is not eigen state of rot^n & dilations.

For : $e^{ik_\mu \chi^\mu}$:

$$:e^{ik_\mu \chi^\mu(\omega, \bar{\omega})}: = \sum_{n=0}^{\infty} \frac{i^n : (k_\mu \chi^\mu(\omega, \bar{\omega}))^n :}{n!}$$

$$:T(z)::e^{ik_\mu \chi^\mu}:$$

$$= \sum_{n=0}^{\infty} \frac{i^n :T(z): : [k_\mu \chi^\mu(\omega, \bar{\omega})]^n :}{n!}$$

$$\sim \sum_{n=0}^{\infty} \frac{i^n n}{n! (z - \omega)} : \partial_z (k_\mu \chi^\mu(z, \bar{z})) [k_\mu \chi^\mu(\omega, \bar{\omega})]^{n-1} :$$

$$- \frac{\alpha' (k_\mu k^\mu)}{4(z - \omega)^2} \sum_{n=0}^{\infty} \frac{i^n n(n-1)}{n!} : [k_\mu \chi^\mu(\omega, \bar{\omega})]^{n-2} :$$

$$\sim \frac{: \partial_z (i k_\mu \chi^\mu(z, \bar{z})) :}{(z - \omega)} \sum_{n=1}^{\infty} \frac{i^{n-1} [k_\mu \chi^\mu(\omega, \bar{\omega})]^{n-1}}{(n-1)!} :$$

$$+ \frac{\alpha' (k_\mu k^\mu)}{4(z - \omega)^2} \sum_{n=2}^{\infty} \frac{i^{n-2}}{(n-2)!} : [k_\mu \chi^\mu(\omega, \bar{\omega})]^{n-2} :$$

$$\rightarrow :T(z):: e^{ik_\mu \chi^\mu(\omega, \bar{\omega})} :$$

$$\sim \frac{\partial_\omega (ik_\mu \chi^\mu(\omega, \bar{\omega}))}{(z - \omega)} e^{ik_\mu \chi^\mu} + \frac{\alpha' (k_\mu k^\mu)}{4(z - \omega)^2} e^{ik_\mu \chi^\mu} :$$

$$\sim \left[\frac{\partial_\omega (:e^{ik_\mu \chi^\mu}:)}{(z - \omega)} + \frac{\alpha' (k_\mu k^\mu)}{4(z - \omega)^2} :e^{ik_\mu \chi^\mu}: \right]$$

In exact same way,

$$:\tilde{T}(\bar{z}):: e^{ik_\mu \chi^\mu(\omega, \bar{\omega})} :$$

$$\sim : \left[\frac{\partial_{\bar{\omega}} (:e^{ik_\mu \chi^\mu}:)}{(\bar{z} - \bar{\omega})} + \frac{\alpha' (k_\mu k^\mu)}{4(\bar{z} - \bar{\omega})^2} :e^{ik_\mu \chi^\mu}: \right]$$

Thus weight of $:e^{ik_\mu \chi^\mu(\omega, \bar{\omega})}: is$

$$(h, \tilde{h}) = \left(\frac{\alpha' k_\mu k^\mu}{4}, \frac{\alpha' k_\mu k^\mu}{4} \right). \text{ And it}$$

is also a primary operator.

For $f_{\mu\nu} : \partial\chi^\mu \partial\chi^\nu e^{ik_\rho \chi^\rho}$:

$$\text{Let } V(\omega, \bar{\omega}) = f_{\mu\nu} \partial\chi^\mu \bar{\partial}\chi^\nu e^{ik_\rho \chi^\rho}(\omega, \bar{\omega})$$

We use the general result for weight of

$$: \prod_i (\partial^{m_i} \chi^{\mu_i}) \prod_j (\bar{\partial}^{n_j} \chi^{\nu_j}) e^{ik_\rho \chi^\rho} :$$

$$\text{to be } \left(\frac{\alpha'}{4} (k_\mu k^\mu) + \sum_i m_i, \frac{\alpha'}{4} (k_\mu k^\mu) + \sum_j n_j \right)$$

Now, $T(z) = -\frac{1}{\alpha'} (\partial_z \chi^\beta) (\partial_z \chi_\beta)$, there are higher order singularities that can arise. In our case we get a 3rd order singularity by the contraction combination.

$$\underbrace{(\partial_z \chi^\beta) (\partial_\omega \chi^\mu)}_{-\frac{\alpha'}{2} \eta^{\beta\mu}} \overline{(\partial_z \chi_\beta) e^{ik_\rho \chi^\rho(\omega, \bar{\omega})}} \times 2 \text{ ways}$$

$$\frac{1}{(z-\omega)^2}$$

$$\overline{(\partial_z \chi_\beta) : e^{ik_\rho \chi^\rho(\omega, \bar{\omega})} :} = -\frac{i\alpha'}{2} k_\beta \frac{: e^{ik_\rho \chi^\rho(\omega, \bar{\omega})} :}{(z-\omega)}$$

Thus the 3rd order singularity is,

$$2f_{\mu\nu} \left(-\frac{\alpha' \eta^{\beta\mu}}{2} \frac{1}{(z-\omega)^2} \right) \bar{\partial} \chi^\nu \left(-\frac{i\alpha' k_\beta}{2} \frac{e^{ik_\rho \chi^\rho(\omega, \bar{\omega})}}{(z-\omega)} \right)$$

$$= \frac{i\alpha'^2 k^\mu f_{\mu\nu} : \bar{\partial} \chi^\nu e^{ik_\rho \chi^\rho}(\omega, \bar{\omega}) :}{(z-\omega)^3}$$

Thus, we get,

$$:T(z)::V(\omega, \bar{\omega}): \sim \frac{\left(\frac{\alpha'}{4} (k_\mu k^\mu) + 1 \right) :V(\omega, \bar{\omega}):}{(z-\omega)^2}$$

$$+ \frac{\partial_\omega :V(\omega, \bar{\omega}):}{(z-\omega)} + \frac{i\alpha'^2 k^\mu f_{\mu\nu} : \bar{\partial} \chi^\nu e^{ik_\rho \chi^\rho}(\omega, \bar{\omega}) :}{(z-\omega)^3}$$

For $:V(\omega, \bar{\omega}):$ to be tensor operator, we need 3rd order singular term to vanish. So we need $k^\mu f_{\mu\nu} = 0$
It has weights

$$(h_V, \tilde{h}_V) = \left(\frac{\alpha'}{4} (k_\mu k^\mu) + 1, \frac{\alpha'}{4} (k_\mu k^\mu) + 1 \right)$$

Conformal Properties of $T(z)$ & $\bar{T}(\bar{z})$

$$T(z) = -\frac{1}{\alpha'} (\partial_z \chi^\nu) (\partial_z \chi_\nu(z, \bar{z})), \quad \bar{T}(\bar{z}) = -\frac{1}{\alpha'} (\partial_{\bar{z}} \chi^\nu) (\partial_{\bar{z}} \chi_\nu(z, \bar{z}))$$

& like previously got,

$$:\partial \hat{\chi}^\mu(z, \bar{z}) \partial \hat{\chi}_\mu(z, \bar{z}): : \partial \hat{\chi}^\nu(w, \bar{w}) \partial \hat{\chi}_\nu(w, \bar{w}):$$

$$\sim \frac{D \alpha'^2}{2} \frac{1}{(z-w)^4} - \frac{2\alpha'}{(z-w)^2} : \partial \hat{\chi}^\nu(w, \bar{w}) \partial \hat{\chi}_\nu(w, \bar{w}):$$

$$- \frac{\alpha'}{(z-w)} \partial [: \hat{\chi}^\nu(w, \bar{w}) \partial \hat{\chi}_\nu(w, \bar{w}) :]$$

$$\rightarrow :T(z)::T(w):$$

$$\sim \frac{1}{(\alpha')^2} \left[\frac{D \alpha'^2}{2} \frac{1}{(z-w)^4} - \frac{2\alpha'}{(z-w)^2} : \partial \hat{\chi}^\nu(w, \bar{w}) \partial \hat{\chi}_\nu(w, \bar{w}): \right.$$

$$\left. - \frac{\alpha'}{(z-w)} \partial [: \hat{\chi}^\nu(w, \bar{w}) \partial \hat{\chi}_\nu(w, \bar{w}) :] \right]$$

$\Rightarrow$

$$:T(z)::T(w): \sim \frac{D}{2(z-w)^4} + \frac{2:T(w):}{(z-w)^2} + \frac{\partial :T(w):}{(z-w)}$$

$$\Rightarrow h \text{ of } :T(w): = 2$$

Similarly,

$$:\tilde{T}(\bar{z}): :T(\bar{w}): \sim \frac{D}{2(\bar{z}-\bar{w})^4} + \frac{2:\tilde{T}(\bar{w}):}{(\bar{z}-\bar{w})^2} + \frac{\partial :T(\bar{w}):}{(\bar{z}-\bar{w})}$$

$\rightarrow \tilde{h}$ of $:T(\bar{w}):$ is '2'.

But $:T(\bar{z}): :T(\bar{w}): \sim 0$ & $:T(\bar{z}): :\tilde{T}(\bar{w}): \sim 0$

Since $:T(\bar{z}):$ is holomorphic & $:T(\bar{z}):$ is anti holomorphic. And OPE b/w holomorphic and anti-holomorphic operators should not have any singularity.

Thus $(h, \tilde{h})$ of $:T(\bar{z}):$ is $(2, 0)$ and $(h, \tilde{h})$ of $:T(\bar{z}):$ is $(0, 2)$.

Moreover, the OPE's of $T(\bar{z})$ & $\tilde{T}(\bar{z})$ with itself has singularities upto 4th order. Thus they are not Primary (or Tensor) operators in the context of Conformal transformations. However, they are eigen states of rotations & dilations.

Since $T(z)$ and $\tilde{T}(\bar{z})$ are not tensor operators, let us find the transformation rule of them under a general conformal transformation $z \rightarrow z + \epsilon \mathcal{V}(z)$.

Then the corresponding Noether current is, $J_z = i\mathcal{V}(z)T(z)$ & $J_{\bar{z}} = i\bar{\mathcal{V}}(\bar{z})\tilde{T}(\bar{z})$

By ward identity, for a general operator

$$\text{Res}_{z \rightarrow \omega} [J_z \hat{\mathcal{O}}(\omega, \bar{\omega})] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{\omega}}} [J_{\bar{z}} \hat{\mathcal{O}}(\omega, \bar{\omega})] = \frac{\delta \hat{\mathcal{O}}(\omega, \bar{\omega})}{i\epsilon}$$

under a transformation $z \rightarrow z + \epsilon \delta z$

First consider ' $T(\omega)$ '.

$$\rightarrow \text{Res}_{z \rightarrow \omega} [J_z T(\omega)] + \overline{\text{Res}_{\bar{z} \rightarrow \bar{\omega}}} [J_{\bar{z}} T(\omega)] = \frac{\delta T(\omega)}{i\epsilon}$$

Zero since T is holomorphic

$$\rightarrow \underline{\underline{\frac{\delta T(\omega)}{\epsilon} = - \text{Res}_{z \rightarrow \omega} [\mathcal{V}(z) T(z) T(\omega)]}}}$$

We use, $\varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(\omega)}{n!} (z-\omega)^n$

&

$$:T(z)::T(\omega): \sim \frac{D}{2(z-\omega)^4} + \frac{2:T(\omega):}{(z-\omega)^2} + \frac{\partial:T(\omega):}{(z-\omega)}$$

$$\varphi(z)T(z)T(\omega) \sim \varphi(z):\tilde{T}(z)::\tilde{T}(\omega):$$

$$\rightarrow \varphi(z)T(z)T(\omega) \sim \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(\omega)}{n!} (z-\omega)^n \times \left[\frac{D}{2(z-\omega)^4} + \frac{2:T(\omega):}{(z-\omega)^2} + \frac{\partial:T(\omega):}{(z-\omega)} \right]$$

$$\rightarrow \text{Res}_{z \rightarrow \omega} [\varphi(z)T(z)T(\omega)] \text{ is given by}$$

$$= \frac{D \partial^3 \varphi(\omega)}{2 \times 3!} + 2:T(\omega):\partial \varphi(\omega) + \varphi(\omega)\partial:T(\omega):$$

$$\rightarrow \underline{\underline{\frac{\partial T(\omega)}{\epsilon} = -\frac{D}{12} \partial^3 \varphi(\omega) - 2:T(\omega):\partial \varphi(\omega) - \varphi(\omega)\partial:T(\omega):}}$$

The finite transformations $z \rightarrow z' = f(z)$ are given by exponentiating the above.

$$\rightarrow T'(z') = \left(\frac{\partial f}{\partial z} \right)^{-2} \left[T(z) - \frac{D}{12} \{f, z\}_S \right]$$

$$\text{where } \{f, z\}_S = (\partial_z^3 f)(\partial_z f)^{-1} - \frac{3}{2} (\partial_z^2 f)^2 (\partial_z f)^{-2}$$

is the Schwarzian derivative of $f(z)$.

$$\text{Consider } f(z) = z + \epsilon v(z)$$

$$\begin{aligned} \rightarrow \{f, z\}_S &= \epsilon (\partial^3 v) (1 + \epsilon \partial v)^{-1} \\ &\quad - \frac{3}{2} (\epsilon \partial^2 v)^2 (1 + \epsilon \partial v)^{-2} \end{aligned}$$

$$\begin{aligned} &= \epsilon (\partial^3 v) \left[1 - \epsilon (\partial v) + \epsilon^2 (\partial v)^2 + \dots \right] \\ &\quad - \frac{3}{2} \epsilon^2 (\partial^2 v)^2 \left[1 - 2\epsilon (\partial v) + 3\epsilon^2 (\partial v)^2 - \dots \right] \end{aligned}$$

$$\rightarrow \{f, z\}_g = \epsilon(\partial^3 v) - \epsilon^2(\partial^3 v)(\partial v) + \epsilon^3(\partial^3 v)(\partial v)^2 \\ - \frac{3}{2}\epsilon^2(\partial^2 v)^2 + 3\epsilon^3(\partial^2 v)^2(\partial v) \\ + \mathcal{O}(\epsilon^4)$$

$$\left(\frac{\partial f}{\partial z}\right)^{-2} = \left[1 - 2\epsilon(\partial v) + 3\epsilon^2(\partial v)^2 - 4\epsilon^3(\partial v)^3 + \dots\right]$$

$$\rightarrow \left(\frac{\partial f}{\partial z}\right)^{-2} \{f, z\}_g =$$

$$= \left[1 - 2\epsilon(\partial v) + 3\epsilon^2(\partial v)^2 - 4\epsilon^3(\partial v)^3 + \dots\right]$$

$$\times \left[\epsilon(\partial^3 v) - \epsilon^2(\partial^3 v)(\partial v) + \epsilon^3(\partial^3 v)(\partial v)^2 \right. \\ \left. - \frac{3}{2}\epsilon^2(\partial^2 v)^2 + 3\epsilon^3(\partial^2 v)^2(\partial v) \right]$$

$$= \epsilon(\partial^3 v) - 2\epsilon^2(\partial v)(\partial^3 v) - \epsilon^2(\partial^3 v)(\partial v) - \frac{3}{2}\epsilon^2(\partial^2 v)^2 \\ + \mathcal{O}(\epsilon^3)$$

$$-\frac{\mathcal{D}}{12} \left(\frac{\partial f}{\partial z}\right)^{-2} \{f, z\}_g = -\frac{\epsilon \mathcal{D}}{12} (\partial^3 v) + \mathcal{O}(\epsilon^2)$$

$$\left(\frac{\partial f}{\partial z}\right)^{-2} T(z) = T(z) - 2\epsilon \partial v T(z) + \mathcal{O}(\epsilon^2)$$

$$\rightarrow T'(z') = \left(\frac{\partial f}{\partial z}\right)^{-2} \left[T(z) - \frac{D}{12} \{f, z\}_S \right]$$

$$= T(z) - 2\epsilon (\partial v) T(z) - \frac{\epsilon D}{12} (\partial^3 v) + \mathcal{O}(\epsilon^2)$$

$$T'(z) = T'(z') \Big|_{z \rightarrow z - \epsilon v}$$

$$= T(z - \epsilon v) - 2\epsilon (\partial v) T(z) - \frac{\epsilon D}{12} (\partial^3 v) + \mathcal{O}(\epsilon^2)$$

$$= T(z) - \epsilon v (\partial T(z)) - 2\epsilon (\partial v) T(z) - \frac{\epsilon D}{12} (\partial^3 v)$$

$$\rightarrow \delta T(z) = T'(z) - T(z) \quad \text{upto 1st order}$$

$$\delta T(z) = -\epsilon v (\partial T(z)) - 2\epsilon (\partial v) T(z) - \frac{\epsilon D}{12} (\partial^3 v)$$

It has correct infinitesimal form.

Now we should show that it has correct form even under composition.

i.e. If $z \rightarrow z' = f(z)$, where

$$T'(z') = \left(\frac{\partial f}{\partial z} \right)^{-2} \left[T(z) - \frac{D}{12} \{f, z\}_S \right]$$

& if $z' \rightarrow z'' = g(z') = g(f(z)) = g \circ f(z)$

where
$$T''(z'') = \left(\frac{\partial g}{\partial z'} \right)^{-2} \left[T'(z') - \frac{D}{12} \{g, z'\}_S \right]$$

Then we need $T''(z'')$ & $T(z)$ to relate as,

$$T''(z'') = \left(\frac{\partial}{\partial z} g \circ f(z) \right)^{-2} \left[T(z) - \frac{D}{12} \{g \circ f, z\}_S \right]$$

$$\text{Let } u(z) = g \circ f(z) = g(f(z))$$

$$\partial u = g'(f(z)) f'(z)$$

$$\partial^2 u = g''(f(z)) [f'(z)]^2 + g'(f(z)) f''(z)$$

$$\partial^3 u = g'''(f(z)) [f'(z)]^3 + 3g''(f(z)) f'(z) f''(z) + g'(f(z)) f'''(z)$$

$$\frac{\partial^3 u}{\partial u} = \frac{g'''(f(z))[f'(z)]^2}{g'(f(z))} + \frac{3g''(f(z))f''(z)}{g'(f(z))} + \frac{f'''(z)}{f'(z)}$$

And also $\frac{\partial^2 u}{\partial u} = \frac{g''(f(z))f'(z)}{g'(f(z))} + \frac{f''(z)}{f'(z)}$

$$\begin{aligned} \rightarrow \frac{3}{2} \left(\frac{\partial^2 u}{\partial u} \right)^2 &= \frac{3}{2} \left[\frac{g''(f(z))}{g'(f(z))} \right]^2 [f'(z)]^2 + \frac{3}{2} \left[\frac{f''(z)}{f'(z)} \right]^2 \\ &\quad + 3 \frac{g''(f(z))f''(z)}{g'(f(z))} \end{aligned}$$

Now,

$$\begin{aligned} \{g \circ f, z\}_s &= \frac{\partial^3 u}{\partial u} - \frac{3}{2} \left(\frac{\partial^2 u}{\partial u} \right)^2 \\ &= \frac{g'''(f(z))[f'(z)]^2}{g'(f(z))} + \frac{3g''(f(z))f''(z)}{g'(f(z))} + \frac{f'''(z)}{f'(z)} \\ &\quad - \frac{3}{2} \left[\frac{g''(f(z))}{g'(f(z))} \right]^2 [f'(z)]^2 - \frac{3}{2} \left[\frac{f''(z)}{f'(z)} \right]^2 \\ &\quad - 3 \frac{g''(f(z))f''(z)}{g'(f(z))} \end{aligned}$$

$$\rightarrow \{g \circ f, z\}_s = \left(\frac{g'''(f(z))}{g'(f(z))} - \frac{3}{2} \left[\frac{g''(f(z))}{g'(f(z))} \right]^2 \right) [f'(z)]^2 + \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left[\frac{f''(z)}{f'(z)} \right]^2$$

$$\Rightarrow \underline{\{g \circ f, z\}_s = \{g, z'\}_s [f'(z)]^2 + \{f, z\}_s}$$

$$\text{Now, } T''(z'') = \left(\frac{\partial g}{\partial z'} \right)^{-2} \left[T'(z') - \frac{D}{12} \{g, z'\}_s \right]$$

$$= [g'(f(z))]^{-2} \left[[f'(z)]^{-2} \left[T(z) - \frac{D}{12} \{f, z\}_s \right] - \frac{D}{12} \{g, z'\}_s \right]$$

$$= [g'(f(z)) f'(z)]^{-2} \left[T(z) - \frac{D}{12} \left[\{g, z'\}_s [f'(z)]^2 + \{f, z\}_s \right] \right]$$

$$\text{Use, } \{g, z'\}_s [f'(z)]^2 + \{f, z\}_s = \{g \circ f, z\}_s$$

$$\& \quad g'(f(z)) f'(z) = \frac{\partial}{\partial z} g(f(z)) = \frac{\partial}{\partial z} g \circ f(z)$$

Thus we get,

$$T''(z'') = \left(\frac{\partial}{\partial z} g \circ f(z) \right)^{-2} \left[T(z) - \frac{D}{12} \{g \circ f, z\}_S \right]$$

where $z'' = g \circ f(z)$. Hence the finite transformation rule we wrote is right.

Thus for infinitesimal $z \rightarrow z + \epsilon v(z)$,

$$\delta T(z) = -\epsilon \left[\frac{D}{12} \partial^3 v + 2T(z) \partial v + v \partial T(z) \right]$$

& for finite $z \rightarrow f(z)$

$$T'(z') = \left(\frac{\partial f}{\partial z} \right)^{-2} \left[T(z) - \frac{D}{12} \{f, z\}_S \right]$$

$$\text{where } \{f, z\}_S = (\partial_z^3 f)(\partial_z f)^{-1} - \frac{3}{2} (\partial_z^2 f)^2 (\partial_z f)^{-2}$$

For $\tilde{T}(\bar{z})$, all results are exactly similar. Just replace $T \leftrightarrow \tilde{T}$, $\partial \leftrightarrow \bar{\partial}$, $z \rightarrow \bar{z}$, & $v \leftrightarrow v^*$, $f \leftrightarrow f^*$, $g \leftrightarrow g^*$, $u \leftrightarrow u^*$, etc.

T for a general CFT & Central Charge

For a general CFT, we know that the Energy-Momentum tensor is traceless in the Euclidean coordinates. This results in Energy momentum tensor to be in the form

$$[T] = \begin{bmatrix} T_{zz} & 0 \\ 0 & T_{\bar{z}\bar{z}} \end{bmatrix} \quad \begin{aligned} T_{zz} &\equiv T(z) \\ T_{\bar{z}\bar{z}} &\equiv \bar{T}(\bar{z}) \\ T_{z\bar{z}} &= T_{\bar{z}z} = 0 \end{aligned}$$

Conservation of Energy-momentum tensor leads to the holomorphic & anti-holomorphic nature of the E.M tensor components.

$$\text{i.e. } \partial_{\bar{z}} T_{zz} = 0 = \partial_z T_{\bar{z}\bar{z}}$$

The Noether current for conformal transformation $z \rightarrow z' = f(z)$ is given by the components $J_z = i\mathcal{V}(z)T(z)$ and $J_{\bar{z}} = i\mathcal{V}^*(\bar{z})\bar{T}(\bar{z})$, where $\mathcal{V}(z)$ is given by the infinitesimal transform $f(z) \sim z + \epsilon\mathcal{V}(z)$.

The Energy momentum tensor components are the generators of translations of any local operator $\hat{\mathcal{O}}(z, \bar{z})$. Thus the OPE of it with T & $\bar{T}$ give a singularity of 1^{st} order with coefficient as $\partial \hat{\mathcal{O}}(z, \bar{z})$ & $\bar{\partial} \hat{\mathcal{O}}(z, \bar{z})$ respectively. In general,

$$T(z) \hat{\mathcal{O}}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{\mathcal{O}}^{(n)}(\omega, \bar{\omega})}{(z - \omega)^{n+1}} + \text{Regular terms}$$

&

$$\bar{T}(\bar{z}) \hat{\mathcal{O}}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{\mathcal{O}}^{(n)}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})^{n+1}} + \text{Regular terms}$$

where,

$$\hat{\mathcal{O}}^{(0)}(\omega, \bar{\omega}) = \partial \hat{\mathcal{O}}(\omega, \bar{\omega}) \quad \& \quad \hat{\mathcal{O}}^{(0)}(\omega, \bar{\omega}) = \bar{\partial} \hat{\mathcal{O}}(\omega, \bar{\omega})$$

Under a infinitesimal conformal transform $z \rightarrow z + \epsilon \mathcal{V}(z)$, the change in operator is,

$$\delta \hat{\mathcal{O}}(\omega, \bar{\omega}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \left[\mathcal{V}^{(n)}(\omega) C_n(\omega, \bar{\omega}) + \mathcal{V}^{*(n)}(\bar{\omega}) \bar{C}_n(\omega, \bar{\omega}) \right]$$

$$\text{where } \mathcal{V}^{(n)}(\omega) = \frac{\partial^n \mathcal{V}(\omega)}{\partial \omega^n} \quad \& \quad \mathcal{V}^{*(n)}(\bar{\omega}) = \frac{\partial^n \mathcal{V}^*(\bar{\omega})}{\partial \bar{\omega}^n}.$$

For rotation & dilation transformation,
 $z \rightarrow z' = \zeta z$, the eigen states of this
 transform are given by those local
 operators which transform as,

$$\hat{A}'(z', \bar{z}') = \zeta^{-h} \zeta^{*\tilde{h}} \hat{A}(z, \bar{z})$$

where $(h, \tilde{h})$ are weights of $\hat{A}(z, \bar{z})$. Then,
 under infinitesimal transformation, where
 $z \rightarrow z + \epsilon z$, the eigen operator transform as

$$\begin{aligned} \delta \hat{A}(z, \bar{z}) = & -(h\epsilon + \tilde{h}\epsilon^*) \hat{A}(z, \bar{z}) \\ & - \epsilon z \partial \hat{A}(z, \bar{z}) - \epsilon^* \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) \end{aligned}$$

The OPE of $\hat{A}(\omega, \bar{\omega})$ with T & $\tilde{T}$ is,

$$\begin{aligned} T(z) \hat{A}(\omega, \bar{\omega}) &= \sum_{n=0}^{\infty} \frac{\hat{A}^{(n)}(\omega, \bar{\omega})}{(z - \omega)^{n+1}} + \text{Regular terms} \\ \& \\ \tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega}) &= \sum_{n=0}^{\infty} \frac{\hat{\tilde{A}}^{(n)}(\omega, \bar{\omega})}{(\bar{z} - \bar{\omega})^{n+1}} + \text{Regular terms} \end{aligned}$$

where,

$$\hat{A}^{(0)}(\omega, \bar{\omega}) = \partial \hat{A}(\omega, \bar{\omega}) \quad \& \quad \hat{\tilde{A}}^{(0)}(\omega, \bar{\omega}) = \bar{\partial} \hat{A}(\omega, \bar{\omega})$$

$$\hat{A}^{(1)}(\omega, \bar{\omega}) = h \hat{A}(\omega, \bar{\omega}) \quad \& \quad \hat{\tilde{A}}^{(1)}(\omega, \bar{\omega}) = \tilde{h} \hat{A}(\omega, \bar{\omega})$$

Primary / tensor operators are those operators which transform under all general $z \rightarrow z' = f(z)$ as,

$$\hat{A}'(z', \bar{z}') = [f'(z)]^{-h} [f'(\bar{z})]^{-\tilde{h}} \hat{A}(z, \bar{z})$$

Under infinitesimal transformation, $\delta z = \epsilon z$, the primary operators transform as,

$$\delta \hat{A}(z, \bar{z}) = -\epsilon \left[\left(h v'(z) + \tilde{h} \tilde{v}'(\bar{z}) \right) \hat{A}(z, \bar{z}) + v(z) \partial \hat{A}(z, \bar{z}) + \tilde{v}(\bar{z}) \bar{\partial} \hat{A}(z, \bar{z}) \right]$$

The OPE of $\hat{A}(\omega, \bar{\omega})$ with T & $\tilde{T}$ is,

$$T(z) \hat{A}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{A}^{(n)}(\omega, \bar{\omega})}{(z-\omega)^{n+1}} + \text{Regular terms}$$

&

$$\tilde{T}(\bar{z}) \hat{A}(\omega, \bar{\omega}) = \sum_{n=0}^{\infty} \frac{\hat{\tilde{A}}^{(n)}(\omega, \bar{\omega})}{(\bar{z}-\bar{\omega})^{n+1}} + \text{Regular terms}$$

where,

$$\hat{A}^{(0)}(\omega, \bar{\omega}) = \partial \hat{A}(\omega, \bar{\omega}) \quad \& \quad \hat{\tilde{A}}^{(0)}(\omega, \bar{\omega}) = \bar{\partial} \hat{A}(\omega, \bar{\omega})$$

$$\hat{A}^{(1)}(\omega, \bar{\omega}) = h \hat{A}(\omega, \bar{\omega}) \quad \& \quad \hat{\tilde{A}}^{(1)}(\omega, \bar{\omega}) = \tilde{h} \hat{A}(\omega, \bar{\omega})$$

$$\& \text{ for all } n > 2, \quad \hat{A}^{(n)}(\omega, \bar{\omega}) = 0 = \hat{\tilde{A}}^{(n)}(\omega, \bar{\omega}).$$

Properties of $T(z)$ & $\bar{T}(\bar{z})$ under a general conformal Transformation

The total energy is obtained by integrating Energy-momentum tensor. Thus, in 2 dimensions

$$\text{Energy} = \int d^2z [T]$$

under scaling $z \rightarrow mz$, where 'm' is a real number. But total energy is independent of conformal transformations.

$\rightarrow d^2z \rightarrow m^2 dz$. So, $[T]$ should transform proportional to m^{-2} .

$\rightarrow T_{zz}$ & $T_{\bar{z}\bar{z}}$, both have scaling dim. of $h + \bar{h} = '2'$.

Now, under rotations the components of T change as a rank-2 object.

If $z \rightarrow z' = e^{i\theta} z$ tensorial behaviour
only for rotations

$$T_{zz} \rightarrow T'_{z'z'} = \left(\frac{\partial z}{\partial z'} \right) \left(\frac{\partial z}{\partial z'} \right) T_{zz} = e^{-2i\theta} T_{zz}$$

$$\rightarrow \underline{T'(z') = e^{-2i\theta} T(z)}$$

Similarly, for $\bar{z} \rightarrow \bar{z}' = e^{-i\theta} \bar{z}$

$$T'_{\bar{z}\bar{z}} \rightarrow T'_{\bar{z}'\bar{z}'} = \left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right) \left(\frac{\partial \bar{z}}{\partial \bar{z}'} \right) T_{\bar{z}\bar{z}} = e^{2i\theta} T_{\bar{z}\bar{z}}$$

$$\rightarrow \underline{\widetilde{T}'(\bar{z}') = e^{2i\theta} \widetilde{T}(\bar{z})}$$

Thus T & $\widetilde{T}$ are eigen state local ops under rotation. The spin is given by $\hbar - \widetilde{\hbar}$. For eigen state of rot^n , the operator transform by $\exp[-(\hbar - \widetilde{\hbar})i\theta]$. So

$\hbar - \widetilde{\hbar}$ for $T(z) = 2$, since $T'(z') = e^{-2i\theta} T(z)$

$\hbar - \widetilde{\hbar}$ for $\widetilde{T}(\bar{z}) = -2$, since $\widetilde{T}'(\bar{z}') = e^{2i\theta} \widetilde{T}(\bar{z})$

Thus for $T(z)$, $h + \tilde{h} = 2$ & $h - \tilde{h} = 2$
 $\rightarrow \underline{T(z) \text{ has } (h, \tilde{h}) = (2, 0)}$

Similarly for $\tilde{T}(\bar{z})$, $h + \tilde{h} = 2$ & $h - \tilde{h} = -2$
 $\rightarrow \underline{\tilde{T}(\bar{z}) \text{ has } (h, \tilde{h}) = (0, 2)}$

So the singular terms of OPE of $T(z)T(w)$ & $\tilde{T}(\bar{z})\tilde{T}(\bar{w})$ are,

$$T(z)T(w) \sim \frac{\partial_w T(w)}{z - w} + \frac{2T(w)}{(z - w)^2} + \text{higher order singular terms}$$

$$\tilde{T}(\bar{z})\tilde{T}(\bar{w}) \sim \frac{\partial_{\bar{w}} \tilde{T}(\bar{w})}{\bar{z} - \bar{w}} + \frac{2\tilde{T}(\bar{w})}{(\bar{z} - \bar{w})^2} + \text{higher order singular terms}$$

In the ' $T(z)T(w)$ ' OPE, all higher order terms will be of the form $\frac{T^{(n)}(w)}{(z - w)^n}$.

Since $h + \tilde{h}$ of $T(w) = 2$, then for the product $T(z)T(w)$, $h + \tilde{h} = 2 \times 2 = 4$.

Now, $h + \tilde{h}$ of $z - w = 1$, then for $(z - w)^{-n}$, its $h + \tilde{h} = -n$. Thus to match the scaling dim of ' $T(z)T(w)$ ' OPE, $h + \tilde{h}$ of $\frac{T^{(n)}(w)}{(z - w)^n}$ should be '4'.

$$\rightarrow h + \tilde{h} \text{ of } T^{(n)}(w) = 4 - n$$

But $h + \tilde{h}$ for any local operator has to be +ve, more precisely $h, \tilde{h} > 0$. This will be proven later.

So, $T^{(4)}(w)$ has to be a scalar with $h = \tilde{h} = 0$. No operator made of T 's or its derivatives. Hence, $T^{(4)}(w)$ should be independent of T & its derivatives.

And only upto $n = 4$, can be present in the expansion of ' $T(z)T(w)$ '. This argument holds even for ' $\tilde{T}(\bar{z})\tilde{T}(\bar{w})$ '.

$$T(z)T(w) \sim \frac{\partial_w T(w)}{z-w} + \frac{2T(w)}{(z-w)^2} + \frac{T^{(3)}(w)}{(z-w)^3} + \frac{T^{(4)}(z)}{(z-w)^4}$$

$$\tilde{T}(\bar{z})\tilde{T}(\bar{w}) \sim \frac{\partial_{\bar{w}} \tilde{T}(\bar{w})}{\bar{z}-\bar{w}} + \frac{2\tilde{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\tilde{T}^{(3)}(\bar{w})}{(\bar{z}-\bar{w})^3} + \frac{\tilde{T}^{(4)}(\bar{z})}{(\bar{z}-\bar{w})^4}$$

$$\& \text{ for all } n > 4, T^{(n)}(w) = 0 = \tilde{T}^{(n)}(\bar{w}).$$

Now, we need the OPE to be invariant under $z \leftrightarrow w$. So, $T(z)T(w)$ should be equal to $T(w)T(z)$ upto singular terms. Consider the OPE,

$$T(w)T(z) \sim \frac{\partial_z T(z)}{w-z} + \frac{2T(z)}{(w-z)^2} + \frac{T^{(3)}(z)}{(w-z)^3} + \frac{T^{(4)}(z)}{(w-z)^4}$$

$$\partial_z T(z) = \partial_w T(w) + (z-w) \partial_w^2 T(w) + \dots$$

$$T(z) = T(w) + (z-w) \partial_w T(w) + (z-w)^2 \partial_w^2 T(w) + \dots$$

$$T^{(3)}(z) = T^{(3)}(w) + (z-w) \partial_w T^{(3)}(w) + (z-w)^2 \partial_w^2 T^{(3)}(w) + \dots$$

$$T^{(4)}(z) = T^{(4)}(w) + (z-w) \partial_w T^{(4)}(w) + (z-w)^2 \partial_w^2 T^{(4)}(w) + (z-w)^3 \partial_w^3 T^{(4)}(w) + \dots$$

Substituting the Taylor expansions, we get,

$$\begin{aligned} \Rightarrow T(\omega)T(z) &\sim \frac{\partial_{\omega}T(\omega)}{\omega-z} + \frac{2T(\omega)}{(\omega-z)^2} + \frac{2(z-\omega)\partial_{\omega}T(\omega)}{(\omega-z)^2} \\ &+ \frac{T^{(3)}(\omega)}{(\omega-z)^3} + \frac{(z-\omega)\partial_{\omega}T^{(3)}(\omega)}{(\omega-z)^3} + \frac{(z-\omega)^2\partial_{\omega}^2T^{(3)}(\omega)}{(\omega-z)^3} \\ &+ \frac{T^{(4)}(\omega)}{(\omega-z)^4} + \frac{(z-\omega)\partial_{\omega}T^{(4)}(\omega)}{(\omega-z)^4} + \frac{(z-\omega)^2\partial_{\omega}^2T^{(4)}(\omega)}{(\omega-z)^4} \\ &+ \frac{(z-\omega)^3\partial_{\omega}^3T^{(4)}(\omega)}{(\omega-z)^4} \end{aligned}$$

$$\begin{aligned} \Rightarrow T(\omega)T(z) &\sim \frac{\partial_{\omega}T(\omega)}{(z-\omega)} + \frac{2T(\omega)}{(z-\omega)^2} - \frac{T^{(3)}(\omega)}{(z-\omega)^3} \\ &+ \frac{\partial_{\omega}T^{(3)}(\omega)}{(z-\omega)^2} - \frac{\partial_{\omega}^2T^{(3)}(\omega)}{(z-\omega)} + \frac{T^{(4)}(\omega)}{(z-\omega)^4} + \frac{\partial_{\omega}T^{(4)}(\omega)}{(z-\omega)^3} \\ &+ \frac{\partial_{\omega}^2T^{(4)}(\omega)}{(z-\omega)^2} + \frac{\partial_{\omega}^3T^{(4)}(\omega)}{(z-\omega)} \end{aligned}$$

$$\begin{aligned} \Rightarrow T(\omega)T(z) &\sim \frac{\partial_{\omega}T(\omega)}{(z-\omega)} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{T^{(3)}(\omega)}{(z-\omega)^3} \\ &+ \frac{T^{(4)}(\omega)}{(z-\omega)^4} - \frac{2T^{(3)}(\omega)}{(z-\omega)^3} + \frac{\partial_{\omega}T^{(3)}(\omega)}{(z-\omega)^2} - \frac{\partial_{\omega}^2T^{(3)}(\omega)}{(z-\omega)} \\ &+ \frac{\partial_{\omega}T^{(4)}(\omega)}{(z-\omega)^3} + \frac{\partial_{\omega}^2T^{(4)}(\omega)}{(z-\omega)^2} + \frac{\partial_{\omega}^3T^{(4)}(\omega)}{(z-\omega)} \end{aligned}$$

$$\Rightarrow T(\omega)T(z) \sim T(z)T(\omega)$$

$$- \frac{2T^{(3)}(\omega)}{(z-\omega)^3} + \frac{\partial_\omega T^{(3)}(\omega)}{(z-\omega)^2} - \frac{\partial_\omega^2 T^{(3)}(\omega)}{(z-\omega)} \\ + \frac{\partial_\omega T^{(4)}(\omega)}{(z-\omega)^3} + \frac{\partial_\omega^2 T^{(4)}(\omega)}{(z-\omega)^2} + \frac{\partial_\omega^3 T^{(4)}(\omega)}{(z-\omega)}$$

The presence of ' $T^{(3)}(\omega)$ ' term & if $T^{(4)}(\omega)$ depends on position, it will not allow $T(\omega)T(z) \sim T(z)T(\omega)$. So we need to set $T^{(3)}$ term zero & $T^{(4)}(\omega)$ should be a constant complex number.

Hence, we get the final $T(z)T(\omega)$ OPE as

$$T(z)T(\omega) \sim \frac{\partial_\omega T(\omega)}{z-\omega} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{c}{2(z-\omega)^4}$$

and similarly for $\tilde{T} \tilde{T}$ we get,

$$\tilde{T}(\bar{z})\tilde{T}(\bar{\omega}) \sim \frac{\partial_{\bar{\omega}} \tilde{T}(\bar{\omega})}{\bar{z}-\bar{\omega}} + \frac{2\tilde{T}(\bar{\omega})}{(\bar{z}-\bar{\omega})^2} + \frac{\tilde{c}}{2(\bar{z}-\bar{\omega})^4}$$

Where the constants c & $\tilde{c}$ are called as the central charges.

$c \rightarrow$ Left moving central charge

$\tilde{c} \rightarrow$ Right moving central charge

This form of OPE, matches exactly with Scalar X^μ CFT whose central charge is $c = \tilde{c} = D$ (dim. of X^μ 's).

Thus by ward identity, for infinitesimal transformation $z \rightarrow z + \epsilon \mathcal{V}(z)$,

$$\frac{\delta T(z)}{\epsilon} = -\frac{c}{12} \partial^3 \mathcal{V}(z) - 2T(z) \partial \mathcal{V}(z) - \mathcal{V}(z) \partial T(z)$$

For finite transformations $z \rightarrow f(z)$,

$$\rightarrow T'(z') = \left(\frac{\partial f}{\partial z} \right)^{-2} \left[T(z) - \frac{c}{12} \{f, z\}_S \right]$$

where $\{f, z\}_S = (\partial_z^3 f)(\partial_z f)^{-1} - \frac{3}{2}(\partial_z^2 f)^2 (\partial_z f)^{-2}$
is the Schwarzian derivative of $f(z)$.