

Massless String with Dilaton background

Consider the massless scalar X^μ action but with arbitrary world sheet metric

$$S_{\text{String}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} (\partial_a X^\mu) (\partial_b X_\mu) g^{ab}$$

Now consider adding an additional term to this action such that the usual symmetries are maintained. One allowed such term is given by Euler characteristic of the world sheet. We came across this in Chapter 1 while discussing the Polyakov action & its modifications.

The Euler characteristic is given by,

$$\chi = \frac{1}{4\pi} \int_W d^2\sigma \sqrt{g} R^{(2)}$$

where $R^{(2)}$ is the Ricci scalar of the world sheet.

But consider a background field $\Phi(X^\mu)$ called as dilaton profile. It is a scalar just like how X^μ is a scalar. So,

$$S_{\text{dilaton}} = \frac{1}{4\pi} \int_W d^2\sigma \sqrt{g} R^{(2)} \Phi(X^\mu)$$

is like a minimal coupling of Dilaton field to String action. It also has all required Symmetries provided $\nabla^2 \Phi = 0$, where $\nabla^2 = \nabla_a \nabla^a$. Then,

$$S = S_{\text{string}} + S_{\text{dilaton}}$$

This theory is basically a free massless Scalar X^μ theory, but with a background dilaton Profile field $\Phi(X^\mu)$

But in the flat metric, $R^{(2)} = 0$, thus we recover String action itself. But it nevertheless modifies the world sheet Energy - Momentum Tensor.

The total action is,

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left[(\partial_a \chi^\mu)(\partial^a \chi_\mu) + \alpha' R^{(2)} \Phi(\chi^\mu) \right]$$

$$S \longrightarrow S_{\text{string}} \text{ if } g_{ab} = \delta_{ab}.$$

Consider the Variation of action w.r.t the metric tensor. As seen previously,

$$\delta S_{\text{string}} = \frac{-1}{4\pi} \int d^2\sigma \frac{\sqrt{g}}{\alpha'} \left[\frac{1}{2} g_{ab} (\partial_c \chi^\mu)(\partial^c \chi_\mu) - (\partial_a \chi^\mu)(\partial_b \chi_\mu) \right] \delta g^{ab}$$

$$\text{Then, } \delta S_{\text{dilaton}} = \frac{1}{4\pi} \int d^2\sigma \Phi(\chi^\mu) \delta [\sqrt{g} R^{(2)}]$$

$$\delta [\sqrt{g} R^{(2)}] = \sqrt{g} [\nabla_a \nabla_b - g_{ab} \nabla_c \nabla^c] \delta g^{ab}$$

$$\text{Now, } \int d^2\sigma \Phi(\chi^\mu) \sqrt{g} [\nabla_a \nabla_b - g_{ab} \nabla_c \nabla^c] \delta g^{ab}$$

$$= (-1)^2 \int d^2\sigma \sqrt{g} \delta g^{ab} [\nabla_a \nabla_b - g_{ab} \nabla_c \nabla^c] \Phi(\chi^\mu)$$

(By integration by parts)

$$\Rightarrow \frac{\partial S}{\partial g^{ab}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} \left[\nabla_a \nabla_b - g_{ab} \nabla_c \nabla^c \right] \Phi(\chi^\mu)$$

$$\Rightarrow \frac{\partial S}{\partial g^{ab}} = \frac{-\sqrt{g}}{4\pi} \left(\frac{1}{\alpha'} \left[\frac{1}{2} g_{ab} (\partial_c \chi^\mu) (\partial^c \chi_\mu) - (\partial_a \chi^\mu) (\partial_b \chi_\mu) \right] - [\nabla_a \nabla_b - g_{ab} \nabla_c \nabla^c] \Phi(\chi^\mu) \right)$$

$$T_{ab} = -\frac{4\pi}{\sqrt{g}} \left(\frac{\partial S}{\partial g^{ab}} \right)$$

Also, $\nabla_c \nabla^c \Phi(\chi^\mu) = 0$ to maintain the symmetries of action.

$$\Rightarrow T_{ab} = T_{ab}^{(\text{string})} - \nabla_a \nabla_b \Phi(\chi^\mu), \text{ where}$$

$$T_{ab}^{(\text{string})} = \frac{1}{\alpha'} \left[\frac{1}{2} g_{ab} (\partial_c \chi^\mu) (\partial^c \chi_\mu) - (\partial_a \chi^\mu) (\partial_b \chi_\mu) \right]$$

Then, for a flat Euclidean metric, we set $g_{ab} = \delta_{ab}$. $\nabla_a \rightarrow \partial_a$. $\sqrt{g} \rightarrow 1$. So, we get back $S \rightarrow S_{\text{string}}$ as

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu) (\partial_b \chi_\mu) \delta^{ab}$$

(since Ricci scalar is zero)

The closed string eqⁿ of motion is

$$\partial_a \partial^a \chi^\mu = 0.$$

In complex coordinates,

$$S = \frac{1}{2\pi\alpha'} \int d^2z (\partial_z \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

The eqⁿ of motion is $\partial_z \partial_{\bar{z}} \chi^\mu = \partial_{\bar{z}} \partial_z \chi^\mu = 0$.

Then, the energy Momentum tensor is,

$$T_{zz} = -\frac{1}{\alpha'} (\partial_z \chi^\mu) (\partial_z \chi_\mu) + \partial_z^2 \Phi(\chi^\mu)$$

$$T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} (\partial_{\bar{z}} \chi^\mu) (\partial_{\bar{z}} \chi_\mu) + \partial_{\bar{z}}^2 \Phi(\chi^\mu)$$

$$T_{z\bar{z}} = \partial_z \partial_{\bar{z}} \Phi(\chi^\mu), \quad T_{\bar{z}z} = \partial_{\bar{z}} \partial_z \Phi(\chi^\mu)$$

Linear Dilaton CFT

The massless χ^μ theory, with arbitrary dilaton field background does not have conformal symmetry. Since,

$$T_{z\bar{z}} = \partial_z \partial_{\bar{z}} \Phi(\chi^\mu), \quad T_{\bar{z}z} = \partial_{\bar{z}} \partial_z \Phi(\chi^\mu)$$

which need not be zero.

But for the theory to have conformal symmetry, we need $T_{z\bar{z}} = T_{\bar{z}z} = 0$. Thus if the dilaton profile is linear w.r.t χ^μ , then by eqⁿ of motion, $T_{z\bar{z}} = T_{\bar{z}z} = 0$ & thus the resulting theory will have conformal invariance.

Choosing a linear dilaton background profile $\Phi(\chi^\mu) = V_\mu \chi^\mu$, where V_μ is a constant D-vector raised and lowered indices by $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, \dots)$, we get $T_{z\bar{z}} = T_{\bar{z}z} = 0$ by eqⁿ of motion.

Thus for Linear Dilaton CFT,

$$S = \frac{1}{2\pi\alpha'} \int d^2z (\partial_z X^\mu) (\partial_{\bar{z}} X_\mu)$$

Eqⁿ of motion is, $\partial_z \partial_{\bar{z}} X^\mu = \partial_{\bar{z}} \partial_z X^\mu = 0$

$\rightarrow \partial_z X^\mu(z, \bar{z})$ is holomorphic

& $\partial_{\bar{z}} X^\mu(z, \bar{z})$ is anti-holomorphic

The Energy momentum tensor components are,

$$T_{zz} = -\frac{1}{\alpha'} (\partial_z X^\mu) (\partial_z X_\mu) + \eta_{\mu\nu} \partial_z^2 X^\mu \equiv T(z)$$

$$T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} (\partial_{\bar{z}} X^\mu) (\partial_{\bar{z}} X_\mu) + \eta_{\mu\nu} \partial_{\bar{z}}^2 X^\mu \equiv \widetilde{T}(\bar{z})$$

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$

The OPE's b/w local operators, normal ordering definition, Contraction terms are all exactly same as what we saw in the previous theory, except for the ones involving $T(z)$ and $\widetilde{T}(\bar{z})$.

$T(z)X^\mu(\omega, \bar{\omega})$ & $\tilde{T}(\bar{z})X^\mu(\omega, \bar{\omega})$ OPEs

$$:T(z)::X^\mu(\omega, \bar{\omega}):$$

$$=:T(z)::X^\mu(\omega, \bar{\omega}): + V_\nu : \partial_z^2 X^\nu :: X^\mu(\omega, \bar{\omega}):$$

string

$$\sim \frac{\partial_\omega X^\mu(\omega, \bar{\omega})}{z - \omega} + V_\nu \partial_z^2 \overline{X^\nu(z, \bar{z}) X^\mu(\omega, \bar{\omega})}$$

$$\text{use } \overline{X^\nu(z, \bar{z}) X^\mu(\omega, \bar{\omega})} = -\frac{\alpha' \eta^{\nu\mu}}{2} \ln|z - \omega|^2$$

$$\rightarrow \partial_z^2 \overline{X^\nu(z, \bar{z}) X^\mu(\omega, \bar{\omega})} = \frac{\alpha' \eta^{\mu\nu}}{2(z - \omega)^2}$$

$$\rightarrow :T(z)::X^\mu(\omega, \bar{\omega}):$$

$$\sim \frac{\partial_\omega X^\mu(\omega, \bar{\omega})}{z - \omega} + \frac{\alpha' V^\mu}{2(z - \omega)^2}$$

The 2nd order singularity term is not proportional to X^μ . Thus, X^μ is not an Eigen state under rot^n & dilations. This also means that X^μ is not a Primary / tensor operator.

Now, consider $:\widetilde{T}(\bar{z})::X^\mu(\omega, \bar{\omega}):$

$$=:\widetilde{T}(\bar{z})::X^\mu(\omega, \bar{\omega}): + V, : \partial_{\bar{z}}^2 X^\nu :: X^\mu(\omega, \bar{\omega}):$$

string

$$\sim \frac{\partial_{\bar{\omega}} X^\mu(\omega, \bar{\omega})}{\bar{z} - \bar{\omega}} + V, \partial_{\bar{z}}^2 \overline{X^\nu(z, \bar{z}) X^\mu(\omega, \bar{\omega})}$$

$$\rightarrow :T(z)::X^\mu(\omega, \bar{\omega}):$$

$$\sim \frac{\partial_{\bar{\omega}} X^\mu(\omega, \bar{\omega})}{\bar{z} - \bar{\omega}} + \frac{\alpha' V^\mu}{2(\bar{z} - \bar{\omega})^2}$$

Let us find the transformation of X^μ under a general infinitesimal conformal transformation $z \rightarrow z + \epsilon \mathcal{V}(z)$. By Ward Identity & T - X OPE using the identity,

$$\delta \hat{\Theta}(\omega, \bar{\omega}) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} \left[\mathcal{V}^{(n)}(\omega) \hat{\Theta}^{(n)}(\omega, \bar{\omega}) + \mathcal{V}^{*(n)}(\bar{\omega}) \hat{\tilde{\Theta}}^{(n)}(\omega, \bar{\omega}) \right]$$

$$\text{where } \mathcal{V}^{(n)}(\omega) = \frac{\partial^n \mathcal{V}(\omega)}{\partial \omega^n} \text{ \& } \mathcal{V}^{*(n)}(\bar{\omega}) = \frac{\partial^n \mathcal{V}^*(\bar{\omega})}{\partial \bar{\omega}^n}$$

$\hat{\Theta}^{(n)}$ & $\hat{\tilde{\Theta}}^{(n)}$ are coefficient of $n+1^{\text{th}}$ order Pole in OPE of T & $\tilde{T}$ with $\hat{\Theta}$.

$$\rightarrow \delta \chi(\omega, \bar{\omega}) = -\epsilon \left[\nu(\omega) \partial \chi^\mu + \nu^*(\bar{\omega}) \bar{\partial} \chi^\mu + \frac{\alpha' V^\mu}{2} (\partial \nu(\omega) + \bar{\partial} \nu^*(\bar{\omega})) \right]$$

Thus the presence of non-zero V^μ , will tell, how the conformal symmetry of the theory changes for the same string action but with a different conformal symmetry generators $T(\bar{z})$ and $\tilde{T}(\bar{z})$.

Thus for each possible V^μ , we get different CFTs.

$T(\bar{z}) \partial_\omega^n \chi^\mu(\omega, \bar{\omega})$ & $\tilde{T}(\bar{z}) \partial_{\bar{\omega}}^n \chi^\mu(\omega, \bar{\omega})$ OPEs

$$:T(\bar{z}): \partial_\omega^n \chi^\mu(\omega, \bar{\omega}):$$

$$=:T(\bar{z}): \partial_\omega^n \chi^\mu(\omega, \bar{\omega}):$$

String

$$+ V_\nu : \partial_{\bar{z}}^2 \chi^\nu : \partial_\omega^n \chi^\mu(\omega, \bar{\omega}):$$

$$\sim \sum_{m=0}^n \frac{n! \partial_\omega^{m+1} \chi^\mu(\omega, \bar{\omega})}{m! (\bar{z} - \omega)^{n+1-m}} - \frac{V^\mu \alpha'}{2} \partial_\omega^n \partial_{\bar{z}}^2 \ln |\bar{z} - \omega|^2$$

$$\rightarrow :T(z)::\partial_{\omega}^n \chi^{\mu}(\omega, \bar{\omega}):$$

$$\sim \sum_{m=0}^n \frac{n! \partial_{\omega}^{m+1} \chi^{\mu}(\omega, \bar{\omega})}{m! (z-\omega)^{n+1-m}} + \frac{V^{\mu} \alpha' (-1)^n (n+1)!}{2 (\omega-z)^{n+2}}$$

Example:

$$\bullet :T(z)::\partial_{\omega} \chi^{\mu}(\omega, \bar{\omega}):$$

$$\sim \frac{\partial_{\omega}^2 \chi^{\mu}(\omega, \bar{\omega})}{(z-\omega)} + \frac{\partial_{\omega} \chi^{\mu}(\omega, \bar{\omega})}{(z-\omega)^2} - \frac{V^{\mu} \alpha'}{(\omega-z)^3}$$

$$\bullet :T(z)::\partial_{\omega}^2 \chi^{\mu}(\omega, \bar{\omega}):$$

$$\sim \frac{\partial_{\omega}^3 \chi^{\mu}(\omega, \bar{\omega})}{(z-\omega)} + \frac{2 \partial_{\omega}^2 \chi^{\mu}(\omega, \bar{\omega})}{(z-\omega)^2} + \frac{2 \partial_{\omega} \chi^{\mu}(\omega, \bar{\omega})}{(z-\omega)^3} + \frac{3 V^{\mu} \alpha'}{(\omega-z)^4}$$

Thus $\partial_{\omega}^n \chi^{\mu}(\omega, \bar{\omega})$ are not tensor operators for any $n \geq 0$. But are eigen states of rot^n & dilations with weights $h = n$ & $\bar{h} = 0$, for $\partial^n \chi$ with $n \geq 1$.

Similarly,

$$:\widetilde{T}(\bar{z})::\partial_{\bar{\omega}}^n \chi^\mu(\omega, \bar{\omega}):$$

$$\sim \sum_{m=0}^n \frac{n! \partial_{\bar{\omega}}^{m+1} \chi^\mu(\omega, \bar{\omega})}{m! (\bar{z} - \bar{\omega})^{n+1-m}} + \frac{V^\mu \alpha' (-1)^n (n+1)!}{2 (\bar{\omega} - \bar{z})^{n+2}}$$

So, $\partial_{\bar{\omega}}^n \chi^\mu(\omega, \bar{\omega})$ are not tensor operators for any $n \geq 0$. But are eigen states of rot^n & dilations with weights $h=0$ & $\tilde{h}=n$, for $\bar{\partial}^n \chi$ with $n \geq 1$.

$T(z)T(w)$ & $\widetilde{T}(\bar{z})\widetilde{T}(\bar{w})$ OPEs

$$:T(z)::T(w):$$

$$= \left[:T(z):: + V_\mu : \partial_z^2 \chi^\mu : \right] \left[:T(w):: + V_\nu : \partial_w^2 \chi^\nu : \right]$$

$$= :T(z)::T(w): + V_\mu : \partial_z^2 \chi^\mu ::T(w):$$

$$+ V_\nu :T(z)::\partial_w^2 \chi^\nu:$$

$$:T(z)::T(w): \sim \frac{D}{2(z-w)^4} + \frac{2:T(w):_{\text{string}}}{(z-w)^2} + \frac{\partial:T(w):_{\text{string}}}{(z-w)} + \dots$$

$$V_\mu : \partial_z^2 \chi^\mu :: T(w):_{\text{string}} \sim -2V_\mu \frac{\partial_w \chi^\mu(w, \bar{w})}{(z-w)^3}$$

$$\begin{aligned} :T(z)::\partial_w^2 \chi^\mu(w, \bar{w}): &\sim \frac{\partial_w^3 \chi^\mu(w, \bar{w})}{(z-w)} \\ &+ \frac{2\partial_w^2 \chi^\mu(w, \bar{w})}{(z-w)^2} + \frac{2\partial_w \chi^\mu(w, \bar{w})}{(z-w)^3} + \frac{3V^\mu \alpha'}{(w-z)^4} \end{aligned}$$

$$\Rightarrow :T(z)::T(w):$$

$$\sim \frac{D}{2(z-w)^4} + \frac{2:T(w):_{\text{string}}}{(z-w)^2} + \frac{\partial:T(w):_{\text{string}}}{(z-w)} + \dots$$

$$+ V_\mu \frac{\partial_w^3 \chi^\mu(w, \bar{w})}{(z-w)} - 2V_\mu \frac{\partial_w \chi^\mu(w, \bar{w})}{(z-w)^3}$$

$$+ 2V_\mu \frac{\partial_w^2 \chi^\mu(w, \bar{w})}{(z-w)^2} + 2V_\mu \frac{\partial_w \chi^\mu(w, \bar{w})}{(z-w)^3}$$

$$+ \frac{3V_\mu V^\mu \alpha'}{(w-z)^4}$$

$$\Rightarrow :T(z)::T(w):$$

$$\sim \frac{D + 6\alpha' V_\mu V^\mu}{2(z-w)^4} + \frac{2 \left[\underset{\text{string}}{:T(w):} + V_\mu \partial_w^2 X^\mu(w, \bar{w}) \right]}{(z-w)^2} + \frac{\left[\partial \underset{\text{string}}{:T(w):} + V_\mu \partial_w^3 X^\mu(w, \bar{w}) \right]}{(z-w)}$$

$$\Rightarrow :T(z)::T(w): \sim \frac{C}{2(z-w)^4} + \frac{2:T(w):}{(z-w)^2} + \frac{\partial :T(w):}{(z-w)}$$

Similarly,

$$\Rightarrow :\widetilde{T}(\bar{z}):\widetilde{T}(\bar{w}): \sim \frac{\widetilde{C}}{2(\bar{z}-\bar{w})^4} + \frac{2:\widetilde{T}(\bar{w}):}{(\bar{z}-\bar{w})^2} + \frac{\partial :\widetilde{T}(\bar{w}):}{(\bar{z}-\bar{w})}$$

Hence, we get the usual form but with a central charge of $C = \widetilde{C} = D + 6\alpha' V_\mu V^\mu$. Also we can see that the weight of $T(z)$ & $\widetilde{T}(\bar{z})$ are $(2,0)$ & $(0,2)$ respectively.

bc - Theory / Ghost Theory

These are types of CFTs which are constructed using anti-commuting or the Grassman type fields.

Consider grassman fields given by $b(z, \bar{z})$, $c(z, \bar{z})$, $\tilde{b}(z, \bar{z})$, and $\tilde{c}(z, \bar{z})$ where

$$bc = -cb, \quad c^2 = b^2 = 0$$
$$\bar{b}\bar{c} = -\bar{c}\bar{b}, \quad \bar{c}^2 = \bar{b}^2 = 0$$
$$b\bar{c} = -\bar{c}b \quad \& \quad \bar{b}c = -c\bar{b}$$

Then the ghost action is given by,

$$S_{\text{ghost}} = S_{bc} + S_{\tilde{b}\tilde{c}}$$

where,

$$S_{bc} = \frac{1}{2\pi} \int d^2z \, b(z, \bar{z}) \partial_{\bar{z}} c(z, \bar{z})$$

$$S_{\tilde{b}\tilde{c}} = \frac{1}{2\pi} \int d^2z \, \tilde{b}(z, \bar{z}) \partial_z \tilde{c}(z, \bar{z})$$

This is like a non-mixing combination two different CFTs ' bc ' & ' $\tilde{b}\tilde{c}$ '.

The eqⁿ of motions are given by,

$$\partial_{\bar{z}} b(z, \bar{z}) = \partial_{\bar{z}} c(z, \bar{z}) = 0$$

$$\& \quad \partial_z \tilde{b}(z, \bar{z}) = \partial_z \tilde{c}(z, \bar{z}) = 0$$

Thus, the ghost fields are such that,

$$b(z, \bar{z}) \longrightarrow b(z) \equiv b \rightarrow \text{Holomorphic}$$

$$c(z, \bar{z}) \longrightarrow c(z) \equiv c \rightarrow \text{Holomorphic}$$

$$\tilde{b}(z, \bar{z}) \longrightarrow \tilde{b}(\bar{z}) \equiv \tilde{b} \rightarrow \text{Anti-Holomorphic}$$

$$\tilde{c}(z, \bar{z}) \longrightarrow \tilde{c}(\bar{z}) \equiv \tilde{c} \rightarrow \text{Anti-Holomorphic}$$

Now, we want this theory to have Conformal Symmetry. So, for that we shall choose $b, c, \tilde{b}$ & $\tilde{c}$ to be tensors with some weights. But they have constraints as S is conformal invariant.

Under $z \rightarrow f(z)$ & $\bar{z} \rightarrow f^*(\bar{z})$

Now, before using eqⁿ of motion, Let,

$$b(z, \bar{z}) \rightarrow [f'(z)]^{-h_b} [f^{*'}(\bar{z})]^{-\tilde{h}_b} b(z, \bar{z})$$

$$c(z, \bar{z}) \rightarrow [f'(z)]^{-h_c} [f^{*'}(\bar{z})]^{-\tilde{h}_c} c(z, \bar{z})$$

$$\tilde{b}(z, \bar{z}) \rightarrow [f'(z)]^{-h_{\tilde{b}}} [f^{*'}(\bar{z})]^{-\tilde{h}_{\tilde{b}}} \tilde{b}(z, \bar{z})$$

$$\tilde{c}(z, \bar{z}) \rightarrow [f'(z)]^{-h_{\tilde{c}}} [f^{*'}(\bar{z})]^{-\tilde{h}_{\tilde{c}}} \tilde{c}(z, \bar{z})$$

Since $S_{\text{ghost}} \rightarrow S_{\text{ghost}}$ (Conformal Symmetry)
we get constraints on the weights.

$$\text{Now, } d^2 z \rightarrow |f'(z)|^2 d^2 z,$$

$$\partial_{\bar{z}} \rightarrow [f^{*'}(\bar{z})]^{-1} \partial_{\bar{z}}$$

$$\partial_{\bar{z}} c \rightarrow [f'(z)]^{-h_c} [f^{*'}(\bar{z})]^{-\tilde{h}_c - 1} \partial_{\bar{z}} c$$

$$b \partial_{\bar{z}} c \rightarrow [f'(z)]^{-(h_b + h_c)} [f^{*'}(\bar{z})]^{-(\tilde{h}_b + \tilde{h}_c + 1)} b \partial_{\bar{z}} c$$

Thus, $d^2 z b \partial_{\bar{z}} C$ transforms as,

$$|f'(z)|^2 [f'(\bar{z})]^{-(h_b+h_c)} [f^{*'}(\bar{z})]^{-(\tilde{h}_b+\tilde{h}_c+1)} d^2 z b \partial_{\bar{z}} C$$

$$\rightarrow d^2 z b(z) \partial_{\bar{z}} C \rightarrow$$

$$[f'(z)]^{-(h_b+h_c-1)} [f^{*'}(\bar{z})]^{-(\tilde{h}_b+\tilde{h}_c)} d^2 z b \partial_{\bar{z}} C$$

Thus to keep S_{bc} invariant, we need

$$h_b + h_c = 1, \quad \tilde{h}_b + \tilde{h}_c = 0$$

Similarly, $d^2 z \tilde{b} \partial_z \tilde{C}$ transforms as,

$$[f'(z)]^{-(h_{\tilde{b}}+h_{\tilde{c}})} [f^{*'}(\bar{z})]^{-(\tilde{h}_{\tilde{b}}+\tilde{h}_{\tilde{c}}-1)} d^2 z \tilde{b} \partial_z \tilde{C}$$

Thus to keep $S_{\tilde{b}\tilde{c}}$ invariant, we need

$$h_{\tilde{b}} + h_{\tilde{c}} = 0, \quad \tilde{h}_{\tilde{b}} + \tilde{h}_{\tilde{c}} = 1$$

Thus just to have conformal S_{ghost} , we need,

$$h_b + h_c = 1 \quad , \quad \tilde{h}_b + \tilde{h}_c = 0$$

$$h_{\tilde{b}} + h_{\tilde{c}} = 0 \quad , \quad \tilde{h}_{\tilde{b}} + \tilde{h}_{\tilde{c}} = 1$$

Once, we get the eqⁿ of motion, we need

$$\partial_{\bar{z}} b(z, \bar{z}) = \partial_{\bar{z}} c(z, \bar{z}) = 0$$

$$\partial_z \tilde{b}(z, \bar{z}) = \partial_z \tilde{c}(z, \bar{z}) = 0$$

This sets $\tilde{h}_b = \tilde{h}_c = h_{\tilde{b}} = h_{\tilde{c}} = 0$

So, the weights of Ghost fields are of the form.

$$(h_b, \tilde{h}_b) = (\lambda, 0)$$

$$(h_c, \tilde{h}_c) = (1 - \lambda, 0)$$

$$(h_{\tilde{b}}, \tilde{h}_{\tilde{b}}) = (0, \tilde{\lambda})$$

$$(h_{\tilde{c}}, \tilde{h}_{\tilde{c}}) = (0, 1 - \tilde{\lambda})$$

Thus λ & $\tilde{\lambda}$ are free parameters that we can vary. Each value of λ & $\tilde{\lambda}$ gives a new CFT. They all belong to a Ghost family of CFT.

By Schwinger-Dyson eqⁿ,

$$\frac{\partial S}{\partial b(z, \bar{z})} b(z', \bar{z}') = \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\frac{\partial S}{\partial c(z, \bar{z})} c(z', \bar{z}') = \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\frac{\partial S}{\partial \tilde{b}(z, \bar{z})} \tilde{b}(z', \bar{z}') = \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\frac{\partial S}{\partial \tilde{c}(z, \bar{z})} \tilde{c}(z', \bar{z}') = \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\begin{aligned} \frac{\partial S}{\partial b(z, \bar{z})} &= \frac{\partial}{\partial b(z, \bar{z})} \frac{1}{2\pi} \int d^2\omega b(\omega, \bar{\omega}) \partial_{\bar{\omega}} c(\omega, \bar{\omega}) \\ &= \frac{1}{2\pi} \int d^2\omega \frac{\partial b(\omega, \bar{\omega})}{\partial b(z, \bar{z})} \partial_{\bar{\omega}} c(\omega, \bar{\omega}) = \frac{1}{2\pi} \partial_{\bar{z}} c(z, \bar{z}) \end{aligned}$$

$$\Rightarrow \underline{\underline{\frac{\partial S}{\partial b(z, \bar{z})} = \frac{1}{2\pi} \partial_{\bar{z}} c(z, \bar{z})}}$$

Similarly,

$$\begin{aligned}
 \frac{\partial S}{\partial C(z, \bar{z})} &= \frac{\partial}{\partial C(z, \bar{z})} \frac{1}{2\pi} \int d^2\omega \, b(\omega, \bar{\omega}) \partial_{\bar{\omega}} c(\omega, \bar{\omega}) \quad \text{Integrate by parts} \\
 &= \frac{\partial}{\partial C(z, \bar{z})} \frac{1}{2\pi} \int d\omega \int d\bar{\omega} \, b(\omega, \bar{\omega}) \partial_{\bar{\omega}} c(\omega, \bar{\omega}) \\
 &= \frac{\partial}{\partial C(z, \bar{z})} \frac{1}{2\pi} \left[\int d\omega \left[b(\omega, \bar{\omega}) c(\omega, \bar{\omega}) \right]_{\bar{\omega}\text{-boundary}} - \int d\bar{\omega} \left[\partial_{\bar{\omega}} b(\omega, \bar{\omega}) \right] c(\omega, \bar{\omega}) \right] \\
 &\quad \left(\begin{array}{l} \text{-ve sign absorbed} \\ \text{by anti-commutation} \end{array} \right) \\
 &= \frac{\partial}{\partial C(z, \bar{z})} \frac{1}{2\pi} \left[\int d^2\omega \, c(\omega, \bar{\omega}) \left[\partial_{\bar{\omega}} b(\omega, \bar{\omega}) \right] \right] \\
 &= \frac{1}{2\pi} \int d^2\omega \, \frac{\partial c(\omega, \bar{\omega})}{\partial C(z, \bar{z})} \left[\partial_{\bar{\omega}} b(\omega, \bar{\omega}) \right] = \frac{1}{2\pi} \partial_{\bar{z}} b(z, \bar{z}) \\
 &\Rightarrow \underline{\underline{\frac{\partial S}{\partial C(z, \bar{z})} = \frac{1}{2\pi} \partial_{\bar{z}} b(z, \bar{z})}}
 \end{aligned}$$

This gives the Green's functions of this theory defined by the eqⁿ.

$$\Rightarrow \partial_{\bar{z}} c(z, \bar{z}) b(z', \bar{z}') = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\partial_{\bar{z}} b(z, \bar{z}) c(z', \bar{z}') = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\& \quad \partial_{\bar{z}} \tilde{c}(z, \bar{z}) \tilde{b}(z', \bar{z}') = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\partial_z \tilde{b}(z, \bar{z}) c(z', \bar{z}') = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

But consider ' $\partial_{\bar{z}} \left(\frac{1}{z - z_0} \right)$ '. It can be shown that

$$\partial_{\bar{z}} \left(\frac{1}{z - z_0} \right) = 2\pi \delta^{(2)}(z - z_0, \bar{z} - \bar{z}_0)$$

This is because ' $\frac{1}{z - z_0}$ ' has a pole at $z \rightarrow z_0$. Thus dirac delta appears. But if $z_0 \neq z$, then $\partial_{\bar{z}} \left(\frac{1}{z - z_0} \right) = 0$.

So, we should think of $\partial_{\bar{z}} \left(\frac{1}{z - z_0} \right)$ in the sense of a distribution.

Consider a smooth test function $\phi(z, \bar{z})$ non-zero inside a region R & zero outside. This is called Compact support of ϕ .

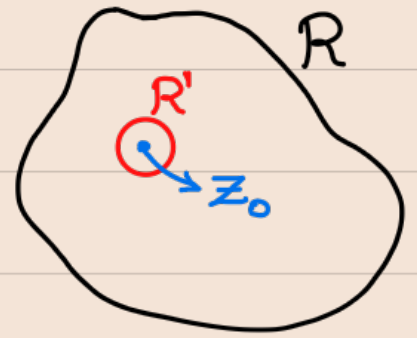
Then we can use integration by parts where boundary term vanishes.

Consider the integration given by,

$$I = \int_R d^2z \phi(z, \bar{z}) \partial_{\bar{z}} \left(\frac{1}{z - z_0} \right)$$

Use integration by parts

$$I = - \int_R d^2z \frac{1}{z - z_0} \partial_{\bar{z}} \phi(z, \bar{z})$$



Now, exclude a small region R' which is a circle of radius ' ϵ '. And then take $\epsilon \rightarrow 0$. So we write

$$I = \lim_{\epsilon \rightarrow 0} (-1) \int_{R-R'} d^2z \frac{1}{z - z_0} \partial_{\bar{z}} \phi(z, \bar{z})$$

Now for region $R-R'$, ' $\frac{1}{z - z_0}$ ' is analytic.
 $\nabla \partial_{\bar{z}} \left(\frac{1}{z - z_0} \right) = 0$ for $R-R'$.

$$\rightarrow I = (-1) \lim_{\epsilon \rightarrow 0} \int_{R-R'} d^2 z \partial_{\bar{z}} \left(\frac{\phi(z, \bar{z})}{z - z_0} \right)$$

(Use green's theorem)

$$\iint_R d^2 z (\partial_z v^{\bar{z}} + \partial_{\bar{z}} v^z) = i \oint_{\partial R} (v^z d\bar{z} - v^{\bar{z}} dz)$$

$$\rightarrow I = \lim_{\epsilon \rightarrow 0} i \int_{\partial(R-R')} dz \left(\frac{\phi(z, \bar{z})}{z - z_0} \right)$$



$$\underbrace{\partial(R-R')}_{\substack{\text{+ve} \\ \text{orientation}}} = \underbrace{\partial R}_{\substack{\text{+ve} \\ \text{orient}}} + L + L' - \underbrace{\partial R'}_{\substack{\text{+ve} \\ \text{orient}}} \quad \& \quad L' \sim -L$$

$$\rightarrow \partial(R-R) = \underbrace{\partial R - \partial R'}_{\text{+ve orientation}}$$

But $\phi(z, \bar{z})$ vanishes at ' ∂R '. Thus

$$\rightarrow I = (-i) \lim_{\epsilon \rightarrow 0} \oint_{\partial R' (+ve \text{ orientation})} dz \frac{\phi(z, \bar{z})}{z - z_0}$$

By Cauchy integral formula,

$$\lim_{\epsilon \rightarrow 0} \oint_{\partial R'} d\bar{z} \frac{\phi(z, \bar{z})}{z - z_0} = 2\pi i \phi(z_0, \bar{z}_0)$$

(+ve orientation)

$$\text{Thus } I = 2\pi \phi(z_0, \bar{z}_0)$$

$$\int_R d^2z \phi(z, \bar{z}) \left[\frac{1}{2\pi} \partial_{\bar{z}} \left(\frac{1}{z - z_0} \right) \right] = \phi(z_0, \bar{z}_0)$$

Thus by our normalization convention of $\int \delta^{(2)}(z, \bar{z}) d^2z = 1$, we can write

$$\int_R d^2z \phi(z, \bar{z}) \delta^{(2)}(z, \bar{z}) = \phi(z_0, \bar{z}_0)$$

$$\Rightarrow \partial_{\bar{z}} \left(\frac{1}{z - z_0} \right) = 2\pi \delta^{(2)}(z - z_0, \bar{z} - \bar{z}_0)$$

Similarly,

$$\partial_z \left(\frac{1}{\bar{z} - \bar{z}_0} \right) = 2\pi \delta^{(2)}(z - z_0, \bar{z} - \bar{z}_0)$$

Thus the green's function like relation for b - c theory,

$$\partial_{\bar{z}} c(z, \bar{z}) b(w, \bar{w}) = 2\pi \delta^{(2)}(z-z', \bar{z}-\bar{w})$$

$$\partial_{\bar{z}} b(z, \bar{z}) c(w, \bar{w}) = 2\pi \delta^{(2)}(z-z', \bar{z}-\bar{w})$$

$$\partial_z \tilde{c}(z, \bar{z}) \tilde{b}(w, \bar{w}) = 2\pi \delta^{(2)}(z-z', \bar{z}-\bar{w})$$

$$\partial_z \tilde{b}(z, \bar{z}) c(w, \bar{w}) = 2\pi \delta^{(2)}(z-z', \bar{z}-\bar{w})$$

can be use to define OPE's b/w the $b, c, \tilde{b}, \tilde{c}$ fields as,

$$c(z, \bar{z}) b(w, \bar{w}) = \frac{1}{z-w} + \text{Non singular}$$

$$b(z, \bar{z}) c(w, \bar{w}) = \frac{1}{z-w} + \text{Non singular}$$

$$\tilde{c}(z, \bar{z}) \tilde{b}(w, \bar{w}) = \frac{1}{\bar{z}-\bar{w}} + \text{Non singular}$$

$$\tilde{b}(z, \bar{z}) c(w, \bar{w}) = \frac{1}{\bar{z}-\bar{w}} + \text{Non singular}$$

But through eqⁿ of motion b, c are holomorphic & $\tilde{b}, \tilde{c}$ are anti-holomorphic.

Thus we finally get

$$c(z)b(w) = \frac{1}{z-w} + \text{Non singular}$$

$$b(z)c(w) = \frac{1}{z-w} + \text{Non singular}$$

$$\tilde{c}(\bar{z})\tilde{b}(\bar{w}) = \frac{1}{\bar{z}-\bar{w}} + \text{Non singular}$$

$$\tilde{b}(\bar{z})\tilde{c}(\bar{w}) = \frac{1}{\bar{z}-\bar{w}} + \text{Non singular}$$

$$\text{Notice, } c(w)b(z) \sim \frac{1}{w-z} = \frac{-1}{z-w}$$

$$\text{But } b(z)c(w) \sim \frac{1}{z-w}$$

Thus even in OPE's, $c(w)b(z) = -b(z)c(w)$

The OPE b/w same fields are non-singular since we need $b(z)b(z) = 0 \neq c(z)c(z) = 0$ due to anti-commutations.

Thus $b(z)b(w) \neq c(z)c(w)$ should tend to zero as $z \rightarrow w$. Hence, they cannot have any singular parts.

In this theory the normal ordered products are, $:c(\underline{z})b(\omega): = c(\underline{z})b(\omega) - \frac{1}{\underline{z} - \omega}$

$$:b(\underline{z})c(\omega): = b(\underline{z})c(\omega) - \frac{1}{\underline{z} - \omega}$$

$$:\tilde{c}(\underline{\bar{z}})\tilde{b}(\bar{\omega}): = \tilde{c}(\underline{\bar{z}})\tilde{b}(\bar{\omega}) - \frac{1}{\underline{\bar{z}} - \bar{\omega}}$$

$$:\tilde{b}(\underline{\bar{z}})\tilde{c}(\bar{\omega}): = \tilde{b}(\underline{\bar{z}})\tilde{c}(\bar{\omega}) - \frac{1}{\underline{\bar{z}} - \bar{\omega}}$$

$$\rightarrow \partial_{\underline{\bar{z}}}:c(\underline{z})b(\omega): = 0 = \partial_{\underline{\bar{z}}}:b(\underline{z})c(\omega):$$

$$\& \partial_{\underline{z}}:\tilde{c}(\underline{\bar{z}})\tilde{b}(\bar{\omega}): = 0 = \partial_{\underline{z}}:\tilde{b}(\underline{\bar{z}})\tilde{c}(\bar{\omega}):$$

This makes all normal ordered products non-singular. We also define the contraction b/w two fields b's & c's as:

$$\overbrace{c(\underline{z})b(\omega)}^{\text{contraction}} = c(\underline{z})b(\omega) - :c(\underline{z})b(\omega): = \frac{1}{\underline{z} - \omega}$$

$$\overbrace{b(\omega)c(\underline{z})}^{\text{contraction}} = b(\omega)c(\underline{z}) - :b(\omega)c(\underline{z}): = \frac{1}{\omega - \underline{z}}$$

$$\rightarrow \underbrace{\overbrace{b(\omega)c(\underline{z})}^{\text{contractions}}}_{\text{Contractions}} = - \overbrace{c(\underline{z})b(\omega)}^{\text{contractions}} \rightarrow \text{-ve sign due to the anti-commuting nature.}$$

$$\text{Similarly, } \overbrace{\tilde{c}(\underline{\bar{z}})\tilde{b}(\bar{\omega})}^{\text{contractions}} = \overbrace{\tilde{b}(\underline{\bar{z}})\tilde{c}(\bar{\omega})}^{\text{contractions}} = \frac{1}{\underline{\bar{z}} - \bar{\omega}}$$

Non-zero Contractions b/w local operators

- $\overline{[\partial_z b(z)] c(w)} = \partial_z \overline{b(z) c(w)} = \frac{-1}{(z-w)^2}$
- $\overline{[\partial_z c(z)] b(w)} = \partial_z \overline{c(z) b(w)} = \frac{-1}{(z-w)^2}$
- $\overline{b(z) [\partial_w c(w)]} = \partial_w \overline{b(z) c(w)} = \frac{1}{(z-w)^2}$
- $\overline{c(z) [\partial_w b(w)]} = \partial_w \overline{c(z) b(w)} = \frac{1}{(z-w)^2}$

$$\rightarrow \overline{[\partial_z c(z)] b(w)} = - \overline{b(w) [\partial_z c(z)]}$$
$$\& \overline{[\partial_z b(z)] c(w)} = - \overline{c(w) [\partial_z b(z)]}$$

- $\overline{[\partial_{\bar{z}} \tilde{b}(\bar{z})] \tilde{c}(\bar{w})} = \partial_{\bar{z}} \overline{\tilde{b}(\bar{z}) \tilde{c}(\bar{w})} = \frac{-1}{(\bar{z}-\bar{w})^2}$
- $\overline{[\partial_{\bar{z}} \tilde{c}(\bar{z})] \tilde{b}(\bar{w})} = \partial_{\bar{z}} \overline{\tilde{c}(\bar{z}) \tilde{b}(\bar{w})} = \frac{-1}{(\bar{z}-\bar{w})^2}$
- $\overline{\tilde{b}(\bar{z}) [\partial_{\bar{w}} \tilde{c}(\bar{w})]} = \partial_{\bar{w}} \overline{\tilde{b}(\bar{z}) \tilde{c}(\bar{w})} = \frac{1}{(\bar{z}-\bar{w})^2}$
- $\overline{\tilde{c}(\bar{z}) [\partial_{\bar{w}} \tilde{b}(\bar{w})]} = \partial_{\bar{w}} \overline{\tilde{c}(\bar{z}) \tilde{b}(\bar{w})} = \frac{1}{(\bar{z}-\bar{w})^2}$

The contractions b/w like-fields are zero.

$$\overline{c(z) c(w)} = \overline{b(z) b(w)} = 0 \quad \&$$

$$\overline{\tilde{c}(\bar{z}) \tilde{c}(\bar{w})} = \overline{\tilde{b}(\bar{z}) \tilde{b}(\bar{w})} = 0$$

Energy Momentum tensor

Requiring that under conformal transform we need $T(z)$ and $\tilde{T}(\bar{z})$ to generate Conformal transformations on the local operators, where, $T(z) = i\mathcal{V}(z)T(z)$ and $\tilde{T}(\bar{z}) = i\mathcal{V}^*(\bar{z})\tilde{T}(\bar{z})$.

Thus, under $z \rightarrow z + \epsilon \mathcal{V}(z)$ & $\bar{z} \rightarrow \bar{z} + \epsilon \mathcal{V}^*(\bar{z})$ the change in action should be such that,

$$\delta S = \frac{1}{2\pi} \int d^2 z \left[T(z) \delta \bar{z} + \tilde{T}(\bar{z}) \delta z \right]$$

By variational principle, the Coefficient of $\delta \bar{z}$ defines the local density that generates the Symmetry transformation in holomorphic component of T-tensor. Similarly coefficient of δz defines the anti-holomorphic component.

$$\Rightarrow \delta S = \frac{1}{2\pi} \int d^2 z \left[T(z) \partial_{\bar{z}} v^*(\bar{z}) + \tilde{T}(\bar{z}) \partial_z v(z) \right]$$

For pure S_{bc} (holomorphic theory),

$$T(z) = (1 - \lambda) (\partial b) c - \lambda b (\partial c)$$

$$\& \tilde{T}(\bar{z}) = 0$$

For pure $S_{\tilde{b}\tilde{c}}$ (anti-holomorphic theory),

$$T(z) = 0$$

$$\& \tilde{T}(\bar{z}) = (1 - \tilde{\lambda}) (\bar{\partial} \tilde{b}) \tilde{c} - \lambda \tilde{b} (\bar{\partial} \tilde{c})$$

But for complete S_{ghost} , we get both $T(z)$ & $\tilde{T}(\bar{z})$.

$$T(z) = (1 - \lambda) (\partial b) c - \lambda b (\partial c)$$

$$\tilde{T}(\bar{z}) = (1 - \tilde{\lambda}) (\bar{\partial} \tilde{b}) \tilde{c} - \lambda \tilde{b} (\bar{\partial} \tilde{c})$$

This can be checked by finding all the combinations of OPE's b/w $T, \tilde{T}$ and all fields $b, c, \tilde{b}$ & $\tilde{c}$ to give correct weights $(h_b, \tilde{h}_b), (h_c, \tilde{h}_c), (h_{\tilde{b}}, \tilde{h}_{\tilde{b}})$ & $(h_{\tilde{c}}, \tilde{h}_{\tilde{c}})$.

$T(z)T(w)$ & $\overline{T}(\bar{z})\overline{T}(\bar{w})$ OPEs

$$:T(z)::T(w):$$

$$= \left[h_c :(\partial b)c: - h_b :b(\partial c): \right]_z \left[h_c :(\partial b)c: - h_b :b(\partial c): \right]_w$$

$$\begin{aligned} &= h_c^2 :(\partial b)c(z)::(\partial b)c(w): \\ &\quad - h_c h_b :(\partial b)c(z)::b(\partial c(w)): \\ &\quad - h_b h_c :b(\partial c(z))::(\partial b)c(w): \\ &\quad + h_b^2 :b(\partial c(z))::b(\partial c(w)): \end{aligned}$$

Consider the first term

$$:(\partial_z b(z))c(z)::(\partial_w b(w))c(w):$$

$$\begin{aligned} &\sim (-1)^2 \overbrace{(\partial_z b(z))c(w)} :c(z)(\partial_w b(w)): \\ &\quad + (-1)^2 \overbrace{c(z)(\partial_w b(w))} :(\partial_z b(z))c(w): \\ &\quad + (-1)^2 \overbrace{(\partial_z b(z))c(w)} \overbrace{c(z)(\partial_w b(w))} \end{aligned}$$

$$\sim -\frac{:(c(z)(\partial_w b(w))):}{(z-w)^2} + \frac{:(\partial_z b(z))c(w):}{(z-w)^2} - \frac{1}{(z-w)^4}$$

Expand about 'w'

$$\begin{aligned}
&\sim - \frac{:c(\omega)(\partial_\omega b(\omega)):_:}{(z-\omega)^2} - \frac{:(\partial_\omega c(\omega))(\partial_\omega b(\omega)):_:}{(z-\omega)} \\
&\quad + \frac{:(\partial_\omega b(\omega))c(\omega):}{(z-\omega)^2} + \frac{:(\partial_\omega^2 b(\omega))c(\omega):}{(z-\omega)} \\
&\quad - \frac{1}{(z-\omega)^4}
\end{aligned}$$

$$\begin{aligned}
&\sim \frac{-1}{(z-\omega)^4} + \frac{:(\partial_\omega b(\omega))c(\omega): - :c(\omega)(\partial_\omega b(\omega)):_:}{(z-\omega)^2} \\
&\quad + \frac{:(\partial_\omega^2 b(\omega))c(\omega): - :(\partial_\omega c(\omega))(\partial_\omega b(\omega)):_:}{(z-\omega)}
\end{aligned}$$

Use anti-commutation property

$$\rightarrow h_c^2 :(\partial_z b(z))c(z)::(\partial_\omega b(\omega))c(\omega):$$

$$\begin{aligned}
&\sim \frac{-h_c^2}{(z-\omega)^4} + \frac{2h_c^2 :(\partial_\omega b(\omega))c(\omega):}{(z-\omega)^2} \\
&\quad + \frac{h_c^2 \partial_\omega [:(\partial_\omega b(\omega))c(\omega):]}{(z-\omega)}
\end{aligned}$$

2nd term

$$:(\partial_z b(z))c(z)::b(\omega)(\partial_\omega c(\omega)):$$

$$\sim \overbrace{(\partial_z b(z))(\partial_\omega c(\omega))} :c(z)b(\omega):$$

$$+ \overbrace{c(z)b(\omega)} :(\partial_z b(z))(\partial_\omega c(\omega)):$$

$$+ \overbrace{(\partial_z b(z))(\partial_\omega c(\omega))} \overbrace{c(z)b(\omega)}$$

$$\sim -\frac{2:c(z)b(\omega):}{(z-\omega)^3} + \frac{:(\partial_z b(z))(\partial_\omega c(\omega)):}{(z-\omega)}$$

Expand about ' ω '

$$\frac{-2}{(z-\omega)^4}$$

$$\sim \frac{-2}{(z-\omega)^4} - \frac{2:c(\omega)b(\omega):}{(z-\omega)^3}$$

$$- \frac{2:\partial_\omega c(\omega)b(\omega):}{(z-\omega)^2} - \frac{:\partial_\omega^2 c(\omega)b(\omega):}{(z-\omega)}$$

$$+ \frac{:(\partial_\omega b(\omega))(\partial_\omega c(\omega)):}{(z-\omega)}$$

Using anti-commutation & regrouping,

$$-h_c h_b :(\partial_z b(z))c(z)::b(w)(\partial_w c(w)):$$

$$\sim \frac{2h_c h_b}{(z-w)^4} + \frac{2h_c h_b :c(w)b(w):}{(z-w)^3}$$

$$+ \frac{2h_c h_b : \partial_w c(w)b(w):}{(z-w)^2}$$

$$- \frac{h_c h_b \partial_w [:b(w)\partial_w c(w):]}{(z-w)}$$

3rd term

$$:b(z)(\partial_z c(z))::(\partial_w b(w))c(w):$$

$$\sim \overbrace{(\partial_z c(z))(\partial_w b(w))} :b(z)c(w):$$

$$+ \overbrace{b(z)c(w)} :(\partial_z c(z))(\partial_w b(w)):$$

$$+ \overbrace{(\partial_z c(z))(\partial_w b(w))} \overbrace{b(z)c(w)}$$

$$\sim - \frac{2 :b(z)c(w):}{(z-w)^3} + \frac{:(\partial_z c(z))(\partial_w b(w)):}{(z-w)}$$

(Expand about 'w')

$$- \frac{2}{(z-w)^4}$$

$$\begin{aligned}
& \sim - \frac{2 :b(\omega)c(\omega):}{(\mathbb{Z}-\omega)^3} - \frac{2 : \partial_\omega b(\omega)c(\omega):}{(\mathbb{Z}-\omega)^2} \\
& - \frac{:\partial_\omega^2 b(\omega)c(\omega):}{(\mathbb{Z}-\omega)} + \frac{:(\partial_\omega c(\omega))(\partial_\omega b(\omega))}{(\mathbb{Z}-\omega)} \\
& \qquad \qquad \qquad - \frac{2}{(\mathbb{Z}-\omega)^4}
\end{aligned}$$

Using anti-commutation & regrouping,

$$\begin{aligned}
& \sim - \frac{2 :b(\omega)c(\omega):}{(\mathbb{Z}-\omega)^3} - \frac{2 : \partial_\omega b(\omega)c(\omega):}{(\mathbb{Z}-\omega)^2} \\
& + \frac{\partial_\omega [:c(\omega)\partial_\omega b(\omega):]}{(\mathbb{Z}-\omega)} - \frac{2}{(\mathbb{Z}-\omega)^4}
\end{aligned}$$

$$\rightarrow -h_b h_c :b(\mathbb{Z})(\partial_{\mathbb{Z}} c(\mathbb{Z}))::(\partial_\omega b(\omega))c(\omega):$$

$$\sim \frac{2h_b h_c}{(\mathbb{Z}-\omega)^4} + \frac{2h_b h_c :b(\omega)c(\omega):}{(\mathbb{Z}-\omega)^3}$$

$$+ \frac{2h_b h_c : \partial_\omega b(\omega)c(\omega):}{(\mathbb{Z}-\omega)^2} - \frac{h_b h_c \partial_\omega [:c(\omega)\partial_\omega b(\omega):]}{(\mathbb{Z}-\omega)}$$

4th term

$$:b(z)(\partial_z c(z))::b(w)(\partial_w c(w)):$$

$$\sim :b(z)(\partial_w c(w)):\overline{(\partial_z c(z))b(w)}$$

$$+ \overline{b(z)(\partial_w c(w))}:(\partial_z c(z))b(w):$$

$$+ \overline{b(z)(\partial_w c(w))}\overline{(\partial_z c(z))b(w)}$$

$$\sim - \frac{:b(z)(\partial_w c(w)):}{(z-w)^2} + \frac{:(\partial_z c(z))b(w):}{(z-w)^2}$$

$$\text{(Expand about 'w')} \quad - \frac{1}{(z-w)^4}$$

$$\sim - \frac{1}{(z-w)^4} - \frac{:b(w)(\partial_w c(w)):}{(z-w)^2}$$

$$- \frac{:(\partial_w b(w))(\partial_w c(w)):}{(z-w)} + \frac{:(\partial_w c(w))b(w):}{(z-w)^2}$$

$$+ \frac{:(\partial_w^2 c(w))b(w):}{(z-w)}$$

Using anti-commutation & regrouping,

$$\sim -\frac{1}{(z-\omega)^4} + \frac{2: (\partial_\omega c(\omega)) b(\omega) :}{(z-\omega)^2} + \frac{\partial_\omega [: (\partial_\omega c(\omega)) b(\omega) :]}{(z-\omega)}$$

$$\rightarrow h_b^2 : b(z) (\partial_z c(z)) :: b(\omega) (\partial_\omega c(\omega)) :$$

$$\sim -\frac{h_b^2}{(z-\omega)^4} + \frac{2h_b^2 : (\partial_\omega c(\omega)) b(\omega) :}{(z-\omega)^2} + \frac{h_b^2 \partial_\omega [: (\partial_\omega c(\omega)) b(\omega) :]}{(z-\omega)}$$

Now to get ' $:T(z)::T(\omega):$ ' we shall add all four terms.

$$\rightarrow :T(z)::T(\omega):$$

$$\sim \frac{-h_c^2}{(z-\omega)^4} + \frac{2h_c^2 :(\partial_\omega b(\omega))c(\omega):}{(z-\omega)^2}$$

$$+ \frac{h_c^2 \partial_\omega [:(\partial_\omega b(\omega))c(\omega):]}{(z-\omega)} + \frac{2h_c h_b}{(z-\omega)^4}$$

$$+ \frac{2h_c h_b :c(\omega)b(\omega):}{(z-\omega)^3} + \frac{2h_c h_b : \partial_\omega c(\omega)b(\omega):}{(z-\omega)^2}$$

$$- \frac{h_c h_b \partial_\omega [:b(\omega)\partial_\omega c(\omega):]}{(z-\omega)} + \frac{2h_b h_c}{(z-\omega)^4}$$

$$+ \frac{2h_b h_c :b(\omega)c(\omega):}{(z-\omega)^3} + \frac{2h_b h_c : \partial_\omega b(\omega)c(\omega):}{(z-\omega)^2}$$

$$- \frac{h_b h_c \partial_\omega [:c(\omega)\partial_\omega b(\omega):]}{(z-\omega)} - \frac{h_b^2}{(z-\omega)^4}$$

$$+ \frac{2h_b^2 :(\partial_\omega c(\omega))b(\omega):}{(z-\omega)^2} + \frac{h_b^2 \partial_\omega [:(\partial_\omega c(\omega))b(\omega):]}{(z-\omega)}$$

$$\rightarrow :T(z)::T(\omega): \sim \frac{-(h_c^2 + h_b^2) + 4h_ch_b}{(z-\omega)^4}$$

$$+ \frac{2(h_c + h_b)h_c :(\partial_\omega b(\omega))c(\omega):}{(z-\omega)^2}$$

$$+ \frac{2(h_c + h_b)h_b :(\partial_\omega c(\omega))b(\omega):}{(z-\omega)^2}$$

$$+ \frac{(h_c + h_b)h_c \partial_\omega [:(\partial_\omega b(\omega))c(\omega):]}{(z-\omega)}$$

$$+ \frac{(h_b + h_c)h_b \partial_\omega [:(\partial_\omega c(\omega))b(\omega):]}{(z-\omega)}$$

Use $h_b = \lambda$ & $h_c = 1 - \lambda$

$$\Rightarrow h_b + h_c = 1,$$

$$\begin{aligned} -(h_c^2 + h_b^2) + 4h_ch_b &= -(h_c + h_b)^2 + 6h_ch_b \\ &= -1 + 6h_ch_b = 6\lambda(1-\lambda) - 1 \end{aligned}$$

$$\text{Let } c = 12\lambda(1-\lambda) - 2 = 1 - 3(2\lambda - 1)^2$$

Thus we get

$$\begin{aligned}
 :T(z)::T(\omega): &\sim \frac{c}{2(z-\omega)^4} \\
 &+ \frac{2[h_c:(\partial_\omega b(\omega))c(\omega): - h_b:b(\omega)(\partial_\omega c(\omega)):] }{(z-\omega)^2} \\
 &+ \frac{\partial_\omega [h_c:(\partial_\omega b(\omega))c(\omega): - h_b:(\partial_\omega c(\omega))b(\omega):]}{(z-\omega)}
 \end{aligned}$$

Since,

$$T(\omega) = [h_c:(\partial_\omega b(\omega))c(\omega): - h_b:b(\omega)(\partial_\omega c(\omega)):]$$

we get the TT OPE in the standard form.

$$\begin{aligned}
 :T(z)::T(\omega): &\sim \frac{c}{2(z-\omega)^4} + \frac{2:T(\omega):}{(z-\omega)^2} \\
 &+ \frac{2\partial_\omega:T(\omega):}{(z-\omega)^2}
 \end{aligned}$$

$$c = 1 - 3(2\lambda - 1)^2$$

$(h_T, \tilde{h}_T) = (2, 0)$ as expected.

Similarly for $\widetilde{T} \widetilde{T}$ OPE, we get it in the same form.

$$\widetilde{T}(\bar{z}) = (1 - \widetilde{\lambda})(\bar{\partial}\widetilde{b})\widetilde{c} - \lambda\widetilde{b}(\bar{\partial}\widetilde{c})$$

$$\begin{aligned} :\widetilde{T}(\bar{z}): :\widetilde{T}(\bar{w}): &\sim \frac{\widetilde{c}}{2(\bar{z} - \bar{w})^4} + \frac{2:\widetilde{T}(\bar{w}):}{(\bar{z} - \bar{w})^2} \\ &+ \frac{2\partial_{\bar{w}}:\widetilde{T}(\bar{w}):}{(\bar{z} - \bar{w})^2} \end{aligned}$$

$$\widetilde{c} = 1 - 3(2\widetilde{\lambda} - 1)^2$$

$(h_{\widetilde{T}}, \widetilde{h}_{\widetilde{T}}) = (0, 2)$ as expected.

Thus S_{ghost} theory has $c \neq \widetilde{c}$ if $\lambda \neq \widetilde{\lambda}$.

Pure S_{bc} theory has $c = 1 - 3(2\lambda - 1)^2 \neq \widetilde{c} = 0$

Pure $S_{\widetilde{b}\widetilde{c}}$ theory has $c = 0 \neq \widetilde{c} = 1 - 3(2\widetilde{\lambda} - 1)^2$

Ghost Number Symmetry

Consider the ghost action given by,

$$S = \frac{1}{2\pi} \int d^2z \left(b(z, \bar{z}) \partial_{\bar{z}} c(z, \bar{z}) + \tilde{b}(z, \bar{z}) \partial_z \tilde{c}(z, \bar{z}) \right)$$

Consider a Global $U(1)$ transformation of the Ghost field 'c' such that $c \rightarrow e^{i\alpha} c$. But to keep the action invariant, we require $b \rightarrow e^{-i\alpha} b$. Similarly, for the $\tilde{c}$ & $\tilde{b}$, if $\tilde{c} \rightarrow e^{i\tilde{\alpha}} \tilde{c}$, then we need $\tilde{b} \rightarrow e^{-i\tilde{\alpha}} \tilde{b}$ to keep action invariant.

Thus the $U(1)$ Symmetry is present in the Ghost action, under

$$c \rightarrow e^{i\alpha} c, \quad b \rightarrow e^{-i\alpha} b$$

&

$$\tilde{c} \rightarrow e^{i\tilde{\alpha}} \tilde{c}, \quad \tilde{b} \rightarrow e^{-i\tilde{\alpha}} \tilde{b}$$

Provided α & $\tilde{\alpha}$ are constants.

For infinitesimal $U(1)$ transformation,
 $\alpha \rightarrow \epsilon \alpha$, $\tilde{\alpha} \rightarrow \epsilon \tilde{\alpha}$, where $\epsilon^2 \rightarrow 0$.

$$\rightarrow c \rightarrow c + i\epsilon \alpha c, \quad b \rightarrow b - i\epsilon \alpha b$$

$$\& \quad \tilde{c} \rightarrow \tilde{c} + i\epsilon \tilde{\alpha} \tilde{c}, \quad \tilde{b} \rightarrow \tilde{b} - i\epsilon \tilde{\alpha} \tilde{b}$$

$$\rightarrow \partial c = +i\epsilon \alpha c, \quad \partial b = -i\epsilon \alpha b$$

$$\& \quad \partial \tilde{c} = +i\epsilon \tilde{\alpha} \tilde{c}, \quad \partial \tilde{b} = -i\epsilon \tilde{\alpha} \tilde{b}$$

To find the conserved current, take
 $\alpha \rightarrow \alpha(z, \bar{z})$ & $\tilde{\alpha} \rightarrow \tilde{\alpha}(z, \bar{z})$ and then
 find the coefficient of $\bar{\partial} \alpha$ & $\partial \tilde{\alpha}$ to
 get the conserved currents j_z & $j_{\bar{z}}$.

$$\delta S = \frac{1}{2\pi} \int d^2 z \left((\partial b)(\bar{\partial} \tilde{c}) + b(\bar{\partial} \partial \tilde{c}) \right. \\ \left. + (\partial \tilde{b})(\bar{\partial} c) + \tilde{b}(\bar{\partial} \partial c) \right)$$

$$= \frac{1}{2\pi} \int d^2 z \left(-i\epsilon \alpha b(\bar{\partial} \tilde{c}) + i\epsilon b(\bar{\partial} \alpha \tilde{c}) \right. \\ \left. - i\epsilon \tilde{\alpha} \tilde{b}(\bar{\partial} c) + i\epsilon \tilde{b}(\bar{\partial} \tilde{\alpha} c) \right)$$

$$\Rightarrow \delta S = \frac{i\epsilon}{2\pi} \int d^2z \left(b c (\partial_{\bar{z}} \alpha) + \tilde{b} \tilde{c} (\partial_z \tilde{\alpha}) \right)$$

In general, we define Noether current by,

$$\delta S = \frac{i\epsilon}{2\pi} \int d^2\sigma \, j^a(\sigma) \partial_a \rho(\sigma)$$

(for us ' $\rho \rightarrow i\alpha$ ')

In complex coordinates, $d^2\sigma = \frac{i}{2} d^2z$

$$\rightarrow \delta S = -\frac{i\epsilon}{4\pi} \int d^2z \left(j^z \partial_{\bar{z}} \alpha + j^{\bar{z}} \partial_z \alpha \right)$$

$$v_z = g_{z\bar{z}} v^{\bar{z}} = \frac{v^{\bar{z}}}{2} \quad \& \quad v_{\bar{z}} = g_{\bar{z}z} v^z = \frac{v^z}{2}$$

$$\rightarrow j^z = 2j_{\bar{z}} \quad , \quad j^{\bar{z}} = 2j_z$$

$$\rightarrow \delta S = -\frac{i\epsilon}{2\pi} \int d^2z \left(j_{\bar{z}} \partial_z \alpha + j_z \partial_{\bar{z}} \alpha \right)$$

Thus by comparison we define the Ghost Noether currents as,

$$\left. \begin{aligned} j_z &= -b(z)c(z) \\ \& \quad j_{\bar{z}} &= -\tilde{b}(\bar{z})\tilde{c}(\bar{z}) \end{aligned} \right\} \begin{array}{l} \text{After using} \\ \text{Eq}^n \text{ of motion} \end{array}$$

These Noether currents are separately conserved. Since, b & c are holomorphic and $\tilde{b}, \tilde{c}$ are anti-holomorphic,

$$j(z) \equiv j_z = -b(z)c(z) \rightarrow \text{holomorphic}$$

$$\tilde{j}(\bar{z}) \equiv j_{\bar{z}} = -\tilde{b}(\bar{z})\tilde{c}(\bar{z}) \rightarrow \text{anti-holomorphic}$$

Since 'b' has weight ' λ ' & 'c' has weight $1-\lambda$. also as b & c are tensor fields, $j(z)$ has weight $(1,0)$.

Similarly $\tilde{j}(\bar{z})$ has weight $(0,1)$.

Consider the T - j OPE.

$$\begin{aligned}
 :T(z)::j(w): & \text{ is given by,} \\
 & = [h_c:(\partial_z b(z))c(z):-h_b:b(z)(\partial_z c(z))]:c(w)b(w): \\
 & = h_c:(\partial_z b(z))c(z)::c(w)b(w): \\
 & \quad - h_b:b(z)(\partial_z c(z))::c(w)b(w):
 \end{aligned}$$

1st term

$$\begin{aligned}
 & h_c:(\partial_z b(z))c(z)::c(w)b(w): \\
 & \sim h_c(-1) \overline{\partial_z b(z) c(w)}:c(z)b(w): \\
 & \quad + h_c(-1): \partial_z b(z) c(w): \overline{c(z) b(w)} \\
 & \quad + h_c(-1) \overline{\partial_z b(z) c(w)} \overline{c(z) b(w)} \\
 & = h_c \frac{:c(z)b(w):}{(z-w)^2} - \frac{h_c:\partial_z b(z) c(w):}{(z-w)} + \frac{h_c}{(z-w)^3} \\
 & = \frac{h_c}{(z-w)^3} + \frac{h_c:c(w)b(w):}{(z-w)^2} + \frac{h_c:\partial_w c(w)b(w):}{(z-w)} \\
 & \quad - \frac{h_c:\partial_w b(w) c(w):}{(z-w)} \\
 & = \frac{h_c}{(z-w)^3} + \frac{h_c:c(w)b(w):}{(z-w)^2} + \frac{h_c \partial_w [:c(w)b(w):]}{(z-w)}
 \end{aligned}$$

2nd term

$$-h_b : b(z) (\partial_z C(z)) :: C(w) b(w) :$$

$$\sim h_b \overline{b(z) C(w)} : (\partial_z C(z)) b(w) :$$

$$+ h_b : b(z) C(w) : \overline{(\partial_z C(z)) b(w)}$$

$$+ h_b \overline{b(z) C(w)} \overline{(\partial_z C(z)) b(w)}$$

$$= \frac{h_b : (\partial_z C(z)) b(w) :}{(z-w)} - \frac{h_b : b(z) C(w) :}{(z-w)^2} - \frac{h_b}{(z-w)^3}$$

$$= \frac{h_b : (\partial_w C(w)) b(w) :}{(z-w)} - \frac{h_b : b(w) C(w) :}{(z-w)^2}$$

$$- \frac{h_b : \partial_w b(w) C(w) :}{(z-w)} - \frac{h_b}{(z-w)^3}$$

$$= - \frac{h_b}{(z-w)^3} + \frac{h_b : C(w) b(w) :}{(z-w)^2} + \frac{h_b \partial_w [: C(w) b(w) :]}{(z-w)}$$

$$\begin{aligned} \Rightarrow :T(z)::j(\omega): &\sim \frac{h_c - h_b}{(\bar{z} - \omega)^3} + \frac{h_c :c(\omega)b(\omega):}{(\bar{z} - \omega)^2} \\ &+ \frac{h_b :c(\omega)b(\omega):}{(\bar{z} - \omega)^2} + \frac{h_c \partial_\omega [:c(\omega)b(\omega):]}{(\bar{z} - \omega)} \\ &+ \frac{h_b \partial_\omega [:c(\omega)b(\omega):]}{(\bar{z} - \omega)} \end{aligned}$$

Use $h_c = 1 - \lambda$ & $h_b = \lambda$ & $j(\omega) = c(\omega)b(\omega)$

$$\Rightarrow :T(z)::j(\omega): \sim \frac{1 - 2\lambda}{(\bar{z} - \omega)^3} + \frac{j(\omega)}{(\bar{z} - \omega)^2} + \frac{\partial_\omega :j(\omega):}{(\bar{z} - \omega)}$$

Thus, ' $j(\omega)$ ' is not a primary operator unless $\lambda = 1/2$ ($h_c = h_b = 1/2$). Nevertheless, it is an eigen state of Rot^n & dilation with weights $(1, 0)$.

Similarly, for $\tilde{j}(\bar{\omega})$ we get,

$$:\tilde{T}(\bar{z}):\tilde{j}(\bar{\omega}): \sim \frac{1 - 2\tilde{\lambda}}{(\bar{z} - \bar{\omega})^3} + \frac{\tilde{j}(\bar{\omega})}{(\bar{z} - \bar{\omega})^2} + \frac{\partial_{\bar{\omega}} :\tilde{j}(\bar{\omega}):}{(\bar{z} - \bar{\omega})}$$

It is primary only if $\tilde{\lambda} = 1/2$. But it is always eigen state of rot^n with weight $(0, 1)$.

Under conformal transformation given by $z \rightarrow z + \epsilon \mathcal{V}(z)$, j_{ghost} transforms as

$$\delta j(\omega) = -\epsilon \sum_{n=0}^{\infty} \frac{1}{n!} [\mathcal{V}^{(n)}(\omega) j^{(n)}(\omega)]$$

where $j^{(n)}$ are coefficients of $n+1^{\text{th}}$ order singularity in the OPE with $T(z)$.

$$\rightarrow \delta j(z) = -\epsilon \left[\mathcal{V}(z) \partial_z j(z) + j(z) \partial_z \mathcal{V}(z) \right] + \frac{1-2\lambda}{2} \partial_z^2 \mathcal{V}(z)$$

$$\rightarrow \frac{\delta j(z)}{\epsilon} = -\partial_z (\mathcal{V}(z) j(z)) + \left(\frac{2\lambda-1}{2} \right) \partial_z^2 \mathcal{V}(z)$$

Thus for finite transformations $z \rightarrow z' = f(z)$

$$j'(z') = (f'(z))^{-1} j(z) + \left(\frac{2\lambda-1}{2} \right) (f'(z))^{-2} f''(z)$$

ψ -theories

For $\lambda = 1/2$, the ghost Noether current becomes a tensor operator. For which the central charge becomes $c = 1$. Now, the bc theory action is given by,

$$S_{bc} = \frac{1}{2\pi} \int d^2z \, b(z, \bar{z}) \partial_{\bar{z}} c(z, \bar{z})$$

where b & c are tensor operators of equal weights. Let us define

$$\psi_1 = \frac{b+c}{\sqrt{2}} \quad \& \quad \psi_2 = \frac{b-c}{i\sqrt{2}}$$

$\Rightarrow \psi_1 \psi_2 = -\psi_2 \psi_1$, they also anti commute.

$\psi_1^2 = 0 = \psi_2^2$. They are Grassman fields.

$$\rightarrow b = \frac{\psi_1 + i\psi_2}{\sqrt{2}}, \quad c = \frac{\psi_1 - i\psi_2}{\sqrt{2}}$$

$\rightarrow$ If $b \rightarrow \psi$ then $c \rightarrow \bar{\psi}$. Substituting these in action, we get the following.

→

$$S_{bc} = \frac{1}{4\pi} \int d^2\bar{z} (\psi_1 + i\psi_2) \partial_{\bar{z}} (\psi_1 - i\psi_2)$$

$$= \frac{1}{4\pi} \int d^2\bar{z} \left[\psi_1 \partial_{\bar{z}} \psi_1 + \psi_2 \partial_{\bar{z}} \psi_2 + i(\psi_2 \partial_{\bar{z}} \psi_1 - \psi_1 \partial_{\bar{z}} \psi_2) \right]$$

$$= \frac{1}{4\pi} \int d^2\bar{z} \left[\psi_1 \partial_{\bar{z}} \psi_1 + \psi_2 \partial_{\bar{z}} \psi_2 + i(\psi_2 \partial_{\bar{z}} \psi_1 + (\partial_{\bar{z}} \psi_2) \psi_1) \right]$$

$$= \frac{1}{4\pi} \int d^2\bar{z} \left[\psi_1 \partial_{\bar{z}} \psi_1 + \psi_2 \partial_{\bar{z}} \psi_2 + i \underbrace{\partial_{\bar{z}} (\psi_2 \psi_1)}_{\text{total derivative term won't contribute}} \right]$$

$$\rightarrow S_{\psi} = \frac{1}{4\pi} \int d^2\bar{z} \left[\psi_1 \partial_{\bar{z}} \psi_1 + \psi_2 \partial_{\bar{z}} \psi_2 \right]$$

Thus we could split bc theory in a conformally invariant way.

$S_\psi = S_1 + S_2$, where $S_1 = \frac{1}{4\pi} \int d^2z \psi_1 \partial_{\bar{z}} \psi_1$
 & $S_2 = \frac{1}{4\pi} \int d^2z \psi_2 \partial_{\bar{z}} \psi_2$ as two independent
 theories. ψ_1 has weight $(1/2, 0)$ &
 ψ_2 has weight $(1/2, 0)$. The equations
 of motion are given by,

$$\partial_{\bar{z}} \psi_1 = 0 = \partial_{\bar{z}} \psi_2$$

So ψ_1 & ψ_2 are both Holomorphic.

By Schwinger-Dyson eqⁿ,

$$\frac{\partial S}{\partial \psi_j(z, \bar{z})} \psi_j(\omega, \bar{\omega}) = \delta^{(2)}(z - \omega, \bar{z} - \bar{\omega}) \quad (j=1, 2)$$

$$\rightarrow \frac{1}{2\pi} \partial_{\bar{z}} \psi_j(z, \bar{z}) \psi_j(\omega, \bar{\omega}) = \delta^{(2)}(z - \omega, \bar{z} - \bar{\omega})$$

$$\rightarrow \partial_{\bar{z}} \psi_j(z, \bar{z}) \psi_j(\omega, \bar{\omega}) = 2\pi \delta^{(2)}(z - \omega, \bar{z} - \bar{\omega})$$

The fundamental OPE's are given by,

$$\rightarrow \underline{\psi_i(z, \bar{z}) \psi_j(w, \bar{w}) = \frac{\delta_{ij}}{z-w} + \text{Non-singular}}$$

Then Normal ordering symbol is,

$$:\psi_i(z, \bar{z}) \psi_j(w, \bar{w}): = \psi_i(z, \bar{z}) \psi_j(w, \bar{w}) - \frac{\delta_{ij}}{z-w}$$

$$\Rightarrow \partial_{\bar{z}} (: \psi_i(z, \bar{z}) \psi_j(w, \bar{w}) :) = 0.$$

The Contraction relations are given by

$$\overline{\psi_i(z, \bar{z}) \psi_j(w, \bar{w})} = \psi_i(z, \bar{z}) \psi_j(w, \bar{w}) - : \psi_i(z, \bar{z}) \psi_j(w, \bar{w}) :$$

$$\rightarrow \underline{\underline{\overline{\psi_i(z, \bar{z}) \psi_j(w, \bar{w})} = \frac{\delta_{ij}}{z-w}}}$$

Thus we observe that ψ_1 & ψ_2 theories are fully independent.

The total energy momentum tensor is then given by,

$$T(\bar{z}) = -\frac{1}{2} [\psi_1(\bar{z}) \partial_{\bar{z}} \psi_1(\bar{z}) + \psi_2(\bar{z}) \partial_{\bar{z}} \psi_2(\bar{z})]$$

$$T(\bar{z}) = T_1(\bar{z}) + T_2(\bar{z})$$

$T(\bar{z})$ has charge $c=1$. But for T_1 & T_2 , they each have charges $\frac{1}{2}$.

This ψ theory represent two Majorana fermions, which are its own anti particle.

If we denote $\vec{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$, then,

$$S_{\psi} = \frac{1}{4\pi} \int d^2 \bar{z} \quad \vec{\psi} \cdot \partial_{\bar{z}} \vec{\psi}$$

$$\& \quad \underline{T(\bar{z}) = -\vec{\psi} \cdot (\partial_{\bar{z}} \vec{\psi})}$$

This theory also has a Ghost number like Symmetry.

S_ψ theory has global $SO(2)$ Symmetry.

(Note: $SO(2)$ is isomorphic to $U(1)$)

$$\text{Under } \vec{\psi} \longrightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{\psi}$$

we get S_ψ invariant. The generator is given by putting $\theta \rightarrow \epsilon\theta$

$$\rightarrow \vec{\psi} \rightarrow \vec{\psi} + \epsilon\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{\psi}$$

$$\rightarrow \delta \vec{\psi} = -\epsilon\theta \begin{bmatrix} \psi_2 \\ -\psi_1 \end{bmatrix}$$

$$\delta S_\psi = \frac{1}{4\pi} \int d^2z \left[(\delta \vec{\psi}) \cdot \partial_{\bar{z}} \vec{\psi} + \vec{\psi} \cdot \partial_{\bar{z}} (\delta \vec{\psi}) \right]$$

$$\delta S_\psi = \frac{1}{4\pi} \int d^2 z \left[(\partial \bar{\psi}) \cdot \partial_{\bar{z}} \bar{\psi} - (\partial_{\bar{z}} \bar{\psi}) \cdot \partial \bar{\psi} \right]$$

$$= \frac{1}{4\pi} \int d^2 z \left[(\partial \bar{\psi}) \cdot \partial_{\bar{z}} \bar{\psi} + \partial \bar{\psi} \cdot (\partial_{\bar{z}} \bar{\psi}) \right]$$

$$\rightarrow \delta S_\psi = \frac{1}{2\pi} \int d^2 z \partial \bar{\psi} \cdot (\partial_{\bar{z}} \bar{\psi})$$

$$= \frac{-\epsilon\theta}{2\pi} \int d^2 z \left[\psi_2 (\partial_{\bar{z}} \psi_1) - \psi_1 (\partial_{\bar{z}} \psi_2) \right]$$

$$= \frac{\epsilon\theta}{2\pi} \int d^2 z \left[(\partial_{\bar{z}} \psi_1) \psi_2 + \psi_1 (\partial_{\bar{z}} \psi_2) \right]$$

$$\rightarrow \delta S_\psi = \frac{\epsilon\theta}{2\pi} \int d^2 z \left[\partial_{\bar{z}} (\psi_1 \psi_2) \right]$$

→ Conserved Ghost Noether current is

$$\underline{j(z) = \psi_1(z) \psi_2(z)} \Rightarrow \text{Holomorphic}$$

β - γ CFT

This family of CFT are given by commuting tensorial fields. The action is,

$$S = \frac{1}{2\pi} \int d^2z \left[B(z, \bar{z}) \partial_{\bar{z}} \gamma(z, \bar{z}) + \tilde{B}(z, \bar{z}) \partial_z \tilde{\gamma}(z, \bar{z}) \right]$$

This can be written as sum of two independent actions of

$$\& \quad S_{\beta\gamma} = \frac{1}{2\pi} \int d^2z \ B(z, \bar{z}) \partial_{\bar{z}} \gamma(z, \bar{z})$$

$$\& \quad S_{\tilde{\beta}\tilde{\gamma}} = \frac{1}{2\pi} \int d^2z \ \tilde{B}(z, \bar{z}) \partial_z \tilde{\gamma}(z, \bar{z})$$

$$\rightarrow S = S_{\beta\gamma} + S_{\tilde{\beta}\tilde{\gamma}}$$

The equation of motion is given by,

$$\partial_{\bar{z}} B(z, \bar{z}) = 0 = \partial_{\bar{z}} \gamma(z, \bar{z})$$

&

$$\partial_z \tilde{B}(z, \bar{z}) = 0 = \partial_z \tilde{\gamma}(z, \bar{z})$$

Thus for fields that satisfy eqⁿ of motion we get

$$\left. \begin{aligned} B &\longrightarrow B(z) \\ \gamma &\longrightarrow \gamma(z) \end{aligned} \right\} \text{Holomorphic}$$

$$\& \left. \begin{aligned} \tilde{B} &\longrightarrow \tilde{B}(\bar{z}) \\ \tilde{\gamma} &\longrightarrow \tilde{\gamma}(\bar{z}) \end{aligned} \right\} \text{Anti-Holomorphic}$$

Now, if we need B & γ to be tensorial operators, then under conformal transformation we need the action to be invariant. Thus same as bc theory, we need $h_B + h_\gamma = 1$
 $\tilde{h}_B + \tilde{h}_\gamma = 0$ & $h_{\tilde{B}} + h_{\tilde{\gamma}} = 0$ & $\tilde{h}_{\tilde{B}} + \tilde{h}_{\tilde{\gamma}} = 1$.

$$\text{Let } h_B = \lambda \Rightarrow h_\gamma = 1 - \lambda$$

$$\& \tilde{h}_{\tilde{B}} = \tilde{\lambda} \Rightarrow \tilde{h}_{\tilde{\gamma}} = 1 - \tilde{\lambda}$$

The fundamental OPEs will be,

$$B(z)\gamma(w) \sim \frac{-1}{z-w} \quad \& \quad \gamma(z)B(w) \sim \frac{1}{z-w}$$

$$\longrightarrow B(z)\gamma(w) \sim \gamma(w)B(z)$$

Similarly,

$$\tilde{B}(z)\tilde{Y}(w) \sim \frac{-1}{z-w} \quad \& \quad \tilde{Y}(z)\tilde{B}(w) \sim \frac{1}{z-w}$$

$$\rightarrow \tilde{B}(z)\tilde{Y}(w) \sim \tilde{Y}(w)\tilde{B}(z)$$

This results in few -ve signs to appear compared to bc theory. But there is no -ve signs when we interchange the places of $B \& Y$ or $\tilde{B} \& \tilde{Y}$.

The normal ordering is given by,

$$:B(z)Y(w): = B(z)Y(w) + \frac{1}{z-w}$$

$$:Y(z)B(w): = Y(z)B(w) - \frac{1}{z-w}$$

&

$$:\tilde{B}(z)\tilde{Y}(w): = \tilde{B}(z)\tilde{Y}(w) + \frac{1}{z-w}$$

$$:\tilde{Y}(z)\tilde{B}(w): = \tilde{Y}(z)\tilde{B}(w) - \frac{1}{z-w}$$

Then the contraction terms are,

$$\overline{B(z) \gamma(w)} = B(z) \gamma(w) - :B(z) \gamma(w):$$

$$\rightarrow \overline{B(z) \gamma(w)} = - \frac{1}{z-w}$$

$$\overline{\gamma(z) B(w)} = \frac{1}{z-w}$$

$$\overline{\tilde{B}(\bar{z}) \tilde{\gamma}(\bar{w})} = - \frac{1}{\bar{z}-\bar{w}}$$

$$\overline{\tilde{\gamma}(\bar{z}) \tilde{B}(\bar{w})} = \frac{1}{\bar{z}-\bar{w}}$$

The energy-momentum tensor is given by

$$T(z) = (1-\lambda) (\partial B) \gamma - \lambda B (\partial \gamma)$$

$$\& \tilde{T}(\bar{z}) = (1-\tilde{\lambda}) (\bar{\partial} \tilde{B}) \tilde{\gamma} - \tilde{\lambda} \tilde{B} (\bar{\partial} \tilde{\gamma})$$

The central charges are,

$$C = 3(2\lambda-1)^2 - 1$$

$$\& \tilde{C} = 3(2\tilde{\lambda}-1)^2 - 1$$

Note:

$$\left(C_{bc}^{\text{theory}} = -C_{B\gamma}^{\text{theory}} \right)$$