

Massless-Closed-free-Scalar Strings

Consider a 2 dim Euclidean world sheet Embedded in a 'D' dimensional Minkowski Space time. Then the Polyakov action for the world sheet is given by,

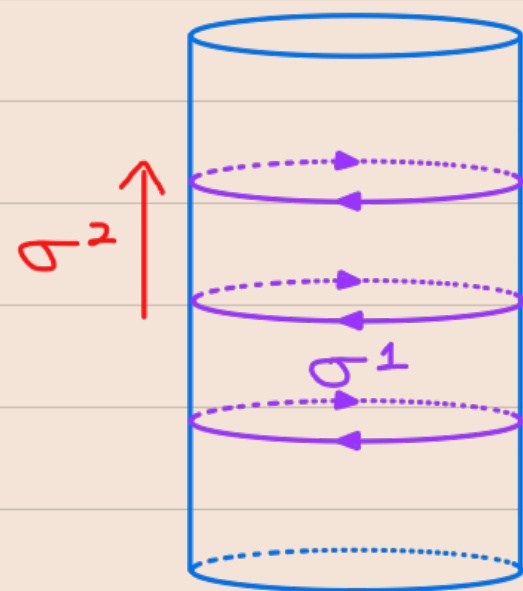
$$S_E = \frac{1}{4\pi\alpha'} \int d^2\sigma (\partial_a \chi^\mu)(\partial_b \chi_\mu) \eta^{ab}$$

where $\sigma^a = (\sigma^1, \sigma^2)$ are world sheet coordinates. $\sigma^1 \rightarrow \sigma$ is space-like coordinate which is periodic in ' 2π ', $\sigma^1 + 2\pi = \sigma^1$. & $\sigma^2 \rightarrow i\tau$ is the Euclidean time where $-\infty < \sigma^2 < +\infty$. The index ' $\mu, \nu, \dots$ ' are Minkowski indices. $\mu, \nu, \dots = 0, 1, 2, \dots, D$.

The fields χ^μ & their derivatives have periodic boundary conditions.

$$\chi^\mu(\sigma^1=0, \sigma^2) = \chi^\mu(\sigma^1=2\pi, \sigma^2)$$

$$\& \partial_a \chi^\mu(\sigma^1=0, \sigma^2) = \partial_a \chi^\mu(\sigma^1=2\pi, \sigma^2)$$



We define complex coordinates on a plane as $\omega = \sigma^1 + i\sigma^2$ & $\bar{\omega} = \sigma^1 - i\sigma^2$ So ω & $\bar{\omega}$ define a point on the sheet.

$\Rightarrow \omega + 2\pi = \omega$ & $\bar{\omega} + 2\pi = \bar{\omega}$ So time translation is got by change of $\sigma^2 = \text{Im}(\omega)$.

In complex ' ω ' coordinates, the action becomes,

$$S_E = \frac{1}{2\pi\alpha'} \int d^2\omega (\partial_\omega \chi^\mu) (\partial_{\bar{\omega}} \chi_\mu)$$

Eqⁿ of motion is

$$\partial_\omega \partial_{\bar{\omega}} \chi^\mu(\omega, \bar{\omega}) = \partial_{\bar{\omega}} \partial_\omega \chi^\mu(\omega, \bar{\omega}) = 0$$

$\Rightarrow \partial_\omega \chi^\mu(\omega, \bar{\omega})$ is holomorphic & $\partial_{\bar{\omega}} \chi^\mu(\omega, \bar{\omega})$ is anti-holomorphic

Thus we can mode expand the above as,

$$\partial_\omega \chi^\mu(\omega, \bar{\omega}) = -\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{in\omega}$$

$$\& \partial_{\bar{\omega}} \chi^\mu(\omega, \bar{\omega}) = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^\mu e^{in\bar{\omega}}$$

we can invert the relation to get α_n^μ & $\tilde{\alpha}_n^\mu$.

$$\partial_\omega \chi^\mu(\omega, \bar{\omega}) = -\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{in\sigma^1} e^{-n\sigma^2}$$

$$\rightarrow \partial_\omega \chi^\mu(\omega, \bar{\omega}) \Big|_{\sigma^2=0} = -\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{in\sigma^1}$$

$$\rightarrow \alpha_n^\mu = -\sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \partial_\omega \chi^\mu(\omega, \bar{\omega}) \Big|_{\sigma^2=0} e^{-in\sigma^1}$$

$$\rightarrow \alpha_n^\mu = -\frac{1}{2} \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} \frac{d\sigma^1}{2\pi} (\partial_1 \chi^\mu - i\partial_2 \chi^\mu) \Big|_{\sigma^2=0} e^{-in\sigma^1}$$

Similarly,

$$\tilde{\alpha}_n^\mu = \frac{1}{2} \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} \frac{d\sigma^1}{2\pi} (\partial_1 \chi^\mu + i\partial_2 \chi^\mu) \Big|_{\sigma^2=0} e^{-in\sigma^1}$$

For the 0^{th} mode

$$\rightarrow \alpha_0^\mu = -\frac{1}{2} \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \underbrace{(\partial_1 \chi^\mu - i \partial_2 \chi^\mu)}_{\text{Zero due to periodic boundary}} \Big|_{\sigma^2=0}$$

$$\rightarrow \alpha_0^\mu = \frac{1}{2} \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} \frac{d\sigma^1}{2\pi} (i \partial_2 \chi^\mu) \Big|_{\sigma^2=0}$$

Similarly,

$$\tilde{\alpha}_0^\mu = \frac{1}{2} \sqrt{\frac{2}{\alpha'}} \int_0^{2\pi} \frac{d\sigma^1}{2\pi} (i \partial_2 \chi^\mu) \Big|_{\sigma^2=0}$$

$$\Rightarrow \boxed{\alpha_0^\mu = \tilde{\alpha}_0^\mu}$$

Equations in z -plane

In z -plane under the conformal transformation $z = e^{-i\omega}$, $\bar{z} = e^{i\bar{\omega}}$ & by tensor property of $\partial_\omega \chi^\mu$ & $\partial_{\bar{\omega}} \chi^\mu$ we can go to z -plane. Also they are tensors of weight $(1,0)$ and $(0,0)$ respectively.

$$\Rightarrow \partial_\omega \chi^\mu(\omega, \bar{\omega}) \longrightarrow \left(\frac{\partial z}{\partial \omega} \right) \partial_z \chi^\mu(z, \bar{z})$$

&

$$\partial_{\bar{\omega}} \chi^\mu(\omega, \bar{\omega}) \longrightarrow \left(\frac{\partial \bar{z}}{\partial \bar{\omega}} \right) \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

$$\rightarrow -i e^{-i\omega} \partial_z \chi^\mu(z, \bar{z}) = -\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{in\omega}$$

$$\rightarrow \boxed{\partial_z \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}}$$

Similarly,

$$i e^{i\bar{\omega}} \partial_{\bar{z}} \chi^\mu(z, \bar{z}) = \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^\mu e^{in\bar{\omega}}$$

$$\Rightarrow \boxed{\partial_{\bar{z}} \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\tilde{\alpha}_n^\mu}{\bar{z}^{n+1}}}$$

Thus the possibility of mode expansion of $\partial_\omega \chi^\mu(\omega, \bar{\omega})$ & $\partial_{\bar{\omega}} \chi^\mu(\omega, \bar{\omega})$ allowed us to write them as Laurent expansion in the z -coordinates in z -plane.

The Laurent coefficients are given by,

$$\alpha_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \partial_z \chi^\mu(z, \bar{z})$$

&

$$\tilde{\alpha}_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_{\tilde{C}} \frac{d\bar{z}}{2\pi} \bar{z}^n \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

where 'C' is a path enclosing the origin anti-clockwise & ' $\tilde{C}$ ' is a path enclosing the origin clockwise.

Consider the Space-time translations, where $\chi^\mu \rightarrow \chi^\mu + c^\mu$, remains invariant. To get the Conserved current, we shall vary the action by keeping c^μ locally dependent.

$$\text{i.e. } c^\mu \rightarrow c^\mu(z, \bar{z})$$

$$\delta S_E = \frac{1}{2\pi\alpha'} \int d^2z \left[(\partial_z \chi^\mu) (\partial_{\bar{z}} \delta \chi_\mu) + (\partial_z \delta \chi^\mu) (\partial_{\bar{z}} \chi_\mu) \right]$$

Noether current is defined by,

$$\delta S = \frac{-i}{2\pi} \int d^2 z \left[T_z \partial_{\bar{z}} c^\mu(z, \bar{z}) + T_{\bar{z}} \partial_z c^\mu(z, \bar{z}) \right]$$

$$\rightarrow T_z^\mu = \frac{i}{\alpha'} \partial_z \chi^\mu(z, \bar{z})$$

$$\& T_{\bar{z}}^\mu = \frac{i}{\alpha'} \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

Thus T_z^μ is holomorphic & $T_{\bar{z}}^\mu$ is anti-holomorphic. Now the conserved charge is given by

$$p^\mu = \oint_C \frac{dz}{2\pi i} T^\mu(z) + \oint_{\tilde{C}} \frac{d\bar{z}}{2\pi i} \tilde{T}^\mu(\bar{z})$$

$$p^\mu = \frac{1}{2\pi\alpha'} \left[\oint_C dz \partial_z \chi^\mu + \oint_{\tilde{C}} d\bar{z} \partial_{\bar{z}} \chi^\mu \right]$$

Consider the 0^{th} mode coefficients,

$$\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} \partial_z \chi^\mu(z, \bar{z})$$

$$\text{and } \tilde{\alpha}_0^\mu = \sqrt{\frac{2}{\alpha'}} \oint_{\tilde{C}} \frac{d\bar{z}}{2\pi} \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

$$\sqrt{\frac{2}{\alpha'}} p^\mu = \frac{1}{\alpha'} \left[\sqrt{\frac{2}{\alpha'}} \oint_{\mathcal{C}} \frac{dz}{2\pi} \partial_z \chi^\mu(z, \bar{z}) + \sqrt{\frac{2}{\alpha'}} \oint_{\tilde{\mathcal{C}}} \frac{d\bar{z}}{2\pi} \partial_{\bar{z}} \chi^\mu(z, \bar{z}) \right]$$

$$\rightarrow \sqrt{\frac{2}{\alpha'}} p^\mu = \frac{\alpha_0^\mu + \tilde{\alpha}_0^\mu}{\alpha'}$$

Since $\alpha_0^\mu = \tilde{\alpha}_0^\mu$, we get,

$$\boxed{p^\mu = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0^\mu}$$

Now, we can get the expression for χ^μ by integrating the laurent series. By the total derivative formula,

$$d\chi^\mu(z, \bar{z}) = dz \partial_z \chi^\mu(z, \bar{z}) + d\bar{z} \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

Thus we get,

$$\chi^\mu(z, \bar{z}) = x^\mu + \int dz \partial_z \chi^\mu(z, \bar{z}) + \int d\bar{z} \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

where x^μ is a constant of Integration.

$$\begin{aligned} \Rightarrow \int d\bar{z} \partial_{\bar{z}} \chi^\mu(z, \bar{z}) &= -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \int \frac{d\bar{z}}{\bar{z}^{n+1}} \\ &= -i \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \ln(\bar{z}) + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu}{n \bar{z}^n} \end{aligned}$$

$$\begin{aligned} \Rightarrow \int d\bar{z} \partial_{\bar{z}} \chi^\mu(z, \bar{z}) &= -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^\mu \int \frac{d\bar{z}}{\bar{z}^{n+1}} \\ &= -i \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \ln(\bar{z}) + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{\alpha}_n^\mu}{n \bar{z}^n} \end{aligned}$$

Thus we get,

$$\begin{aligned} \chi^\mu(z, \bar{z}) &= x^\mu - i \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \ln(z) - i \sqrt{\frac{\alpha'}{2}} \tilde{\alpha}_0^\mu \ln(\bar{z}) \\ &\quad + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu}{n z^n} + i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\tilde{\alpha}_n^\mu}{n \bar{z}^n} \end{aligned}$$

Substitute, $p^\mu = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0^\mu$

Thus we finally get,

$$\chi^\mu(z, \bar{z}) = x^\mu - \frac{i\alpha' p^\mu}{2} \ln(|z|^2) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\alpha_n^\mu}{z^n} + \frac{\tilde{\alpha}_n^\mu}{\bar{z}^n} \right]$$

When $|z|^2 = 1 \rightarrow \sigma^2 = 0$, $z = e^{-i\sigma^1}$

$$\chi^\mu(\sigma^1, 0) = x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\alpha_n^\mu e^{in\sigma^1} + \tilde{\alpha}_n^\mu e^{-in\sigma^1} \right]$$

Now, $\int_0^{2\pi} \frac{d\sigma^1}{2\pi} \chi^\mu(\sigma^1, 0)$ is given by,

$$= x^\mu + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\underbrace{\alpha_n^\mu \int_0^{2\pi} \frac{d\sigma^1}{2\pi} e^{in\sigma^1}}_{\delta_{n0}} + \tilde{\alpha}_n^\mu \underbrace{\int_0^{2\pi} \frac{d\sigma^1}{2\pi} e^{-in\sigma^1}}_{\delta_{n0}} \right]$$

$\Rightarrow x^\mu = \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \chi^\mu(\sigma^1, 0)$ is the Initial center of mass location.

Canonical Commutation Relations

$$\text{Now, } \hat{\alpha}_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \partial_z \hat{\chi}^\mu(z, \bar{z})$$

Let

$$\hat{Q}_1 = \oint_{C_1} \frac{dz_1}{2\pi i} \hat{T}_1(z_1) \quad \& \quad \hat{Q}_2 = \oint_{C_2} \frac{dz_2}{2\pi i} \hat{T}_2(z_2)$$

where T_1 & T_2 are holomorphic. Then previously, we saw that the commutations b/w two such objects can be written as,

$$[Q_1, Q_2] = \oint_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} \underbrace{[\hat{T}_1(z_1) \hat{T}_2(z_2)]}_{\text{OPE}}$$

$\rightarrow [\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu]$ is given by,

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \left(\frac{2}{\alpha'} \text{Res}_{z_1 \rightarrow z_2} [z_1^n z_2^m \partial_{z_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{z_2} \hat{\chi}^\nu(z_2, \bar{z}_2)] \right)$$

w.k.t.

$$\partial_{z_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{z_2} \hat{\chi}^\nu(z_2, \bar{z}_2) \sim -\frac{\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)^2}$$

Expand z_1^n about z_2 .

$$z_1^n = \sum_{k=0}^{\infty} \frac{z_{12}^k}{k!} \partial_{z_2}^k (z_2^n) = \sum_{k=0}^n \frac{z_{12}^k z_2^{n-k}}{k!} \frac{n!}{(n-k)!}$$

$$\rightarrow \text{Res}_{z_1 \rightarrow z_2} [z_1^n z_2^m \partial_{z_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{z_2} \hat{\chi}^\nu(z_2, \bar{z}_2)]$$

$$= \text{Res}_{z_1 \rightarrow z_2} \left[\frac{(-\alpha') \eta^{\mu\nu} z_2^m}{2(z_1 - z_2)^2} \sum_{k=0}^n \frac{(z_1 - z_2)^k z_2^{n-k}}{k!} \frac{n!}{(n-k)!} \right]$$

$$= \frac{(-\alpha') \eta^{\mu\nu} z_2^m z_2^{n-1} n}{2}$$

$$\rightarrow [\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = \oint_{C_2} \frac{dz_2}{2\pi i} \left(\frac{2}{\alpha'} \frac{(-\alpha') \eta^{\mu\nu} z_2^m z_2^{n-1} n}{2} \right)$$

$$= n \eta^{\mu\nu} \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n-1} \quad \left. \vphantom{\oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m+n-1}} \right\} \delta_{m,-n}$$

$$\rightarrow \boxed{[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = n \eta^{\mu\nu} \delta_{m,-n}}$$

In the exact same way for $\hat{\tilde{\alpha}}_n^\mu, \hat{\tilde{\alpha}}_m^\nu$ we get,

$$\boxed{[\hat{\tilde{\alpha}}_n^\mu, \hat{\tilde{\alpha}}_m^\nu] = n \eta^{\mu\nu} \delta_{m,-n}}$$

Also since $\partial_{z_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{\bar{z}_2} \hat{\chi}^\nu(z_2, \bar{z}_2)$ has no singular parts, $\boxed{[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = 0}$

Commutations with $\hat{p}^\mu$

$$\text{Now, } \hat{p}^\mu = \sqrt{\frac{2}{\alpha'}} \hat{\alpha}_0^\mu = \frac{2}{\alpha'} \oint_C \frac{dz_1}{2\pi} \partial_z \hat{\chi}^\mu(z_1, \bar{z}_1)$$

$$\rightarrow [\hat{p}^\mu, \hat{\alpha}_n^\nu] = \sqrt{\frac{2}{\alpha'}} [\hat{\alpha}_0^\mu, \hat{\alpha}_n^\nu] = 0$$

$$\rightarrow \boxed{[\hat{p}^\mu, \hat{\alpha}_n^\nu] = 0}$$

$$[\hat{p}^\mu, \hat{p}^\nu] = \frac{2}{\alpha'} [\hat{\alpha}_0^\mu, \hat{\alpha}_0^\nu] = 0$$

$$\rightarrow \boxed{[\hat{p}^\mu, \hat{p}^\nu] = 0}$$

∴ Since $[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = 0$, we directly get,

$$\boxed{[\hat{p}^\mu, \hat{\alpha}_n^\nu] = 0}$$

To find $[\hat{x}^\mu, \hat{p}^\nu]$, where

$$\hat{p}^\mu = \sqrt{\frac{2}{\alpha'}} \hat{\alpha}_0^\mu = \frac{2}{\alpha'} \oint_C \frac{dz_1}{2\pi} \partial_z \hat{X}^\mu(z_1, \bar{z}_1)$$

consider, $[\hat{p}^\mu, \hat{X}^\nu(z_2, \bar{z}_2)]$ given by,

$$= \frac{2}{\alpha'} \oint_C \frac{dz_1}{2\pi} [\partial_{z_1} \hat{X}^\mu(z_1, \bar{z}_1), \hat{X}^\nu(z_2, \bar{z}_2)]$$

Now, $[\partial_{z_1} \hat{X}^\mu(z_1, \bar{z}_1), \hat{X}^\nu(z_2, \bar{z}_2)] = \text{Singular Part of OPE}$

$$\rightarrow [\partial_{z_1} \hat{X}^\mu(z_1, \bar{z}_1), \hat{X}^\nu(z_2, \bar{z}_2)] = \frac{-\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)}$$

$$\text{Then, } [\hat{p}^\mu, \hat{X}^\nu(z_2, \bar{z}_2)] = \frac{2}{\alpha'} \oint_C \frac{dz_1}{2\pi} \left[\frac{-\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)} \right]$$

If $(z_2, \bar{z}_2)$ lies inside the Contour,

$$\rightarrow \underline{[\hat{X}^\nu(z_2, \bar{z}_2), \hat{p}^\mu] = i\eta^{\mu\nu}}$$

So, right hand side is independent of the position $(z_2, \bar{z}_2)$. So, by using the mode expansion of $\hat{\chi}^\nu(z_2, \bar{z}_2)$, the only term which can contribute to the commutator is the position independent x^μ term.

$$\rightarrow \boxed{[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}}$$

The commutators b/w $\hat{x}^\mu$ & the oscillator modes $\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu$ can be found by the OPEs as follows,

$$[\hat{\alpha}_n^\nu, \hat{\chi}^\mu(z_2, \bar{z}_2)]$$

$$= \sqrt{\frac{2}{\alpha'}} \oint_{C_1} \frac{dz_1}{2\pi} z_1^n [\partial_{z_1} \hat{\chi}^\mu(z_1, \bar{z}_1), \hat{\chi}^\nu(z_2, \bar{z}_2)]$$

$$= \sqrt{\frac{2}{\alpha'}} \oint_{C_1} \frac{dz_1}{2\pi} z_1^n \left[\frac{-\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)} \right]$$

$$= -\sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} \oint_{C_1} \frac{dz_1}{2\pi} \frac{z_1^n}{(z_1 - z_2)}$$

If the contour encloses both z_2 & origin,

$$\oint_{C_1} \frac{dz_1 z_1^n}{2\pi (z_1 - z_2)} = \begin{cases} 0, & n < 0 \\ i z_2^n, & n \geq 0 \end{cases}$$

$$\rightarrow [\hat{\alpha}_n^\nu, \hat{\chi}^\mu(z_2, \bar{z}_2)] = \begin{cases} -i \sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} z_2^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Now, χ^μ is the Zeroth mode part of $\hat{\chi}^\mu$.

$$\rightarrow \hat{\chi}^\mu = \oint_C \frac{dz}{2\pi i} \frac{\hat{\chi}^\mu(z, \bar{z})}{z}$$

$$\rightarrow [\hat{\alpha}_n^\nu, \hat{\chi}^\mu] = \begin{cases} -i \sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} \underbrace{\oint_C \frac{dz z^n}{2\pi i z}}_{\delta_{n,0}}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\rightarrow \underline{[\hat{\chi}^\mu, \hat{\alpha}_n^\nu] = i \sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} \delta_{n,0}}$$

$$\underline{\text{Similarly, } [\hat{\chi}^\mu, \hat{\tilde{\alpha}}_n^\nu] = i \sqrt{\frac{\alpha'}{2}} \eta^{\mu\nu} \delta_{n,0}}$$

Virasoro generators

The energy momentum tensor components are given by,

$$T(z) = T_{zz} = -\frac{1}{\alpha'} (\partial_z \chi^\mu)(\partial_z \chi_\mu)$$

$$\tilde{T}(\bar{z}) = T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} (\partial_{\bar{z}} \chi^\mu)(\partial_{\bar{z}} \chi_\mu)$$

Consider $T(z)$ & we substitute the expansions of $\partial_z \chi^\mu$ & $\partial_z \chi_\mu$.

$$\partial_z \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\hat{\alpha}_n^\mu}{z^{n+1}}$$

$$\begin{aligned} \Rightarrow -\frac{1}{\alpha'} \partial_z \chi^\mu(z, \bar{z}) \partial_z \chi_\mu(z, \bar{z}) \\ = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{\hat{\alpha}_n^\mu}{z^{n+1}} \sum_{k=-\infty}^{\infty} \frac{\hat{\alpha}_k^\mu}{z^{k+1}} \end{aligned}$$

$$\Rightarrow T(z) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\hat{\alpha}_n^\mu \hat{\alpha}_k^\nu \eta_{\mu\nu}}{z^{n+k+2}}$$

Now, $\hat{L}_m = \oint_C \frac{dz}{2\pi i} z^{m+1} T(z)$

For $m \neq 0$

$$\hat{L}_m = \oint_C \frac{dz}{2\pi i} z^{m+1} \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\hat{\alpha}_n^\mu \hat{\alpha}_k^\nu \eta_{\mu\nu}}{z^{n+k+2}}$$

$$= \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{\alpha}_n^\mu \hat{\alpha}_k^\nu \eta_{\mu\nu} \underbrace{\oint_C \frac{dz}{2\pi i} \frac{1}{z^{n+k-m+1}}}_{\delta_{k, m-n}}$$

$$\rightarrow \hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{\alpha}_n^\mu \hat{\alpha}_k^\nu \eta_{\mu\nu} \delta_{k, m-n}$$

$$\rightarrow \hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_n^\mu \hat{\alpha}_{m-n}^\nu \eta_{\mu\nu}$$

Since $m \neq 0$, $\hat{\alpha}_n^\mu \hat{\alpha}_{m-n}^\nu = \hat{\alpha}_{m-n}^\nu \hat{\alpha}_n^\mu$

$$\Rightarrow \boxed{\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}} \quad , \text{ for } m \neq 0$$

For $m=0$

$$\hat{L}_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \hat{\alpha}_n^\mu \hat{\alpha}_k^\nu \eta_{\mu\nu} \underbrace{\oint_C \frac{dz}{2\pi i} \frac{1}{z^{n+k+1}}}_{\delta_{n,-k}}$$

$$\hat{L}_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}$$

Separate the Zero mode & expand,

$$\hat{L}_0 = \frac{1}{2} \hat{\alpha}_0^\mu \hat{\alpha}_0^\nu \eta_{\mu\nu} + \frac{1}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}$$

$$\text{Use, } \hat{p}^\mu = \sqrt{\frac{2}{\alpha'}} \hat{\alpha}_0^\mu$$

$$\rightarrow \hat{L}_0 = \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \frac{1}{2} \sum_{n=1}^{\infty} [\hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n} + \hat{\alpha}_{\mu n} \hat{\alpha}_{-n}^\mu]$$

$$\rightarrow \hat{L}_0 = \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n} + A^X$$

where ' A^X ' is some ordering constant to be determined.

Similarly, for the anti-holomorphic part we get $\tilde{T}(\bar{z}) = T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} (\partial_{\bar{z}} \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$

where, $\partial_{\bar{z}} \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}^{n+1}}$

For $m \neq 0$, we get,

$$\hat{\tilde{L}}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\tilde{\alpha}}_{m-n}^\mu \hat{\tilde{\alpha}}_n^\nu \eta_{\mu\nu}$$

For $m=0$,

$$\rightarrow \hat{\tilde{L}}_0 = \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\tilde{\alpha}}_{-n}^\mu \hat{\tilde{\alpha}}_{\mu n} + \tilde{A}^x$$

where $\tilde{A}^x$ is some ordering constant to be determined.

Now, the hamiltonian is given by,

$$\hat{H} = \hat{L}_0 + \hat{\tilde{L}}_0 - \frac{c + \tilde{c}}{24}$$

$$\rightarrow \hat{H} = \frac{\alpha'}{2} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} (\hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n} + \hat{\tilde{\alpha}}_{-n}^\mu \hat{\tilde{\alpha}}_{\mu n}) + A^x + \tilde{A}^x - \frac{c + \tilde{c}}{24}$$

Consider commutators of $\hat{H}$ with the oscillator modes.

$$[\hat{H}, \hat{\alpha}_n^\mu] = -n \hat{\alpha}_n^\mu, \quad [\hat{H}, \hat{\alpha}_{-n}^\mu] = n \hat{\alpha}_{-n}^\mu$$

$$[\hat{H}, \hat{\tilde{\alpha}}_n^\mu] = -n \hat{\tilde{\alpha}}_n^\mu, \quad [\hat{H}, \hat{\tilde{\alpha}}_{-n}^\mu] = n \hat{\tilde{\alpha}}_{-n}^\mu$$

Thus we see that ' $\hat{\alpha}_n^\mu$ ' & ' $\hat{\tilde{\alpha}}_n^\mu$ ' decreases the energy. ' $\hat{\alpha}_{-n}^\mu$ ' & ' $\hat{\tilde{\alpha}}_{-n}^\mu$ ' increases the energy.

$\rightarrow \hat{\alpha}_{-n}^\mu$ & $\hat{\tilde{\alpha}}_{-n}^\mu \rightarrow$ Creation Operator

$\hat{\alpha}_n^\mu$ & $\hat{\tilde{\alpha}}_n^\mu \rightarrow$ Annihilation Operator

Let us consider a Fock space lowest energy ground state given by $|0\rangle$ such that

$$\hat{\alpha}_n^\mu |0\rangle = \hat{\tilde{\alpha}}_n^\mu |0\rangle = 0, \text{ for all } n > 1 \text{ \& } \hat{p}^\mu |0\rangle = 0$$

$$\text{Consider } \hat{L}_1 = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{1-n}^\mu \hat{\alpha}_{\mu n}$$

$$= \underbrace{\frac{\hat{\alpha}_1^\mu \hat{\alpha}_{\mu 0}}{2}}_{n=0} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\underbrace{\hat{\alpha}_{1-n}^\mu \hat{\alpha}_{\mu n}}_{\text{Commutates}} + \hat{\alpha}_{1+n}^\mu \underbrace{\hat{\alpha}_{\mu -n}}_{\text{Commutates}} \right)$$

$$\rightarrow \hat{L}_1 = \frac{\hat{\alpha}_1^\mu \hat{\alpha}_{\mu 0}}{2} + \frac{\hat{\alpha}_{\mu 0} \hat{\alpha}_1^\mu}{2} + \frac{1}{2} \sum_{n=2}^{\infty} \underbrace{\hat{\alpha}_{1-n}^\mu \hat{\alpha}_{\mu n}}_{n \rightarrow n+1} + \frac{1}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{\mu-n} \hat{\alpha}_{1+n}^\mu$$

$\hat{\alpha}_{\mu 0} = \sqrt{\frac{\alpha'}{2}} \hat{p}_\mu$

$$\Rightarrow \boxed{\hat{L}_1 = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_1^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{\mu-n} \hat{\alpha}_{n+1}^\mu}$$

Now, $\hat{L}_{-1} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{-1-n}^\mu \hat{\alpha}_{\mu n}$

$$= \frac{1}{2} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{\mu 0} + \frac{1}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{-1-n}^\mu \hat{\alpha}_{\mu n} + \hat{\alpha}_{-1+n}^\mu \hat{\alpha}_{\mu-n})$$

$$= \frac{\hat{\alpha}_{-1}^\mu \hat{\alpha}_{\mu 0}}{2} + \frac{\hat{\alpha}_{\mu 0} \hat{\alpha}_{-1}^\mu}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-1-n}^\mu \hat{\alpha}_{\mu n} + \frac{1}{2} \sum_{n=2}^{\infty} \underbrace{\hat{\alpha}_{-1+n}^\mu \hat{\alpha}_{\mu-n}}_{n \rightarrow n+1}$$

$\hat{\alpha}_{\mu 0} = \sqrt{\frac{\alpha'}{2}} \hat{p}_\mu$

$$\Rightarrow \boxed{\hat{L}_{-1} = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n-1}^\mu \hat{\alpha}_{\mu n}}$$

Thus, if $\hat{\alpha}_n^\mu |0\rangle = 0$ for all $n > 0$ & if $\hat{p}^\mu |0\rangle = 0$, we get

$$\underline{\hat{L}_1 |0\rangle = 0 \text{ \& \& } \hat{L}_{-1} |0\rangle = 0}$$

By Virasoro algebra b/w $\hat{L}_0$ & $\hat{L}_{\pm 1}$ we get, $[\hat{L}_1, \hat{L}_{-1}] = 2\hat{L}_0$

$$\Rightarrow (\underbrace{\hat{L}_1 \hat{L}_{-1}}_0 + \underbrace{\hat{L}_{-1} \hat{L}_1}_0) |0\rangle = 2\hat{L}_0 |0\rangle$$

$$\Rightarrow \underline{\underline{\hat{L}_0 |0\rangle = 0}}$$

Thus,

$$\underbrace{\frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu}_0 |0\rangle + \sum_{n=1}^{\infty} \underbrace{\hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n}}_0 |0\rangle + A^X |0\rangle = 0$$

$$\Rightarrow \boxed{A^X = 0}$$

Thus Normal ordering constant is finite.

In the exact same way for the anti-holomorphic part,

$$\boxed{\hat{L}_1 = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_1^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{\mu-n} \hat{\alpha}_{n+1}^\mu}$$

&

$$\boxed{\hat{L}_{-1} = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n-1}^\mu \hat{\alpha}_{\mu n}}$$

By Virasoro algebra b/w $\hat{L}_0$ & $\hat{L}_{\pm 1}$ we get, $[\hat{L}_1, \hat{L}_{-1}] = 2\hat{L}_0$

$$\Rightarrow (\hat{L}_1 \hat{L}_{-1} + \hat{L}_{-1} \hat{L}_1) |0\rangle = 2\hat{L}_0 |0\rangle$$

$$\Rightarrow \underline{\underline{\hat{L}_0 |0\rangle = 0}}$$

Thus,

$$\frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu |0\rangle + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n} |0\rangle + \tilde{A} |0\rangle = 0$$

$$\Rightarrow \boxed{\tilde{A}^x = 0}$$

Thus we finally get,

$\hat{L}_0 = \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n}$
$\hat{\tilde{L}}_0 = \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\tilde{\alpha}}_{-n}^\mu \hat{\tilde{\alpha}}_{\mu n}$

In previous results we saw that the central charge for X^μ theory was $c = \tilde{c} = D$. We can show this even by Virasoro algebra.

We know that the generators satisfy

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m,-n}$$

$$\rightarrow [L_m, L_{-m}] = 2mL_0 + \frac{c}{12} m(m^2-1)$$

For $m > 0$

$$\rightarrow \hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}, \text{ for } m \neq 0$$

$$\hat{L}_m = \frac{1}{2} \eta_{\mu\nu} \left[\hat{\alpha}_m^\mu \hat{\alpha}_0^\nu + \sum_{n=1}^{\infty} (\hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{m+n}^\mu) \right]$$

Now, when $\hat{L}_m$ acts on $|0\rangle$ all terms vanish since $\hat{\alpha}_0^\nu$, $\hat{\alpha}_n^\nu$, $\hat{\alpha}_{m+n}^\mu$ acting on it leads to Zero.

$$\text{for } m > 0 \rightarrow \hat{L}_m |0\rangle = 0$$

For $m < 0$, put $m \rightarrow -m$

$$\Rightarrow \hat{L}_{-m} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \hat{\alpha}_{-m-n}^{\mu} \hat{\alpha}_n^{\nu} \eta_{\mu\nu}, \quad m > 0$$

$$\hat{L}_{-m} = \frac{1}{2} \eta_{\mu\nu} \left[\hat{\alpha}_{-m}^{\mu} \hat{\alpha}_0^{\nu} + \sum_{n=1}^{\infty} (\hat{\alpha}_{-m-n}^{\mu} \hat{\alpha}_n^{\nu} + \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu}) \right]$$

Now, when $\hat{L}_m$ acts on $|0\rangle$ all terms when $n \geq m$ vanish. But if $n < m$ then last term contributes. moreover for $m = \pm 1$, we previously saw, the result vanishes

$$\Rightarrow \hat{L}_{-m} |0\rangle = \frac{1}{2} \eta_{\mu\nu} \sum_{n=1}^{m-1} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} |0\rangle, \quad m > 1$$

$$\Rightarrow [\hat{L}_m, \hat{L}_{-m}] |0\rangle = \hat{L}_m \hat{L}_{-m} |0\rangle - \underbrace{\hat{L}_{-m} \hat{L}_m |0\rangle}_0$$

But also,

$$[\hat{L}_m, \hat{L}_{-m}] |0\rangle = \frac{c}{12} m(m^2 - 1)$$

$$\text{Since } \hat{L}_0 |0\rangle = 0$$

$$\Rightarrow L_m L_{-m} |0\rangle = \frac{c}{12} m(m^2-1) |0\rangle, \quad m > 1$$

Now,

$$\begin{aligned} L_m L_{-m} |0\rangle &= \frac{1}{2} \sum_{k=-\infty}^{\infty} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} \eta_{\rho\sigma} \\ &\quad \times \frac{1}{2} \eta_{\mu\nu} \sum_{n=1}^{m-1} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \\ &= \frac{\eta_{\mu\nu} \eta_{\rho\sigma}}{4} \sum_{k=-\infty}^{\infty} \sum_{n=1}^{m-1} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \end{aligned}$$

consider,

$$\begin{aligned} &\hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \\ &= \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_k^{\sigma} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \\ &\quad + k \delta_{nk} \eta^{\sigma\nu} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \\ &= \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_k^{\sigma} |0\rangle \\ &\quad + k \delta_{k, m-n} \eta^{\mu\sigma} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_{-n}^{\nu} |0\rangle \\ &\quad + k \delta_{nk} \eta^{\sigma\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-k}^{\rho} |0\rangle \\ &\quad + k \delta_{nk} \eta^{\sigma\nu} (m-k) \eta^{\mu\rho} |0\rangle \end{aligned}$$

$$\begin{aligned}
&= \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_k^{\sigma} |0\rangle \\
&\quad + (m-k) \eta^{\rho\nu} \delta_{m-k,n} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_k^{\sigma} |0\rangle \\
&\quad + k \delta_{k,m-n} \eta^{\mu\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{m-k}^{\rho} |0\rangle \\
&\quad + k \delta_{k,m-n} \eta^{\mu\sigma} \eta^{\sigma\nu} (m-k) \delta_{m-k,n} |0\rangle \\
&\quad + k \delta_{nk} \eta^{\sigma\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-k}^{\rho} |0\rangle \\
&\quad + k \delta_{nk} \eta^{\sigma\nu} (m-k) \eta^{\mu\rho} |0\rangle
\end{aligned}$$

$$\begin{aligned}
&= \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} |0\rangle \\
&\quad + (m-k) \eta^{\mu\rho} \delta_{nk} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_k^{\sigma} |0\rangle \\
&\quad + (m-k) \eta^{\rho\nu} \delta_{m-k,n} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_k^{\sigma} |0\rangle \\
&\quad + k \delta_{k,m-n} \eta^{\mu\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{m-k}^{\rho} |0\rangle \\
&\quad + k \delta_{k,m-n} \eta^{\mu\sigma} \eta^{\sigma\nu} (m-k) \delta_{m-k,n} |0\rangle \\
&\quad + k \delta_{nk} \eta^{\sigma\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-k}^{\rho} |0\rangle \\
&\quad + k \delta_{nk} \eta^{\sigma\nu} (m-k) \eta^{\mu\rho} |0\rangle
\end{aligned}$$

Thus,

$$\begin{aligned}
 & \sum_{k=-\infty}^{\infty} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \\
 &= \sum_{k=-\infty}^{\infty} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} |0\rangle \\
 &+ (m-n) \eta^{\mu\rho} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_n^{\sigma} |0\rangle \quad \times \\
 &+ n \eta^{\rho\nu} \hat{\alpha}_{n-m}^{\mu} \hat{\alpha}_{m-n}^{\sigma} |0\rangle \quad \times \\
 &+ (m-n) \eta^{\mu\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_n^{\rho} |0\rangle \quad \times \\
 &+ (m-n)n \eta^{\mu\sigma} \eta^{\sigma\nu} |0\rangle \\
 &+ n \eta^{\sigma\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-n}^{\rho} |0\rangle \quad \times \\
 &+ n(m-n) \eta^{\sigma\nu} \eta^{\mu\rho} |0\rangle
 \end{aligned}$$

1st term

$$\begin{aligned}
 & \sum_{n=1}^{m-1} \sum_{k=-\infty}^{\infty} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} |0\rangle \\
 &= \sum_{n=1}^{m-1} \sum_{k=1}^{\infty} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} \hat{\alpha}_{m+k}^{\rho} \hat{\alpha}_{-k}^{\sigma} |0\rangle \quad \begin{array}{c} \swarrow \text{Commutates} \end{array} \\
 &= 0 //
 \end{aligned}$$

$$\begin{aligned} \text{Thus, } \sum_{n=1}^{m-1} \sum_{k=-\infty}^{\infty} \hat{\alpha}_{m-k}^{\rho} \hat{\alpha}_k^{\sigma} \hat{\alpha}_{-n}^{\nu} \hat{\alpha}_{-m+n}^{\mu} |0\rangle \\ = \sum_{n=1}^{m-1} 2(m-n)n \eta^{\mu\sigma} \eta^{\sigma\nu} |0\rangle \end{aligned}$$

Hence we get,

$$L_m L_{-m} |0\rangle = \frac{\eta_{\mu\nu} \eta_{\rho\sigma}}{2} \sum_{n=1}^{m-1} (m-n)n \eta^{\mu\sigma} \eta^{\sigma\nu} |0\rangle$$

$$\eta_{\mu\nu} \eta_{\rho\sigma} \eta^{\mu\sigma} \eta^{\sigma\nu} = \eta_{\nu}^{\nu} = D$$

$$\& \sum_{n=1}^{m-1} (m-n)n = \frac{(m-1)m(m+1)}{6} = \frac{m(m^2-1)}{6}$$

$$\Rightarrow L_m L_{-m} |0\rangle = \frac{D}{12} m(m^2-1) |0\rangle$$

$\rightarrow \underline{C = D}$. Similarly even $\underline{\tilde{C} = D}$

Thus the hamiltonian is,

$$\hat{H} = \frac{\alpha'}{2} \hat{p}^{\mu} \hat{p}_{\mu} + \sum_{n=1}^{\infty} (\hat{\alpha}_{-n}^{\mu} \hat{\alpha}_{\mu n} + \hat{\tilde{\alpha}}_{-n}^{\mu} \hat{\tilde{\alpha}}_{\mu n}) - \frac{D}{12}$$

Creation & Annihilation Normal Ordering

This is the familiar normal ordering procedure in canonical quantization where we place the lowering operators to the right of raising operators. This is denoted by the symbol $::$. For fermionic operators whose fundamental relations are given by anti-commutations, we pick up a -ve sign whenever we exchange two such operators.

Note that previously we had defined a normal ordering symbol $::$ for the local operators. This is called as Conformal Normal ordering.

Now, $\hat{p}^\mu |0\rangle = 0$. So, ' $\hat{p}^\mu$ ' can be thought of lowering operator.

Since $[\hat{x}^\mu, \hat{p}^\mu] = i\eta^{\mu\nu}$, ' $\hat{x}^\mu$ ' can be thought of raising operator.

Relation b/w Conformal Normal ordering & Creation - Annihilation Normal ordering

Consider the product of $\hat{\chi}^\mu(z_1, \bar{z}_1)$ with $\hat{\chi}^\nu(z_2, \bar{z}_2)$ where $|z_1|^2 > |z_2|^2$

$$\begin{aligned}\hat{\chi}^\mu(z_1, \bar{z}_1) = & \hat{x}^\mu - \frac{i\alpha' \hat{p}^\mu}{2} \ln(|z_1|^2) \\ & + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right]\end{aligned}$$

$$\begin{aligned}\hat{\chi}^\nu(z_2, \bar{z}_2) = & \hat{x}^\nu - \frac{i\alpha' \hat{p}^\nu}{2} \ln(|z_2|^2) \\ & + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\hat{\alpha}_n^\nu}{z_2^n} + \frac{\hat{\tilde{\alpha}}_n^\nu}{\bar{z}_2^n} \right]\end{aligned}$$

So after multiplying them, we can move the lower operators to the right of raising operators & keep track of the commutations properly.

$$\begin{aligned}\rightarrow \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) = & :\hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2): \\ & + \text{Commutations}\end{aligned}$$

$\hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2)$ is given by,

$$= \hat{\chi}^\mu \hat{\chi}^\nu - \frac{i\alpha' \hat{p}^\mu \hat{\chi}^\nu}{2} \ln(|z_1|^2) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\hat{\alpha}_n^\mu \hat{\chi}^\nu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu \hat{\chi}^\nu}{\bar{z}_1^n} \right]$$

$$- \frac{i\alpha' \hat{\chi}^\mu \hat{p}^\nu}{2} \ln(|z_2|^2) - \left[\frac{\alpha'}{2} \right]^2 \hat{p}^\mu \hat{p}^\nu \ln(|z_1|^2 + |z_2|^2)$$

$$+ \left(\frac{\alpha'}{2} \right)^{3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\hat{\alpha}_n^\mu \hat{p}^\nu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu \hat{p}^\nu}{\bar{z}_1^n} \right] \ln(|z_2|^2)$$

$$+ i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\hat{\chi}^\mu \hat{\alpha}_n^\nu}{z_2^n} + \frac{\hat{\chi}^\mu \hat{\tilde{\alpha}}_n^\nu}{\bar{z}_2^n} \right]$$

$$+ \left(\frac{\alpha'}{2} \right)^{3/2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\hat{p}^\mu \hat{\alpha}_n^\nu}{z_2^n} + \frac{\hat{p}^\mu \hat{\tilde{\alpha}}_n^\nu}{\bar{z}_2^n} \right] \ln(|z_1|^2)$$

$$- \frac{\alpha'}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_m^\nu}{z_2^m} + \frac{\hat{\tilde{\alpha}}_m^\nu}{\bar{z}_2^m} \right]$$

The objects that don't commute are in the 2nd term $-\frac{i\alpha' \hat{p}^\mu \hat{x}^\nu \ln(|z_1|^2)}{2}$ & the

last term

$$-\frac{\alpha'}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_m^\nu}{z_2^m} + \frac{\hat{\tilde{\alpha}}_m^\nu}{\bar{z}_2^m} \right]$$

So we can exchange $\hat{p}^\mu$ & $\hat{x}^\nu$ in 1st term to get,

$$\begin{aligned} & -\frac{i\alpha' \hat{p}^\mu \hat{x}^\nu \ln(|z_1|^2)}{2} \\ &= -\frac{i\alpha' \hat{x}^\nu \hat{p}^\mu \ln(|z_1|^2)}{2} - \underbrace{\frac{\eta^{\mu\nu} \alpha' \ln(|z_1|^2)}{2}}_{\text{Extra term}} \end{aligned}$$

and in the last term can be expanded keeping summing indices +ve and then take care of the commutations.

$$\begin{aligned}
& -\frac{\alpha'}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_m^\nu}{z_2^m} + \frac{\hat{\tilde{\alpha}}_m^\nu}{\bar{z}_2^m} \right] \\
&= -\frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left(\left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right] - \left[\frac{\hat{\alpha}_{-n}^\mu}{z_1^{-n}} + \frac{\hat{\tilde{\alpha}}_{-n}^\mu}{\bar{z}_1^{-n}} \right] \right) \\
&\quad \times \left(\left[\frac{\hat{\alpha}_m^\nu}{z_2^m} + \frac{\hat{\tilde{\alpha}}_m^\nu}{\bar{z}_2^m} \right] - \left[\frac{\hat{\alpha}_{-m}^\nu}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^\nu}{\bar{z}_2^{-m}} \right] \right) \\
&= -\frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\begin{aligned}
& \left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_m^\nu}{z_2^m} + \frac{\hat{\tilde{\alpha}}_m^\nu}{\bar{z}_2^m} \right] \\
& - \left[\frac{\hat{\alpha}_n^\mu}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^\mu}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_{-m}^\nu}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^\nu}{\bar{z}_2^{-m}} \right] \\
& - \left[\frac{\hat{\alpha}_{-n}^\mu}{z_1^{-n}} + \frac{\hat{\tilde{\alpha}}_{-n}^\mu}{\bar{z}_1^{-n}} \right] \left[\frac{\hat{\alpha}_m^\nu}{z_2^m} + \frac{\hat{\tilde{\alpha}}_m^\nu}{\bar{z}_2^m} \right] \\
& + \left[\frac{\hat{\alpha}_{-n}^\mu}{z_1^{-n}} + \frac{\hat{\tilde{\alpha}}_{-n}^\mu}{\bar{z}_1^{-n}} \right] \left[\frac{\hat{\alpha}_{-m}^\nu}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^\nu}{\bar{z}_2^{-m}} \right]
\end{aligned} \right]
\end{aligned}$$

Only 2nd term in the above expression gives a commutation relation.

$$\begin{aligned}
& \frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \\
&= \frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n} \right] \\
&+ \frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n}, \frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \\
&\left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n}, \frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \\
&= \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n}, \frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} \right] + \left[\frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n}, \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \\
&= \frac{n \eta^{\mu\nu} \delta_{nm}}{z_1^n z_2^{-m}} + \frac{n \eta^{\mu\nu} \delta_{nm}}{\bar{z}_1^n \bar{z}_2^{-m}} \\
&\rightarrow \frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n}, \frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \\
&= \frac{\alpha'}{2} \eta^{\mu\nu} \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{z_2^m}{z_1^m} + \frac{\bar{z}_2^m}{\bar{z}_1^m} \right]
\end{aligned}$$

$$\begin{aligned}
& \frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n} \right] \left[\frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \\
&= \frac{\alpha'}{2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{nm} \left[\frac{\hat{\alpha}_{-m}^{\nu}}{z_2^{-m}} + \frac{\hat{\tilde{\alpha}}_{-m}^{\nu}}{\bar{z}_2^{-m}} \right] \left[\frac{\hat{\alpha}_n^{\mu}}{z_1^n} + \frac{\hat{\tilde{\alpha}}_n^{\mu}}{\bar{z}_1^n} \right] \\
&\quad + \frac{\alpha'}{2} \eta^{\mu\nu} \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{z_2^m}{z_1^m} + \frac{\bar{z}_2^m}{\bar{z}_1^m} \right]
\end{aligned}$$

Thus the extra terms appearing from normal ordering are

$${}^{ee} -\frac{\eta^{\mu\nu} \alpha'}{2} \ln(|z_1|^2) + \frac{\alpha'}{2} \eta^{\mu\nu} \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{z_2^m}{z_1^m} + \frac{\bar{z}_2^m}{\bar{z}_1^m} \right]$$

So " $\hat{\chi}^{\mu}(z_1, \bar{z}_1) \hat{\chi}^{\nu}(z_2, \bar{z}_2)$ " can be written as

$$\hat{\chi}^{\mu}(z_1, \bar{z}_1) \hat{\chi}^{\nu}(z_2, \bar{z}_2) = :\hat{\chi}^{\mu}(z_1, \bar{z}_1) \hat{\chi}^{\nu}(z_2, \bar{z}_2):$$

$$-\frac{\alpha' \eta^{\mu\nu}}{2} \left(\ln(|z_1|^2) - \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{z_2^m}{z_1^m} + \frac{\bar{z}_2^m}{\bar{z}_1^m} \right] \right)$$

But the extra terms can be simplified

Since $|\bar{z}_1|^2 > |\bar{z}_2|^2$,

$$\begin{aligned}
 \ln(|\bar{z}_1|^2) - \sum_{m=1}^{\infty} \frac{1}{m} \left[\frac{z_2^m}{\bar{z}_1^m} + \frac{\bar{z}_2^m}{\bar{z}_1^m} \right] \\
 = \ln(|\bar{z}_1|^2) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z_2^m}{\bar{z}_1^m} \right) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\bar{z}_2^m}{\bar{z}_1^m} \right) \\
 = \ln(|\bar{z}_1|^2) + \ln\left(1 - \frac{z_2}{\bar{z}_1}\right) + \ln\left(1 - \frac{\bar{z}_2}{\bar{z}_1}\right) \\
 = \ln(z_1) + \ln(\bar{z}_1) + \ln\left(1 - \frac{z_2}{\bar{z}_1}\right) + \ln\left(1 - \frac{\bar{z}_2}{\bar{z}_1}\right) \\
 = \ln(z_1 - z_2) + \ln(\bar{z}_1 - \bar{z}_2) = \ln(|z_1 - z_2|^2)
 \end{aligned}$$

$$\begin{aligned}
 \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) = & \circ \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) \circ \\
 & - \frac{\alpha' \eta^{\mu\nu}}{2} \ln(|z_1 - z_2|^2)
 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
 \circ \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) \circ = & \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) \\
 & + \frac{\alpha' \eta^{\mu\nu}}{2} \ln(|z_1 - z_2|^2)
 \end{aligned}$$

The above relation is true only if the Local operators are radially ordered.

But even the Conformal Normal ordering what we defined previously was

$$\begin{aligned} :\hat{\chi}^\mu(\bar{z}_1, \bar{\bar{z}}_1) \hat{\chi}^\nu(\bar{z}_2, \bar{\bar{z}}_2): &= \hat{\chi}^\mu(\bar{z}_1, \bar{\bar{z}}_1) \hat{\chi}^\nu(\bar{z}_2, \bar{\bar{z}}_2) \\ &\quad + \frac{\alpha' \eta^{\mu\nu}}{2} \ln(|\bar{z}_1 - \bar{z}_2|^2) \end{aligned}$$

→ For massless χ^μ CFT,

$$:\hat{\chi}^\mu(\bar{z}_1, \bar{\bar{z}}_1) \hat{\chi}^\nu(\bar{z}_2, \bar{\bar{z}}_2): = :\hat{\chi}^\mu(\bar{z}_1, \bar{\bar{z}}_1) \hat{\chi}^\nu(\bar{z}_2, \bar{\bar{z}}_2):$$

This relation need not hold for general CFTs. But special for massless χ^μ CFT.

Thus we can also get

$$\begin{aligned} :\partial_{\bar{z}_1} \hat{\chi}^\mu(\bar{z}_1, \bar{\bar{z}}_1) \hat{\chi}^\nu(\bar{z}_2, \bar{\bar{z}}_2): &= \partial_{\bar{z}_1} \hat{\chi}^\mu(\bar{z}_1, \bar{\bar{z}}_1) \hat{\chi}^\nu(\bar{z}_2, \bar{\bar{z}}_2) \\ &\quad + \frac{\alpha' \eta^{\mu\nu}}{2(\bar{z}_1 - \bar{z}_2)} \end{aligned}$$

$$\text{And, } \circ \partial_{\bar{z}_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{\bar{z}_2} \hat{\chi}^\nu(z_2, \bar{z}_2) \circ$$

$$= \partial_{\bar{z}_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{\bar{z}_2} \hat{\chi}^\nu(z_2, \bar{z}_2) + \frac{\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)^2}$$

The Wick's contractions are,

$$\overline{\hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2)} = \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) - \circ \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2) \circ$$

$$\rightarrow \overline{\hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2)} = -\frac{\alpha' \eta^{\mu\nu}}{2} \ln(|z_1 - z_2|^2)$$

Similarly,

$$\& \quad \overline{\partial_{\bar{z}_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2)} = -\frac{\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)}$$

$$\overline{\partial_{\bar{z}_1} \hat{\chi}^\mu(z_1, \bar{z}_1) \partial_{\bar{z}_2} \hat{\chi}^\nu(z_2, \bar{z}_2)} = -\frac{\alpha' \eta^{\mu\nu}}{2(z_1 - z_2)^2}$$

Thus we can write Wick's theorem even for Creation / annihilation normal ordering. Let 'F' be a general function of operators.

$$\text{Then, } \Rightarrow \underbrace{F}_{\text{Time Order}} = \circ \left[\underbrace{F}_{\text{Normal Order}} + \sum (\text{All possible Contractions}) \right] \circ$$

Let, $\Delta^{\mu\nu}(z_1, \bar{z}_1, z_2, \bar{z}_2) = \overline{\hat{\chi}^\mu(z_1, \bar{z}_1) \hat{\chi}^\nu(z_2, \bar{z}_2)}$
 be the 2 point function. Then,

$$\mathcal{F} = \text{Exp} \left[\frac{1}{2} \int d^2\bar{z}_1 d^2\bar{z}_2 \Delta^{\mu\nu}(z_1, \bar{z}_1, z_2, \bar{z}_2) \frac{\partial}{\partial \hat{\chi}^\mu(z_1, \bar{z}_1)} \frac{\partial}{\partial \hat{\chi}^\nu(z_2, \bar{z}_2)} \right] \circ \mathcal{F} \circ$$

Time Ordered       Normal Ordered

Since the normal ordering constants are Zero in the expansion of Virasoro generators, we can write,

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \circ \hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu \circ \eta_{\mu\nu}$$

$$\hat{\tilde{L}}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \circ \hat{\tilde{\alpha}}_{m-n}^\mu \hat{\tilde{\alpha}}_n^\nu \circ \eta_{\mu\nu}$$

Linear Dilation CFT

Here, all eqⁿs change according to

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \circ \hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu \circ \eta_{\mu\nu} + i \sqrt{\frac{\alpha'}{2}} (m+1) V_\mu \hat{\alpha}_m^\mu$$

bc CFT

Here we have the bc-ghost action on the $\mathbb{Z}$ -plane given by

$$S_{bc} = \frac{1}{2\pi} \int d^2z \, \hat{b}(z, \bar{z}) \partial_{\bar{z}} \hat{c}(z, \bar{z})$$

The eqⁿ of motions are given by,

$$\partial_{\bar{z}} \hat{b}(z, \bar{z}) = \partial_{\bar{z}} \hat{c}(z, \bar{z}) = 0$$

Thus, the ghost fields are such that,

$$\begin{aligned} \hat{b}(z, \bar{z}) &\longrightarrow \hat{b}(z) \longrightarrow \text{Holomorphic} \\ \hat{c}(z, \bar{z}) &\longrightarrow \hat{c}(z) \longrightarrow \text{Holomorphic} \end{aligned}$$

Here we shall consider only the bc holomorphic part. But still we should know that anti-holomorphic part can always incorporated & written analogously.

Since we have $\hat{b}(z)$ & $\hat{c}(z)$ as holomorphic tensor operators with weights $h_b = \lambda$ & $h_c = 1 - \lambda$ respectively, we can Laurent expand them as,

$$\hat{b}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{b}_n}{z^{n+\lambda}} \quad \& \quad \hat{c}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{c}_n}{z^{n+1-\lambda}}$$

$$\left. \begin{aligned} \hat{b}_n &= \oint_{C'} \frac{dz}{2\pi i} z^{n+\lambda-1} \hat{b}(z) \\ \hat{c}_n &= \oint_{C'} \frac{dz}{2\pi i} z^{n-\lambda} \hat{c}(z) \end{aligned} \right\} \quad C' \rightarrow \text{Contour enclosing the origin}$$

The expansions work only for integer value of ' λ '. We shall consider ' λ ' to be integer. The OPEs b/w the local operators is,

$$\bullet \quad \hat{b}(z_1) \hat{c}(z_2) \sim \frac{1}{z_1 - z_2} \sim \hat{c}(z_1) \hat{b}(z_2)$$

• $\hat{b}(z_1) \hat{b}(z_2)$ & $\hat{c}(z_1) \hat{c}(z_2)$ has no singular part as $z_1 \rightarrow z_2$.

Canonical Anti Commutation Relations

$$\text{Now, } \hat{b}_n = \oint_{C_1} \frac{dz_1}{2\pi i} z_1^{n+\lambda-1} \hat{b}(z_1) \text{ and}$$

$$\hat{c}_m = \oint_{C_2} \frac{dz_2}{2\pi i} z_2^{m-\lambda} \hat{c}(z_2) \text{ are fermionic operators.}$$

$$\{\hat{b}_n, \hat{c}_m\} = \oint_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} \left[z_1^{n+\lambda-1} z_2^{m-\lambda} \underbrace{\hat{b}(z_1) \hat{c}(z_2)}_{\text{OPE}} \right]$$

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} \left[\frac{z_1^{n+\lambda-1} z_2^{m-\lambda}}{z_1 - z_2} \right]$$

(Expand z_1 about z_2)

$$= \oint_{C_2} \frac{dz_2}{2\pi i} \text{Res}_{z_1 \rightarrow z_2} \left[\frac{z_2^{n+\lambda-1} z_2^{m-\lambda}}{z_1 - z_2} \right]$$

$$\rightarrow \{\hat{b}_n, \hat{c}_m\} = \oint_{C_2} \frac{dz_2}{2\pi i} \underbrace{z_2^{n+m-1}}_{\delta_{n,-m}}$$

$$\rightarrow \boxed{\{\hat{b}_n, \hat{c}_m\} = \delta_{n,-m}}$$

Since $\hat{b}(z_1)\hat{b}(z_2) \neq \hat{c}(z_1)\hat{c}(z_2)$ has no singular part as $z_1 \rightarrow z_2$, the Laurent modes follow,

$$\boxed{\{\hat{b}_n, \hat{b}_m\} = 0 = \{\hat{c}_n, \hat{c}_m\}}$$

The 0^{th} mode oscillators follow,

$$\{\hat{b}_0, \hat{c}_0\} = 1 \neq \{\hat{b}_0, \hat{b}_0\} = \{\hat{c}_0, \hat{c}_0\} = 0$$

They follow the usual fermionic algebra. But there can be two ground states since $\hat{b}_0 \neq \hat{c}_0$ are equivalent. Moreover, we have

$$\hat{b}_0^2 = \hat{c}_0^2 = 0$$

$\rightarrow$ So ' $\hat{b}_0$ ' has at least one non-zero state which it can annihilate. Similarly, even ' $\hat{c}_0$ ' has at least one non-zero state which it can annihilate. But since the commutation $[\hat{b}_0, \hat{c}_0]$ need not be zero, we can have two distinct such states.

We can construct a two dimensional vector space spanned by two primary states denoted by $|\uparrow\rangle$ & $|\downarrow\rangle$. They can be linearly independent & satisfy,

$$\hat{c}_0|\uparrow\rangle = 0 \quad \& \quad \hat{b}_0|\downarrow\rangle = 0$$

$$\text{Consider, } \hat{c}_0|\uparrow\rangle = 0 \Rightarrow \hat{b}_0\hat{c}_0|\uparrow\rangle = 0$$

$$\Rightarrow (1 - \hat{c}_0\hat{b}_0)|\uparrow\rangle = 0$$

$$\Rightarrow \hat{c}_0\hat{b}_0|\uparrow\rangle = |\uparrow\rangle. \text{ But, } \hat{c}_0\hat{b}_0|\downarrow\rangle \neq |\downarrow\rangle$$

$$\Rightarrow \hat{c}_0\hat{b}_0 \text{ is not Identity operator.}$$

$$\text{Let } |\nu\rangle = \hat{b}_0|\uparrow\rangle \neq 0 \Rightarrow \hat{c}_0|\nu\rangle = |\uparrow\rangle$$

$$\text{Now } \hat{b}_0|\nu\rangle = \hat{b}_0^2|\uparrow\rangle = 0$$

$$\Rightarrow \hat{b}_0|\nu\rangle = 0 \Rightarrow |\nu\rangle = |\downarrow\rangle$$

$$\text{Thus, } \underline{\hat{c}_0|\downarrow\rangle = |\uparrow\rangle}$$

Similarly, $\hat{b}_0|\downarrow\rangle = 0 \Rightarrow \hat{c}_0\hat{b}_0|\downarrow\rangle = 0$
 $\Rightarrow (1 - \hat{b}_0\hat{c}_0)|\downarrow\rangle = 0$

$\Rightarrow \hat{b}_0\hat{c}_0|\downarrow\rangle = |\downarrow\rangle$. But, $\hat{b}_0\hat{c}_0|\uparrow\rangle \neq |\uparrow\rangle$

$\Rightarrow \hat{b}_0\hat{c}_0$ is not Identity operator.

Let $|u\rangle = \hat{c}_0|\uparrow\rangle \neq 0 \Rightarrow \hat{b}_0|u\rangle = |\downarrow\rangle$

Now $\hat{c}_0|u\rangle = \hat{c}_0^2|\uparrow\rangle = 0$

$\Rightarrow \hat{c}_0|u\rangle = 0 \Rightarrow |u\rangle = |\uparrow\rangle$

Thus, $\hat{b}_0|\uparrow\rangle = |\downarrow\rangle$

Thus these operators have a two dimensional representation as follows.

Let $|\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $|\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\rightarrow \hat{b}_0$ act as lowering operator
 since $\hat{b}_0 |\downarrow\rangle = 0$ & $\hat{c}_0$ act as raising operator since $\hat{c}_0 |\uparrow\rangle = 0$. Thus we can write $\hat{b}_0$ & $\hat{c}_0$ in matrix form as.

$$\hat{b}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \& \quad \hat{c}_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

But if include other operators $\hat{b}_n$ & $\hat{c}_n$ then all these cannot be included in the 2-dim repⁿ. So, we need to extend the dimensions to infinity, to repⁿ all b_n 's & c_n 's as matrices. But we need not do that.

For $n > 0$

Consider $\{\hat{b}_n, \hat{c}_m\} = \delta_{n,-m}$

$$\Rightarrow \{\hat{b}_{-n}, \hat{c}_n\} = 1$$

Similarly, $\{\hat{c}_{-n}, \hat{b}_n\} = 1$

Thus by convention we can assign
 $\hat{b}_{-n} \neq \hat{c}_{-n}$ as raising operators.
 $\hat{b}_n \neq \hat{c}_n$ as lowering operators.

Since $\hat{b}_0|\downarrow\rangle=0$ & since ' $\hat{b}_0$ ' is lowering operator. We set,

$$\hat{b}_n|\downarrow\rangle=0=\hat{c}_n|\downarrow\rangle, n>0$$

$\Rightarrow |\downarrow\rangle$ is the bc-ghost vacuum state.

This automatically fixes $\hat{b}_n|\uparrow\rangle=0=\hat{c}_n|\uparrow\rangle$ for all $n>0$. Because,

$$\begin{aligned}\hat{b}_n|\uparrow\rangle &= \hat{b}_n\hat{c}_0|\downarrow\rangle = -\hat{c}_0\underbrace{\hat{b}_n|\downarrow\rangle}_0 \quad (\text{for } n>0) \\ &\Rightarrow \underline{\underline{\hat{b}_n|\uparrow\rangle=0}}\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{c}_n|\uparrow\rangle &= \hat{c}_n\hat{c}_0|\downarrow\rangle = -\hat{c}_0\underbrace{\hat{c}_n|\downarrow\rangle}_0 \quad (\text{for } n>0) \\ &\Rightarrow \underline{\underline{\hat{c}_n|\uparrow\rangle=0}}\end{aligned}$$

Thus we can summarize the results in bc CFT as,

$$\begin{array}{c|c}
 \hat{c}_0 |\uparrow\rangle = 0 & \hat{b}_0 |\downarrow\rangle = 0 \\
 \hat{c}_0 |\downarrow\rangle = |\uparrow\rangle & \hat{b}_0 |\uparrow\rangle = |\downarrow\rangle \\
 \hline
 \hat{b}_n |\downarrow\rangle = 0 = \hat{c}_n |\downarrow\rangle, n > 0 & \\
 \hat{b}_n |\uparrow\rangle = 0 = \hat{c}_n |\uparrow\rangle, n > 0 &
 \end{array}$$

Energy Momentum Tensor & Virasoro generators

The energy momentum tensor's holomorphic component is given by,

$$\hat{T}(z) = h_c (\partial_z \hat{b}(z)) \hat{c}(z) - h_b \hat{b}(z) (\partial_z \hat{c}(z))$$

where $h_c = \lambda$, $h_b = 1 - \lambda$,

$$\hat{b}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{b}_n}{z^{n+h_b}} \quad \& \quad \hat{c}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{c}_n}{z^{n+h_c}}$$

$$\Rightarrow \partial_z \hat{b}(z) = - \sum_{n=-\infty}^{\infty} \frac{(n+h_b) \hat{b}_n}{z^{n+h_b+1}}$$

$$\partial_z \hat{c}(z) = - \sum_{n=-\infty}^{\infty} \frac{(n+h_c) \hat{c}_n}{z^{n+h_c+1}}$$

$$\begin{aligned} \hat{T}(z) = & -h_c \sum_{n=-\infty}^{\infty} \frac{(n+h_b) \hat{b}_n}{z^{n+h_b+1}} \sum_{m=-\infty}^{\infty} \frac{\hat{c}_m}{z^{m+h_c}} \\ & + h_b \sum_{n=-\infty}^{\infty} \frac{\hat{b}_n}{z^{n+h_b}} \sum_{m=-\infty}^{\infty} \frac{(m+h_c) \hat{c}_m}{z^{m+h_c+1}} \end{aligned}$$

Thus, we get,

$$\hat{T}(z) = \sum_{n,m} \frac{-h_c(n+h_b) \hat{b}_n \hat{c}_m + h_b(m+h_c) \hat{b}_n \hat{c}_m}{z^{n+m+2}}$$

$$\rightarrow \hat{L}_k = \oint_C \frac{dz}{2\pi i} z^{k+1} T(z)$$

$$= \sum_{n,m} \left[-h_c(n+h_b) + h_b(m+h_c) \right] \hat{b}_n \hat{c}_m \delta_{k, m+n}$$

$$= \sum_{n=-\infty}^{\infty} \left[-h_c(n+h_b) - h_b(k-n+h_c) \right] \hat{b}_n \hat{c}_{k-n}$$

$$\rightarrow \hat{L}_m = \sum_{n=-\infty}^{\infty} [m\lambda - n] \hat{b}_n \hat{c}_{m-n}$$

$$\text{Now, } \{\hat{b}_n, \hat{c}_{m-n}\} = \delta_{m,0}$$

Thus only the 0^{th} mode has the normal ordering constant. So, we can write

$$\hat{L}_m = \sum_{n=-\infty}^{\infty} [m\lambda - n] : \hat{b}_n \hat{c}_{m-n} : + \delta_{m,0} A^g$$

where ' A^g ' is a normal ordering constant to be determined. It should be kept in mind that whenever the fermionic operators are exchanged, they pickup a -ve sign.

$$\begin{aligned} \text{For } m \neq 0, \quad \hat{L}_m &= \sum_{n=-\infty}^{\infty} [m\lambda - n] : \hat{b}_n \hat{c}_{m-n} : \\ &= m\lambda : \hat{b}_0 \hat{c}_m : + \sum_{n=1}^{\infty} \left([m\lambda - n] \hat{b}_n \hat{c}_{m-n} + [m\lambda + n] \hat{b}_{-n} \hat{c}_{m+n} \right) \\ &= -m\lambda \hat{c}_m \hat{b}_0 + \sum_{n=1}^{\infty} \left([n - m\lambda] \hat{c}_{m-n} \hat{b}_n + [n + m\lambda] \hat{b}_{-n} \hat{c}_{m+n} \right) \end{aligned}$$

Thus if $m > 0$, we see that the states $|\uparrow\rangle$ & $|\downarrow\rangle$ are annihilated by all $\hat{L}_m$ ($m > 0$). So these states are the primary states of the bc-theory. (where we have set ' $|\downarrow\rangle$ ' to represent bc-vacuum, which is annihilated by all the lowering operators.

Consider the generators,

$$\hat{L}_{-1} = \lambda \hat{c}_{-1} \hat{b}_0 + \sum_{n=1}^{\infty} \left([n+\lambda] \hat{c}_{-1-n} \hat{b}_n + [n-\lambda] \hat{b}_{-n} \hat{c}_{n-1} \right)$$

&

$$\hat{L}_1 = -\lambda \hat{c}_1 \hat{b}_0 + \sum_{n=1}^{\infty} \left([n-\lambda] \hat{c}_{1-n} \hat{b}_n + [n+\lambda] \hat{b}_{-n} \hat{c}_{n+1} \right)$$

$$\Rightarrow \hat{L}_{-1} |\downarrow\rangle = [1-\lambda] \hat{b}_{-1} \hat{c}_0 |\downarrow\rangle = [1-\lambda] \hat{b}_{-1} |\uparrow\rangle$$

$$\text{Then, } \hat{L}_1 \hat{L}_{-1} |\downarrow\rangle = [1-\lambda] \hat{L}_1 \hat{b}_{-1} |\uparrow\rangle$$

$$= \lambda [\lambda-1] \hat{c}_1 \hat{b}_0 \hat{b}_{-1} |\uparrow\rangle$$

$$+ [1-\lambda] \sum_{n=1}^{\infty} \left([n-\lambda] \hat{c}_{1-n} \hat{b}_n \hat{b}_{-1} |\uparrow\rangle + [n+\lambda] \hat{b}_{-n} \hat{c}_{1+n} \hat{b}_{-1} |\uparrow\rangle \right)$$

$$\begin{aligned} \Rightarrow \hat{L}_1 \hat{L}_{-1} |\downarrow\rangle &= \lambda[1-\lambda] \hat{c}_1 \hat{b}_{-1} |\downarrow\rangle \\ &+ \sum_{n=1}^{\infty} [n+\lambda] \left(\hat{b}_{-n} \hat{c}_{n,0} - \hat{b}_{-1} \underbrace{\hat{c}_{1+n}}_{0 \text{ as } n \neq 0} \right) |\uparrow\rangle \end{aligned}$$

$$\Rightarrow \hat{L}_1 \hat{L}_{-1} |\downarrow\rangle = \lambda[1-\lambda] |\downarrow\rangle$$

But $\hat{L}_{-1} \hat{L}_1 |\downarrow\rangle = 0$, since $|\downarrow\rangle$ is primary.

$$\text{Thus } \underline{[\hat{L}_1, \hat{L}_{-1}] |\downarrow\rangle = \lambda[1-\lambda] |\downarrow\rangle}$$

But by Virasoro algebra, $[\hat{L}_1, \hat{L}_{-1}] = 2\hat{L}_0$

$$\Rightarrow \hat{L}_0 |\downarrow\rangle = \frac{\lambda[1-\lambda]}{2} |\downarrow\rangle$$

$$\text{But } \hat{L}_0 = \sum_{n=-\infty}^{\infty} n : \hat{c}_{-n} \hat{b}_n : + A^g$$

$$\Rightarrow \underline{A^g = \frac{\lambda[1-\lambda]}{2}}$$

Thus, we finally get,

$$\boxed{\hat{L}_m = \sum_{n=-\infty}^{\infty} [m\lambda - n] : \hat{b}_n \hat{c}_{m-n} : + \frac{\lambda[1-\lambda]}{2} \delta_{m,0}}$$

Relation b/w Conformal Normal Ordering and Creation - Annihilation Normal Ordering

Consider the product of Laurent expansion of $\hat{b}(z_1)$ & $\hat{c}(z_2)$ where $|z_1|^2 > |z_2|^2$

$$\hat{b}(z_1) = \frac{\hat{b}_0}{z_1^{h_b}} + \sum_{n=1}^{\infty} \left[\frac{\hat{b}_n}{z_1^{n+h_b}} + \frac{\hat{b}_{-n}}{z_1^{-n+h_b}} \right]$$

$$\& \hat{c}(z_2) = \frac{\hat{c}_0}{z_2^{h_c}} + \sum_{m=1}^{\infty} \left[\frac{\hat{c}_m}{z_2^{m+h_c}} + \frac{\hat{c}_{-m}}{z_2^{-m+h_c}} \right]$$

where $h_c = \lambda$, $h_b = 1 - \lambda$. Then,

$$\begin{aligned} \hat{b}(z_1)\hat{c}(z_2) &= \frac{\hat{b}_0\hat{c}_0}{z_1^{h_b}z_2^{h_c}} + \overbrace{\frac{\hat{b}_0}{z_1^{h_b}} \sum_{m=1}^{\infty} \left[\frac{\hat{c}_m}{z_2^{m+h_c}} + \frac{\hat{c}_{-m}}{z_2^{-m+h_c}} \right]}^{\text{anti commutes}} \\ &\quad + \sum_{n=1}^{\infty} \left[\frac{\hat{b}_n}{z_1^{n+h_b}} + \frac{\hat{b}_{-n}}{z_1^{-n+h_b}} \right] \overbrace{\frac{\hat{c}_0}{z_2^{h_c}}}^{\text{anti commutes}} \\ &\quad + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\frac{\hat{b}_n}{z_1^{n+h_b}} + \frac{\hat{b}_{-n}}{z_1^{-n+h_b}} \right] \left[\frac{\hat{c}_m}{z_2^{m+h_c}} + \frac{\hat{c}_{-m}}{z_2^{-m+h_c}} \right] \end{aligned}$$

Now we shall get all creation operators to the left of all annihilation operators but keep track of anti-commutations.

$$\begin{aligned}\hat{b}(z_1)\hat{c}(z_2) &= \frac{1}{z_1^{h_b} z_2^{h_c}} - \frac{:\hat{c}_0\hat{b}_0:}{z_1^{h_b} z_2^{h_c}} \\ &\quad - \sum_{m=1}^{\infty} : \left[\frac{\hat{c}_m}{z_2^{m+h_c}} + \frac{\hat{c}_{-m}}{z_2^{-m+h_c}} \right] \frac{\hat{b}_0}{z_1^{h_b}} : \\ &\quad - \frac{:\hat{c}_0}{z_2^{h_c}} \sum_{n=1}^{\infty} \left[\frac{\hat{b}_n}{z_1^{n+h_b}} + \frac{\hat{b}_{-n}}{z_1^{-n+h_b}} \right] :\end{aligned}$$

$$+ \sum_{n,m} \left[\overbrace{\frac{\hat{b}_n \hat{c}_m}{z_1^{n+h_b} z_2^{m+h_c}}}^{\text{anti commutes}} + \overbrace{\frac{\hat{b}_{-n} \hat{c}_{-m}}{z_1^{-n+h_b} z_2^{-m+h_c}}}^{\text{anti commutes}} + \underbrace{\frac{\hat{b}_n \hat{c}_{-m}}{z_1^{n+h_b} z_2^{-m+h_c}}}_{\text{Use anti-commutation relation \& then apply normal ordering}} + \underbrace{\frac{\hat{b}_{-n} \hat{c}_m}{z_1^{-n+h_b} z_2^{m+h_c}}}_{\text{apply Normal ordering}} \right]$$

$$\begin{aligned}
&= \frac{1}{z_1^{h_b} z_2^{h_c}} + \frac{\hat{b}_0}{z_1^{h_b}} \left[\sum_{m=-\infty}^{\infty} \frac{\hat{c}_m}{z_2^{m+h_c}} \right] \\
&\quad + \sum_{n=1}^{\infty} \left[\frac{\hat{b}_n}{z_1^{n+h_b}} + \frac{\hat{b}_{-n}}{z_1^{-n+h_b}} \right] \frac{\hat{c}_0}{z_2^{h_c}} \\
&\quad + \sum_{n,m} \left[\frac{\hat{b}_n \hat{c}_m}{z_1^{n+h_b} z_2^{m+h_c}} + \frac{\hat{b}_{-n} \hat{c}_{-m}}{z_1^{-n+h_b} z_2^{-m+h_c}} \right. \\
&\quad \left. + \frac{\hat{b}_n \hat{c}_{-m}}{z_1^{n+h_b} z_2^{-m+h_c}} + \frac{\hat{b}_{-n} \hat{c}_m}{z_1^{-n+h_b} z_2^{m+h_c}} \right] \\
&\quad + \sum_{n,m} \frac{\delta_{nm}}{z_1^{n+h_b} z_2^{-m+h_c}}
\end{aligned}$$

Thus the 1st and last terms are extra after exchanging necessary operators & effectively capturing normal ordering.

$$\begin{aligned}
\Rightarrow \hat{b}(z_1) \hat{c}(z_2) &= \hat{b}(z_1) \hat{c}(z_2) + \frac{1}{z_1^{h_b} z_2^{h_c}} \\
&\quad + \sum_{n=1}^{\infty} \left[\frac{z_2^{n-h_c}}{z_1^{n+h_b}} \right]
\end{aligned}$$

$$\Rightarrow \hat{b}(z_1)\hat{c}(z_2) - :\hat{b}(z_1)\hat{c}(z_2): = \sum_{n=1}^{\infty} \left[\frac{z_2^{n-h_c}}{z_1^{n+h_b}} \right]$$

$$= \frac{1}{z_1^{h_b} z_2^{h_c}} \sum_{n=0}^{\infty} \left[\frac{z_2}{z_1} \right]^n = \frac{1}{z_1^{h_b} z_2^{h_c}} \frac{1}{1 - \frac{z_2}{z_1}}$$

$$= \frac{1}{z_1^{\lambda} z_2^{1-\lambda}} \frac{z_1}{z_1 - z_2} = \left[\frac{z_1}{z_2} \right]^{1-\lambda} \frac{1}{z_1 - z_2}$$

$$\Rightarrow \hat{b}(z_1)\hat{c}(z_2) - :\hat{b}(z_1)\hat{c}(z_2): = \left[\frac{z_1}{z_2} \right]^{1-\lambda} \frac{1}{z_1 - z_2}$$

But conformal normal ordering is defined by

$$\Rightarrow \hat{b}(z_1)\hat{c}(z_2) - :\hat{b}(z_1)\hat{c}(z_2): = \frac{1}{z_1 - z_2}$$

$$:\hat{b}(z_1)\hat{c}(z_2): - :\hat{b}(z_1)\hat{c}(z_2): = \frac{1}{z_1 - z_2} \left(\left[\frac{z_1}{z_2} \right]^{1-\lambda} - 1 \right)$$

Using this we can also find the normal ordering constants for m^{th} mode.

Finding A_m^g constant by $:T(z): - :T(z):$

Now, as $z_1 \rightarrow z_2$,

$$\lim_{z_1 \rightarrow z_2} \frac{\left(\left[\frac{z_1}{z_2}\right]^{1-\lambda} - 1\right)}{z_1 - z_2} = \lim_{\epsilon \rightarrow 0} \frac{\left(\left[1 + \frac{\epsilon}{z_2}\right]^{1-\lambda} - 1\right)}{\epsilon}$$

$\underbrace{\hspace{10em}}_{z_1 = z_2 + \epsilon}$

$$= \lim_{\epsilon \rightarrow 0} \left(\frac{1 + (1-\lambda) \frac{\epsilon}{z_2} - 1}{\epsilon} \right) = \frac{(1-\lambda)}{z_2}$$

$$\Rightarrow \hat{b}(z_1) \hat{c}(z_2) - : \hat{b}(z_1) \hat{c}(z_2) : = \left[\frac{z_1}{z_2}\right]^{1-\lambda} \frac{1}{z_1 - z_2}$$

$$: \hat{b}(z) \hat{c}(z) : - : \hat{b}(z) \hat{c}(z) : = \frac{1-\lambda}{z}$$

Now, $: \partial_{z_1} \hat{b}(z_1) \hat{c}(z_2) : - : \partial_{z_1} \hat{b}(z_1) \hat{c}(z_2) :$

$$= \frac{1-\lambda}{(z_1 - z_2) z_2} \left[\frac{z_1}{z_2}\right]^{-\lambda} - \frac{1}{(z_1 - z_2)^2} \left(\left[\frac{z_1}{z_2}\right]^{1-\lambda} - 1\right)$$

$$= \frac{1-\lambda}{(z_1 - z_2) z_1} \left[\frac{z_1}{z_2}\right]^{1-\lambda} - \frac{1}{(z_1 - z_2)^2} \left(\left[\frac{z_1}{z_2}\right]^{1-\lambda} - 1\right)$$

$$\text{As } \lim_{z_1 \rightarrow z_2} : \partial_{z_1} \hat{b}(z_1) \hat{c}(z_2) : - : \partial_{z_1} \hat{b}(z_1) \hat{c}(z_2) :$$

$$= \lim_{z_1 \rightarrow z_2} \left[\frac{(1-\lambda)}{(z_1 - z_2) z_1} \left[\frac{z_1}{z_2} \right]^{1-\lambda} - \frac{1}{(z_1 - z_2)^2} \left(\left[\frac{z_1}{z_2} \right]^{1-\lambda} - 1 \right) \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{(1-\lambda)}{\epsilon(z_2 + \epsilon)} \left[1 + \frac{\epsilon}{z_2} \right]^{1-\lambda} - \frac{1}{\epsilon^2} \left(\left[1 + \frac{\epsilon}{z_2} \right]^{1-\lambda} - 1 \right) \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{(1-\lambda)}{\epsilon(z_2 + \epsilon)} \left(1 + (1-\lambda) \frac{\epsilon}{z_2} \right) - \frac{1}{\epsilon^2} \left(1 + (1-\lambda) \frac{\epsilon}{z_2} - \frac{(1-\lambda)\lambda \epsilon^2}{2z_2^2} - 1 \right) \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{(1-\lambda)}{\epsilon z_2} \left(1 - \frac{\epsilon}{z_2} \right) \left(1 + (1-\lambda) \frac{\epsilon}{z_2} \right) - \frac{(1-\lambda)}{\epsilon z_2} + \frac{(1-\lambda)\lambda}{2z_2^2} \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\left(\frac{(1-\lambda)}{\epsilon z_2} - \frac{(1-\lambda)}{z_2^2} \right) \left(1 + (1-\lambda) \frac{\epsilon}{z_2} \right) - \frac{(1-\lambda)}{\epsilon z_2} + \frac{(1-\lambda)\lambda}{2z_2^2} \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{(1-\lambda)}{\epsilon \bar{z}_2} - \frac{(1-\lambda)}{\bar{z}_2^2} + \frac{(1-\lambda)^2}{\bar{z}_2^2} - \frac{(1-\lambda)^2 \epsilon}{\bar{z}_2^3} - \frac{(1-\lambda)}{\epsilon \bar{z}_2} + \frac{(1-\lambda)\lambda}{2\bar{z}_2^2} \right]$$

$$= -\frac{(1-\lambda)}{\bar{z}_2^2} + \frac{(1-\lambda)^2}{\bar{z}_2^2} + \frac{(1-\lambda)\lambda}{2\bar{z}_2^2}$$

$$= \left[\frac{-1}{\bar{z}_2^2} + \frac{(1-\lambda)}{\bar{z}_2^2} + \frac{\lambda}{2\bar{z}_2^2} \right] (1-\lambda) = -\frac{\lambda(1-\lambda)}{2\bar{z}_2^2}$$

$$\Rightarrow \underline{\underline{:\partial_{\bar{z}} \hat{b}(z) \hat{c}(z): - :\partial_{\bar{z}} \hat{b}(z) \hat{c}(z): = -\frac{\lambda(1-\lambda)}{2\bar{z}^2}}}$$

Similarly we can find,

$$:\hat{b}(z_1) \partial_{\bar{z}_2} \hat{c}(z_2): - :\hat{b}(z_1) \partial_{\bar{z}_2} \hat{c}(z_2):$$

$$= \partial_{\bar{z}_2} \left[\frac{1}{z_1 - \bar{z}_2} \left(\left(\frac{z_1}{\bar{z}_2} \right)^{1-\lambda} - 1 \right) \right]$$

$$= -\frac{(1-\lambda)}{(z_1 - \bar{z}_2) \bar{z}_2} \left(\frac{z_1}{\bar{z}_2} \right)^{1-\lambda} + \frac{1}{(z_1 - \bar{z}_2)^2} \left(\left(\frac{z_1}{\bar{z}_2} \right)^{1-\lambda} - 1 \right)$$

$$\text{As } \lim_{z_1 \rightarrow z_2} : \hat{b}(z_1) \partial_{z_2} \hat{c}(z_2) : - : \hat{b}(z_1) \partial_{z_2} \hat{c}(z_2) :$$

$$= \lim_{z_1 \rightarrow z_2} \left[\frac{-(1-\lambda)}{(z_1 - z_2) z_2} \left[\frac{z_1}{z_2} \right]^{1-\lambda} + \frac{1}{(z_1 - z_2)^2} \left(\left[\frac{z_1}{z_2} \right]^{1-\lambda} - 1 \right) \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{-(1-\lambda)}{\epsilon z_2} \left[1 + \frac{\epsilon}{z_2} \right]^{1-\lambda} + \frac{1}{\epsilon^2} \left(\left[1 + \frac{\epsilon}{z_2} \right]^{1-\lambda} - 1 \right) \right]$$

$$= \lim_{\epsilon \rightarrow 0} \left[\frac{-(1-\lambda)}{\epsilon z_2} \left(1 + (1-\lambda) \frac{\epsilon}{z_2} \right) + \frac{1}{\epsilon^2} \left(1 + (1-\lambda) \frac{\epsilon}{z_2} - \frac{(1-\lambda)\lambda \epsilon^2}{2 z_2^2} - 1 \right) \right]$$

$$= \left[-\frac{(1-\lambda)\lambda}{2 z_2^2} - \frac{(1-\lambda)^2}{z_2^2} \right] = \left[\lambda - \frac{\lambda}{2} - 1 \right] \frac{(1-\lambda)}{z_2^2}$$

$$\Rightarrow \underline{: \hat{b}(z) \partial_z \hat{c}(z) : - : \hat{b}(z) \partial_z \hat{c}(z) :} = \frac{(\lambda - 2)(1-\lambda)}{2 z^2}$$

Now $:T(z): - :T(z):$ is given by

$$\begin{aligned}
 &= \left[(1-\lambda) : \partial_z \hat{b}(z) \hat{c}(z) : - \lambda : \hat{b}(z) \partial_z \hat{c}(z) : \right] \\
 &\quad - \left[(1-\lambda) : \partial_z \hat{b}(z) \hat{c}(z) : - \lambda : \hat{b}(z) \partial_z \hat{c}(z) : \right] \\
 &= (1-\lambda) \left[: \partial_z \hat{b}(z) \hat{c}(z) : - : \partial_z \hat{b}(z) \hat{c}(z) : \right] \\
 &\quad - \lambda \left[: \hat{b}(z) \partial_z \hat{c}(z) : - : \hat{b}(z) \partial_z \hat{c}(z) : \right] \\
 &= \frac{-\lambda(1-\lambda)(1-\lambda)}{2z^2} - \lambda \frac{(\lambda-2)(1-\lambda)}{2z^2} = \frac{\lambda(1-\lambda)}{2z^2}
 \end{aligned}$$

$$\Rightarrow \boxed{:T(z): - :T(z): = \frac{\lambda(1-\lambda)}{2z^2}}$$

The Normal ordering constant for the m^{th} mode Virasoro generator can be got by the difference of $:T(z): - :T(z):$, where,

$$A_m = \oint_C \frac{dz}{2\pi i} z^{m+1} [:T(z): - :T(z):]$$

$$\rightarrow \underline{A_{m=0}^g = \frac{\lambda(1-\lambda)}{2}}$$

bc-Ghost Number Operators

The ghost number current is given by $j_w(w) = -b(w)c(w)$ in the complex w -coordinates on the cylinder. The charge corresponding to this current is the Ghost number charge defined by,

$$N^g = \int_0^{2\pi} \frac{d\sigma^1}{2\pi i} :j_w(\sigma^1, \sigma^2):$$

Since ' j_w ' is conserved, N^g is a Constant. So we can choose $\sigma^2 = 0$ & then do the integral

$$N^g = \int_0^{2\pi} \frac{dw}{2\pi i} :j_w(w):, \text{ such that } \sigma^2 = 0.$$

Now, to go to z -plane, we should do a conformal transformation given by

$$z = e^{-iw} \rightarrow \partial_w z = -iz \quad \& \quad \partial_w^2 z = -z$$

Under this transformation,

$$j_z(z) = (\partial_\omega z)^{-1} j_\omega(\omega) + \left(\frac{2\lambda-1}{2}\right) (\partial_\omega z)^{-2} (\partial_\omega^2 z)$$

$$\Rightarrow j_z(z) = \frac{i}{z} j_\omega(\omega) + \left(\frac{2\lambda-1}{2}\right) \frac{1}{z}$$

$$\Rightarrow j_\omega(\omega) = -iz j_z(z) + i\left(\frac{2\lambda-1}{2}\right)$$

and $dz = -iz d\omega$. Also the integral over σ^2 from 0 to 2π becomes a Contour integral enclosing the origin. It is contour independent as $j_z(z)$ is holomorphic.

$$\Rightarrow N^g = \oint_C \frac{dz}{2\pi i} \left(\frac{i}{z}\right) \left[-iz:j_z(z): + i\left(\frac{2\lambda-1}{2}\right)\right]$$

$$\Rightarrow N^g = \oint_C \frac{dz}{2\pi i} \left[:j_z(z): - \left(\frac{2\lambda-1}{2z}\right)\right]$$

$$\Rightarrow N^g = \oint_C \frac{dz}{2\pi i} :j_z(z): - \left(\frac{2\lambda-1}{2}\right)$$

$$\Rightarrow N^g = - \oint_C \frac{dz}{2\pi i} :b(z)c(z): - \left(\frac{2\lambda-1}{2} \right)$$

We previously found that,

$$:\hat{b}(z)\hat{c}(z): - :\hat{b}(z)\hat{c}(z): = \frac{1-\lambda}{z}$$

$$\Rightarrow N^g = - \oint_C \frac{dz}{2\pi i} :\hat{b}(z)\hat{c}(z): + \lambda - 1 - \left(\frac{2\lambda-1}{2} \right)$$

$$\Rightarrow N^g = - \oint_C \frac{dz}{2\pi i} :\hat{b}(z)\hat{c}(z): - \frac{1}{2}$$

$$= - \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} :\hat{b}_n \hat{c}_m: \oint_C \frac{dz}{2\pi i} \underbrace{\frac{1}{z^{n+m+1}}}_{\delta_{n,-m}} - \frac{1}{2}$$

$$= - \sum_{n=-\infty}^{\infty} :\hat{b}_n \hat{c}_{-n}: - \frac{1}{2}$$

$$= - :\hat{b}_0 \hat{c}_0: - \sum_{n=1}^{\infty} :(\hat{b}_n \hat{c}_{-n} + \hat{b}_{-n} \hat{c}_n): - \frac{1}{2}$$

$$\Rightarrow \boxed{N^g = \hat{c}_0 \hat{b}_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{b}_n - \hat{b}_{-n} \hat{c}_n)}$$

We can calculate the commutation relations b/w ' N^g ' & mode operators are

$$[N^g, \hat{b}_m] = N^g \hat{b}_m - \hat{b}_m N^g$$

$$N^g \hat{b}_m = \hat{C}_0 \hat{b}_0 \hat{b}_m - \frac{1}{2} \hat{b}_m + \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{b}_n \hat{b}_m - \hat{b}_{-n} \hat{C}_n \hat{b}_m)$$

$$= -\hat{C}_0 \hat{b}_m \hat{b}_0 - \frac{1}{2} \hat{b}_m + \sum_{n=1}^{\infty} (-\hat{C}_{-n} \hat{b}_m \hat{b}_n - \delta_{m,-n} \hat{b}_{-n} + \hat{b}_{-n} \hat{b}_m \hat{C}_n)$$

$$= -\delta_{m,0} \hat{b}_0 - \sum_{n=1}^{\infty} (\delta_{n,m} \hat{b}_n + \delta_{m,-n} \hat{b}_{-n}) + \hat{b}_m \hat{C}_0 \hat{b}_0 - \frac{1}{2} \hat{b}_m + \hat{b}_m \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{b}_n - \hat{b}_{-n} \hat{C}_n)$$

$$\Rightarrow N^g \hat{b}_m = - \sum_{n=-\infty}^{\infty} \delta_{n,m} \hat{b}_n + \hat{b}_m N^g$$

$$\Rightarrow \boxed{[N^g, \hat{b}_m] = -\hat{b}_m} \quad \text{for all 'm'}$$

Similarly, we get

$$[N^g, \hat{C}_m] = N^g \hat{C}_m - \hat{C}_m N^g$$

$$N^g \hat{C}_m = \hat{C}_0 \hat{b}_0 \hat{C}_m - \frac{1}{2} \hat{C}_m + \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{b}_n \hat{C}_m - \hat{b}_{-n} \hat{C}_n \hat{C}_m)$$

$$= \delta_{m,0} \hat{C}_0 - \hat{C}_0 \hat{C}_m \hat{b}_0 - \frac{1}{2} \hat{C}_m + \sum_{n=1}^{\infty} (\delta_{m,-n} \hat{C}_{-n} - \hat{C}_{-n} \hat{C}_m \hat{b}_n + \hat{b}_{-n} \hat{C}_m \hat{C}_n)$$

$$= \delta_{m,0} \hat{C}_0 + \hat{C}_m \hat{C}_0 \hat{b}_0 - \frac{1}{2} \hat{C}_m + \sum_{n=1}^{\infty} (\delta_{m,-n} \hat{C}_{-n} + \hat{C}_m \hat{C}_{-n} \hat{b}_n + \delta_{m,n} \hat{C}_n - \hat{C}_m \hat{b}_{-n} \hat{C}_n)$$

$$= \delta_{m,0} \hat{C}_0 + \sum_{n=1}^{\infty} (\delta_{m,-n} \hat{C}_{-n} + \delta_{m,n} \hat{C}_n) + \hat{C}_m \left[\hat{C}_0 \hat{b}_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{b}_n - \hat{b}_{-n} \hat{C}_n) \right]$$

$\hat{C}_m N^g$

$$\Rightarrow [N^g, \hat{C}_m] = \delta_{m,0} \hat{C}_0 + \sum_{n=1}^{\infty} (\delta_{m,-n} \hat{C}_{-n} + \delta_{m,n} \hat{C}_n)$$

$$\Rightarrow [N^g, \hat{c}_m] = \sum_{n=-\infty}^{\infty} \delta_{n,m} \hat{c}_n$$

$$\Rightarrow \boxed{[N^g, \hat{c}_m] = \hat{c}_m} \quad , \text{ for all 'm'}$$

$$\text{Thus, we get, } N^g \hat{b}_m = \hat{b}_m (N^g - 1)$$

$$\& N^g \hat{c}_m = \hat{c}_m (N^g + 1)$$

Hence, $\hat{c}_m$ increases the eigen value of N^g .
And $\hat{b}_m$ decreases the eigen value of N^g .

Thus, ' N^g ' counts the difference between the number of c-type excitations & the b-type excitations.

$$\text{Now, } N^g = \hat{c}_0 \hat{b}_0 - \frac{1}{2} + \sum_{n=1}^{\infty} \underbrace{(\hat{c}_{-n} \hat{b}_n - \hat{b}_{-n} \hat{c}_n)}_{\substack{\text{annihilates} \\ \text{both } |\uparrow\rangle \& |\downarrow\rangle}}$$

$$\Rightarrow N^g |\uparrow\rangle = \hat{c}_0 \hat{b}_0 |\uparrow\rangle - \frac{1}{2} |\uparrow\rangle$$

$$= \hat{c}_0 |\downarrow\rangle - \frac{1}{2} |\uparrow\rangle = |\uparrow\rangle - \frac{1}{2} |\uparrow\rangle$$

$$\Rightarrow \boxed{N^g |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle}$$

$$\text{Similarly, } \Rightarrow N^g |\downarrow\rangle = \hat{c}_0 \underbrace{\hat{b}_0}_0 |\downarrow\rangle - \frac{1}{2} |\downarrow\rangle$$

$$\Rightarrow \boxed{N^g |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle}$$

Complete bc Ghost Action

The complete ghost action is given by,

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2 z \left[b(z, \bar{z}) \partial_{\bar{z}} c(z, \bar{z}) + \tilde{b}(z, \bar{z}) \partial_z \tilde{c}(z, \bar{z}) \right]$$

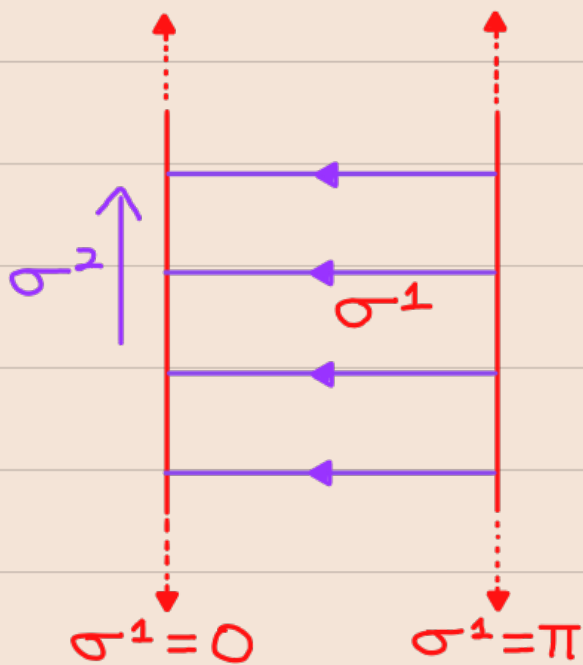
$$\rightarrow S_{\text{ghost}} = S_{bc} + S_{\tilde{b}\tilde{c}}$$

On shell, b & c are holomorphic. But $\tilde{b}$ & $\tilde{c}$ are anti-holomorphic. The $\tilde{b}$ & $\tilde{c}$ fields are completely independent of b & c . So we can say that there are two independent theories one is fully holomorphic & other is completely anti-holomorphic. All the results for $\tilde{b}\tilde{c}$ theory are similar to that of bc theory.
 $b \leftrightarrow \tilde{b}, \tilde{c} \leftrightarrow c, z \leftrightarrow \bar{z}, T \leftrightarrow \tilde{T}, L_m \leftrightarrow \tilde{L}_m, j \leftrightarrow \tilde{j}, \text{etc.}$

Open String CFT mode expansion

The open string world sheet can be thought of as an infinitely long strip whose Euclidean time coordinate $\sigma^2 = i\tau$ runs from $-\infty$ to $+\infty$, where we go from the bottom to top of the strip. Then the spatial σ^1 coordinate runs from $0 < \sigma^1 < \pi$. And the time coordinate $-\infty < \sigma^2 < +\infty$.

Infinite Strip World Sheet



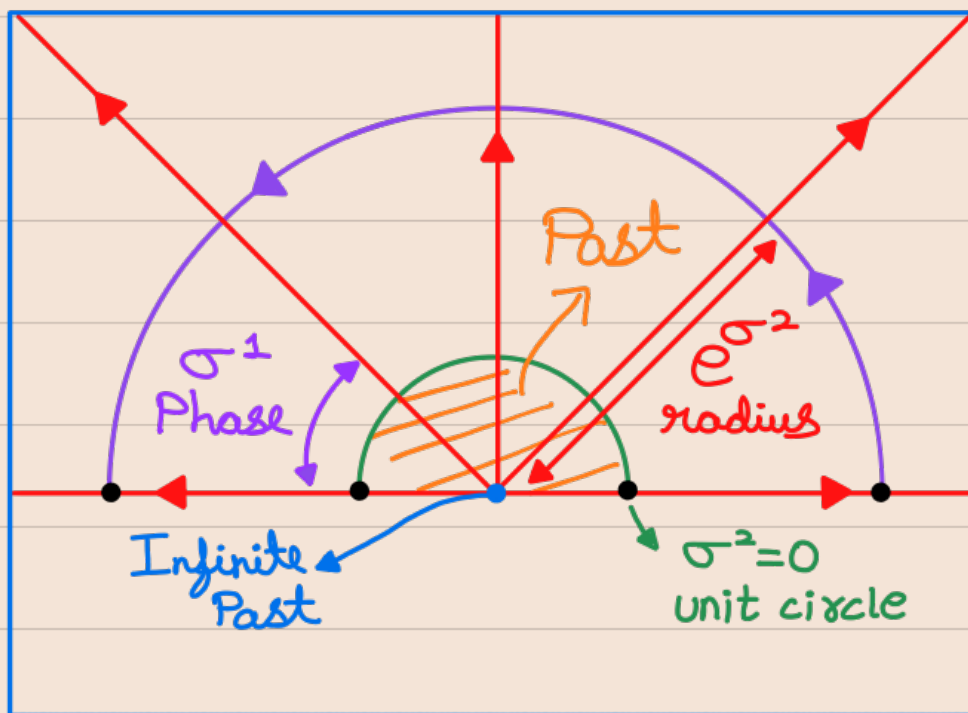
We define the complex coordinates on the strip as $\omega = \sigma^1 + i\sigma^2$ & $\bar{\omega} = \sigma^1 - i\sigma^2$. So, ω & $\bar{\omega}$ define a point on the world sheet.

We perform a conformal map of the strip to a plane by defining $z = -e^{-i\omega}$

$$\rightarrow z = -e^{(-i\sigma^1 + \sigma^2)} = e^{\sigma^2} [-\cos(\sigma^1) + i\sin(\sigma^1)]$$

The radial distance in $(z, \bar{z})$ plane relates to the euclidean time by, $\tau = e^{\sigma^2}$. So $\sigma^2 = -\infty$ (Infinite past) is the origin. The Entire Strip is mapped to just the upper half of z -plane. The end points of the String is at $\text{Im}(z)=0$.

z -plane



The doubling trick allows us to define the Energy-momentum tensor to the full z -plane by defining, $T(z) = \begin{cases} T_{zz}(z), & \text{Im}(z) > 0 \\ T_{\bar{z}\bar{z}}(z^*), & \text{Im}(z) < 0 \end{cases}$

Massless X^μ CFT on Open String

The Neumann boundary condition for the X^μ theory on a strip is given by,

$$n^a \partial_a X^\mu (\sigma^1 = \pi \text{ or } 0, \sigma^2) = 0$$

where n^a is the outward normal to boundary.
So at $\sigma^1 = 0$, $n^a = (-1, 0)$

& at $\sigma^1 = \pi$, $n^a = (1, 0)$

$$\Rightarrow \partial_1 X^\mu (\sigma^1 = 0, \sigma^2) = \partial_1 X^\mu (\sigma^1 = \pi, \sigma^2) = 0$$

$$\Rightarrow (\partial_\omega + \partial_{\bar{\omega}}) X^\mu (\omega = \pi + i\sigma^2) = 0$$

$$\& (\partial_\omega + \partial_{\bar{\omega}}) X^\mu (\omega = 0 + i\sigma^2) = 0$$

If we do a conformal map, $z = -e^{-i\omega}$
& $\bar{z} = -e^{i\bar{\omega}}$. $\partial_\omega z = -iz$, $\partial_{\bar{\omega}} \bar{z} = i\bar{z}$

$$\partial_\omega + \partial_{\bar{\omega}} = (\partial_\omega z) \partial_z + (\partial_{\bar{\omega}} \bar{z}) \partial_{\bar{z}} = -iz \partial_z + i\bar{z} \partial_{\bar{z}}$$

when $\omega = 0 + i\sigma^2$, $z = -e^{\sigma^2} < 0$ (-ve real)
 $\omega = \pi + i\sigma^2$, $z = e^{\sigma^2} > 0$ (+ve real)

Thus boundary condition in z -plane is,

$$(-i z \partial_z + i \bar{z} \partial_{\bar{z}}) \chi^\mu(z, \text{Im}(z)=0) = 0$$

$$z = \bar{z} \text{ when } \text{Im}(z) = 0$$

$$\Rightarrow \left. \partial_z \chi^\mu(z, \bar{z}) \right|_{\text{Im}(z)=0} = \left. \partial_{\bar{z}} \chi^\mu(z, \bar{z}) \right|_{\text{Im}(z)=0}$$

Thus, by doubling trick we shall consider only the holomorphic part $\partial \chi^\mu(z)$ where,

$$\partial \chi^\mu(z) = \begin{cases} \partial \chi^\mu(z, \bar{z}) & , \text{Im}(z) > 0 \\ \bar{\partial} \chi^\mu(z, z^*) & , \text{Im}(z) < 0 \end{cases}$$

In terms of Laurent modes,

$$\partial_z \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}$$

$$\partial_{\bar{z}} \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\tilde{\alpha}_n^\mu}{\bar{z}^{n+1}}$$

When $\text{Im}(z)=0$, i.e. $\bar{z}=z$, we get,

$$\partial_z \chi^\mu(z=\gamma) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{\gamma^{n+1}}$$

$$\partial_{\bar{z}} \chi^\mu(z=\gamma) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\tilde{\alpha}_n^\mu}{\gamma^{n+1}}$$

Since $\partial_z \chi^\mu(z=\gamma) = \partial_{\bar{z}} \chi^\mu(z=\gamma)$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{\gamma^{n+1}} = \sum_{n=-\infty}^{\infty} \frac{\tilde{\alpha}_n^\mu}{\gamma^{n+1}}$$

$$\Rightarrow \boxed{\alpha_n^\mu = \tilde{\alpha}_n^\mu}$$

Thus χ^μ theory on open String has only one set of oscillator modes. And the allowed total possible states & excitations have reduced by half. So, we get,

$$\partial_z \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}$$

&

$$\partial_{\bar{z}} \chi^\mu(z, \bar{z}) = -i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{\bar{z}^{n+1}}$$

$$\text{Now, } d\chi^\mu = dz \partial_z \chi^\mu(z, \bar{z}) + d\bar{z} \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

$$\Rightarrow \chi^\mu(z, \bar{z}) = \chi^\mu + \int dz \partial_z \chi^\mu(z, \bar{z}) + \int d\bar{z} \partial_{\bar{z}} \chi^\mu(z, \bar{z})$$

Thus we get,

$$\chi^\mu(z, \bar{z}) = \chi^\mu - i \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \left[\int dz \frac{1}{z^{n+1}} + \int d\bar{z} \frac{1}{\bar{z}^{n+1}} \right]$$

$$= \chi^\mu - i \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \left[\int dz \frac{1}{z} + \int d\bar{z} \frac{1}{\bar{z}} \right]$$

$$- i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n^\mu \left[\int dz \frac{1}{z^{n+1}} + \int d\bar{z} \frac{1}{\bar{z}^{n+1}} \right]$$

$$= \chi^\mu - i \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \left[\ln(z) + \ln(\bar{z}) \right]$$

$$- i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n^\mu \left[\frac{z^{-n}}{(-n)} + \frac{\bar{z}^{-n}}{(-n)} \right]$$

$$\chi^\mu(z, \bar{z}) = \chi^\mu - i \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu \ln|z|^2$$

$$+ i \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu}{n} \left[\frac{1}{z^n} + \frac{1}{\bar{z}^n} \right]$$

Now, when $|\mathbb{Z}|^2 = 1 \rightarrow \sigma^2 = 0$, $\mathbb{Z} = -e^{-i\sigma^1}$.

$$\Rightarrow \chi^\mu(\sigma^1, 0) = x^\mu + i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu}{n} \cos(n\sigma^1)$$

Then, $\int_0^\pi \frac{d\sigma^1}{\pi} \chi^\mu(\sigma^1, 0)$ is given by,

$$= x^\mu + i\sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu}{n} \underbrace{\int_0^\pi \frac{d\sigma^1}{\pi} \cos(n\sigma^1)}_{0 \text{ if } n \neq 0}$$

$$\Rightarrow x^\mu = \int_0^\pi \frac{d\sigma^1}{\pi} \chi^\mu(\sigma^1, 0) \text{ is the Initial center of mass location.}$$

To find the Center of mass momentum, we shall use the Noether currents arising due to Space-time translations. It is given by,

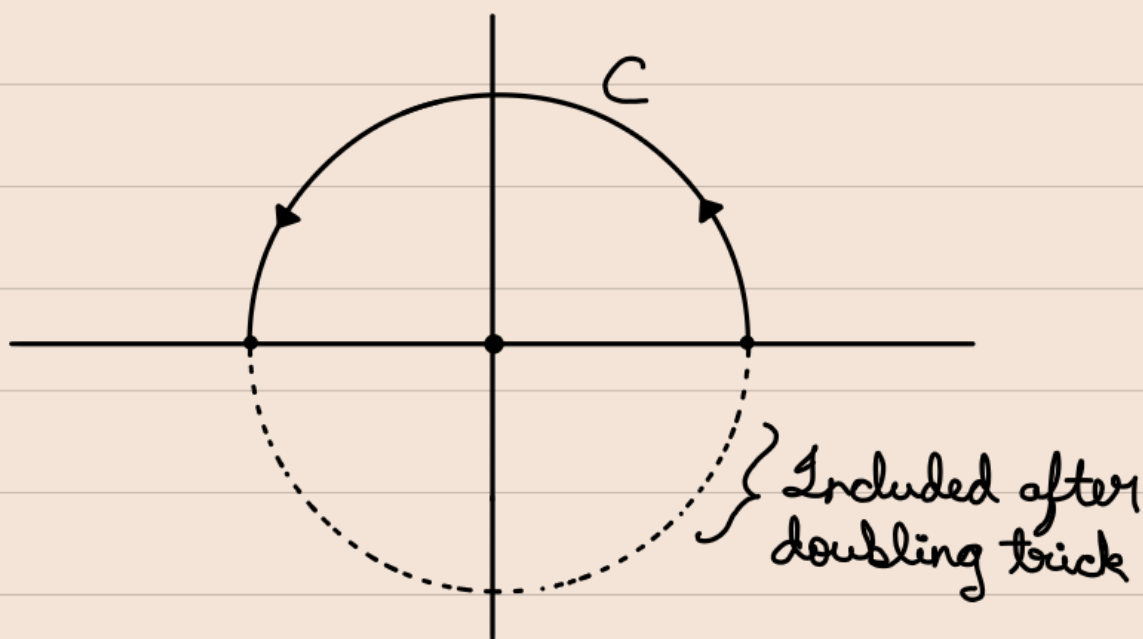
$$J_{\mathbb{Z}}^\mu = \frac{i}{\alpha'} \partial_{\mathbb{Z}} \chi^\mu(\mathbb{Z}, \bar{\mathbb{Z}}) \quad \& \quad J_{\bar{\mathbb{Z}}}^\mu = \frac{i}{\alpha'} \partial_{\bar{\mathbb{Z}}} \chi^\mu(\mathbb{Z}, \bar{\mathbb{Z}})$$

The boundary condition fixes $J_{\mathbb{Z}}^\mu$ & $J_{\bar{\mathbb{Z}}}^\mu$ at $\text{Im}(\mathbb{Z}) = 0$ real axis.

$$\rightarrow J_z^\mu(z) = J_{\bar{z}}^\mu(\bar{z}) \text{ at } \text{Im}(z) = 0 \text{ axis.}$$

Now the conserved charge is given by

$$p^\mu = \int_C \frac{dz}{2\pi i} J_z^\mu(z) - \int_C \frac{d\bar{z}}{2\pi i} J_{\bar{z}}^\mu(\bar{z})$$



Where $C: z = \gamma e^{i\theta}$, $0 < \theta < \pi$,

$\gamma = \text{constant}$, $\bar{z} = \gamma e^{-i\theta}$, $0 < \theta < \pi$

We can do the doubling trick for $J_z^\mu(z)$ and $J_{\bar{z}}^\mu(\bar{z})$ since they are equal at the boundary where we define,

$$J^\mu(z) = \begin{cases} J_z^\mu(z) & , \text{Im}(z) > 0 \\ J_{\bar{z}}^\mu(z^*) & , \text{Im}(z) < 0 \end{cases}$$

$$\rightarrow p^\mu = \frac{1}{2\pi\alpha'} \left[\int_C d\bar{z} \partial_{\bar{z}} \chi^\mu - \int_C d\bar{z} \partial_{\bar{z}} \chi^\mu \right]$$

$$\rightarrow p^\mu = \frac{i}{2\pi\alpha'} \int_0^\pi d\theta \left(z \partial_z \chi^\mu + \bar{z} \partial_{\bar{z}} \chi^\mu \right)$$

By doubling trick, for $\pi < \theta < 2\pi$ we have

$$z \partial_z \chi^\mu = z^* \partial_{z^*} \chi^\mu,$$

$$\rightarrow p^\mu = \frac{i}{2\pi\alpha'} \int_0^\pi d\theta z \partial_z \chi^\mu + \int_\pi^{2\pi} d\theta z \partial_z \chi^\mu$$

$$\rightarrow p^\mu = \frac{1}{2\pi\alpha'} \oint d\bar{z} \partial_{\bar{z}} \chi^\mu$$

$C: 0 < \theta < 2\pi$, full circle

Consider the 0^{th} mode coefficient,

$$\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{d\bar{z}}{2\pi} \partial_{\bar{z}} \chi^\mu$$

$$\rightarrow \sqrt{\frac{\alpha'}{2}} \alpha_0^\mu = \alpha' p^\mu \Rightarrow p^\mu = \underline{\underline{\frac{\alpha_0^\mu}{\sqrt{2\alpha'}}}}$$

Thus we get,

$$\chi^\mu(z, \bar{z}) = x^\mu - i\alpha' p^\mu \ln(|z|^2) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\alpha_n^\mu}{n} \left[\frac{1}{z^n} + \frac{1}{\bar{z}^n} \right]$$

and its derivatives are given by,

$$\partial_z \chi^\mu(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}$$

&

$$\partial_{\bar{z}} \chi^\mu(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{\bar{z}^{n+1}}$$

The results of χ^μ theory for open String are exactly similar to that of closed string except that the number of oscillators' modes are reduced by half. And the entire Open String theory can be described just by the holomorphic components through doubling trick which mimics the holomorphic components of the closed string theory.

The commutator relations are exactly similar to the results we got before.

$$[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = n \eta^{\mu\nu} \delta_{m, -n}, \quad [\hat{p}^\mu, \hat{\alpha}_n^\nu] = 0$$

$$[\hat{p}^\mu, \hat{p}^\nu] = 0, \quad [\hat{x}^\mu, \hat{p}^\nu] = i \eta^{\mu\nu}$$

$$[\hat{x}^\mu, \hat{\alpha}_n^\nu] = i \sqrt{2\alpha'} \eta^{\mu\nu} \delta_{n,0}.$$

The energy momentum tensor component in the entire z -plane after doubling trick is

$$T(z) = -\frac{1}{\alpha'} (\partial_z X^\mu)(\partial_z X_\mu). \text{ Where, } :T(z): = :\bar{T}(z):$$

Thus this theory has $A^x = 0$ for any mode. Since, both the types of normal orderings $:::$ & $:\circ:$ are equivalent.

So, the Virasoro Generators are

$$\underline{\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu: \eta_{\mu\nu}}$$

$$\Rightarrow \hat{L}_0 = \alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu$$

And thus the Hamiltonian is given by,

$$\hat{H} = \hat{L}_0 - \frac{C}{24} \quad \hat{L}_m \text{ \& } \tilde{C} \text{ doesn't exist}$$

$$\Rightarrow \hat{H} = \alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu - \frac{D}{24}$$

where the central charge for X^μ theory is given by 'D'.

bc - CFT on Open Strings

To define a consistent open string bc CFT, we shall work with the full ghost action given by,

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2 z \left[b(z, \bar{z}) \partial_{\bar{z}} c(z, \bar{z}) + \tilde{b}(z, \bar{z}) \partial_z \tilde{c}(z, \bar{z}) \right]$$

The equation of motion is given by,

$$\delta S = \frac{1}{2\pi} \int d^2 z \left[\delta b (\partial_{\bar{z}} c) + b \partial_{\bar{z}} (\delta c) + \delta \tilde{b} (\partial_z \tilde{c}) + \tilde{b} \partial_z (\delta \tilde{c}) \right]$$

For 2nd & 4th term do integration by parts,

$$\delta S = \frac{1}{2\pi} \int d^2 z \left[\delta b (\partial_{\bar{z}} c) - (\partial_{\bar{z}} b) \delta c + \delta \tilde{b} (\partial_z \tilde{c}) - (\partial_z \tilde{b}) \delta \tilde{c} \right] + \frac{1}{2\pi} \underbrace{\int d\bar{z} \left[b \delta c \right]_{\bar{z}\text{-boundary}} + \int d\bar{z} \left[\tilde{b} \delta \tilde{c} \right]_{\bar{z}\text{-boundary}}}_{\text{boundary term}}$$

The boundary term in $x = \text{Re}(z)$ and $y = \text{Im}(z)$ coordinates is given by

$$\frac{i}{4\pi} \int_{-\infty}^{\infty} dx (b \delta c - \tilde{b} \delta \tilde{c}) \Big|_{y=0}$$

($\bar{z}$ -boundary = z -boundary is $y=0$ line.)

On the ends of open string, $\sigma^1 = 0, \pi$.
Since, $\omega = \sigma^1 + i\sigma^2$ & $\bar{z} = -e^{-i\omega}$, the
String boundary correspond to $\text{Im}(z) = 0$

Thus for the boundary term to vanish, we
need to set $b(z) = \tilde{b}(\bar{z})$ & $c(z) = \tilde{c}(\bar{z})$ at
the string boundary, $\text{Im}(z) = 0$.

This sets b, c holomorphic & $\tilde{b}, \tilde{c}$ as
anti-holomorphic. Instead of having both
 $b(z), \tilde{b}(\bar{z})$ and $c(z), \tilde{c}(\bar{z})$ only in the
upper-half of z plane, we shall do
the doubling trick and define one set
of b & $\tilde{c}$ fields throughout z -plane.

For $\text{Im}(z) < 0$, we set $b(z) = \tilde{b}(z^*)$
and $c(z) = \tilde{c}(z^*)$. $\rightarrow$ Doubling trick.

Similarly, the Energy momentum tensor
components are given by T_{zz} & $T_{\bar{z}\bar{z}}$ defined
only on the upper half plane.

$$T_{zz} = (1 - \lambda)(\partial b)c - \lambda b(\partial c)$$

$$T_{\bar{z}\bar{z}} = (1 - \bar{\lambda})(\bar{\partial}\bar{b})\bar{c} - \lambda \bar{b}(\bar{\partial}\bar{c})$$

But to not have any current flowing out of the boundary we need to set, $T_{\omega\omega} = T_{\bar{\omega}\bar{\omega}}$ at $\sigma^1 = 0, \pi$ on the cylinder. In z -plane this becomes, $T_{zz} = T_{\bar{z}\bar{z}}$ at $\text{Im}(z) = 0$ with only one central charge.

Instead of having both T_{zz} & $T_{\bar{z}\bar{z}}$ on the z -upper plane with a boundary condition $T_{zz} = T_{\bar{z}\bar{z}}$ at $\text{Im}(z) = 0$, we shall do the doubling trick and define one function $T(z)$ throughout the whole plane given by

$$T(z) = \begin{cases} T_{zz} & , \text{Im}(z) > 0 \\ T_{\bar{z}\bar{z}} & , \text{Im}(z) < 0 \end{cases}$$

This CFT has every result same as that of a pure S_{bc} theory.