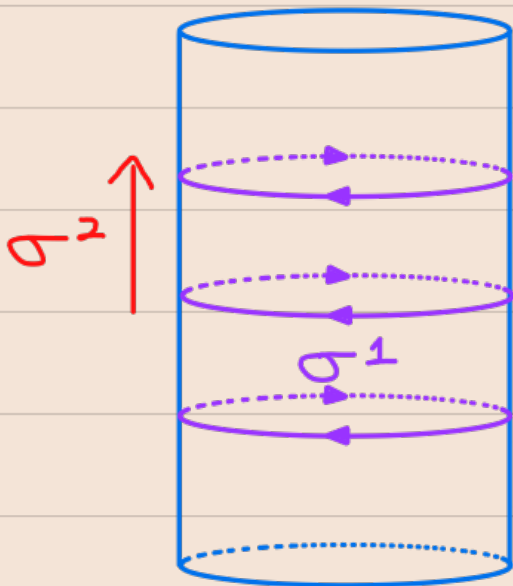


Closed String Virasoro Algebra

The closed string world sheet can be thought of a cylinder whose Euclidean time coordinate $\sigma^2 = i\tau$ runs from $-\infty$ to $+\infty$, where we go from the bottom to top of the cylinder. Then the spatial σ^1 coordinate is periodic in ' 2π '.

$$\sigma^1 + 2\pi = \sigma^1 \quad \& \quad -\infty < \sigma^2 < +\infty$$

Cylindrical World Sheet



We define the complex coordinates on cylinder as $\omega = \sigma^1 + i\sigma^2$ & $\bar{\omega} = \sigma^1 - i\sigma^2$. So, ω & $\bar{\omega}$ define a point on the world sheet.

$$\Rightarrow \omega + 2\pi = \omega \quad \& \quad \bar{\omega} + 2\pi = \bar{\omega}$$

So time translation is got by change of $\sigma^2 = \text{Im}(\omega)$.

But we can perform a conformal map of Cylinder to a plane by defining

$$z = e^{-i\omega} \quad \& \quad \bar{z} = e^{+i\omega}$$

$$\rightarrow z = e^{(-i\sigma^1 + \sigma^2)} \quad \& \quad \bar{z} = e^{(i\sigma^1 + \sigma^2)}$$

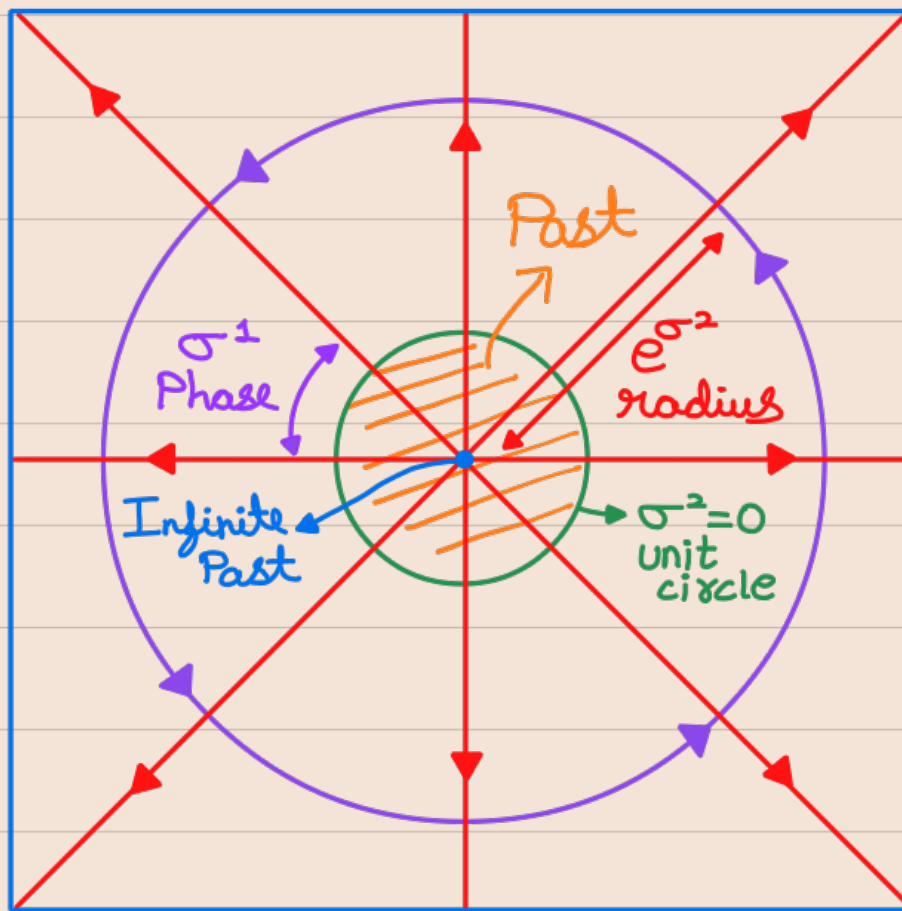
In terms of $(z, \bar{z})$ coordinates,

$$z = e^{\sigma^2} e^{-i\sigma^1} \quad \& \quad \bar{z} = e^{\sigma^2} e^{i\sigma^1}$$

The radial distance in $(z, \bar{z})$ plane relates to the euclidean time by, $\tau = e^{\sigma^2}$.

So $\sigma^2 = -\infty$ (Infinite past) is $(0,0)$ the origin of ' $z, \bar{z}$ ' plane & $\sigma^2 = 0$ is the unit circle. Then the spatial σ^1 coordinate of the cylinder gives the phase of the $(z, \bar{z})$ plane. This is also natural since $\sigma^1 + 2\pi = \sigma^1$.

$(z, \bar{z})$ plane



The time evolution in cylindrical world sheet is done by Hamiltonian operator given by ' ∂_{σ^2} ' which changes σ^2 of states. The same in ' ω ' coordinates is got by $\partial_{\sigma^2} = i(\partial_{\omega} - \partial_{\bar{\omega}})$. But in the z coordinates $\partial_{\omega} = \frac{\partial z}{\partial \omega} \partial_z$, $\partial_{\bar{\omega}} = \frac{\partial \bar{z}}{\partial \bar{\omega}} \partial_{\bar{z}}$

$\rightarrow \partial_{\sigma^2} = z \partial_z + \bar{z} \partial_{\bar{z}} \Rightarrow$ Dilation Operator
 $\& i \partial_{\sigma^1} = z \partial_z - \bar{z} \partial_{\bar{z}} \Rightarrow$ Rotation Operator

Thus the time evolution in world sheet cylinder corresponds to the dilation in $(z, \bar{z})$ plane (radial evolution) & spatial evolution / translation in cylinder correspond to phase rotation in $(z, \bar{z})$ plane.

Time ordering in cylinder becomes radial ordering in $(z, \bar{z})$ plane. Operators away from origin are arranged to left of the operators nearer to the origin.

Energy Momentum Tensor in Cylinder & Plane

The conformal map b/w cylinder $(\omega, \bar{\omega})$ & the plane $(z, \bar{z})$ is given by $z = e^{-i\omega}$ & $\bar{z} = e^{i\bar{\omega}}$. Then the Schwarzian derivative is

$$\{z, \omega\}_S = (\partial_\omega^3 z)(\partial_\omega z)^{-1} - \frac{3}{2}(\partial_\omega^2 z)^2(\partial_\omega z)^{-2}$$

$$= (-i)^3(-i)^{-1} - \frac{3}{2}(-i)^4(-i)^{-2} = -1 + \frac{3}{2}$$

$$\Rightarrow \{z, \omega\}_S = 1/2$$

$$\rightarrow T_{\text{plane}}(z) = (\partial_w z)^{-2} \left[T_{\text{cylinder}}(w) - \frac{c}{12} \{z, w\}_g \right]$$

$$T_{\text{Plane}}(z) = \frac{(-i)^2}{z^2} \left[T_{\text{cylinder}}(w) - \frac{c}{24} \right]$$

$$\Rightarrow T_{\text{Plane}}(z) = \frac{-1}{z^2} \left[T_{\text{cylinder}}(w) - \frac{c}{24} \right]$$

$$\rightarrow \underline{T_{\text{cylinder}}(w) = -z^2 T_{\text{Plane}}(z) + \frac{c}{24}}$$

Similarly for $\tilde{T}_{\text{cylinder}}(\bar{w})$ we get,

$$\rightarrow \underline{\tilde{T}_{\text{cylinder}}(\bar{w}) = -\bar{z}^2 \tilde{T}_{\text{Plane}}(\bar{z}) + \frac{\tilde{c}}{24}}$$

For a Holomorphic / Anti-Holomorphic function we can always perform a Laurent expansion of a holomorphic / anti-holomorphic function. So, on the $(z, \bar{z})$ plane, we can expand $T(z)$ & $\tilde{T}(\bar{z})$ about the origin as,

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \quad \& \quad \tilde{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

The coefficients of expansion are known as the 'Virasoro Generators'.

$$\text{Then } z^{n+1} T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m z^{n+1}}{z^{m+2}}$$

$$\Rightarrow z^{n+1} T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m-n+1}}$$

Then perform a closed Contour Integral where the contour (C) encloses the origin. This gives residue of ' $z^{n+1} T(z)$ '.

$$\begin{aligned} \rightarrow \oint_C z^{n+1} T(z) dz &= \sum_{m=-\infty}^{\infty} \oint_C \frac{L_m dz}{z^{m-n+1}} \\ &= \sum_{m=-\infty}^{\infty} L_m (2\pi i \delta_{mn}) = 2\pi i L_n \end{aligned}$$

$$\rightarrow \boxed{L_n = \frac{1}{2\pi i} \oint_C z^{n+1} T(z) dz}$$

$$\text{where, } T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

This expansion in $(\omega, \bar{\omega})$ coordinates is,

$$\begin{aligned} T(\omega)_{\text{cylinder}} &= -z^2 T_{\text{Plane}}(z) + \frac{c}{24} \\ &= -z^2 \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} + \frac{c}{24} \\ &= - \sum_{m=-\infty}^{\infty} \left(\frac{L_m}{z^m} - \frac{c\delta_{m0}}{24} \right) \\ &= - \sum_{m=-\infty}^{\infty} \left(L_m e^{im\omega} - \frac{c\delta_{m0}}{24} \right) \\ &= - \sum_{m=-\infty}^{\infty} \left(L_m e^{im\omega} - \frac{c\delta_{m0}}{24} e^{im\omega} \right) \end{aligned}$$

$$\rightarrow T(\omega)_{\text{cylinder}} = - \sum_{m=-\infty}^{\infty} \left(L_m - \frac{c\delta_{m0}}{24} \right) e^{im\omega}$$

$$\rightarrow \boxed{T(\omega)_{\text{cylinder}} = - \sum_{m=-\infty}^{\infty} T_m e^{im\omega}}$$

where $\underline{T_m = L_m - \frac{c\delta_{m0}}{24}}$

By substituting $\omega = \sigma^1 + i\sigma^2$

$$\begin{aligned}\Rightarrow T(\omega)_{\text{cylinder}} &= - \sum_{m=-\infty}^{\infty} T_m e^{im(\sigma^1 + i\sigma^2)} \\ &= - \sum_{m=-\infty}^{\infty} T_m e^{im\sigma^1} e^{-m\sigma^2}\end{aligned}$$

Thus at $\sigma^2 = 0$, i.e. when Euclidean time is zero, $T(\omega)_{\text{cylinder}}$ is just the spatial fourier transform with T_m as fourier coefficients. When $\sigma^2 = 0$, $\omega = \bar{\omega}$.

$$\Rightarrow T(\sigma^1, \sigma^2 = 0)_{\text{cylinder}} = - \sum_{m=-\infty}^{\infty} T_m e^{im\sigma^1}$$

We can invert the above relation to get T_m in terms of T_{cylinder} as,

$$T(\sigma^1, \sigma^2 = 0)_{\text{cylinder}} e^{-in\sigma^1} = - \sum_{m=-\infty}^{\infty} T_m e^{i(m-n)\sigma^1}$$

Integrate w.r.t ' σ^1 ' with measure $\frac{d\sigma^1}{2\pi}$ from $0 < \sigma^1 < 2\pi$.

$$\rightarrow \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$= - \sum_{m=-\infty}^{\infty} T_m \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \underbrace{e^{i(m-n)\sigma^1}}_{\delta_{mn}} = - \sum_{m=-\infty}^{\infty} T_m \delta_{mn}$$

$$\rightarrow T_n = - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

In terms of Virasoro generators,

$$L_m = T_m + \frac{c\delta_{m0}}{24}$$

$$L_m = \frac{c\delta_{m0}}{24} - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T(\sigma^1, \sigma^2=0) e^{-im\sigma^1}$$

We can relate the Virasoro generators to the Physical observables using the above eqⁿ.

Now, for the anti-holomorphic part,

$$\tilde{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

We can get $\tilde{L}_n$ by performing closed contour integral enclosing origin on $\bar{z}^{n+1} \tilde{T}(\bar{z})$.

$$\tilde{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

$$\rightarrow \bar{z}^{n+1} \tilde{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m-n+1}}$$

$$\rightarrow \oint_C \bar{z}^{n+1} \tilde{T}(\bar{z}) d\bar{z} = \sum_{m=-\infty}^{\infty} \tilde{L}_m \oint_C \frac{d\bar{z}}{\bar{z}^{m-n+1}}$$

The orientation of contour is clock-wise.

$$\text{So, } \oint_C \frac{d\bar{z}}{\bar{z}^{m-n+1}} = 2\pi i \delta_{mn}$$

$$\rightarrow \boxed{\tilde{L}_n = \frac{1}{2\pi i} \oint_C \bar{z}^{n+1} \tilde{T}(\bar{z}) d\bar{z}}$$

Now, we can relate $\tilde{T}(\bar{z})$ with $\tilde{T}(\bar{\omega})_{\text{cylinder}}$

$$\tilde{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}$$

$$\tilde{T}_{\text{cylinder}}(\bar{\omega}) = -\bar{z}^2 \tilde{T}_{\text{Plane}}(\bar{z}) + \frac{\tilde{c}}{24}$$

$$= - \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^m} + \frac{\tilde{c}}{24}$$

$$= - \sum_{m=-\infty}^{\infty} \left(\tilde{L}_m - \frac{\tilde{c} \delta_{m0}}{24} \right) e^{-im\bar{\omega}}$$

$$\rightarrow \boxed{\tilde{T}_{\text{cylinder}}(\bar{\omega}) = - \sum_{m=-\infty}^{\infty} \tilde{T}_m e^{-im\bar{\omega}}}$$

$$\text{where, } \underline{\tilde{T}_m = \tilde{L}_m - \frac{\tilde{c} \delta_{m0}}{24}}$$

By substituting $\bar{\omega} = \sigma^1 - i\sigma^2$

$$\rightarrow \tilde{T}_{\text{cylinder}}(\bar{\omega}) = - \sum_{m=-\infty}^{\infty} \tilde{T}_m e^{-im(\sigma^1 - i\sigma^2)}$$

$$\rightarrow \tilde{T}_{\text{cylinder}}(\bar{\omega}) = - \sum_{m=-\infty}^{\infty} \tilde{T}_m e^{-im\sigma^1} e^{-m\sigma^2}$$

Thus at $\sigma^2 = 0$, i.e. when Euclidean time is Zero, $\tilde{T}_{\text{cylinder}}(\bar{\omega})$ is just the spatial fourier transform with $\tilde{T}_m$ as fourier coefficients.

$$\rightarrow \tilde{T}_{\text{cylinder}}(\sigma^1, \sigma^2 = 0) = - \sum_{m=-\infty}^{\infty} \tilde{T}_m e^{-im\sigma^1}$$

We can invert the above relation to get $\tilde{T}_m$ in terms of $\tilde{T}_{\text{cylinder}}$.

$$\tilde{T}_{\text{cylinder}}(\sigma^1, \sigma^2 = 0) e^{in\sigma^1} = - \sum_{m=-\infty}^{\infty} \tilde{T}_m e^{-i(m-n)\sigma^1}$$

Integrate w.r.t ' σ^1 ' with measure $\frac{d\sigma^1}{2\pi}$ from $0 < \sigma^1 < 2\pi$.

$$\rightarrow \tilde{T}_n = - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \tilde{T}_{\text{cylinder}}(\sigma^1, \sigma^2 = 0) e^{in\sigma^1}$$

In terms of Virasoro generators,

$$\tilde{L}_m = \tilde{T}_m + \frac{\tilde{c}\tilde{\Delta}_{m0}}{24}$$

$$\tilde{L}_m = \frac{\tilde{c}\tilde{\Delta}_{m0}}{24} - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \tilde{T}(\sigma^1, \sigma^2=0) e^{im\sigma^1}$$

Consider the Energy momentum tensor relation b/w the $(\omega, \bar{\omega})$ coordinates and (σ^1, σ^2) coordinates.

$$\text{Now, } T_{\text{cylinder}}(\omega) = T_{\omega\omega}$$

$$T_{\omega\omega} = \frac{\partial\sigma^a}{\partial\omega} \frac{\partial\sigma^b}{\partial\omega} T_{ab}$$

$$= \frac{1}{4} T_{11} + \frac{1}{4i} T_{12} + \frac{1}{4i} T_{21} - \frac{1}{4} T_{22}$$

(Since $T_{12} = T_{21}$)

$$\Rightarrow T_{\omega\omega} = \frac{T_{11} - T_{22} - 2iT_{12}}{4}$$

Similarly, $T_{\bar{\omega}\bar{\omega}} = \frac{T_{11} - T_{22} + 2iT_{12}}{4}$

But for CFT, $T_{\omega\bar{\omega}} = T_{\bar{\omega}\omega} = 0$
 This is got by $T_{11} + T_{12} = \text{Trace}(T_{ab}) = 0$.

Thus $T_{11} = -T_{22} \Rightarrow T_{\omega\omega} = \frac{-T_{22} - iT_{12}}{2}$

& $T_{\bar{\omega}\bar{\omega}} = \frac{-T_{22} + iT_{12}}{2}$

$\rightarrow \underline{T_{22} = -(T_{\omega\omega} + T_{\bar{\omega}\bar{\omega}}) = -T_{11}}$

And, $\underline{T_{21} = T_{12} = i(T_{\omega\omega} - T_{\bar{\omega}\bar{\omega}})}$

The Hamiltonian of world sheet which will generate the σ^2 translation is,

$$H = \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{22}$$

$\rightarrow H = - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} (T_{\omega\omega} + T_{\bar{\omega}\bar{\omega}})$

Now, the 0^{th} mode Virasoro generators are,

$$L_0 = \frac{c}{24} - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{\omega\omega}(\sigma^1, \sigma^2=0)$$

$$\tilde{L}_0 = \frac{\tilde{c}}{24} - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{\bar{\omega}\bar{\omega}}(\sigma^1, \sigma^2=0)$$

$$L_0 + \tilde{L}_0 = \frac{c + \tilde{c}}{24} - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \left[T_{\omega\omega}(\sigma^1, \sigma^2=0) + T_{\bar{\omega}\bar{\omega}}(\sigma^1, \sigma^2=0) \right]$$

$$L_0 + \tilde{L}_0 = \frac{c + \tilde{c}}{24} + \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{22} = \frac{c + \tilde{c}}{24} + H$$

$$\rightarrow \boxed{H = L_0 + \tilde{L}_0 - \frac{c + \tilde{c}}{24}}$$

$$\text{Now, } H = -i\partial_\tau = \partial_{\sigma_2} = z\partial_z + \bar{z}\partial_{\bar{z}}$$

But ' $z\partial_z + \bar{z}\partial_{\bar{z}}$ ' is the dilation operator. So, time evolution in world sheet is the radial evolution in $(z, \bar{z})$ Plane.

Now in the sense of Path integrals & Hilbert Space formalism, we know that the expectation value of time ordered products of Operators called as Correlations are given by the Cutting & Sewing procedure at a particular time. This defines the Hilbert Space at that time slice.

Similarly, in the $(z, \bar{z})$ plane, when we want to find the Correlations / Operator products, we need to do Cutting & Sewing procedure at a given time i.e. at a contour of fixed radius in $z, \bar{z}$ plane. Here we define a Hilbert Space on that circle. This is known as Radial Quantization. So, when we define

$$L_n = \frac{1}{2\pi i} \oint_{C: |z|=\rho} z^{n+1} T(z) dz$$

&

$$\tilde{L}_n = \frac{1}{2\pi i} \oint_{C: |\bar{z}|=\rho} \bar{z}^{n+1} \tilde{T}(\bar{z}) d\bar{z}$$

we describe an operator at a radius ' ρ '.

But since $T(z)$ is holomorphic & $\tilde{T}(\bar{z})$ is anti-holomorphic, L_n & $\tilde{L}_n$ are independent of the Contour used for integration. This implies they are independent of radius, proper time ' τ ' and Euclidean time σ^2 .

This can also be stated as,
"Virasoro Generators are Invariant under rescaling or time translation."

Thus L_n & $\tilde{L}_n$ are conserved charges which are the generators of Conformal transformations. To see this, consider the infinitesimal conformal transformation $z \rightarrow z + \epsilon \mathcal{V}(z)$ & $\bar{z} \rightarrow \bar{z} + \epsilon \mathcal{V}^*(\bar{z})$.

The Noether current is given by,
 $J(z) = i\mathcal{V}(z)T(z)$ & $\tilde{J}(\bar{z}) = i\mathcal{V}^*(\bar{z})\tilde{T}(\bar{z})$

Then the corresponding charges are given by integrating the current around a Contour.

The charges are given by,

$$Q = \frac{-1}{2\pi} \oint_C T(z) dz = \frac{1}{2\pi i} \oint_C \mathcal{V}(z) T(z) dz$$

&

$$\tilde{Q} = \frac{-1}{2\pi} \oint_C \tilde{T}(\bar{z}) d\bar{z} = \frac{1}{2\pi i} \oint_C \mathcal{V}^*(\bar{z}) \tilde{T}(\bar{z}) d\bar{z}$$

$$\text{If } \mathcal{V}(z) = z^{n+1} \quad \& \quad \mathcal{V}^*(\bar{z}) = \bar{z}^{n+1}$$

$$\text{then, } \underline{Q \rightarrow L_n} \quad \& \quad \underline{\tilde{Q} \rightarrow \tilde{L}_n}$$

Thus the Virasoro Generators L_n & $\tilde{L}_n$ are the conserved Noether charges for infinitesimal conformal transformation $z \rightarrow z + \epsilon z^{n+1}$ & $\bar{z} \rightarrow \bar{z} + \epsilon \bar{z}^{n+1}$ respectively.

Expand ' $\mathcal{V}(z)$ ' about the origin. Thus,

$$\mathcal{V}(z) = \sum_{n=-\infty}^{\infty} \mathcal{V}_n z^{n+1}, \quad \mathcal{V}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}(z) dz}{z^{n+2}}$$

$$Q = \frac{1}{2\pi i} \oint_C \mathcal{V}(z) T(z) dz$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \mathcal{V}_n \oint_C z^{n+1} T(z) dz$$

$$\rightarrow \boxed{Q = \sum_{n=-\infty}^{\infty} \mathcal{V}_n L_n}$$

Similarly, for the anti-holomorphic part, Expand ' $\mathcal{V}^*(\bar{z})$ ' about the origin. Thus,

$$\mathcal{V}^*(\bar{z}) = \sum_{m=-\infty}^{\infty} \tilde{\mathcal{V}}_m \bar{z}^{m+1}, \quad \tilde{\mathcal{V}}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}^*(\bar{z}) d\bar{z}}{\bar{z}^{n+2}}$$

$$\tilde{Q} = \frac{1}{2\pi i} \oint_C \mathcal{V}^*(\bar{z}) \tilde{T}(\bar{z}) d\bar{z}$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \tilde{\mathcal{V}}_n \oint_C \bar{z}^{n+1} \tilde{T}(\bar{z}) d\bar{z}$$

$$\rightarrow \boxed{\tilde{Q} = \sum_{n=-\infty}^{\infty} \tilde{\mathcal{V}}_n \tilde{L}_n}$$

Thus any conserved charge for any conformal transformations can be written as the linear combinations of Virasoro Generators.

For Translations in $(z, \bar{z})$ plane, we need $\mathcal{V}(z) = z_0$, $\mathcal{V}^*(\bar{z}) = \bar{z}_0$. Then,

$$\mathcal{V}(z) = z_0 z^{(-1+1)} \rightarrow \mathcal{V}_{-1} = z_0 \rightarrow \mathcal{V}_{n, n \neq -1} = 0$$

$$\mathcal{V}^*(\bar{z}) = \bar{z}_0 \bar{z}^{(-1+1)} \rightarrow \tilde{\mathcal{V}}_{-1} = \bar{z}_0 \rightarrow \tilde{\mathcal{V}}_{n, n \neq -1} = 0$$

$\Rightarrow L_{-1}$ generates translations in z &
 $\tilde{L}_{-1}$ generates translations in $\bar{z}$

If $z \rightarrow z + \epsilon z_0$, then in the cylinder, this will correspond to the following.

$$z = e^{-i\omega} \Rightarrow \omega = i \ln(z). \text{ So, under the translations in } z, \omega \rightarrow i \ln(z + \epsilon z_0)$$

$$\rightarrow \omega \rightarrow i \ln(z) + i \ln\left(1 + \frac{\epsilon z_0}{z}\right) = \omega + i \frac{\epsilon z_0}{z}$$

$$\Rightarrow \delta\omega = i \epsilon z_0 e^{i\omega}$$

$$\rightarrow \delta\sigma^1 + i\delta\sigma^2 = i\epsilon(a_0 + ib_0)e^{-\sigma^2} \begin{bmatrix} \cos(\sigma^1) \\ + i \sin(\sigma^1) \end{bmatrix}$$

$$(\text{Let } z_0 = a_0 + ib_0)$$

$$\delta\sigma^1 = \epsilon e^{-\sigma^2} [-a_0 \sin(\sigma^1) - b_0 \cos(\sigma^1)]$$

$$\delta\sigma^2 = \epsilon e^{-\sigma^2} [a_0 \cos(\sigma^1) - b_0 \sin(\sigma^1)]$$

This is some non linear transformation.

For rotations in $(z, \bar{z})$ plane, we need

$$z \rightarrow e^{i\theta} z \quad \& \quad \bar{z} \rightarrow e^{-i\theta} \bar{z}.$$

$$\rightarrow \mathcal{V}(z) = iz \quad \& \quad \mathcal{V}^*(\bar{z}) = -i\bar{z}$$

$$\text{Thus } Q = iL_0 \quad \& \quad \tilde{Q} = -i\tilde{L}_0$$

$$z = e^{-i\omega} \Rightarrow e^{i\theta} z = e^{i(\theta - \omega)}$$

$$\Rightarrow e^{i\theta} z = e^{\sigma^2} e^{-i(\sigma^1 - \theta)}$$

$$\Rightarrow \sigma^2 \rightarrow \sigma^2 \quad \& \quad \sigma^1 \rightarrow (\sigma^1 - \theta)_{\text{mod } 2\pi}$$

$$\text{Similarly, } \bar{z} = e^{i\omega} \Rightarrow e^{-i\theta} \bar{z} = e^{i(\omega - \theta)}$$

$$\Rightarrow e^{-i\theta} \bar{z} = e^{-\sigma^2} e^{i(\sigma^1 - \theta)}$$

$$\Rightarrow \sigma^2 \rightarrow \sigma^2 \quad \& \quad \sigma^1 \rightarrow (\sigma^1 - \theta)_{\text{mod } 2\pi}$$

Rotational invariance of $(z, \bar{z})$ correspond to invariance in selecting a starting point for the σ^1 spatial coordinate of the string.

For dilations in $(z, \bar{z})$ plane, we need

$$z \rightarrow mz \quad \& \quad \bar{z} \rightarrow m\bar{z}. \quad \left(\begin{array}{l} m \Rightarrow \text{real} \\ \& +ve \end{array} \right)$$

$$\rightarrow \mathcal{V}(z) = z \quad \& \quad \mathcal{V}^*(\bar{z}) = \bar{z}$$

$$\text{Thus } Q = L_0 \quad \& \quad \tilde{Q} = \tilde{L}_0$$

$$z = e^{-i\omega} \Rightarrow mz = e^{\ln(m)} e^{-i\omega}$$

$$\Rightarrow mz = e^{[\sigma^2 + \ln(m)]} e^{-i\sigma^1}$$

$$\Rightarrow \sigma^2 \rightarrow \sigma^2 + \ln(m) \quad \& \quad \sigma^1 \rightarrow \sigma^1$$

Similarly,

$$\bar{z} = e^{i\omega} \Rightarrow m\bar{z} = e^{\ln(m)} e^{i\omega}$$

$$\Rightarrow m\bar{z} = e^{[\sigma^2 + \ln(m)]} e^{i\sigma^1}$$

$$\Rightarrow \sigma^2 \rightarrow \sigma^2 + \ln(m) \quad \& \quad \sigma^1 \rightarrow \sigma^1$$

Dilations in the $(z, \bar{z})$ plane correspond to time translation in string world sheet.

Virasoro Algebra Commutation relations

Consider $[L_m, L_n] = L_m L_n - L_n L_m$

This is an operator equation which sits inside the time ordered or in this case the radial ordered correlator brackets.

Thus, the local operators at larger radial value should be to the left & the ones with smaller radial value to the right.

Now, we can write these virasoro generators as contour integrals. But these objects are contour independent.

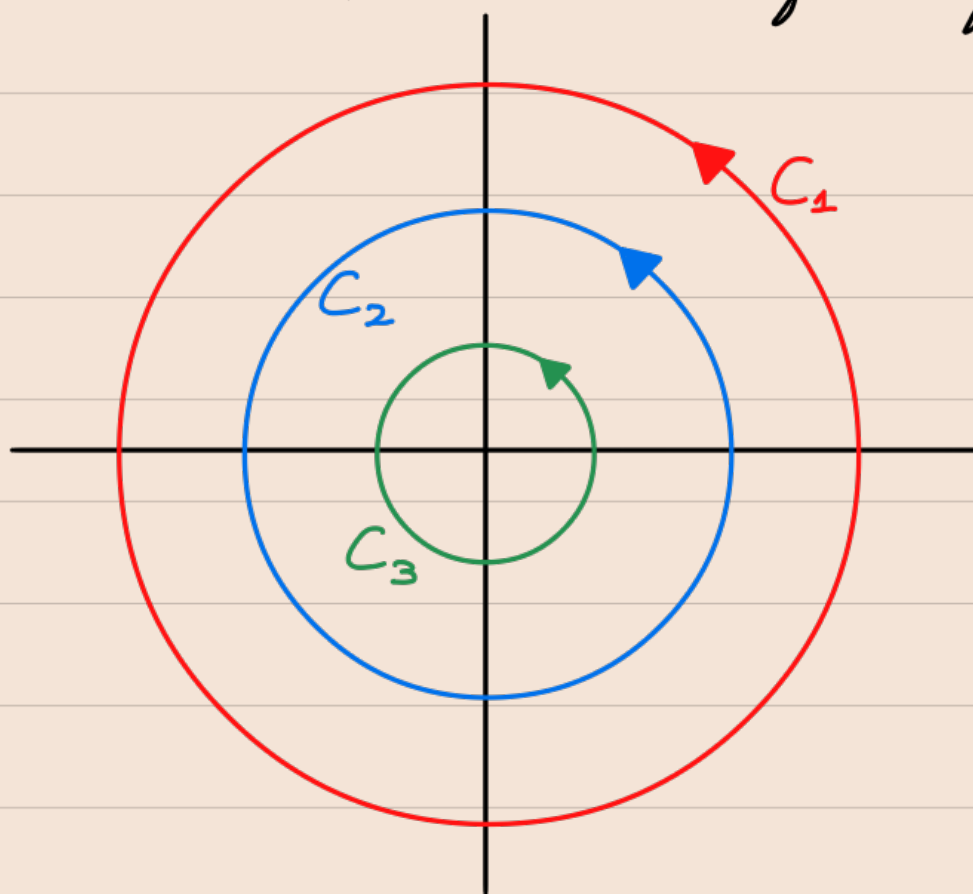
In general,

$$L_m = \oint_C \frac{dz}{2\pi i} z^{m+1} T(z)$$

where 'C' is any contour enclosing the Origin $z=0$. And $T(z)$ lives on the Contour 'C'.

$$\rightarrow [L_m, L_n] = L_m(C_1) L_n(C_2) - L_n(C_2) L_m(C_3)$$

The contours C_1, C_2, C_3 are of the form,

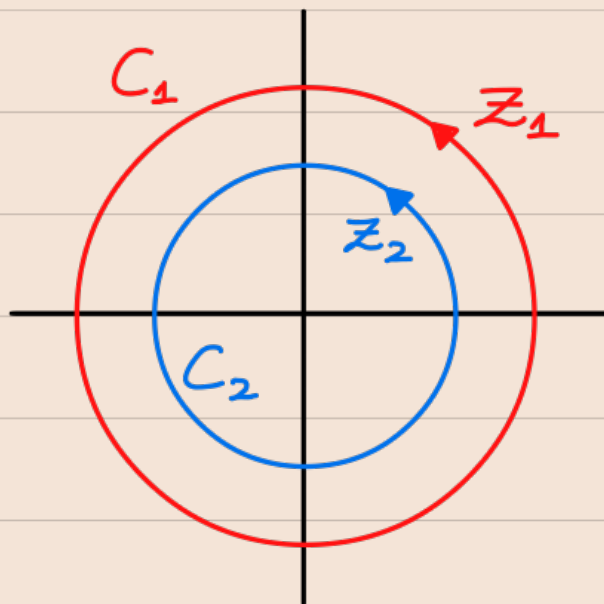


This structure of contours are needed to make correct sense of radial ordering of $T(z)$ inside the contour integral. Thus,

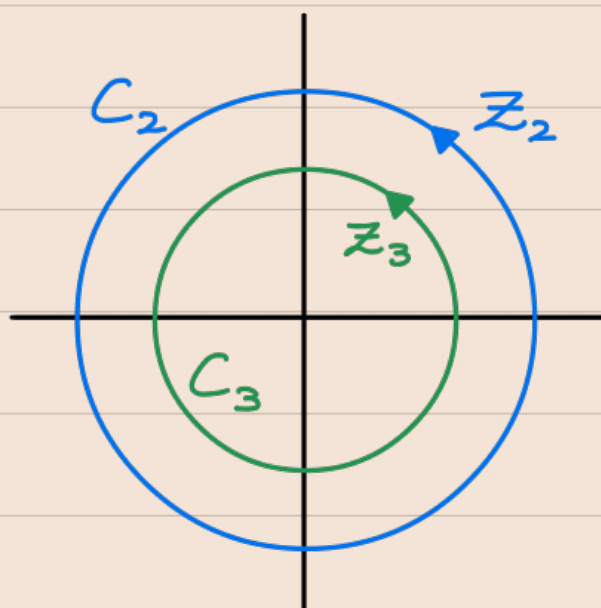
$$\begin{aligned} [L_m, L_n] = & \oint_{C_1} \frac{dz_1}{2\pi i} \oint_{C_2} \frac{dz_2}{2\pi i} z_1^{m+1} z_2^{n+1} T(z_1) T(z_2) \\ & - \oint_{C_2} \frac{dz_2}{2\pi i} \oint_{C_3} \frac{dz_3}{2\pi i} z_2^{n+1} z_3^{m+1} T(z_2) T(z_3) \end{aligned}$$

where C_1 encloses C_2 , and C_2 encloses C_3

Thus $L_m L_n$ means



& $L_n L_m$ means



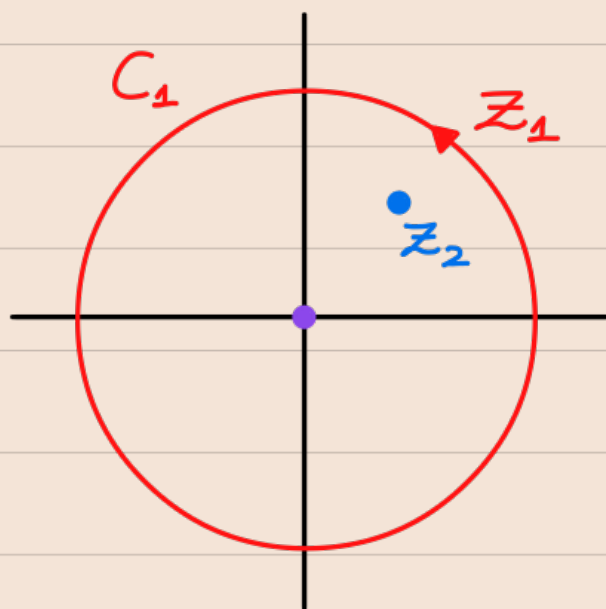
$$\rightarrow [L_m, L_n] = L_m L_n - L_n L_m$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \left(\oint_{C_1} \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} T(z_1) T(z_2) - \oint_{C_3} \frac{dz_3}{2\pi i} z_2^{n+1} z_3^{m+1} T(z_2) T(z_3) \right)$$

Now, to do this integral, we shall first fix a point ' z_2 ' in the inner integrals.

Observe that, there are two poles in each term in the inner integral. The first one being the pole at origin $(0,0)$ & the other at z_2 because of the singularities in the TT OPE.

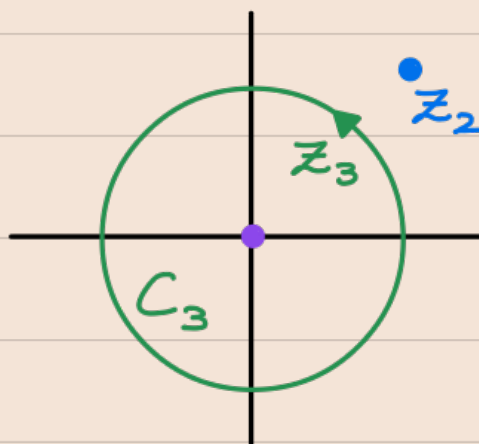
Poles in $L_m L_n$ term



Thus,

$$\oint_{C_1} \frac{dz_1}{2\pi i} z_1^{m+1} z_2^{n+1} T(z_1) T(z_2) = z_2^{n+1} L_m + \text{Res}_{z_1 \rightarrow z_2} [z_1^{m+1} z_2^{n+1} T(z_1) T(z_2)]$$

Poles in $L_n L_m$ term



Thus,

$$\oint_{C_3} \frac{dz_3}{2\pi i} z_3^{m+1} z_2^{n+1} T(z_2) T(z_3) = L_m z_2^{n+1}$$

(z_2 is not enclosed by C_3)

$$\rightarrow [L_m, L_n] = \oint_{C_2} \frac{dz_2}{2\pi i} \left(\cancel{z_2^{n+1}} L_m + \text{Res}_{z_1 \rightarrow z_2} [\cancel{z_1^{m+1}} z_2^{n+1} T(z_1) T(z_2)] - \cancel{L_m z_2^{n+1}} \right)$$

Thus finally we get,

$$[L_m, L_n] = \oint_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} [z_1^{m+1} z_2^{n+1} T(z_1) T(z_2)]$$

Now, we shall find the above residue.
We can use

$$T(z_1) T(z_2) \sim \frac{\partial_{z_2} T(z_2)}{z_{12}} + \frac{2T(z_2)}{z_{12}^2} + \frac{c}{2z_{12}^4}$$

$$(z_{12} = z_1 - z_2)$$

& write z_1^{m+1} in powers of z_2 .

$$z_1^{m+1} = \sum_{k=0}^{\infty} \frac{z_{12}^k}{k!} \partial_{z_2}^k (z_2^{m+1}) = \sum_{k=0}^{m+1} \frac{z_{12}^k}{k!} z_2^{m+1-k} \frac{(m+1)!}{(m+1-k)!}$$

$$z_1^{m+1} T(z_1) T(z_2) \sim$$

$$\sum_{k=0}^{m+1} \frac{z_{12}^k z_2^{m+1-k}}{k!} \frac{(m+1)!}{(m+1-k)!} \left[\frac{\partial_{z_2} T(z_2)}{z_{12}} + \frac{2 T(z_2)}{z_{12}^2} + \frac{c}{2 z_{12}^4} \right]$$

$$\text{Res}_{z_1 \rightarrow z_2} [z_1^{m+1} z_2^{n+1} T(z_1) T(z_2)]$$

$$= z_2^{n+1} \text{Res}_{z_1 \rightarrow z_2} [z_1^{m+1} T(z_1) T(z_2)]$$

$$= z_2^{n+1} \left[z_2^{m+1} \partial_{z_2} T(z_2) + 2 z_2^m (m+1) T(z_2) + \frac{c}{12} m(m^2-1) z_2^{m-2} \right]$$

$$\Rightarrow \text{Res}_{z_1 \rightarrow z_2} [z_1^{m+1} z_2^{n+1} T(z_1) T(z_2)]$$

$$= z_2^{m+n+2} \partial_{z_2} T(z_2) + 2 z_2^{m+n+1} (m+1) T(z_2) + \frac{c}{12} m(m^2-1) z_2^{m+n-1}$$

Now, we get $[L_m, L_n]$ as contour integral enclosing origin.

$$[L_m, L_n] = \oint_{C_2} \frac{dz_2}{2\pi i} \operatorname{Res}_{z_1 \rightarrow z_2} [z_1^{m+1} z_2^{n+1} T(z_1) T(z_2)]$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n+2} \partial_{z_2} T(z_2) + 2(m+1) \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n+1} T(z_2) + \frac{c}{12} m(m^2-1) \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n-1} \underbrace{\quad}_{\delta_{m,-n}}$$

$$\text{Use } T(z_2) = \sum_{k=-\infty}^{\infty} \frac{L_k}{z_2^{k+2}}$$

$$\oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n+2} \partial_{z_2} T(z_2) = - \sum_{k=-\infty}^{\infty} (k+2) L_k \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n-k-1} \underbrace{\quad}_{\delta_{k,m+n}}$$

$$\oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n+2} \partial_{z_2} T(z_2) = -(m+n+2) L_{m+n}$$

$$\oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n+1} T(z_2) = \sum_{k=-\infty}^{\infty} L_k \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n-k-1} \underbrace{\quad}_{\delta_{m+n,k}}$$

$$2(m+1) \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n+1} T(z_2) = 2(m+1) L_{m+n}$$

$$\Rightarrow [L_m, L_n] = -(m+n+2) L_{m+n} + 2(m+1) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m,-n}$$

Thus the Virasoro algebra for closed string is given by,

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m,-n}$$

Even $\tilde{L}_m$ satisfies the same algebra but replacing c by $\tilde{c}$.

$$[\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{m+n} + \frac{\tilde{c}}{12} m(m^2-1) \delta_{m,-n}$$

Since $T(z_1) \tilde{T}(\bar{z}_2)$ has no singularities as $z_1 \rightarrow z_2$, we have

$$[L_m, \tilde{L}_n] = 0$$

Relation b/w Commutators & OPEs

The fact that the singularities of OPE's are the ones which contribute to the commutation relations should be obvious by now. To generalize this consider, a general conformal transformation given by $z \rightarrow z + \epsilon \mathcal{V}(z)$ & $\bar{z} \rightarrow \bar{z} + \epsilon \mathcal{V}^*(\bar{z})$.

The conserved currents are given by

$$Q = \frac{-1}{2\pi} \oint_C \mathcal{T}(z) dz \quad \& \quad \tilde{Q} = \frac{-1}{2\pi} \oint_C \tilde{\mathcal{T}}(\bar{z}) d\bar{z}$$

where $\mathcal{T}(z) = i\mathcal{V}(z)T(z)$ & $\tilde{\mathcal{T}}(\bar{z}) = i\mathcal{V}^*(\bar{z})\tilde{T}(\bar{z})$

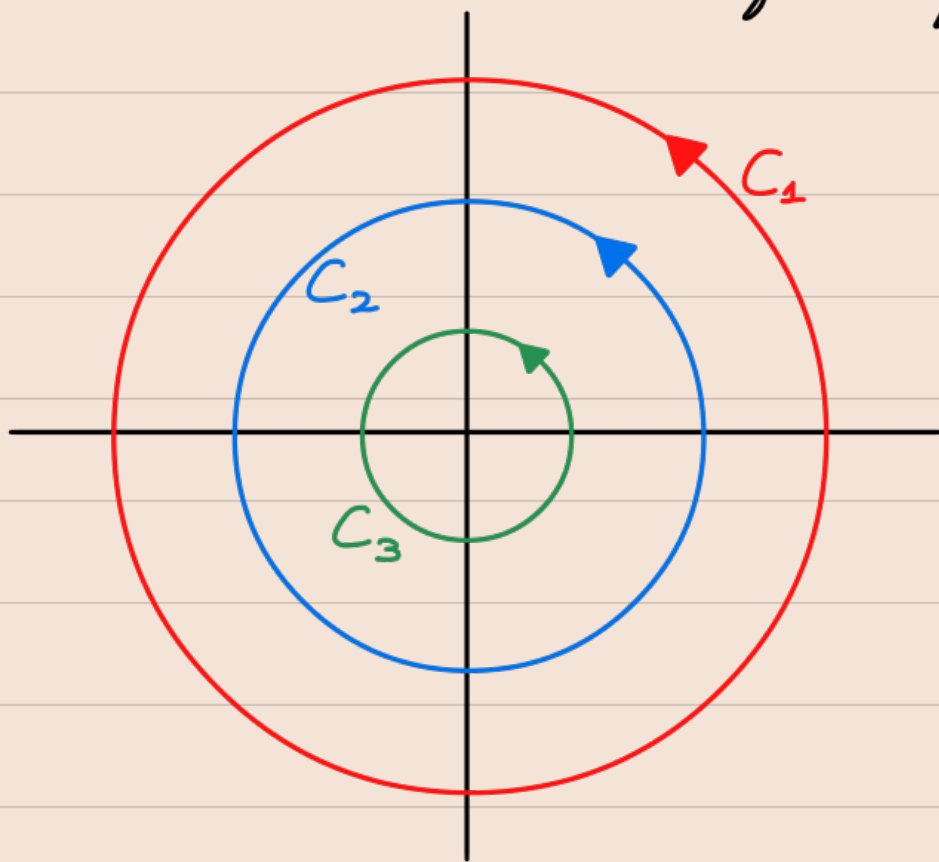
Now, we wish to find out the commutator $[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1$. This is also an operator eqⁿ inside a radially ordered correlation brackets. Thus same like before we shall choose three contours, to find the above eqⁿ.

$$\rightarrow [Q_1, Q_2] = Q_1(C_1)Q_2(C_2) - Q_2(C_2)Q_1(C_3)$$

$$Q_1(C_1) = - \oint_{C_1} \frac{dz_1}{2\pi} J_1(z_1), \quad Q_2(C_2) = - \oint_{C_2} \frac{dz_2}{2\pi} J_2(z_2)$$

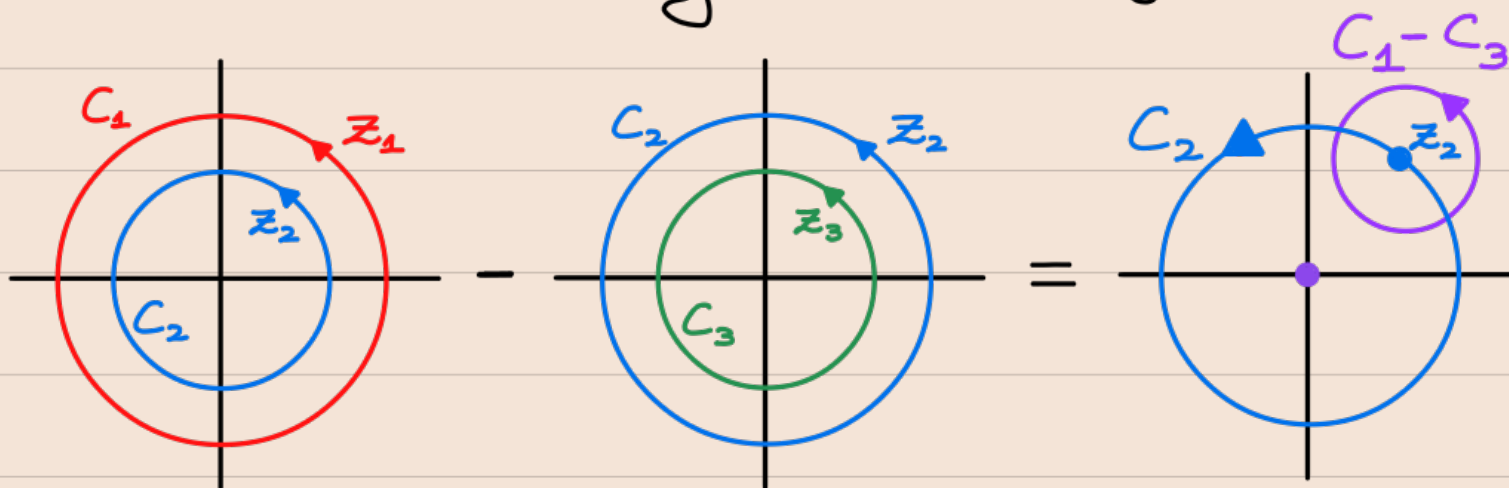
$$\& \quad Q_1(C_3) = - \oint_{C_3} \frac{dz_3}{2\pi} J_1(z_3)$$

The contours C_1, C_2, C_3 are of the form,



$$\rightarrow [Q_1, Q_2] = \oint_{C_2} \frac{dz_2}{2\pi} \left(\oint_{C_1} \frac{dz_1}{2\pi} J_1(z_1) J_2(z_2) - \oint_{C_3} \frac{dz_3}{2\pi} J_2(z_2) J_1(z_3) \right)$$

Now, Since T_1 is holomorphic, Q_1 's value is independent of the contour. So by some way what we did before we can replace the inner integrals by single contour integration, where we fix the z_2 value during inner integration.



$$\Rightarrow [Q_1, Q_2] = - \oint_{C_2} \frac{dz_2}{2\pi} \text{Res}_{z_1 \rightarrow z_2} \underbrace{[T_1(z_1) T_2(z_2)]}_{\text{OPE}}$$

Thus, for conserved charges, we get the commutators to depend on OPEs of currents.

Similarly for the anti-holomorphic part,

$$[\tilde{Q}_1, \tilde{Q}_2] = - \oint_{C_2} \frac{d\bar{z}_2}{2\pi} \overline{\text{Res}}_{\bar{z}_1 \rightarrow \bar{z}_2} \underbrace{[\tilde{T}_1(\bar{z}_1) \tilde{T}_2(\bar{z}_2)]}_{\text{OPE}}$$

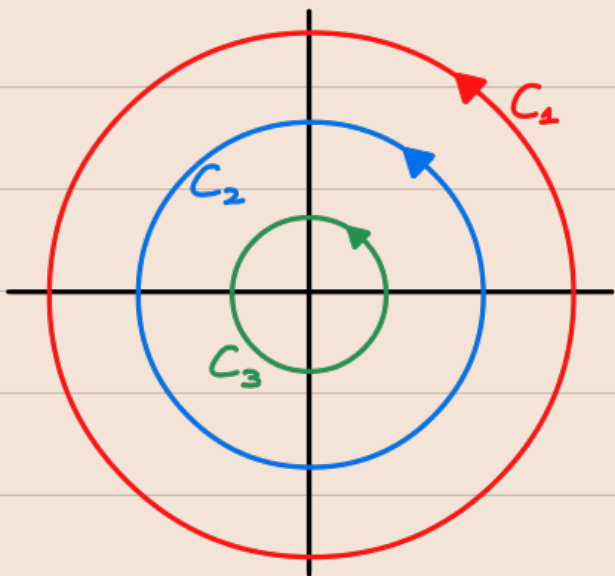
But observe that in the entire derivation we could have replaced Q_2 by some general operator ' $\hat{\Theta}$ ' which is got by the contour integration of some local operator $\hat{\Theta}(z, \bar{z})$. But this operator ' $\hat{\Theta}$ ' depends on the contour of integration. Let,

$$Q(C_1) = - \oint_{C_1} \frac{dz_1}{2\pi} J(z_1) , \quad Q(C_3) = - \oint_{C_3} \frac{dz_3}{2\pi} J(z_3)$$

$$\& \hat{\Theta}(C) = \oint_C \frac{dz}{2\pi i} \hat{\Theta}(z, \bar{z})$$

Now, Q is independent of Contour as the current J is holomorphic. But consider,

$$\rightarrow [Q, \hat{\Theta}](C) = Q(C_1) \hat{\Theta}(C) - \hat{\Theta}(C) Q(C_3)$$

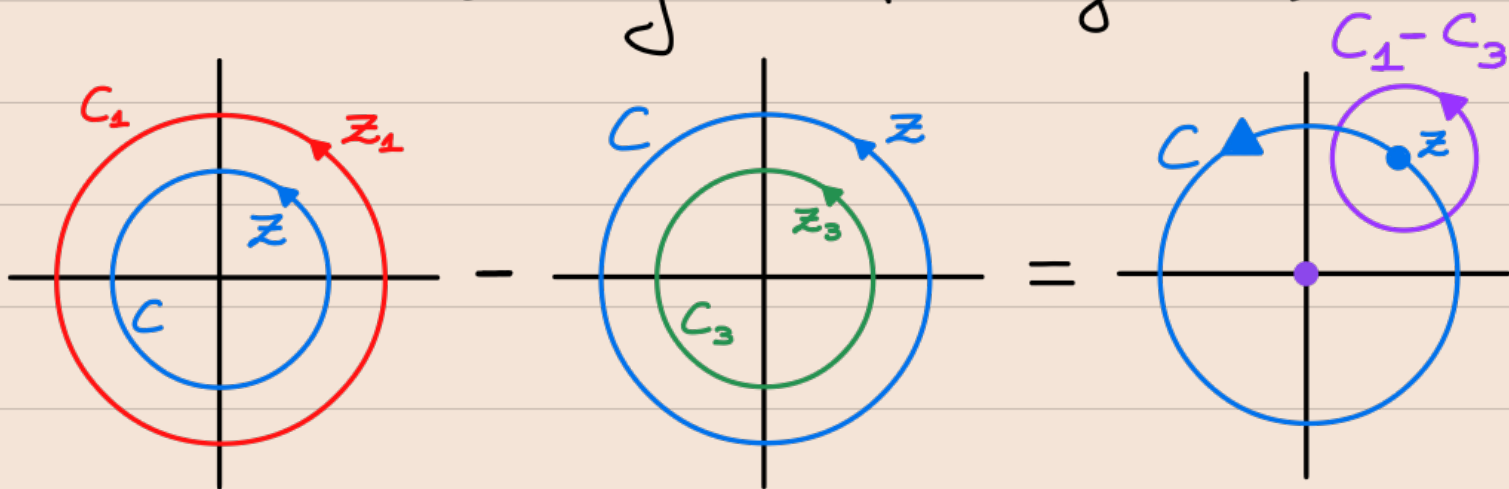


This is needed because the above eqⁿ is operator eqⁿ living inside radial ordered correlation brackets.

Thus,

$$[Q, \hat{\Theta}](C) = - \oint_C \frac{dz}{2\pi i} \left(\oint_{C_1} \frac{dz_1}{2\pi} T(z_1) \hat{\Theta}(z, \bar{z}) - \oint_{C_3} \frac{dz_3}{2\pi} \hat{\Theta}(z, \bar{z}) T(z_3) \right)$$

Since T is holomorphic, Q 's value will not depend on the contour. So, we can replace the inner integrals by a single contour integration, where we fix the z value during inner integration.



$$\Rightarrow [Q, \hat{\Theta}](C) = - \oint_C \frac{dz}{2\pi i} \left(\oint_{C_1 - C_3} \frac{dz_1}{2\pi i} T(z_1) \hat{\Theta}(z, \bar{z}) \right)$$

$$\Rightarrow [Q, \hat{\Theta}](C) = - \oint_C \frac{dz}{2\pi} \text{Res}_{z_1 \rightarrow z} \underbrace{[T(z_1) \hat{\Theta}(z, \bar{z})]}_{\text{OPE}}$$

$$\begin{aligned} \Rightarrow \oint_C \frac{dz}{2\pi i} [Q, \hat{\mathcal{O}}(z, \bar{z})] \\ = - \oint_C \frac{dz}{2\pi} \operatorname{Res}_{z_1 \rightarrow z} [\mathcal{T}(z_1) \hat{\mathcal{O}}(z, \bar{z})] \end{aligned}$$

$$\Rightarrow [Q, \hat{\mathcal{O}}(z, \bar{z})] = -i \operatorname{Res}_{z_1 \rightarrow z} [\mathcal{T}(z_1) \hat{\mathcal{O}}(z, \bar{z})]$$

In the exact same way, the commutator of the local operator $\hat{\mathcal{O}}(z, \bar{z})$ with charge $\tilde{Q}$,

$$[\tilde{Q}, \hat{\mathcal{O}}(z, \bar{z})] = -i \operatorname{Res}_{\bar{z}_1 \rightarrow \bar{z}} [\tilde{\mathcal{T}}(\bar{z}_1) \hat{\mathcal{O}}(z, \bar{z})]$$

$$\Rightarrow [Q, \hat{\mathcal{O}}(z, \bar{z})] + [\tilde{Q}, \hat{\mathcal{O}}(z, \bar{z})]$$

$$= -i \left(\operatorname{Res}_{z_1 \rightarrow z_2} [\mathcal{T}(z_1) \hat{\mathcal{O}}(z, \bar{z})] + \operatorname{Res}_{\bar{z}_1 \rightarrow \bar{z}_2} [\tilde{\mathcal{T}}(\bar{z}_1) \hat{\mathcal{O}}(z, \bar{z})] \right)$$

$$\Rightarrow [Q + \tilde{Q}, \hat{\mathcal{O}}(z, \bar{z})]$$

$$= \left(\operatorname{Res}_{z_1 \rightarrow z_2} [\mathcal{V}(z_1) \mathcal{T}(z_1) \hat{\mathcal{O}}(z, \bar{z})] + \operatorname{Res}_{\bar{z}_1 \rightarrow \bar{z}_2} [\mathcal{V}^*(\bar{z}_1) \tilde{\mathcal{T}}(\bar{z}_1) \hat{\mathcal{O}}(z, \bar{z})] \right)$$

But by ward identity,

$$-\frac{\delta \hat{\mathcal{O}}(\omega, \bar{\omega})}{\epsilon} = \left(\text{Res}_{\substack{z \rightarrow \omega}} [\mathcal{V}(z) T(z) \hat{\mathcal{O}}(\omega, \bar{\omega})] + \overline{\text{Res}}_{\substack{\bar{z} \rightarrow \bar{\omega}}} [\mathcal{V}^*(\bar{z}) \bar{T}(\bar{z}) \hat{\mathcal{O}}(\omega, \bar{\omega})] \right)$$

$$\rightarrow \boxed{[Q + \tilde{Q}, \hat{\mathcal{O}}(z, \bar{z})] = -\frac{\delta \hat{\mathcal{O}}(z, \bar{z})}{\epsilon}}$$

where $\delta \hat{\mathcal{O}}(z, \bar{z}) = \hat{\mathcal{O}}'(z, \bar{z}) - \hat{\mathcal{O}}(z, \bar{z})$ is the local change in $\hat{\mathcal{O}}(z, \bar{z})$ under the conformal transformation $\delta z = \epsilon \mathcal{V}(z) \neq \delta \bar{z} = \epsilon \mathcal{V}^*(\bar{z})$

Thus it is the conserved charges that generate the transformation of any local operator.

They are given in terms of Virasoro generators as,

$$Q = \sum_{n=-\infty}^{\infty} \mathcal{V}_n L_n \quad \text{where} \quad \mathcal{V}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}(z) dz}{z^{n+2}}$$

$$\tilde{Q} = \sum_{n=-\infty}^{\infty} \tilde{\mathcal{V}}_n \tilde{L}_n \quad \text{where} \quad \tilde{\mathcal{V}}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}^*(\bar{z}) d\bar{z}}{\bar{z}^{n+2}}$$

For Translations in $(z, \bar{z})$ plane, we need $\mathcal{V}(z) = z_0$, $\mathcal{V}^*(\bar{z}) = \bar{z}_0$. Then,

$$\mathcal{V}(z) = z_0 z^{(-1+1)} \rightarrow \mathcal{V}_{-1} = z_0 \rightarrow \mathcal{V}_{n, n \neq -1} = 0$$

$$\mathcal{V}^*(\bar{z}) = \bar{z}_0 \bar{z}^{(-1+1)} \rightarrow \tilde{\mathcal{V}}_{-1} = \bar{z}_0 \rightarrow \tilde{\mathcal{V}}_{n, n \neq -1} = 0$$

$\Rightarrow z_0 L_{-1}$ generates translations $z \rightarrow z + \epsilon z_0$

& $\bar{z}_0 \tilde{L}_{-1}$ generates translations $\bar{z} \rightarrow \bar{z} + \epsilon \bar{z}_0$

$$\Rightarrow [z_0 L_{-1} + \bar{z}_0 \tilde{L}_{-1}, \hat{\mathcal{O}}(z, \bar{z})] = -\frac{\partial \hat{\mathcal{O}}(z, \bar{z})}{\epsilon}$$

Any general local operator $\hat{\mathcal{O}}(z, \bar{z})$ will transform passively under translations as,

$$\Rightarrow \partial \hat{\mathcal{O}} = -\epsilon [z_0 \partial_z \hat{\mathcal{O}}(z, \bar{z}) + \bar{z}_0 \partial_{\bar{z}} \hat{\mathcal{O}}(z, \bar{z})]$$

$$\begin{aligned} \rightarrow [z_0 L_{-1} + \bar{z}_0 \tilde{L}_{-1}, \hat{\mathcal{O}}(z, \bar{z})] \\ = z_0 \partial \hat{\mathcal{O}}(z, \bar{z}) + \bar{z}_0 \bar{\partial} \hat{\mathcal{O}}(z, \bar{z}) \end{aligned}$$

But Since $T(z)$ & $\tilde{T}(\bar{z})$ are independent,

$$\Rightarrow \underline{[L_{-1}, \hat{\mathcal{O}}(z, \bar{z})] = \partial \hat{\mathcal{O}}(z, \bar{z})}$$

$$\& \underline{[\tilde{L}_{-1}, \hat{\mathcal{O}}(z, \bar{z})] = \bar{\partial} \hat{\mathcal{O}}(z, \bar{z})}$$

For rotations in $(z, \bar{z})$ plane, we need

$$z \rightarrow e^{i\epsilon} z \quad \& \quad \bar{z} \rightarrow e^{-i\epsilon} \bar{z}$$

$$\rightarrow \partial z = i\epsilon z \quad \partial \bar{z} = -i\epsilon \bar{z}$$

$$\rightarrow \mathcal{V}(z) = iz \quad \& \quad \mathcal{V}^*(\bar{z}) = -i\bar{z}$$

$$\rightarrow \mathcal{V}_0 = i \quad \& \quad \tilde{\mathcal{V}}_0 = -i$$

$$\text{Thus } Q = iL_0 \quad \& \quad \tilde{Q} = -i\tilde{L}_0$$

$$\rightarrow [L_0 - \tilde{L}_0, \hat{\mathcal{O}}(z, \bar{z})] = -\frac{\partial \hat{\mathcal{O}}(z, \bar{z})}{i\epsilon}$$

If $\hat{A}(z, \bar{z})$ is an eigen state under rotations, then under $z \rightarrow z' = \mathcal{U}_\epsilon z$ and $\bar{z} \rightarrow \bar{z}' = \mathcal{U}_\epsilon^* \bar{z}$,

$$\hat{A}'(z', \bar{z}') = \mathcal{U}_\epsilon^{-h} \mathcal{U}_\epsilon^{*\tilde{h}} \hat{A}(z, \bar{z})$$

where $(h, \tilde{h})$ are weights of $\hat{A}(z, \bar{z})$.

For pure infinitesimal rotations,

$$\begin{aligned} \partial \hat{A}(z, \bar{z}) &= -i\epsilon z \partial \hat{A}(z, \bar{z}) + i\epsilon \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) \\ &\quad - i\epsilon(h - \tilde{h}) \hat{A}(z, \bar{z}) \end{aligned}$$

$\rightarrow$

$$\begin{aligned} [L_0 - \tilde{L}_0, \hat{A}(z, \bar{z})] &= z \partial \hat{A}(z, \bar{z}) - \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) \\ &\quad + (h - \tilde{h}) \hat{A}(z, \bar{z}) \end{aligned}$$

$$\rightarrow [L_0, \hat{A}(z, \bar{z})] = z \partial \hat{A}(z, \bar{z}) + \hbar \hat{A}(z, \bar{z})$$

&

$$[\tilde{L}_0, \hat{A}(z, \bar{z})] = \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) + \hbar \hat{A}(z, \bar{z})$$

Since, $[L_{-1}, \hat{A}(z, \bar{z})] = \partial \hat{A}(z, \bar{z})$ and

$$[\tilde{L}_{-1}, \hat{A}(z, \bar{z})] = \bar{\partial} \hat{A}(z, \bar{z})$$

$$\rightarrow [L_0 - z L_{-1}, \hat{A}(z, \bar{z})] = \hbar \hat{A}(z, \bar{z})$$

$$\& \quad \underline{[\tilde{L}_0 - \bar{z} \tilde{L}_{-1}, \hat{A}(z, \bar{z})] = \hbar \hat{A}(z, \bar{z})}$$

For dilations in $(z, \bar{z})$ plane, we need

$$z \rightarrow z + \epsilon z \quad \& \quad \bar{z} \rightarrow \bar{z} + \epsilon \bar{z}$$

$$\rightarrow \partial z = \epsilon z \quad \partial \bar{z} = \epsilon \bar{z}$$

$$\rightarrow \mathcal{V}(z) = z \quad \& \quad \mathcal{V}^*(\bar{z}) = \bar{z}$$

$$\rightarrow \mathcal{V}_0 = 1 \quad \& \quad \tilde{\mathcal{V}}_0 = 1$$

Thus $Q = L_0$ & $\tilde{Q} = \tilde{L}_0$

& $\hat{D} = \hat{L}_0 + \hat{\tilde{L}}_0$ is the dilation operator

$$\rightarrow [L_0 + \tilde{L}_0, \hat{\mathcal{O}}(z, \bar{z})] = -\frac{\partial \hat{\mathcal{O}}(z, \bar{z})}{\epsilon}$$

If $\hat{A}(z, \bar{z})$ is an eigen state under dilation, then under $z \rightarrow z' = \zeta z$ and $\bar{z} \rightarrow \bar{z}' = \zeta^* \bar{z}$, $\hat{A}'(z', \bar{z}') = \zeta^{-h} \zeta^{*\bar{h}} \hat{A}(z, \bar{z})$ where $(h, \bar{h})$ are weights of $\hat{A}(z, \bar{z})$. Then, for pure infinitesimal dilation,

$$\delta \hat{A}(z, \bar{z}) = -\epsilon (h + \bar{h}) \hat{A}(z, \bar{z}) - \epsilon [z \partial \hat{A}(z, \bar{z}) + \bar{z} \bar{\partial} \hat{A}(z, \bar{z})]$$

$$\rightarrow [L_0 + \tilde{L}_0, \hat{A}(z, \bar{z})] = z \partial \hat{A}(z, \bar{z}) + \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) + (h + \bar{h}) \hat{A}(z, \bar{z})$$

$$\rightarrow [L_0, \hat{A}(z, \bar{z})] = z \partial \hat{A}(z, \bar{z}) + h \hat{A}(z, \bar{z})$$

$$\& [\tilde{L}_0, \hat{A}(z, \bar{z})] = \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) + \bar{h} \hat{A}(z, \bar{z})$$

$\Rightarrow i(L_0 - \tilde{L}_0)$ generates Rotations

$\& L_0 + \tilde{L}_0$ generates Dilations

Thus, all possible conformal transformations $\delta z = \epsilon \mathcal{V}(z)$ $\&$ $\delta \bar{z} = \epsilon \mathcal{V}^*(\bar{z})$ gives rise to an infinite set of Conserved charges $Q = \sum_{m=-\infty}^{\infty} \mathcal{V}_m L_m$ $\&$ $\tilde{Q} = \sum_{m=-\infty}^{\infty} \tilde{\mathcal{V}}_m \tilde{L}_m$, where the Virasoro generators themselves are in an infinite number.

CFT has an infinite generators that act on the Hilbert Space. This space consists of the states obtained by the radial quantization of the theory.

$sl(2, \mathbb{C})$ closed algebra

Consider $\hat{L}_0$ & $\hat{\bar{L}}_0$.

$$[\hat{L}_0, \hat{L}_n] = -n\hat{L}_n \quad \& \quad [\hat{\bar{L}}_0, \hat{\bar{L}}_n] = -n\hat{\bar{L}}_n$$
$$\rightarrow \hat{L}_0 \hat{L}_n = \hat{L}_n (\hat{L}_0 - n), \quad \hat{\bar{L}}_0 \hat{\bar{L}}_n = \hat{\bar{L}}_n (\hat{\bar{L}}_0 - n)$$

Thus $\hat{L}_n$ ($n > 0$) decreases the eigen value of $\hat{L}_0$ by $-n$ & $\hat{L}_{-n}$ ($n > 0$) increases the eigen value of $\hat{L}_0$ by $+n$. Same statement holds for anti-holomorphic generators.

Now, consider the generators given by, $\hat{L}_{\pm 1}$ & $\hat{L}_0$. They form a $sl(2, \mathbb{C})$ algebra.

$$\left. \begin{aligned} [\hat{L}_0, \hat{L}_1] &= -\hat{L}_1 \\ [\hat{L}_1, \hat{L}_{-1}] &= 2\hat{L}_0 \\ [\hat{L}_{-1}, \hat{L}_0] &= +\hat{L}_{-1} \end{aligned} \right\} \begin{aligned} \left[\hat{L}_0, \frac{\hat{L}_1}{\sqrt{2}} \right] &= -\frac{\hat{L}_1}{\sqrt{2}} \\ \left[\frac{\hat{L}_1}{\sqrt{2}}, \frac{\hat{L}_{-1}}{\sqrt{2}} \right] &= \hat{L}_0 \\ \left[\frac{\hat{L}_{-1}}{\sqrt{2}}, \hat{L}_0 \right] &= +\frac{\hat{L}_{-1}}{\sqrt{2}} \end{aligned}$$

Thus $\hat{L}_0, \frac{\pm \hat{L}_{\pm 1}}{\sqrt{2}}$ form a closed $sl(2, \mathbb{C})$ algebra. And it doesn't involve central charge. This is true for any CFT with any $c \neq \bar{c}$.

Similarly, $\hat{\bar{L}}_0, \frac{\pm \hat{\bar{L}}_{\pm 1}}{\sqrt{2}}$ form $sl(2, \mathbb{C})$ algebra.

Now, $\hat{L}_{-1}$ generates translations: $z \rightarrow z + z_0$
 $\hat{L}_0$ generates rotⁿ & dilations: $z \rightarrow \mathbb{C}_3 z$
 $\hat{L}_{+1}$ generates $z \rightarrow z + \epsilon z^2$: $z \rightarrow \frac{1}{1 - \epsilon z}$

Thus $\hat{L}_0, \pm \hat{L}_{\pm 1}$ together generates the general Möbius transformations. $z \rightarrow \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$ & $ad - bc \neq 0$. This is a repⁿ of $GL(2, \mathbb{C})$ group.

Commutators of Holomorphic & Anti-Holomorphic Primary operators with Virasoro Generators

Consider a holomorphic tensor operator $\hat{A}(z)$ with weight $(h, 0)$. So under a general conformal transformation $z \rightarrow z' = f(z)$, the operator transforms as, $\hat{A}'(z') = [\partial f(z)]^{-h} \hat{A}(z)$

We can do a Laurent expansion of a holomorphic tensor field as,

$$\hat{A}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{A}_n}{z^{n+h}}$$

where,

$$\hat{A}_n = \oint_C \frac{dz}{2\pi i} z^{n+h-1} \hat{A}(z)$$

'C' is any contour enclosing origin.

So under a general infinitesimal conformal transformation $z \rightarrow z + \epsilon \mathcal{V}(z)$, the conserved charges are,

$$Q = \sum_{n=-\infty}^{\infty} \mathcal{V}_n L_n$$

where $\mathcal{V}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}(z) dz}{z^{n+2}}$

The commutator relation b/w the charge 'Q' with $\hat{A}(z)$ is given by,

$$\rightarrow [Q, \hat{A}(z)] = -\frac{\partial \hat{A}(z)}{\epsilon}$$

The primary operator $\hat{A}$ changes as,

$$\rightarrow \partial \hat{A}(z) = -\epsilon \left[\varphi(z) \partial \hat{A}(z) + \hbar \varphi'(z) \hat{A}(z) \right]$$

$$\rightarrow [Q, \hat{A}(z)] = \varphi(z) \partial \hat{A}(z) + \hbar \varphi'(z) \hat{A}(z)$$

$$\text{Use } Q = \sum_{n=-\infty}^{\infty} \varphi_n L_n, \quad \varphi(z) = \sum_{n=-\infty}^{\infty} \varphi_n z^{n+1}$$

$$\text{and } \varphi'(z) = \sum_{n=-\infty}^{\infty} (n+1) \varphi_n z^n$$

$$\begin{aligned} \rightarrow \sum_{n=-\infty}^{\infty} \varphi_n [L_n, \hat{A}(z)] &= \sum_{n=-\infty}^{\infty} \varphi_n z^{n+1} \partial \hat{A}(z) \\ &\quad + \hbar \sum_{n=-\infty}^{\infty} (n+1) \varphi_n z^n \hat{A}(z) \end{aligned}$$

$$\rightarrow \underline{[L_n, \hat{A}(z)] = z^{n+1} \partial \hat{A}(z) + \hbar (n+1) z^n \hat{A}(z)}$$

$$\text{But } \hat{A}(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{A}_m}{z^{m+h}}$$

$$\& \partial \hat{A}(z, \bar{z}) = - \sum_{m=-\infty}^{\infty} (m+h) \frac{\hat{A}_m}{z^{m+h+1}}$$

$$\begin{aligned} \rightarrow \sum_{m=-\infty}^{\infty} \frac{[L_n, \hat{A}_m]}{z^{m+h}} &= - \sum_{m=-\infty}^{\infty} (m+h) \frac{\hat{A}_m z^{n+1}}{z^{m+h+1}} \\ &\quad + \sum_{m=-\infty}^{\infty} \frac{h(n+1) \hat{A}_m z^n}{z^{m+h}} \end{aligned}$$

$$\begin{aligned} \rightarrow \sum_{m=-\infty}^{\infty} \frac{[L_n, \hat{A}_m]}{z^{m+h}} &= - \sum_{m=-\infty}^{\infty} (m+h) \frac{\hat{A}_m z^n}{z^{m+h}} \\ &\quad + \sum_{m=-\infty}^{\infty} \frac{h(n+1) \hat{A}_m z^n}{z^{m+h}} \end{aligned}$$

$$\rightarrow \sum_{m=-\infty}^{\infty} \frac{[L_n, \hat{A}_m]}{z^{m+h}} = \sum_{m=-\infty}^{\infty} \frac{(hn-m) \hat{A}_m}{z^{m-n+h}}$$



 Put $m-n=k$

$$\rightarrow \sum_{m=-\infty}^{\infty} \frac{[L_n, \hat{A}_m]}{z^{m+h}} = \sum_{k=-\infty}^{\infty} \frac{(hn-n-k) \hat{A}_{n+k}}{z^{k+h}}$$

replace $k \rightarrow m$

$$\rightarrow [L_n, \hat{A}_m] = (hn-n-m) \hat{A}_{n+m}$$

$$\rightarrow [L_n, \hat{A}_m] = [(h-1)n-m] \hat{A}_{n+m}$$

This is satisfied by all holomorphic tensor operators.

Similarly, the anti-holomorphic tensor operators $\hat{A}(\bar{z})$ with weights $(0, \tilde{h})$,

$$\hat{A}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\hat{A}_m}{\bar{z}^{m+\tilde{h}}}, \quad \hat{A}_n = \oint_C \frac{d\bar{z}}{2\pi i} z^{n+\tilde{h}-1} \hat{A}(\bar{z})$$

they satisfy,

$$\rightarrow [\tilde{L}_n, \hat{A}_m] = [(\tilde{h}-1)n-m] \hat{A}_{n+m}$$

For the holomorphic / anti-holomorphic operators that are eigen states of the rot^n / dilation, we get the commutation to be neat only with the L_0 generator

$$[L_0, \hat{A}_m] = -m \hat{A}_m \Rightarrow L_0 \hat{A}_m = \hat{A}_m (L_0 - m)$$

$$\& [\tilde{L}_0, \hat{\tilde{A}}_m] = -m \hat{\tilde{A}}_m \Rightarrow \tilde{L}_0 \hat{\tilde{A}}_m = \hat{\tilde{A}}_m (\tilde{L}_0 - m)$$

Exercise 2.16 - Polchinski

To show that a $(1,1)$ tensor operator transforms as a total derivative under any conformal transform.

We previously got

$$[L_n, \hat{A}(z)] = z^{n+1} \partial \hat{A}(z) + \hbar(n+1) z^n \hat{A}(z)$$

$$\& [\tilde{L}_n, \hat{\tilde{A}}(\bar{z})] = \bar{z}^{n+1} \bar{\partial} \hat{\tilde{A}}(\bar{z}) + \hbar(n+1) \bar{z}^n \hat{\tilde{A}}(\bar{z})$$

But these statements are true for any general tensor operator $\hat{A}(z, \bar{z})$ whose weights are $(h, \tilde{h})$. In that case,

$$[L_n, \hat{A}(z, \bar{z})] = z^{n+1} \partial \hat{A}(z, \bar{z}) + h(n+1) z^n \hat{A}(z, \bar{z})$$

$$\& [\tilde{L}_n, \hat{A}(z, \bar{z})] = \bar{z}^{n+1} \bar{\partial} \hat{A}(z, \bar{z}) + \tilde{h}(n+1) \bar{z}^n \hat{A}(z, \bar{z})$$

If $(h, \tilde{h}) = (1, 1)$, then we get

$$[L_n, \hat{A}(z, \bar{z})] = \partial (z^{n+1} \hat{A}(z, \bar{z}))$$

$$\& [\tilde{L}_n, \hat{A}(z, \bar{z})] = \bar{\partial} (\bar{z}^{n+1} \hat{A}(z, \bar{z}))$$

Under the conformal transformation $\partial z = \epsilon \mathcal{V}(z)$
 $\& \partial \bar{z} = \epsilon \mathcal{V}^*(\bar{z})$ the conserved charges that generate the transformation of any local operator is given in terms of Virasoro generators

$$Q = \sum_{n=-\infty}^{\infty} \mathcal{V}_n L_n \quad \text{where} \quad \mathcal{V}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}(z) dz}{z^{n+2}}$$

$$\tilde{Q} = \sum_{n=-\infty}^{\infty} \tilde{\mathcal{V}}_n \tilde{L}_n \quad \text{where} \quad \tilde{\mathcal{V}}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}^*(\bar{z}) d\bar{z}}{\bar{z}^{n+2}}$$

By Ward Identity,

$$\rightarrow [Q + \tilde{Q}, \hat{A}(z, \bar{z})] = -\frac{\partial \hat{A}(z, \bar{z})}{\epsilon}$$

where $\partial \hat{A}(z, \bar{z}) = \hat{A}'(z, \bar{z}) - \hat{A}(z, \bar{z})$ is the local change in $\hat{A}(z, \bar{z})$.

$$\Rightarrow [Q + \tilde{Q}, \hat{A}(z, \bar{z})]$$

$$= \sum_{n=-\infty}^{\infty} [(\mathcal{V}_n L_n + \tilde{\mathcal{V}}_n \tilde{L}_n), \hat{A}(z, \bar{z})]$$

$$= \sum_{n=-\infty}^{\infty} (\mathcal{V}_n [L_n, \hat{A}(z, \bar{z})] + \tilde{\mathcal{V}}_n [\tilde{L}_n, \hat{A}(z, \bar{z})])$$

$$= \sum_{n=-\infty}^{\infty} (\mathcal{V}_n \partial(z^{n+1} \hat{A}(z, \bar{z})) + \tilde{\mathcal{V}}_n \bar{\partial}(\bar{z}^{n+1} \hat{A}(z, \bar{z})))$$

$$= \partial \left(\sum_{n=-\infty}^{\infty} \mathcal{V}_n z^{n+1} \hat{A}(z, \bar{z}) \right) + \bar{\partial} \left(\sum_{n=-\infty}^{\infty} \tilde{\mathcal{V}}_n \bar{z}^{n+1} \hat{A}(z, \bar{z}) \right)$$

$$\Rightarrow [Q + \tilde{Q}, \hat{A}(z, \bar{z})] = \partial(\mathcal{V}(z) \hat{A}(z, \bar{z})) + \bar{\partial}(\mathcal{V}^*(\bar{z}) \hat{A}(z, \bar{z}))$$

where,

$$\mathcal{V}(z) = \sum_{n=-\infty}^{\infty} \mathcal{V}_n z^{n+1} \quad \& \quad \mathcal{V}^*(\bar{z}) = \sum_{n=-\infty}^{\infty} \tilde{\mathcal{V}}_n \bar{z}^{n+1}$$

$$\Rightarrow [Q, \hat{A}(z, \bar{z})] = \partial(\psi(z) \hat{A}(z, \bar{z}))$$

$$\nabla [\tilde{Q}, \hat{A}(z, \bar{z})] = \bar{\partial}(\psi^*(\bar{z}) \hat{A}(z, \bar{z}))$$

$$\text{Thus } \frac{-\partial \hat{A}(z, \bar{z})}{\epsilon} = \partial(\psi(z) \hat{A}(z, \bar{z})) + \bar{\partial}(\psi^*(\bar{z}) \hat{A}(z, \bar{z}))$$

This is true only for a $(1, 1)$ tensor operator.

Representations of Virasoro Algebra

We previously saw that the Hamiltonian operator which generates evolution of 'T' is given by, $\hat{H} = \hat{L}_0 + \hat{\tilde{L}}_0 - \frac{c + \tilde{c}}{24}$

Thus physically ' $\hat{H}$ ' should correspond to the closed string energy.

Suppose we have an eigen state of operators $\hat{L}_0$ and $\hat{\tilde{L}}_0$ given by $|h, \tilde{h}\rangle$.

Then,

$$\hat{L}_0|h, \tilde{h}\rangle = h|h, \tilde{h}\rangle \neq \hat{\tilde{L}}_0|h, \tilde{h}\rangle = \tilde{h}|h, \tilde{h}\rangle$$

$$\Rightarrow \hat{H}|h, \tilde{h}\rangle = \left(h + \tilde{h} - \frac{c + \tilde{c}}{24}\right)|h, \tilde{h}\rangle$$

$$\Rightarrow E(h, \tilde{h}) = \langle h, \tilde{h} | \hat{H} | h, \tilde{h} \rangle = h + \tilde{h} - \frac{c + \tilde{c}}{24}$$

Now, the operators $\hat{L}_n$ & $\hat{\tilde{L}}_n$ change the Eigen states such that their energy will change by a defined unit $\pm n$.

$$\hat{L}_0 \hat{L}_n = \hat{L}_n (\hat{L}_0 - n), \quad \hat{\tilde{L}}_0 \hat{\tilde{L}}_n = \hat{\tilde{L}}_n (\hat{\tilde{L}}_0 - n)$$

$$\text{i.e. } \hat{H}(\hat{L}_n |h, \tilde{h}\rangle)$$

$$= \left(\hat{L}_0 \hat{L}_n + \hat{\tilde{L}}_0 \hat{\tilde{L}}_n - \frac{c + \tilde{c}}{24} \hat{L}_n \right) |h, \tilde{h}\rangle$$

$$= \left(\hat{L}_n (\hat{L}_0 - n) + \hat{L}_n \hat{\tilde{L}}_0 - \frac{c + \tilde{c}}{24} \hat{L}_n \right) |h, \tilde{h}\rangle$$

$$= \hat{L}_n \left(\hat{L}_0 + \hat{\tilde{L}}_0 - \frac{c + \tilde{c}}{24} - n \right) |h, \tilde{h}\rangle$$

$$= \left(h + \tilde{h} - \frac{c + \tilde{c}}{24} - n \right) \hat{L}_n |h, \tilde{h}\rangle$$

$$\Rightarrow \underline{\hat{H}(\hat{L}_n |h, \tilde{h}\rangle) = (E(h, \tilde{h}) - n)(\hat{L}_n |h, \tilde{h}\rangle)}$$

Similarly,

$$\underline{\hat{H}(\hat{\tilde{L}}_n |h, \tilde{h}\rangle) = (E(h, \tilde{h}) - n)(\hat{\tilde{L}}_n |h, \tilde{h}\rangle)}$$

Thus $\hat{L}_n$ & $\hat{\tilde{L}}_n$ are ladder operators. For $n > 0$, they lower the energy. And for $n < 0$, they raise the energy.

If the energy spectrum is to be bounded below, then there should be some state which is annihilated by all $\hat{L}_n$ & $\hat{\tilde{L}}_n$ for $n > 0$. Such states are called as the primary states, denoting it by $|\psi\rangle$.

$$\Rightarrow \hat{L}_n |\psi\rangle = 0 = \hat{\tilde{L}}_n |\psi\rangle \text{ for all } n > 0.$$

These are the states with lowest energy. So they represent ground state.

Representations of Virasoro algebra can be done by creating the descendants by acting creation operators on the primary states. This results in an infinite tower of states called as descendants.

$$\text{Level 0} \rightarrow |\psi\rangle$$

$$\text{Level 1} \rightarrow \hat{L}_{-1}|\psi\rangle$$

$$\text{Level 2} \rightarrow \hat{L}_{-1}^2|\psi\rangle \quad \hat{L}_{-2}|\psi\rangle$$

$$\vdots \quad \hat{L}_{-1}^3|\psi\rangle \quad \hat{L}_{-1}\hat{L}_{-2}|\psi\rangle \quad \hat{L}_{-3}|\psi\rangle$$

$$\hat{L}_{-1}^4|\psi\rangle \quad \hat{L}_{-1}\hat{L}_{-3}|\psi\rangle \quad \hat{L}_{-1}^2\hat{L}_{-2}|\psi\rangle \quad \hat{L}_{-2}^2|\psi\rangle \quad \hat{L}_{-4}|\psi\rangle$$

$\vdots$ So on

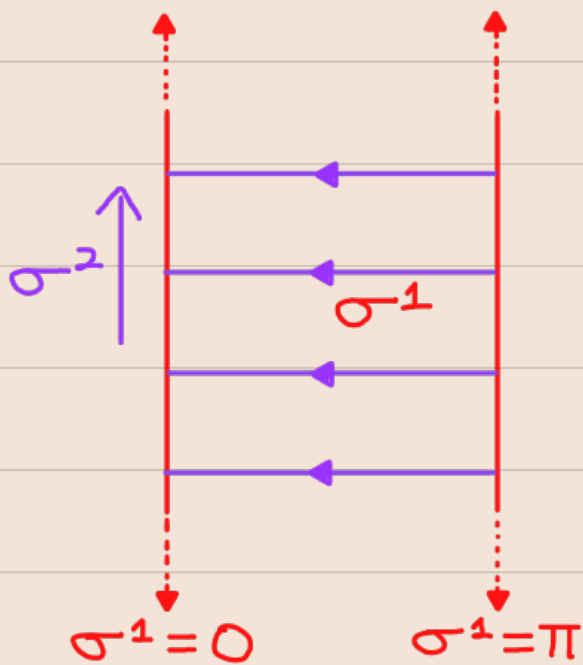
A primary state & their descendants form an irreducible repⁿ of Virasoro algebra.

Thus if we know the Spectrum of all the primary states and their descendants then the Spectrum of whole theory can be found.

Open String Virasoro Algebra

The open string world sheet can be thought of as an infinitely long strip whose Euclidean time coordinate $\sigma^2 = i\tau$ runs from $-\infty$ to $+\infty$, where we go from the bottom to top of the strip. Then the spatial σ^1 coordinate runs from $0 < \sigma^1 < \pi$. And the time coordinate $-\infty < \sigma^2 < +\infty$.

Infinite Strip World Sheet



We define the complex coordinates on the strip as $\omega = \sigma^1 + i\sigma^2$ & $\bar{\omega} = \sigma^1 - i\sigma^2$. So, ω & $\bar{\omega}$ define a point on the world sheet. By analytic continuation $\omega = \sigma^1 - \tau$ and $\bar{\omega} = \sigma^1 + \tau$. To match with the existing terminology of Holomorphic $\sim$ Left moving

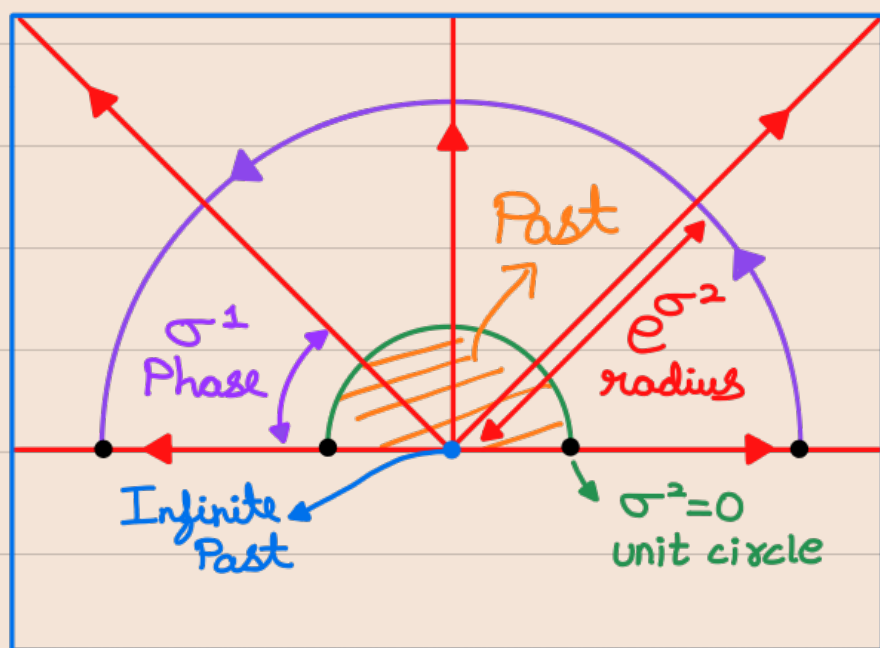
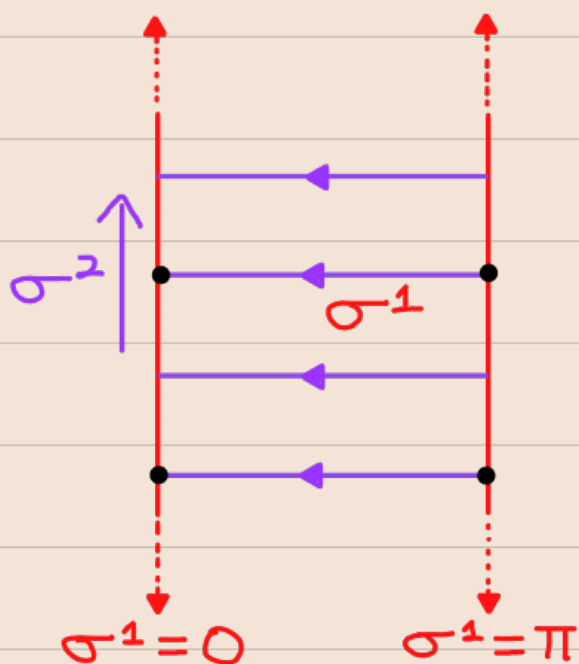
⌘ Anti-Holomorphic ~ Right moving, we should consider σ_1 to increase from right to left.

When we perform a conformal map of Strip to a plane by defining $z = -e^{-i\omega}$
 $\rightarrow z = -e^{(-i\sigma^1 + \sigma^2)} = e^{\sigma^2} [-\cos(\sigma^1) + i\sin(\sigma^1)]$

The radial distance in $(z, \bar{z})$ plane relates to the euclidean time by, $r = e^{\sigma^2}$. So $\sigma^2 = -\infty$ (Infinite past) is the origin. The Entire Strip is mapped to just the upper half of z -plane. The end points of the String is at $\text{Im}(z) = 0$.

Infinite Strip World Sheet

z -plane



Presence of boundary breaks translation invariance in the direction normal to the boundary. But there is still translational invariance in the direction tangential to the boundary.

Now, in euclidean coordinates $\sigma^a = (\sigma^1, \sigma^2)$,
Let $n^a(\sigma^1)$ give the unit normal, & let $t^a(\sigma^1)$ give the unit tangent at any σ^1 .

Since there is still translational invariance in the direction tangential to the boundary. ' $t^a T_{ab}$ ' is always conserved.

$$\rightarrow \underline{\partial^b (t^a T_{ab}) = 0}$$

To implement neumann boundary to the strip we don't want any current to flow out of the boundary.

$$\rightarrow \underline{\underline{n^a (t^b T_{ab}) = 0}}, \text{ at boundary}$$

So at $\sigma^1 = 0$, $n^a = (-1, 0)$, $t^a = (0, -1)$
& at $\sigma^1 = \pi$, $n^a = (1, 0)$, $t^a = (0, 1)$

at $\sigma^1 = 0$,

$$\rightarrow n^1 t^1 T_{11} + n^2 t^2 T_{22} + n^1 t^2 T_{12} + n^2 t^1 T_{21} = 0$$
$$\Rightarrow 0 + 0 + T_{12}(\sigma^1 = 0) + 0 = 0$$

Since $T_{ab} = T_{ba}$, $T_{12}(\sigma^1 = 0) = T_{21}(\sigma^1 = 0) = 0$

at $\sigma^1 = \pi$,

$$\rightarrow n^1 t^1 T_{11} + n^2 t^2 T_{22} + n^1 t^2 T_{12} + n^2 t^1 T_{21} = 0$$

Since $T_{ab} = T_{ba}$, $T_{12}(\sigma^1 = \pi) = T_{21}(\sigma^1 = \pi) = 0$

Thus, finally we get, $T_{12} = T_{21} = 0$ at both the boundary points, at $\sigma^1 = 0, \pi$.

This implies $T_{12} = T_{21} = 0$ at real line in the $\mathbb{Z}$ -plane. (at $\text{Im}(\mathbb{Z}) = 0$).

In complex $(\omega, \bar{\omega})$ coordinates,

$$T_{\omega\omega} = \frac{1}{4} (T_{11} - iT_{12} - iT_{21} - T_{22})$$

$$T_{\bar{\omega}\bar{\omega}} = \frac{1}{4} (T_{11} + iT_{12} + iT_{21} - T_{22})$$

$$T_{\omega\bar{\omega}} = \frac{1}{4} (T_{11} + iT_{12} - iT_{21} + T_{22})$$

$$T_{\bar{\omega}\omega} = \frac{1}{4} (T_{11} - iT_{12} + iT_{21} + T_{22})$$

at σ^1 boundary,

$$\begin{aligned} T_{\omega\omega} &= \frac{1}{4} (T_{11} - T_{22}) \\ T_{\bar{\omega}\bar{\omega}} &= \frac{1}{4} (T_{11} - T_{22}) \end{aligned} \quad \leftarrow \text{Equal}$$

$$T_{\omega\bar{\omega}} = \frac{1}{4} (T_{11} + T_{22}) = \frac{\text{Trace}(T)}{4}$$

$$T_{\bar{\omega}\omega} = \frac{1}{4} (T_{11} + T_{22}) = \frac{\text{Trace}(T)}{4}$$

Thus $T_{\omega\omega} = T_{\bar{\omega}\bar{\omega}}$ (when $\sigma^1 = 0, \pi$)

Energy Momentum Tensor in Plane is got by

$$\rightarrow T_{zz} = (\partial_\omega z)^{-2} \left[T_{\omega\omega} - \frac{c}{12} \{z, \omega\}_g \right]$$

The conformal map b/w cylinder $(\omega, \bar{\omega})$ & the plane is given by $z = -e^{-i\omega}$. Then the Schwarzian derivative is

$$\begin{aligned} \{z, \omega\}_g &= (\partial_\omega^3 z)(\partial_\omega z)^{-1} - \frac{3}{2}(\partial_\omega^2 z)^2(\partial_\omega z)^{-2} \\ &= (i)^3(i)^{-1} - \frac{3}{2}(i)^4(i)^{-2} = -1 + \frac{3}{2} = 1/2 \end{aligned}$$

$$\Rightarrow \{z, \omega\}_g = 1/2$$

$$\Rightarrow \underline{T_{zz} = \frac{-1}{z^2} \left[T_{\omega\omega} - \frac{c}{24} \right]}$$

$$\text{Similarly, } \underline{T_{\bar{z}\bar{z}} = \frac{-1}{\bar{z}^2} \left[T_{\bar{\omega}\bar{\omega}} - \frac{\tilde{c}}{24} \right]}$$

Now, $\sigma^1 = 0, \pi$ correspond to the line $z = \bar{z}$ in the plane. Since $T_{\omega\omega} = T_{\bar{\omega}\bar{\omega}}$, we get $[\text{Re}(z)]^2 (T_{zz} - T_{\bar{z}\bar{z}}) = \frac{c - \tilde{c}}{24}$.

But if we need no momentum flux in the boundary, we need $c = \bar{c}$ & $T_{zz} = T_{\bar{z}\bar{z}}$ at the line $\text{Im}(z) = 0$.

Thus open string theory has 1 central charge.

In the current set up, we have access only in the upper half of the z -plane. i.e. $\text{Im}(z) \geq 0$. We have $T_{zz}(z)$ & $T_{\bar{z}\bar{z}}(\bar{z})$ both to live independently in upper plane. But at $\text{Im}(z) = 0$, $z = \bar{z}$ we have $T_{zz}(z) = T_{\bar{z}\bar{z}}(\bar{z})$. This is the Gluing condition for Energy momentum tensor.

This is inconvenient as we have two chiral components (need not be equal) in the bulk ($\text{Im}(z) > 0$), but equal when $\text{Im}(z) = 0$. So, we can use the doubling trick.

"For $\text{Im}(z) < 0$, we shall define T_{zz} at ' z ' ($\text{Im}(z) < 0$), with the value of $T_{\bar{z}\bar{z}}$ at z^* ($\text{Im}(z^*) > 0$) which is at the mirror point above the real line."

Thus after the doubling trick, we shall work in full $\mathbb{C}$ plane, where we define $\bar{z} = z^*$, they are not independent any more. And thus we have only one Component of Energy momentum tensor $T(z)$ which is given by,

$$T(z) = \begin{cases} T_{zz}(z) & , \text{Im}(z) > 0 \\ T_{\bar{z}\bar{z}}(z^*) & , \text{Im}(z) < 0 \end{cases}$$

Thus $T(z)$ now is defined for the entire complex plane but we do not work with the anti holomorphic part of E.M tensor.

Now, there is no boundary condition for $T(z)$ & it is conserved. And it is holomorphic in the entire plane.

$$\bar{\partial} T(z) = 0 \text{ for all } \mathbb{C}$$

The consequence of the doubling trick is, now there is only one set of Virasoro generators defined in the entire z -plane.

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}$$

where $L_n = \oint_C \frac{dz}{2\pi i} z^{n+1} T(z)$

There is no independent set of $\tilde{L}_n$ for the Open strings.

The above L_n can be expanded using the original T_{zz} & $T_{\bar{z}\bar{z}}$ as,

$$L_n = \frac{1}{2\pi i} \left[\underbrace{\int_C}_{\substack{\text{upper} \\ \text{semi circle} \\ \text{anti-clockwise}}} dz z^{n+1} T_{zz} + \underbrace{\int_C}_{\substack{\text{upper} \\ \text{semi circle} \\ \text{clockwise}}} d\bar{z} \bar{z}^{n+1} T_{\bar{z}\bar{z}} \right]$$

The open string Virasoro algebra has the same form as before because of the doubling trick.

Now, using the doubling trick, we can also get E.M. tensor in ω -coordinates as,

$$\Rightarrow \underline{T_{\text{strip}}(\omega) = -\underset{\text{Plane}}{z^2} T(z) + \frac{c}{24}}$$

In terms of Virasoro generators,

$$\begin{aligned} T_{\text{strip}}(\omega) &= -z^2 \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} + \frac{c}{24} \\ &= -\sum_{m=-\infty}^{\infty} (-1)^m e^{im\omega} L_m + \frac{c}{24} \end{aligned}$$

$$\Rightarrow \underline{T_{\text{strip}}(\omega) = -\sum_{m=-\infty}^{\infty} T_m e^{im\omega}}$$

where $T_m = (-1)^m L_m - \frac{c\delta_{m0}}{24}$

Thus at $\sigma^2 = 0$, i.e. when Euclidean time is zero, $T_{\text{strip}}(\omega)$ is just the spatial fourier transform with T_m as fourier coefficients. When $\sigma^2 = 0$, $\omega = \bar{\omega}$.

$$\Rightarrow \underline{T_{\text{strip}}(\sigma^1, \sigma^2 = 0) = -\sum_{m=-\infty}^{\infty} T_m e^{im\sigma^1}}$$

We can invert the above relation to get T_m in terms of T_{cylinder} .

Now the physical strip correspond to the $0 < \sigma^1 < \pi$. Under doubling trick we define $T_{\text{strip}}(\omega)$ for $0 < \sigma^1 < 2\pi$, where the values of $T_{\text{strip}}(\omega)$ at $\pi < \sigma^1 < 2\pi$ are got by the image of those at $0 < \sigma^1 < \pi$. So,

$$\int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$= \int_0^{\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$+ \underbrace{\int_{\pi}^{2\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}}_{\text{Here we need to use image of } T_{\text{strip}} \text{ at } \sigma^1 \in [0, \pi]}$$

Here we need to use image of T_{strip} at $\sigma^1 \in [0, \pi]$

Substitute $u = 2\pi - \sigma^1$

$$\int_{\pi}^{2\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$= \int_0^{\pi} \frac{du}{2\pi} T_{\text{strip}}(u, \sigma^2=0) e^{-in(2\pi-u)}$$

replace $u \rightarrow \sigma^1$
(dummy variable)

$$\rightarrow \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$= \int_0^{\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$+ \int_0^{\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{in\sigma^1}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$= \int_0^{\pi} \frac{d\sigma^1}{\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) \cos(n\sigma^1)$$

But,

$$\int_0^{2\pi} \frac{d\sigma^1}{2\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) e^{-in\sigma^1}$$

$$= - \sum_{m=-\infty}^{\infty} T_m \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \underbrace{e^{im\sigma^1} e^{-in\sigma^1}}_{\delta_{mn}}$$

$$= -T_n$$

$$\rightarrow T_n = - \int_0^{\pi} \frac{d\sigma^1}{\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) \cos(n\sigma^1)$$

Thus, we get L_m 's, w.r.t T_{strip} as

$$L_m = (-1)^m \frac{c\alpha_{m0}}{24} - (-1)^m \int_0^{\pi} \frac{d\sigma^1}{\pi} T_{\text{strip}}(\sigma^1, \sigma^2=0) \cos(m\sigma^1)$$

Now, $T_{22} = -(T_{\omega\omega} + T_{\bar{\omega}\bar{\omega}})$

By doubling trick $T_{22} = -T_{\text{strip}}(\omega)$

The hamiltonian which generates σ^2 translation is

$$H = \int_0^\pi \frac{d\sigma^1}{\pi} T_{22}$$

$$\begin{aligned} L_0 &= \frac{c}{24} - \int_0^\pi \frac{d\sigma^1}{\pi} T(\sigma^1, \sigma^2=0) \\ &= \frac{c}{24} + \underbrace{\int_0^\pi \frac{d\sigma^1}{\pi} T_{22}(\sigma^2=0)}_H \end{aligned}$$

Thus the Hamiltonian of open string at $\sigma^2=0$ is,

$$H = L_0 - \frac{c}{24}$$

Thus even in open strings, the 0^{th} mode virasoro generators which generate rot^n & dilation in z -plane is what contributes to the Hamiltonian. But now, there are only one set of ' L_0 '.

Now, $H = -i\partial_\tau = \partial_{\sigma_2} = \bar{z}\partial_{\bar{z}}$ ($\bar{z}$ must appear)

But ' $\bar{z}\partial_{\bar{z}}$ ' is the $\bar{z}$ -dilation operator.
So, time evolution in world sheet is the radial evolution in $\bar{z}$ -Plane.

Now, L_n are conserved charges as they are independent of radial evolution in the Radially quantized CFT.

Thus L_n are the generators of Conformal transformations.

Under the infinitesimal conformal transformation $\bar{z} \rightarrow \bar{z} + \epsilon \mathcal{V}(\bar{z})$

The Noether current is $J(\bar{z}) = i\mathcal{V}(\bar{z})T(\bar{z})$

Then the corresponding charges are given by integrating the current around a contour.

$$Q = \frac{-1}{2\pi} \oint_C T(z) dz = \frac{1}{2\pi i} \oint_C \mathcal{V}(z) T(z) dz$$

If $\mathcal{V}(z) = z^{n+1}$ then, $Q \rightarrow L_n$

Thus the Virasoro Generators L_n are the conserved Noether charges for infinitesimal conformal transformation $z \rightarrow z + \epsilon z^{n+1}$

For a general $\mathcal{V}(z)$, expand it about the origin. Thus,

$$\mathcal{V}(z) = \sum_{n=-\infty}^{\infty} \mathcal{V}_n z^{n+1}, \quad \mathcal{V}_n = \frac{1}{2\pi i} \oint_C \frac{\mathcal{V}(z) dz}{z^{n+2}}$$

$$\begin{aligned} Q &= \frac{1}{2\pi i} \oint_C \mathcal{V}(z) T(z) dz \\ &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \mathcal{V}_n \oint_C z^{n+1} T(z) dz \end{aligned}$$

$$\rightarrow \boxed{Q = \sum_{n=-\infty}^{\infty} \mathcal{V}_n L_n}$$

Even in open string theory we can relate the OPE's with the commutators of generators as,

$$\underline{[Q, \hat{\mathcal{O}}(z)] = -i \operatorname{Res}_{z_1 \rightarrow z} [\mathcal{T}(z_1) \hat{\mathcal{O}}(z)]}$$

& by ward identity, $[Q, \hat{\mathcal{O}}(z)] = -\frac{\partial \hat{\mathcal{O}}(z)}{\epsilon}$

where $\partial \hat{\mathcal{O}}(z) = \hat{\mathcal{O}}'(z) - \hat{\mathcal{O}}(z)$ under the conformal transformation $\partial z = \epsilon \mathcal{V}(z)$.

So even here we see that L_{-1} generates the translation $[L_{-1}, \hat{\mathcal{O}}(z)] = \partial \hat{\mathcal{O}}(z)$

And for a local operator which is an eigen state under rot^n /dilations with weight 'h' we get, $[L_0 - zL_{-1}, \hat{A}(z)] = h \hat{A}(z)$

iL_0 generates Rotations & L_0 generates Dilations. If we Laurent expand ' $\hat{A}(z)$ ' as,

$$\hat{A}(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{A}_m}{z^{m+h}} \rightarrow \underline{\underline{[L_0, \hat{A}_m] = -m \hat{A}_{n+m}}}$$

Any tensor operator $\hat{A}(z)$ with weight 'h', where

$$\hat{A}(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{A}_m}{z^{m+h}}$$

$$\& \hat{A}_m = \oint_C \frac{dz}{2\pi i} z^{n+h-1} \hat{A}(z)$$

we get,

$$\underline{[L_n, \hat{A}_m] = [(h-1)n - m] \hat{A}_{n+m}}$$

Representations of Virasoro Algebra

We previously saw that the Hamiltonian operator which generates evolution of 'z' is given by, $\hat{H} = \hat{L}_0 - \frac{c}{24}$

Thus physically ' $\hat{H}$ ' should correspond to the closed string energy.

Suppose we have an eigen state of operators $\hat{L}_0$ given by $|h\rangle$. Where $\hat{L}_0|h\rangle = h|h\rangle$

$$\Rightarrow \hat{H}|h\rangle = \left(h - \frac{c}{24}\right)|h\rangle$$

$$\Rightarrow E(h) = \langle h|\hat{H}|h\rangle = h - \frac{c}{24}$$

Now, the operators $\hat{L}_n$ change the Eigen states such that their energy will change by a defined unit $\pm n$.

$$\hat{L}_0 \hat{L}_n = \hat{L}_n (\hat{L}_0 - n)$$

$$\begin{aligned} \text{i.e. } \hat{H}(\hat{L}_n|h\rangle) &= \left(\hat{L}_0 \hat{L}_n - \frac{c}{24} \hat{L}_n\right)|h\rangle \\ &= \left(\hat{L}_n (\hat{L}_0 - n) - \frac{c}{24} \hat{L}_n\right)|h\rangle \\ &= \left(h - \frac{c}{24} - n\right) \hat{L}_n|h, \vec{h}\rangle \end{aligned}$$

$$\Rightarrow \underline{\hat{H}(\hat{L}_n|h, \vec{h}\rangle) = (E(h) - n)(\hat{L}_n|h\rangle)}$$

Thus $\hat{L}_n$ are ladder operators. For $n > 0$, they lower the energy. And for $n < 0$, they raise the energy.

If the energy spectrum is to be bounded below, then there should be some state which is annihilated by all $\hat{L}_n$ for $n > 0$. Such states are called as the primary states, denoting it by $|\psi\rangle$.

$$\Rightarrow \hat{L}_n |\psi\rangle = 0, \text{ for all } n > 0.$$

Representations of Virasoro algebra can be done by creating the descendants by acting creation operators on the primary states. This results in an infinite tower of states called as descendants.

$$\text{Level 0} \rightarrow |\psi\rangle$$

$$\text{Level 1} \rightarrow \hat{L}_{-1} |\psi\rangle$$

$$\text{Level 2} \rightarrow \hat{L}_{-1}^2 |\psi\rangle \quad \hat{L}_{-2} |\psi\rangle$$

$$\vdots \quad \hat{L}_{-1}^3 |\psi\rangle \quad \hat{L}_{-1} \hat{L}_{-2} |\psi\rangle \quad \hat{L}_{-3} |\psi\rangle$$

$$\hat{L}_{-1}^4 |\psi\rangle \quad \hat{L}_{-1} \hat{L}_{-3} |\psi\rangle \quad \hat{L}_{-1}^2 \hat{L}_{-2} |\psi\rangle \quad \hat{L}_{-2}^2 |\psi\rangle \quad \hat{L}_{-4} |\psi\rangle$$

$\vdots$ So on