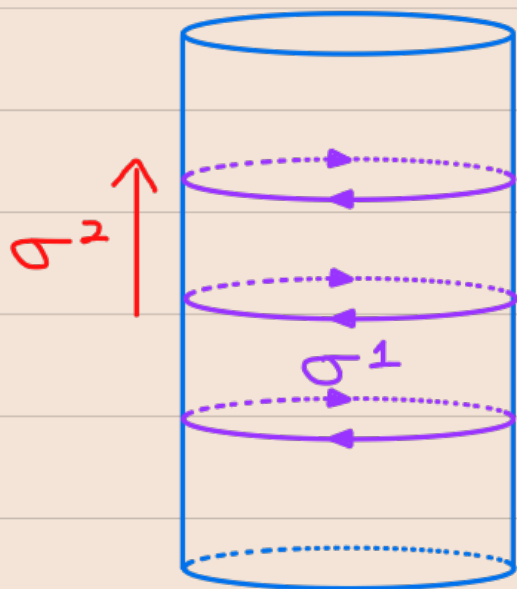


Closed String State Operator Map

Here, we will demonstrate an Isomorphism between the local operators of a CFT with the Hilbert Space of States of the same CFT under radial quantization.

Consider a CFT defined on a Cylinder. The euclidean time coordinate $\sigma^2 = i\tau$ runs from $-\infty$ to $+\infty$, which corresponds to ' σ^2 ' taking different cross sectional time slices where we go from the bottom to the top of the cylinder. Then the spatial coordinate ' σ^1 ' takes values b/w 0 to 2π which gives the vertical line with a specific angular coordinate of the surface of the cylinder. So σ^1 is periodic in ' 2π '.

$$\sigma^1 + 2\pi = \sigma^1 \quad \& \quad -\infty < \sigma^2 < +\infty$$



We define the complex coordinates on cylinder as $\omega = \sigma^1 + i\sigma^2$ & $\bar{\omega} = \sigma^1 - i\sigma^2$. So, ω & $\bar{\omega}$ define a point on the world sheet.

$\Rightarrow \omega + 2\pi = \omega$ & $\bar{\omega} + 2\pi = \bar{\omega}$
So time translation is got by change of $\sigma^2 = \text{Im}(\omega)$.

But we can perform a conformal map of Cylinder to a plane by defining

$$z = e^{-i\omega} \quad \& \quad \bar{z} = e^{+i\bar{\omega}}$$

$$\rightarrow z = e^{(-i\sigma^1 + \sigma^2)} \quad \& \quad \bar{z} = e^{(i\sigma^1 + \sigma^2)}$$

In terms of $(z, \bar{z})$ coordinates,

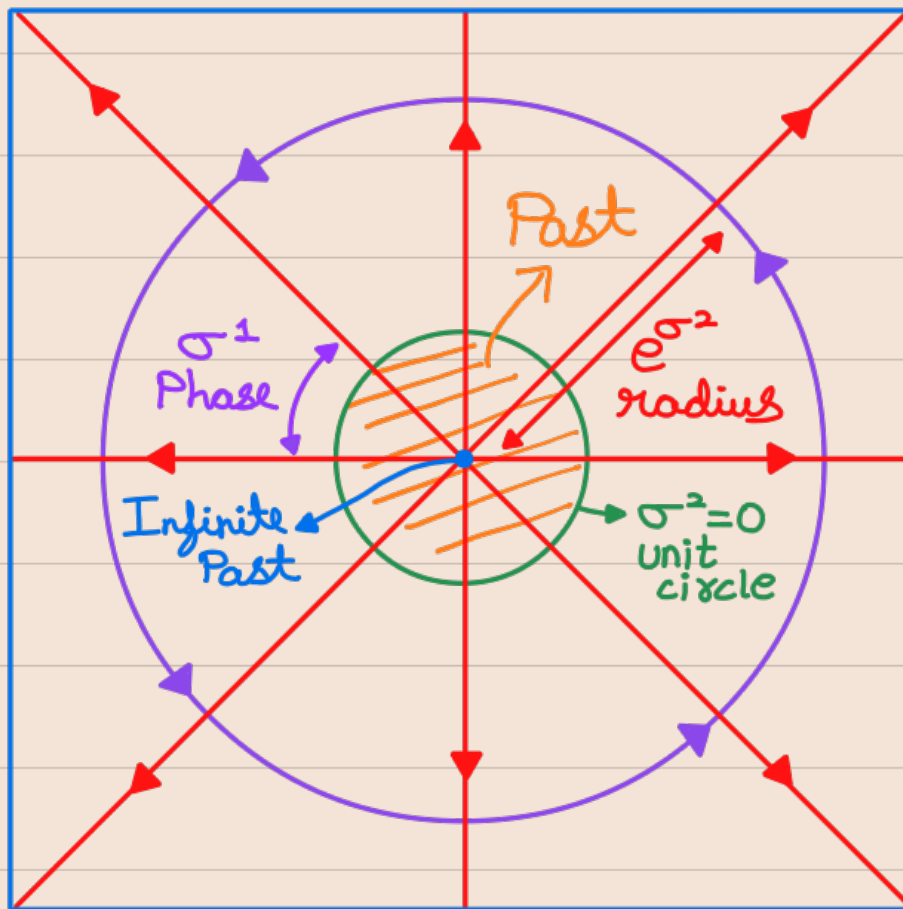
$$z = e^{\sigma^2} e^{-i\sigma^1} \quad \& \quad \bar{z} = e^{\sigma^2} e^{i\sigma^1}$$

The radial distance in $(z, \bar{z})$ plane relates to the euclidean time by, $\tau = e^{\sigma^2}$.

So $\sigma^2 = -\infty$ (Infinite past) is the origin

$\sigma^2 = 0$ is the unit circle. Then the spatial σ^1 coordinate of the cylinder gives the phase of the $(z, \bar{z})$ plane.

$(z, \bar{z})$ plane



Thus the time evolution in world Sheet cylinder corresponds to the dilation in $(z, \bar{z})$ plane (radial evolution) & Spatial evolution / translation in cylinder correspond to phase rotation in $(z, \bar{z})$ plane.

Wave Function's evolution in Path integrals

In Quantum Mechanics, the propagator is given by,

$$G(x_f, t_f, x_i, t_i) = \int_{x_i(t_i)}^{x_f(t_f)} \mathcal{D}[x(t)] e^{iS[x(t)]}$$

$G(x_f, t_f, x_i, t_i)$ propagates a particle from position $x_i(t_i)$ at initial time ' t_i ' to a position $x_f(t_f)$ at a final time ' t_f '. This actually evolves the wavefunction of the particle.

$$\Psi_f(x_f, t_f) = \int dx_i G(x_f, t_f, x_i, t_i) \Psi_i(x_i, t_i)$$

$$\Rightarrow \Psi_f(x_f, t_f) = \int dx_i \int_{x_i=x(t_i)}^{x_f=x(t_f)} \mathcal{D}[x(t)] e^{iS[x(t)]} \Psi_i(x_i, t_i)$$

So, to determine the value of the wavefunction at (x_f, t_f) , we evaluate the path integral restricting to only those paths that satisfy $x(t_f) = x_f$. And the initial wavefunction

acts as the weight factor for the integral over the initial boundary condition $x_i = x(t_i)$.

Thus, to get ' $\Psi_f(x_f, t_f)$ ' we do a path integral with all possible initial boundary condition but a fixed final boundary, where the initial wave function acts as weight.

Wave Functional evolution on a Cylinder

A wave functional is a quantum field theory equivalent of a quantum wave function. It describes the 'state' or the configuration of fields as a function of the fields itself.

Consider the quantum field theory on the cylinder described by fields ' $\phi(\sigma^a)$ ', where ' σ^a ' represents the coordinates of the cylinder. $\sigma^1 = \sigma = \text{Spatial angle}$, $\sigma^2 = \text{Euclidean time}$.

Now, the wave functional $\Psi[\phi(\sigma), \sigma^2]$ gives the complex probability amplitude for the field to take the configuration $\phi(\sigma^a)$ across the cylinder at some euclidean time ' σ^2 '.

Same like how we have the propagator in quantum mechanics which evolves the wave function, here in field theory, path integral over field configurations does the job.

Consider the initial configuration of the quantum field $\phi_i(\sigma)$ at a given euclidean time σ_i^2 . The probability amplitude for this configuration is given by the wave functional $\Psi_i[\phi_i(\sigma), \sigma_i^2]$.

The evolution of the wave functional to a final time σ_f^2 is represented by $\Psi_f[\phi_f(\sigma), \sigma_f^2]$ which gives the probability amplitude for the field to take a configuration $\phi_f(\sigma)$ at a final time σ_f^2 .

It is given by,

$$\Psi_f[\phi_f(\sigma), \sigma_f^2] = \int_{\phi_i(\sigma) = \phi(\sigma_f^1, \sigma_f^2)} \mathcal{D}[\phi_i(\sigma)] \int_{\phi_f(\sigma) = \phi(\sigma_f^1, \sigma_f^2)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \Psi_i[\phi_i(\sigma), \sigma_i^2]$$

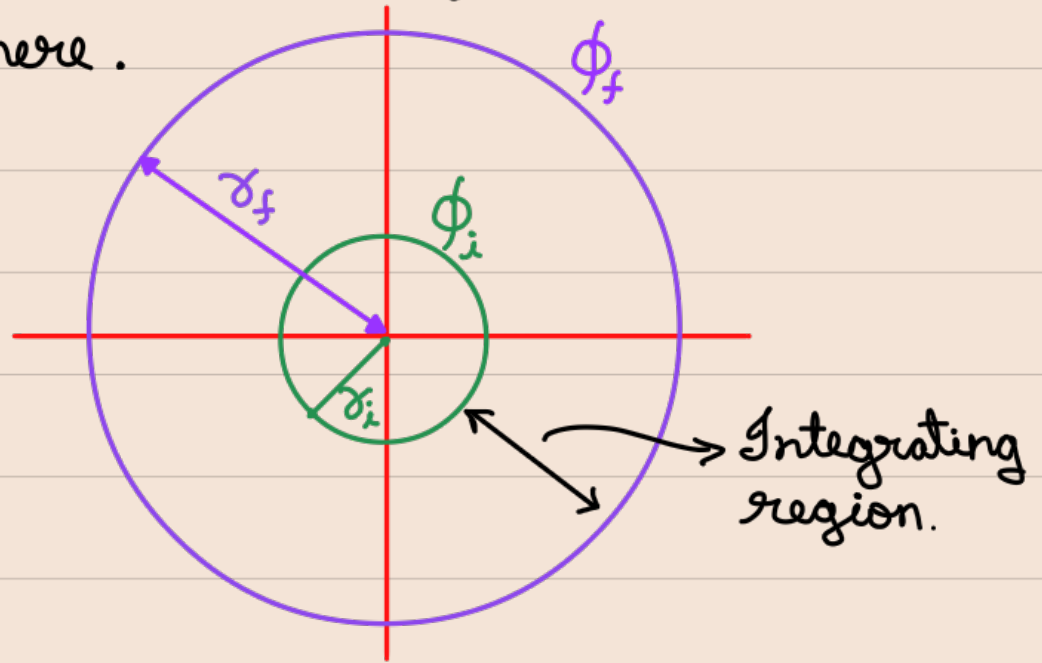
Where the quantum field propagator is

$$\mathcal{Z}(\phi_f(\sigma), \sigma_f^2, \phi_i(\sigma), \sigma_i^2) = \int_{\phi_i(\sigma) = \phi(\sigma_f^1, \sigma_f^2)}^{\phi_f(\sigma) = \phi(\sigma_f^1, \sigma_f^2)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]}$$

State operator Isomorphism: From operator to State

Consider a quantum field on the cylinder at a particular euclidean time σ_2 . This slice on the cylinder corresponds to a Circle of constant radius $r = e^{\sigma_2}$ on the $\mathbb{Z}$ -plane. So, here the quantum fields live on this circle. The time evolution in the cylinder becomes radial evolution on $\mathbb{Z}$ -plane.

Suppose the initial field configuration ϕ_i is defined on some radius ' γ_i ' $\rightarrow \phi_i = \phi(\gamma_i, \sigma) \neq$ if we want to evolve it to a final radius ' γ_f ' $\rightarrow \phi_f = \phi(\gamma_f, \sigma)$. This forms an annulus as shown here.



So the radial evolution is given by,

$$\Psi_f[\phi_f(\sigma), \gamma_f] = \int_{\phi_i(\sigma) = \phi(\gamma_i, \sigma)}^{\phi_f(\sigma) = \phi(\gamma_f, \sigma)} \mathcal{D}[\phi_i(\sigma)] \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \Psi_i[\phi_i(\sigma), \gamma_i]$$

Now, the effect of initial state is to change the weighting of the path integral over all possible initial field configurations at ' γ_i '.

But if we take the initial state to far past and ultimately to $\tau_i = 0$ ($\sigma_2 = -\infty$), then the effect of initial state is only at a point $\tau = 0$. Then $\Psi_f[\phi_f(\sigma), \tau_f]$ is got by integrating the weighted path integral over all possible field configurations $\phi_i(\sigma)$ at $\tau = 0$.

$$\Psi_f[\phi_f(\sigma), \tau_f] = \int_{\phi_i(\sigma) = \phi(0, \sigma)}^{\phi_f(\sigma) = \phi(\tau_f, \sigma)} \mathcal{D}[\phi_i(\sigma)] \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \Psi_i[\phi_i(\sigma), 0]$$

$$\Rightarrow \Psi_f[\phi_f(\sigma), \tau_f] = \int_{\phi_i(\sigma) = \phi(0, \sigma)}^{\phi_f(\sigma) = \phi(\tau_f, \sigma)} \mathcal{D}[\phi_i(\sigma)] e^{-S[\phi]} \Psi_i[\phi_i(\sigma), 0]$$

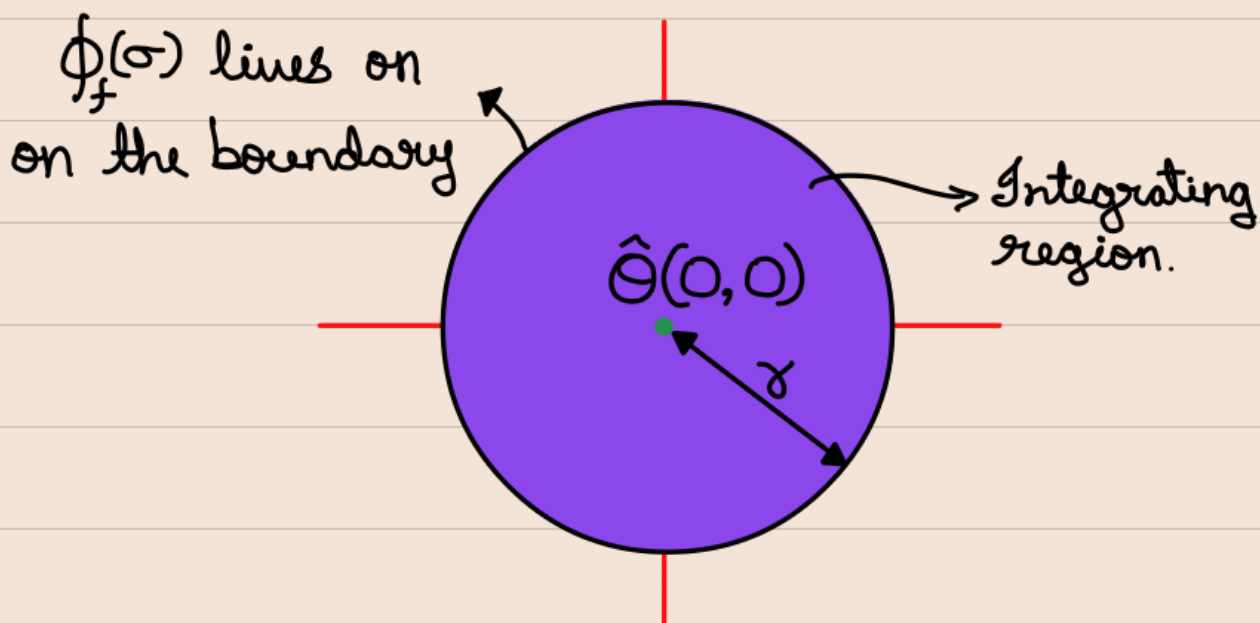
Here we are integrating over all possible field configurations within the entire disk $|z| \leq \tau_f$. This is analogous to integration $\int \mathcal{D}[\phi_i(\sigma)]$ at inner boundary. And the weighting of the path integral is changed by something acting at a single point $z = 0$. This is exactly what we mean by a Local operator.

Now, we can replace ' $\Psi_i[\phi_i(\sigma), 0]$ ' with some local operator at the origin ' $\hat{\Theta}(0, 0)$ '. Then, we get a wave functional $\Psi_{\Theta}[\phi_f(\sigma), \gamma]$ at some radius ' γ '.

$$\Psi_{\Theta}[\phi_f(\sigma), \gamma] = \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{\Theta}(0, 0)$$

$\phi_f(\sigma) = \phi(\gamma, \sigma)$

' $\phi(\gamma, \sigma)$ ' is the field configuration at the outer boundary circle of the disk of radius ' γ '.



So a wave functional of the above form got by a local operator at the origin is like the Schrodinger picture of a state. This is the State operator Isomorphism from operator to states (wave functional).

State operator Isomorphism: From State to operator

Consider an annular region $\gamma_i < |z| < \gamma_o$ with outer radius ' γ_o ', boundary $\phi_o(\sigma) = \phi(\gamma_o, \sigma)$ & inner radius ' γ_i ', boundary $\phi_i(\sigma) = \phi(\gamma_i, \sigma)$.

Let us consider some state functional ' Ψ '. In the outer boundary the state is $\Psi[\phi_o(\sigma), \gamma_o]$ & state at inner boundary is $\Psi[\phi_i(\sigma), \gamma_i]$.

The time evolution operator is given by $e^{-i\hat{H}\tau}$ for a Lorentzian cylinder. But in the euclidean cylinder where $\tau \rightarrow -i\sigma^2$, we get the euclidean time evolution operator as $e^{-\hat{H}\sigma^2}$. Now, in the z -plane we have radius given by ' e^{σ^2} '.

For $\sigma_i^2 \rightarrow \sigma_o^2$ evolution, the evolution operator is given by,

$$e^{-\hat{H}(\sigma_o^2 - \sigma_i^2)} = e^{-\hat{H}\sigma_o^2} e^{\hat{H}\sigma_i^2} = \gamma_o^{-\hat{H}} \gamma_i^{\hat{H}} = \left(\frac{\gamma_o}{\gamma_i}\right)^{-\hat{H}}$$

In terms of Virasoro generators, we have

$$\hat{H} = L_0 + \hat{\tilde{L}}_0 - \frac{c + \tilde{c}}{24}$$

So, finally the radial evolution operator from

$$\gamma_i \rightarrow \gamma_0 \text{ is } \left(\frac{\gamma_0}{\gamma_i}\right)^{-(L_0 + \hat{\tilde{L}}_0)} \left(\frac{\gamma_0}{\gamma_i}\right)^{\frac{c + \tilde{c}}{24}}$$

& radial evolution from $\gamma_0 \rightarrow \gamma_i$ is

$$\left(\frac{\gamma_0}{\gamma_i}\right)^{(L_0 + \hat{\tilde{L}}_0)} \left(\frac{\gamma_0}{\gamma_i}\right)^{\frac{-c - \tilde{c}}{24}}$$

Now, consider the path integral over the annular region given by,

$$\int \mathcal{D}[\phi_i(\sigma)] \int_{\phi_i(\sigma)}^{\phi_0(\sigma)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \left(\frac{\gamma_i}{\gamma_0}\right)^{\hat{H}} \Psi[\phi(\sigma), \gamma_i]$$

This is a path integral over the annular region from $\tau_i \rightarrow \tau_o$ with boundary conditions $\phi_i(\sigma)$ at ' τ_i ' & $\phi_o(\sigma)$ at ' τ_o ' weighted by a factor of

$$\left(\frac{\tau_i}{\tau_o}\right)^{\hat{H}} \Psi[\phi_i(\sigma), 0] = \left(\frac{\tau_o}{\tau_i}\right)^{(L_o + \hat{L}_o)} \left(\frac{\tau_o}{\tau_i}\right)^{\frac{-c - \tilde{c}}{24}} \Psi[\phi_i(\sigma), 0]$$

moreover we also integrate over all the possible field configurations $\phi_i(\sigma)$ at inner circle.

Here, the path integral, $\int_{\phi_i(\sigma)}^{\phi_o(\sigma)} \mathcal{D}[\phi_i(\sigma)] \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]}$

takes ' $\left(\frac{\tau_i}{\tau_o}\right)^{-\hat{H}} \Psi[\phi_i(\sigma), \tau_i]$ ' and brings it

to the outer circle state ' $\Psi[\phi_o(\sigma), \tau_o]$ ' with boundary condition ' $\phi_o(\sigma) = \phi(\tau_o, \sigma)$ '. But this evolution is done by the evolution operator

$$\left(\frac{\tau_o}{\tau_i}\right)^{-\hat{H}} = \left(\frac{\tau_o}{\tau_i}\right)^{-(L_o + \hat{L}_o)} \left(\frac{\tau_o}{\tau_i}\right)^{\frac{c + \tilde{c}}{24}}$$

Thus the path integral

$$\int \mathcal{D}[\phi_i(\sigma)] \int_{\phi_i(\sigma)}^{\phi_o(\sigma)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \left(\frac{\gamma_i}{\gamma_o} \right)^{-\hat{H}} \Psi[\phi_i(\sigma), \gamma_i]$$

is equivalent to operator $\left(\frac{\gamma_o}{\gamma_i} \right)^{-\hat{H}}$ acting on $\left(\frac{\gamma_i}{\gamma_o} \right)^{-\hat{H}} \Psi[\phi_i(\sigma), \gamma_i]$ & taking it to ' $\Psi[\phi_o(\sigma), \gamma_o]$ '

$$\left(\frac{\gamma_o}{\gamma_i} \right)^{-\hat{H}} \left(\frac{\gamma_i}{\gamma_o} \right)^{-\hat{H}} \Psi[\phi_i(\sigma), \gamma_i] \longrightarrow \Psi[\phi_o(\sigma), \gamma_o]$$

Thus, we can write,

$$\Psi[\phi_o(\sigma), \gamma_o] = \int \mathcal{D}[\phi_i(\sigma)] \int_{\phi_i(\sigma)}^{\phi_o(\sigma)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \left(\frac{\gamma_i}{\gamma_o} \right)^{-\hat{H}} \Psi[\phi_i(\sigma), \gamma_i]$$

Now if we take $\gamma_i \rightarrow 0$, then the annular region becomes a disk & we can write the above eqⁿ as,

$$\begin{aligned}
 \Psi[\phi_0(\sigma), \gamma_0] &= \int_{\phi_i(\sigma)}^{\phi_0(\sigma)} \mathcal{D}[\phi_i(\sigma)] \int_{\phi_i(\sigma)}^{\phi_0(\sigma)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \left(\frac{\gamma_i}{\gamma_0}\right)^{-\hat{H}} \Psi[\phi_i(\sigma), 0] \\
 &= \int_{\phi_i(\sigma)}^{\phi_0(\sigma)} \mathcal{D}[\phi_i(\sigma)] e^{-S[\phi_i]} \lim_{\gamma_i \rightarrow 0} \left(\frac{\gamma_i}{\gamma_0}\right)^{-\hat{H}} \Psi[\phi_i(\sigma), 0] \\
 &\quad \underbrace{\hspace{10em}}_{\phi_i(\sigma) \rightarrow \phi(\sigma) \text{ (dummy variable)}}
 \end{aligned}$$

incorporate

Let $\gamma_0 = \gamma$, outer radius of the disk

$$\Rightarrow \Psi[\phi_0(\sigma), \gamma] = \int_{\phi_i(\sigma)}^{\phi_0(\sigma)} \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \lim_{\gamma_i \rightarrow 0} \left(\frac{\gamma_i}{\gamma}\right)^{-\hat{H}} \Psi[\phi_i(\sigma), 0]$$

Now, the path integral over the inner circle in this limit can be seen as the defⁿ of some local operator at $z=0$, where

$$\lim_{\gamma_i \rightarrow 0} \left(\frac{\gamma_i}{\gamma}\right)^{-\hat{H}} \Psi[\phi_i(\sigma), 0] \approx \text{Some local operator}$$

State operator map to OPEs

2 operator product

Consider any two local operators $\hat{A}_i(z_1, \bar{z}_1)$ & $\hat{A}_j(z_2, \bar{z}_2)$ that are eigen states of rot^n & dilation. The OPE b/w them was found as,

$$\hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0) = \sum_k C_{ij}^k z^{(h_k - h_i - h_j)} \bar{z}^{(\tilde{h}_k - \tilde{h}_i - \tilde{h}_j)} \hat{A}_k(0, 0)$$

Where $\{\hat{A}_k\}$ are a set of local operators with weights $(h_k, \tilde{h}_k)$.

The above is an operator equation hence they live inside correlator brackets or defined inside the path integral.

Without loss of generality we shall work with unit circle as boundary dividing the z -plane to interior & exterior. This is possible because of the conformal invariance of the theory.

Let $|z| < 1$. So the path integral can be divided into integral over the fields ϕ_i on the interior of the disk, an integral over fields ϕ_b on the unit circle & an integral over fields ϕ_e on the exterior of the disk. This is the Cutting technique.

So, the LHS $\hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0)$ can be written as,

$$\int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0)$$

$$\begin{aligned} \text{Now, } S[\phi] &= \int_{\mathbb{C}^2} d^2z \mathcal{L}(\{\phi\}) \\ &= \underbrace{\int_{|z| < 1} d^2z \mathcal{L}(\{\phi_b\})}_{S_i} + \underbrace{\int_{|z| > 1} d^2z \mathcal{L}(\{\phi_b\})}_{S_e} \end{aligned}$$

$$\Rightarrow S[\phi] = S_i + S_e$$

$$\Rightarrow e^{-S[\phi]} = e^{-S_i[\phi_i]} e^{-S_e[\phi_e]}$$

Thus,

$$\begin{aligned}
 & \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0) \\
 &= \int \mathcal{D}[\phi_b(\sigma)] \int_{\phi_b(\sigma)} \mathcal{D}[\phi_e(\sigma)] e^{-S_e[\phi_e]} \underbrace{\int_{\phi_b(\sigma)}^{\phi_b(\sigma)} \mathcal{D}[\phi_i(\sigma)] e^{-S[\phi_i]} \hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0)}_{\text{This is some functional of } \phi_b(\sigma)}
 \end{aligned}$$

$$= \int \mathcal{D}[\phi_b(\sigma)] \int_{\phi_b(\sigma)} \mathcal{D}[\phi_e(\sigma)] e^{-S_e[\phi_e]} \Psi_{i,j,z,\bar{z}}[\phi_b(\sigma), \gamma=1]$$

But by state operator correspondance, the state / functional $\Psi_{i,j,z,\bar{z}}[\phi_b(\sigma), \gamma=1]$ can be replaced by a single local operator at $(0,0)$ (or anywhere unless there is no other operator)
So,

$$\Psi_{i,j,z,\bar{z}}[\phi_b(\sigma), \gamma=1] = \int_{\phi_b(\sigma)}^{\phi_b(\sigma)} \mathcal{D}[\phi_i(\sigma)] e^{-S[\phi_i]} \hat{A}_{i,j,z,\bar{z}}(0,0)$$

$$\rightarrow \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0)$$

$$= \int \mathcal{D}[\phi_b(\sigma)] \int_{\phi_b(\sigma)} \mathcal{D}[\phi_e(\sigma)] e^{-S_e[\phi_e]} \int_{\phi_b(\sigma)}^{\phi_b(\sigma)} \mathcal{D}[\phi_i(\sigma)] e^{-S[\phi_i]} \hat{A}_{i,j,z,\bar{z}}(0, 0)$$

$$\rightarrow \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{A}_i(z, \bar{z}) \hat{A}_j(0, 0) = \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{A}_{i,j,z,\bar{z}}(0, 0)$$

To put this in a standard form, expand ' $\hat{A}_{i,j,z,\bar{z}}(0, 0)$ ' in a complete set of $\hat{L}_0$, $\hat{\bar{L}}_0$ eigen states. This gives

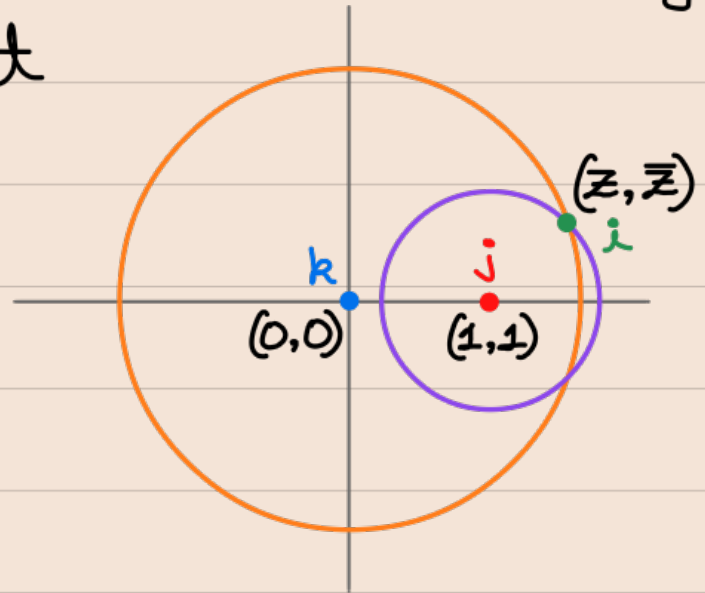
$$\hat{A}_{i,j,z,\bar{z}}(0, 0) = \sum_k C_{ij}^k z^{(h_k - h_i - h_j)} \bar{z}^{(\bar{h}_k - \bar{h}_i - \bar{h}_j)} \hat{A}_k(0, 0)$$

The convergence of this OPE is just the usual convergence of a complete set in quantum mechanics.

Three operator Product $\rightarrow$ Type 1

The product of three operators can be written as $\hat{A}_i(z, \bar{z}) \hat{A}_j(1, 1) \hat{A}_k(0, 0)$ because of the translational & scale invariance. They can be set such that

$$|z - 1| < 1 \quad \text{and} \\ |z| > 1$$



Now, we can do the $\hat{A}_i(z, \bar{z}) \hat{A}_j(1, 1)$ OPE

$$\hat{A}_i(z, \bar{z}) \hat{A}_j(1, 1) = \sum_{\ell} C_{ij}^{\ell} (z-1)^{(h_{\ell}-h_i-h_j)} (\bar{z}-1)^{(\tilde{h}_{\ell}-\tilde{h}_i-\tilde{h}_j)} \hat{A}_{\ell}(1, 1)$$

Then we can do the OPE $\hat{A}_{\ell}(1, 1) \hat{A}_k(0, 0)$

$$\hat{A}_{\ell}(1, 1) \hat{A}_k(0, 0) = \sum_m C_{\ell k}^m \hat{A}_m(0, 0)$$

$$\rightarrow \hat{A}_i(z, \bar{z}) \hat{A}_j(1, 1) \hat{A}_k(0, 0)$$

$$= \sum_{\ell} \sum_m C_{ij}^{\ell} C_{\ell k}^m (z-1)^{(h_{\ell}-h_i-h_j)} (\bar{z}-1)^{(\tilde{h}_{\ell}-\tilde{h}_i-\tilde{h}_j)} \hat{A}_m(0, 0)$$

Due to associativity, we could first do the $\hat{A}_j(1,1)\hat{A}_k(0,0)$ OPE & then with $\hat{A}_i(z,\bar{z})$.

$$\rightarrow \hat{A}_j(1,1)\hat{A}_k(0,0) = \sum_l C_{jk}^l \hat{A}_l(0,0) \quad \&$$

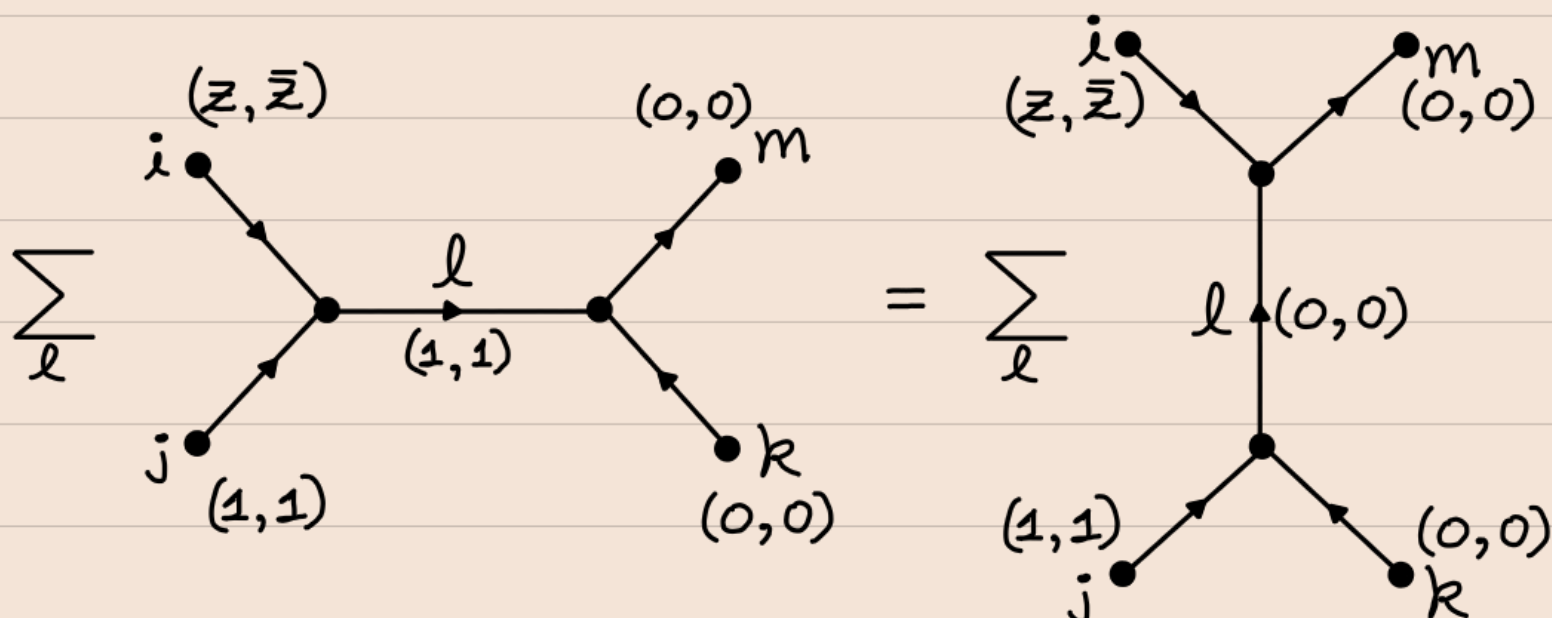
$$\hat{A}_i(z,\bar{z})\hat{A}_l(0,0) = \sum_m C_{il}^m z^{(h_m-h_i-h_l)} \bar{z}^{(\tilde{h}_m-\tilde{h}_i-\tilde{h}_l)} \hat{A}_m(0,0)$$

$$\rightarrow \hat{A}_i(z,\bar{z})\hat{A}_j(1,1)\hat{A}_k(0,0)$$

$$= \sum_l \sum_m C_{jk}^l C_{il}^m z^{(h_m-h_i-h_l)} \bar{z}^{(\tilde{h}_m-\tilde{h}_i-\tilde{h}_l)} \hat{A}_m(0,0)$$

Since the basis is complete,

$$\sum_l C_{ij}^l C_{lk}^m (z-1)^{(h_l-h_i-h_j)} (\bar{z}-1)^{(\tilde{h}_l-\tilde{h}_i-\tilde{h}_j)} = \sum_l C_{jk}^l C_{il}^m z^{(h_m-h_i-h_l)} \bar{z}^{(\tilde{h}_m-\tilde{h}_i-\tilde{h}_l)}$$

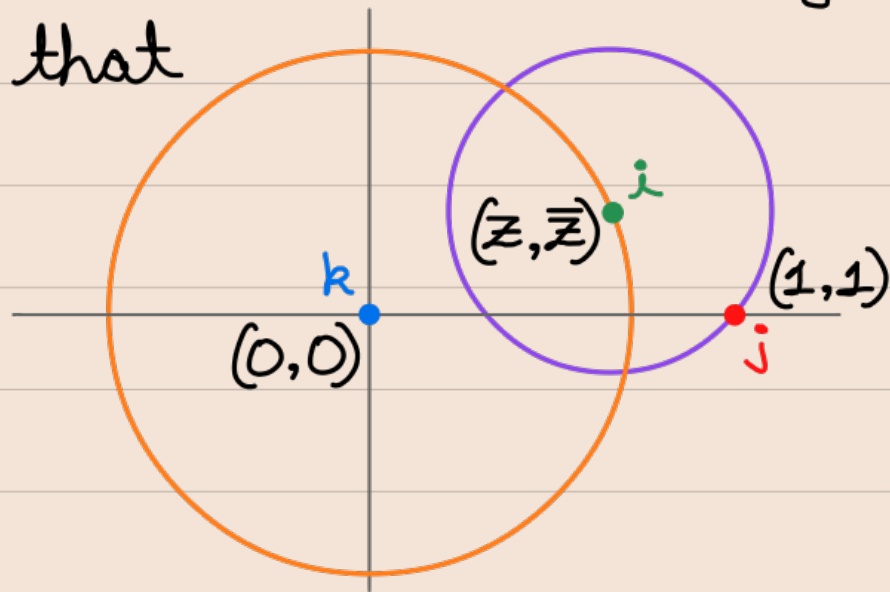


Three operator Product $\rightarrow$ Type 2

The product of three operators can also be written as $\hat{A}_j(1,1)\hat{A}_i(z,\bar{z})\hat{A}_k(0,0)$ because of the translational & Scale invariance. They can be set such that

$$|1-z| < 1 \text{ and}$$

$$|\bar{z}| < 1$$



Now, we can do the $\hat{A}_j(1,1)\hat{A}_i(z,\bar{z})$ OPE

$$\hat{A}_j(1,1)\hat{A}_i(z,\bar{z}) = \sum_{\ell} C_{ji}^{\ell} (1-z)^{(h_{\ell}-h_i-h_j)} (1-\bar{z})^{(\tilde{h}_{\ell}-\tilde{h}_i-\tilde{h}_j)} \hat{A}_{\ell}(z,\bar{z})$$

Then we can do the OPE $\hat{A}_{\ell}(z,\bar{z})\hat{A}_k(0,0)$

$$\hat{A}_{\ell}(z,\bar{z})\hat{A}_k(0,0) = \sum_m C_{\ell k}^m z^{(h_m-h_{\ell}-h_k)} \bar{z}^{(\tilde{h}_m-\tilde{h}_{\ell}-\tilde{h}_k)} \hat{A}_m(0,0)$$

$$\rightarrow \hat{A}_j(1,1)\hat{A}_i(z,\bar{z})\hat{A}_k(0,0)$$

$$= \sum_{\ell} \sum_m C_{ji}^{\ell} C_{\ell k}^m (1-z)^{(h_{\ell}-h_i-h_j)} (1-\bar{z})^{(\tilde{h}_{\ell}-\tilde{h}_i-\tilde{h}_j)} \times z^{(h_m-h_{\ell}-h_k)} \bar{z}^{(\tilde{h}_m-\tilde{h}_{\ell}-\tilde{h}_k)} \hat{A}_m(0,0)$$

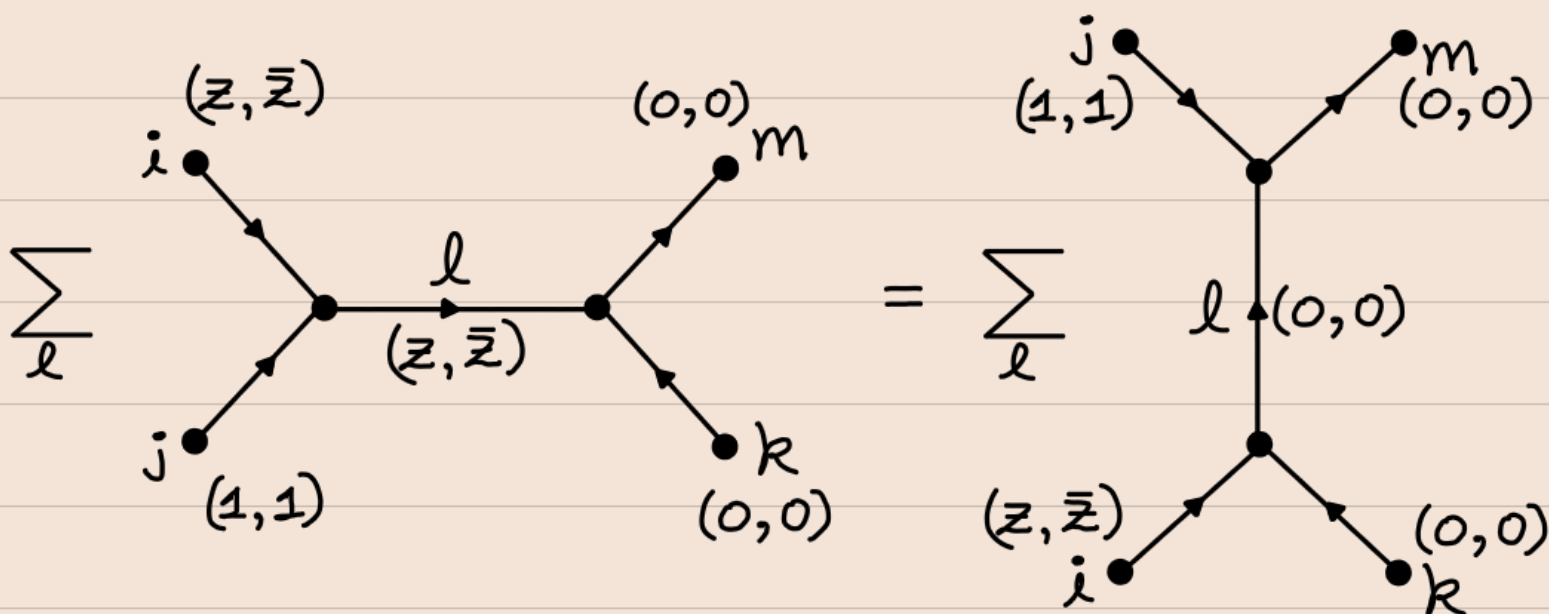
Due to associativity, we could first do the $\hat{A}_i(z, \bar{z}) \hat{A}_k(0,0)$ OPE & then with $\hat{A}_j(1,1)$.

$$\hat{A}_i(z, \bar{z}) \hat{A}_k(0,0) = \sum_l C_{ik}^l z^{(h_l - h_i - h_k)} \bar{z}^{(\tilde{h}_l - \tilde{h}_i - \tilde{h}_k)} \hat{A}_l(0,0)$$

$$\begin{aligned} &\rightarrow \hat{A}_j(1,1) \hat{A}_i(z, \bar{z}) \hat{A}_k(0,0) \\ &= \sum_l \sum_m C_{ik}^l C_{jl}^m z^{(h_m - h_j - h_l)} \bar{z}^{(\tilde{h}_m - \tilde{h}_j - \tilde{h}_l)} \hat{A}_m(0,0) \end{aligned}$$

Since the basis is complete,

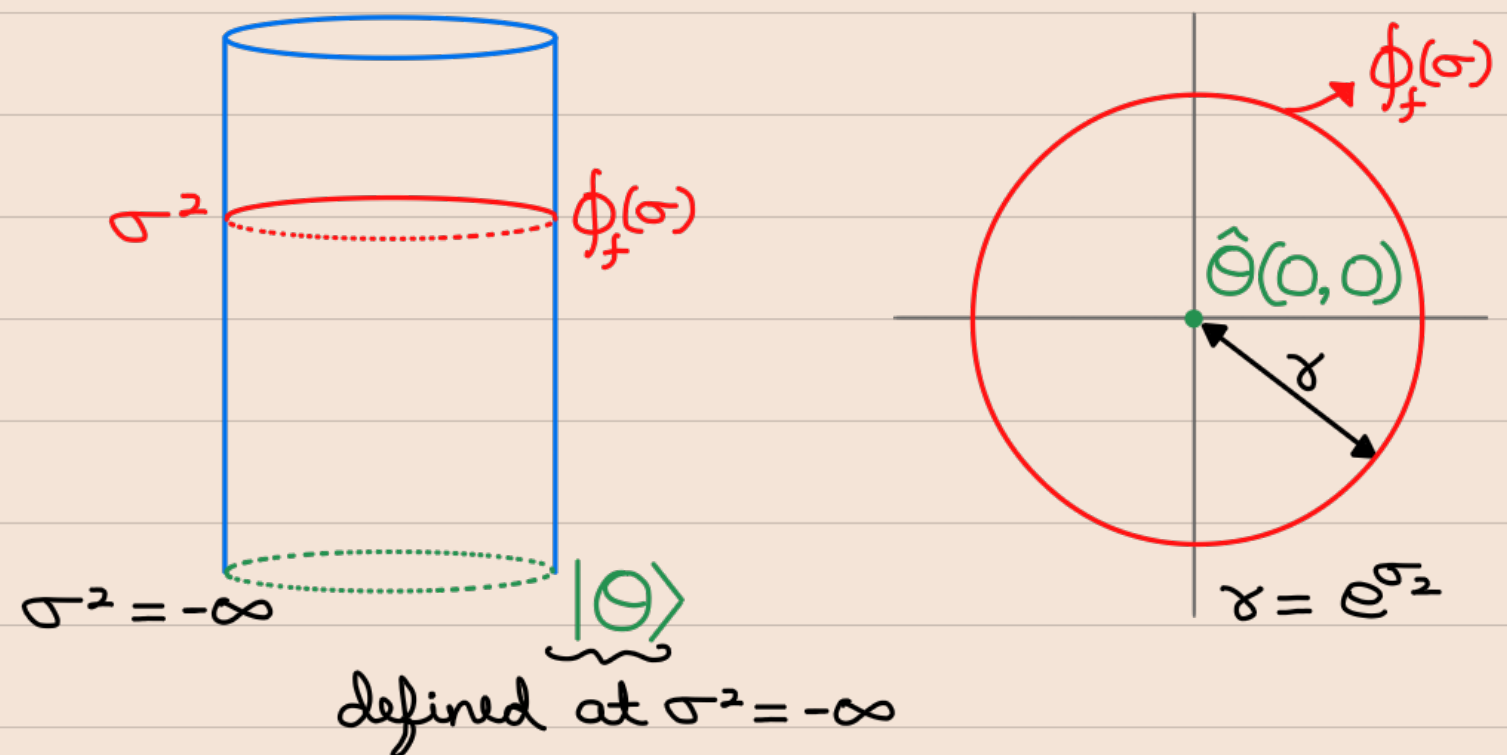
$$\begin{aligned} &\sum_l C_{ji}^l C_{lk}^m (1-z)^{(h_l - h_i - h_j)} (1-\bar{z})^{(\tilde{h}_l - \tilde{h}_i - \tilde{h}_j)} z^{(h_m - h_l - h_k)} \bar{z}^{(\tilde{h}_m - \tilde{h}_l - \tilde{h}_k)} \\ &= \sum_l C_{ik}^l C_{jl}^m z^{(h_m - h_j - h_l)} \bar{z}^{(\tilde{h}_m - \tilde{h}_j - \tilde{h}_l)} \end{aligned}$$



Vertex Operators

State-operator map is an isomorphism b/w the states & operators which is special for CFTs.

Since in CFTs the initial state is present at the origin point & the time evolution is given by radial evaluation from origin through hamiltonian & the inverse is also true since the state can be evolved back by the same hamiltonian & reach the origin. This defines a local operator at the origin called as the Vertex operator associated with that state.



Thus, path integral on a disk centered at the origin with a local operator $\hat{\Theta}(0,0)$ on the origin maps to the path integral on the cylinder with a specified initial state $|\Theta\rangle$
 $\hat{\Theta}(0,0) \Rightarrow$ Vertex Operator

Then, the action of any charge operator $\hat{Q}$ on the initial state $|\Theta\rangle$ can be written as an OPE.

$$\text{Let } \hat{Q} = \oint_C \frac{dz}{2\pi i} T(z) \Rightarrow \text{Conserved charge}$$

Since, the OPE $T(z)\hat{\Theta}(0,0)$ already lives inside the path integral, we can write the action of $\hat{Q}$ on $|\Theta\rangle$ as the contour integral of OPE.

$$\Rightarrow \hat{Q}|\Theta\rangle = \oint_C \frac{dz}{2\pi i} T(z)\hat{\Theta}(0,0) \simeq \hat{Q} \cdot \hat{\Theta}(0,0)$$

This is allowed because of the state operator isomorphism because every charge operator acting on the Hilbert space of states has an image acting on the space of local operators.

So, $\hat{Q}$ acting on $|\Theta\rangle$ maps states to states. Similarly, $\hat{Q} \cdot$ acting on $\hat{\Theta}(0,0)$ maps local operators to local operators. This also means that if instead the local operator is at some another point i.e. $\hat{\Theta}(z', \bar{z}')$, then $\hat{Q} \cdot \hat{\Theta}(z', \bar{z}')$ means ' $\hat{Q}$ ' is got by a contour integral centered around ' $(z', \bar{z}')$ ' & this is got by

$$\oint_{C: |z-z'|=\epsilon} \frac{dz}{2\pi i} T(z) \underbrace{\hat{\Theta}(z', \bar{z}')}_{\text{OPE}} \simeq \hat{Q} \cdot \hat{\Theta}(z', \bar{z}')$$

Thus, $\hat{Q} \cdot \hat{\Theta}(z', \bar{z}')$ defines its own equivalent copy of charge expanded around $(z', \bar{z}')$.

Conformal Transformations

Consider an infinitesimal conformal transformation given by, $\delta z = \epsilon \mathcal{V}(z)$ & $\delta \bar{z} = \epsilon \mathcal{V}^*(\bar{z})$. Then, by ward identity, the change in any local operator $\hat{\mathcal{O}}(z, \bar{z})$ at origin, is given by,

$$\delta \hat{\mathcal{O}}(0,0) = -\epsilon \left(\text{Res}_{z \rightarrow 0} [\mathcal{V}(z) \hat{T}(z) \hat{\mathcal{O}}(0,0)] + \overline{\text{Res}_{\bar{z} \rightarrow 0} [\mathcal{V}^*(\bar{z}) \hat{\bar{T}}(\bar{z}) \hat{\mathcal{O}}(0,0)]} \right)$$

Then, we can Laurent expand $\mathcal{V}, T$ & $\bar{T}$ about 0. Let,

$$\mathcal{V}(z) = \sum_{n=-\infty}^{\infty} \mathcal{V}_n z^{n+1} \quad \text{where} \quad \mathcal{V}_n = \oint_C \frac{dz}{2\pi i} \frac{\mathcal{V}(z)}{z^{n+2}}$$

$$\& \quad \mathcal{V}^*(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\mathcal{V}}_n \bar{z}^{n+1} \quad \text{where} \quad \bar{\mathcal{V}}_n = \oint_C \frac{d\bar{z}}{2\pi i} \frac{\mathcal{V}^*(\bar{z})}{\bar{z}^{n+2}}$$

Similarly,

$$\hat{T}(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m}{z^{m+2}} \Rightarrow \hat{L}_m = \oint_C \frac{dz}{2\pi i} z^{m+1} \hat{T}(z)$$

$$\hat{\hat{T}}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\hat{\hat{L}}_m}{\bar{z}^{m+2}} \rightarrow \hat{\hat{L}}_m = \oint_C \frac{d\bar{z}}{2\pi i} \bar{z}^{m+1} \hat{\hat{T}}(\bar{z})$$

$\rightarrow$

$$\nu(z) \hat{T}(z) \hat{\Theta}(0,0) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \nu_n z^{n+1} \frac{\hat{L}_m \cdot \hat{\Theta}(0,0)}{z^{m+2}}$$

$$\rightarrow \text{Res}_{z \rightarrow 0} [\nu(z) \hat{T}(z) \hat{\Theta}(0,0)] = \sum_{n=-\infty}^{\infty} \nu_n \hat{L}_n \cdot \hat{\Theta}(0,0)$$

Similarly,

$$\overline{\text{Res}_{\bar{z} \rightarrow 0}} [\bar{\nu}^*(\bar{z}) \hat{\hat{T}}(\bar{z}) \hat{\Theta}(0,0)] = \sum_{n=-\infty}^{\infty} \bar{\nu}_n \hat{\hat{L}}_n \cdot \hat{\Theta}(0,0)$$

$$\rightarrow \underline{\underline{\frac{-\partial \hat{\Theta}(0,0)}{\epsilon} = \sum_{n=-\infty}^{\infty} (\nu_n \hat{L}_n + \bar{\nu}_n \hat{\hat{L}}_n) \cdot \hat{\Theta}(0,0)}}$$

But $\hat{Q} = \sum_{n=-\infty}^{\infty} \nu_n \hat{L}_n \Rightarrow \text{generates } \partial z$

& $\hat{\hat{Q}} = \sum_{n=-\infty}^{\infty} \bar{\nu}_n \hat{\hat{L}}_n \Rightarrow \text{generates } \partial \bar{z}$

$$\rightarrow \boxed{\frac{-\partial \hat{\Theta}(0,0)}{\epsilon} = (\hat{Q} + \hat{\hat{Q}}) \cdot \hat{\Theta}(0,0)}$$

For Translations in $(z, \bar{z})$ plane, we need $\mathcal{V}(z) = z_0$, $\mathcal{V}^*(\bar{z}) = \bar{z}_0$. Then,
 $\mathcal{V}(z) = z_0 z^{(-1+1)} \rightarrow \mathcal{V}_{-1} = z_0 \rightarrow \mathcal{V}_n, n \neq -1 = 0$
 $\mathcal{V}^*(\bar{z}) = \bar{z}_0 \bar{z}^{(-1+1)} \rightarrow \tilde{\mathcal{V}}_{-1} = \bar{z}_0 \rightarrow \tilde{\mathcal{V}}_n, n \neq -1 = 0$

Thus ' $z_0 \hat{L}_{-1} + \bar{z}_0 \hat{\tilde{L}}_{-1}$ ' generates translations $\delta z = \epsilon z_0$ & $\delta \bar{z} = \epsilon \bar{z}_0$.

$$\rightarrow (z_0 \hat{L}_{-1} + \bar{z}_0 \hat{\tilde{L}}_{-1}) \cdot \hat{\Theta}(0,0) = -\frac{\delta \hat{\Theta}(0,0)}{\epsilon}$$

$\Rightarrow z_0 \hat{L}_{-1}$ generates translations $z \rightarrow z + \epsilon z_0$

& $\bar{z}_0 \hat{\tilde{L}}_{-1}$ generates translations $\bar{z} \rightarrow \bar{z} + \epsilon \bar{z}_0$

$$\text{But, } \delta \hat{\Theta} = -\epsilon [z_0 \partial \hat{\Theta} + \bar{z}_0 \bar{\partial} \hat{\Theta}]$$

$$\Rightarrow (z_0 \hat{L}_{-1} + \bar{z}_0 \hat{\tilde{L}}_{-1}) \cdot \hat{\Theta}(0,0) = z_0 \partial \hat{\Theta} + \bar{z}_0 \bar{\partial} \hat{\Theta}$$

$$\Rightarrow \boxed{\hat{L}_{-1} \cdot \hat{\Theta}(0,0) = \partial \hat{\Theta}} \quad \& \quad \boxed{\hat{\tilde{L}}_{-1} \cdot \hat{\Theta}(0,0) = \bar{\partial} \hat{\Theta}}$$

Thus these virasoro generators can be represented as

$$\underline{\hat{L}_{-1} \cdot = \partial} \quad \& \quad \underline{\hat{\tilde{L}}_{-1} \cdot = \bar{\partial}}$$

Thus, under State-operator Isomorphism every vertex operator correspond to a state in Radial quantization. Thus, for any general local operator,

$$\Rightarrow \underline{\hat{L}_{-1}|\theta\rangle = |\partial\theta\rangle \quad \& \quad \hat{\tilde{L}}_{-1}|\theta\rangle = |\bar{\partial}\hat{\theta}\rangle}$$

This is also evident from the OPE b/w $\hat{T}(z)$ & $\hat{\tilde{T}}(\bar{z})$ with any ' $\hat{\theta}(0,0)$ '. The 1st order Singular term has Coefficient to be $\partial\hat{\theta}(0,0)$ & $\bar{\partial}\hat{\theta}(0,0)$ respectively.

For rotations in $(z, \bar{z})$ plane, we need

$$z \rightarrow e^{i\epsilon} z \quad \& \quad \bar{z} \rightarrow e^{-i\epsilon} \bar{z}$$

$$\rightarrow \delta z = i\epsilon z \quad \delta \bar{z} = -i\epsilon \bar{z}$$

$$\rightarrow \mathcal{V}(z) = i\epsilon z \quad \& \quad \mathcal{V}^*(\bar{z}) = -i\epsilon \bar{z}$$

$$\rightarrow \mathcal{V}_0 = i \quad \& \quad \tilde{\mathcal{V}}_0 = -i$$

$$\text{Thus } \hat{Q} = i\hat{L}_0 \quad \& \quad \hat{\tilde{Q}} = -i\hat{\tilde{L}}_0$$

$$\rightarrow i(\hat{L}_0 - \hat{\tilde{L}}_0) \cdot \hat{\theta}(0,0) = \frac{-\delta\hat{\theta}(0,0)}{\epsilon} \quad \left. \vphantom{\frac{-\delta\hat{\theta}(0,0)}{\epsilon}} \right\} \text{Rotation}$$

Thus ' $i(\hat{L}_0 - \hat{\tilde{L}}_0)$ ' generates rotations.

For dilations in $(z, \bar{z})$ plane, we need

$$z \rightarrow z + \epsilon z \quad \& \quad \bar{z} \rightarrow \bar{z} + \epsilon \bar{z}$$

$$\rightarrow \delta z = \epsilon z \quad \delta \bar{z} = \epsilon \bar{z}$$

$$\rightarrow \mathcal{V}(z) = z \quad \& \quad \mathcal{V}^*(\bar{z}) = \bar{z}$$

$$\rightarrow \mathcal{V}_0 = 1 \quad \& \quad \tilde{\mathcal{V}}_0 = 1$$

$$\text{Thus } Q = L_0 \quad \& \quad \tilde{Q} = \tilde{L}_0$$

$$\rightarrow (\hat{L}_0 + \hat{\tilde{L}}_0) \cdot \hat{\Theta}(0,0) = -\frac{\delta \hat{\Theta}(0,0)}{\epsilon} \Big\} \text{Rotation}$$

$\& \hat{L}_0 + \hat{\tilde{L}}_0$ is the generator of dilations.

Eigen States of Rotations & Dilations

If $\hat{A}(z, \bar{z})$ is an eigen state under rotations, then under $z \rightarrow z' = \mathcal{C}_\zeta z$ and $\bar{z} \rightarrow \bar{z}' = \mathcal{C}_\zeta^* \bar{z}$,

$$\hat{A}'(z', \bar{z}') = \mathcal{C}_\zeta^{-h} \mathcal{C}_\zeta^{*\tilde{h}} \hat{A}(z, \bar{z})$$

where $(h, \tilde{h})$ are weights of $\hat{A}(z, \bar{z})$.

For pure infinitesimal rotations, $\mathcal{C}_\zeta = 1 + i\epsilon$
 $\& \quad \mathcal{C}_\zeta^* = 1 - i\epsilon$

This gives,

$$\partial \hat{A}(z, \bar{z}) = -i\epsilon z \partial \hat{A}(z, \bar{z}) + i\epsilon \bar{z} \bar{\partial} \hat{A}(z, \bar{z}) - i\epsilon(h - \tilde{h}) \hat{A}(z, \bar{z})$$

$$\rightarrow \partial \hat{A}(0, 0) = -i\epsilon(h - \tilde{h}) \hat{A}(0, 0)$$

$$\rightarrow (\hat{L}_0 - \tilde{L}_0) \cdot \hat{A}(0, 0) = (h - \tilde{h}) \hat{A}(0, 0)$$

Similarly, for pure dilations, $\mathcal{L}_3 = 1 + \epsilon$
 $\mathcal{L}_3^* = 1 + \epsilon$

This gives,

$$\partial \hat{A}(z, \bar{z}) = -\epsilon(h + \tilde{h}) \hat{A}(z, \bar{z}) - \epsilon[z \partial \hat{A}(z, \bar{z}) + \bar{z} \bar{\partial} \hat{A}(z, \bar{z})]$$

$$\rightarrow \partial \hat{A}(0, 0) = -\epsilon(h + \tilde{h}) \hat{A}(0, 0)$$

$$\rightarrow (\hat{L}_0 + \tilde{L}_0) \cdot \hat{A}(0, 0) = (h + \tilde{h}) \hat{A}(0, 0)$$

Thus by combining the results, we get,

$$\hat{L}_0 \cdot \hat{A}(0, 0) = h \hat{A}(0, 0)$$

$$\& \tilde{L}_0 \cdot \hat{A}(0, 0) = \tilde{h} \hat{A}(0, 0)$$

By state operator formalism, every vertex operator has a corresponding state. Thus for those operators that transform properly under rotations & dilations, their corresponding states are Eigen states of Rotations & dilations

$$\underline{\hat{L}_0 |A\rangle = h |A\rangle \quad \& \quad \tilde{\hat{L}}_0 |A\rangle = \tilde{h} |A\rangle}$$

Since $\hat{L}_0 - \tilde{\hat{L}}_0$ generates rotⁿ,

$$(\hat{L}_0 - \tilde{\hat{L}}_0) |A\rangle = (h - \tilde{h}) |A\rangle$$

$\Rightarrow S = h - \tilde{h}$ is called Spin of $\hat{A}$

Similarly, since, ' $\hat{L}_0 + \tilde{\hat{L}}_0$ ' generates Scaling,

$$(\hat{L}_0 + \tilde{\hat{L}}_0) |A\rangle = (h + \tilde{h}) |A\rangle$$

$\Rightarrow \Delta = h + \tilde{h}$ is called Scaling dimensions of $\hat{A}$

Along with this, we can also write,

$$\hat{L}_{-1} |A\rangle = |\partial A\rangle \quad \& \quad \tilde{\hat{L}}_{-1} |A\rangle = |\bar{\partial} A\rangle$$

Primary States & Primary Operators

Consider a primary operator given by $\hat{A}(0,0)$ at the origin. The OPE of $\hat{T}(z)$ with this is of the form $\hat{T}(z)\hat{A}(0,0)$ given by,

$$\hat{T}(z)\hat{A}(0,0) = \frac{h\hat{A}(0,0)}{z^2} + \frac{(\partial\hat{A})(0,0)}{z} + \text{Non Singular}$$

where $\hat{A}$ has weight $(h, \tilde{h})$. Now,

$$\hat{T}(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m}{z^{m+2}} \Rightarrow \hat{L}_m = \oint_C \frac{dz}{2\pi i} z^{m+1} \hat{T}(z)$$

$\Rightarrow \hat{L}_m$ is like the conserved charge operator
So, we get,

$$\hat{L}_m |A\rangle = \oint_C \frac{dz}{2\pi i} z^{m+1} \hat{T}(z) \hat{A}(0,0) \simeq \hat{L}_m \cdot \hat{A}(0,0)$$

So, $\hat{L}_m$ acting on $|A\rangle$ maps states to states.
Similarly, $\hat{L}_m \cdot$ acting on $\hat{A}(0,0)$ maps local operators to local operators.

Similarly, $\hat{L}_m \cdot \hat{A}(z, \bar{z})$ gives

$$\hat{L}_m \cdot \hat{A}(z, \bar{z}) \simeq \oint_{C: |z-z'|=\epsilon} \frac{dz}{2\pi i} (z-z')^{m+1} \underbrace{\hat{T}(z) \hat{A}(z, \bar{z})}_{\text{OPE}}$$

Thus every local operator at different points sees its own copy of Virasoro generators which is obtained by a Laurent expansion centered on the position of local operators.

So, we can write,

$$\rightarrow \underbrace{\hat{T}(z) \hat{A}(0,0)}_{\text{OPE}} = \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m |A\rangle}{z^{m+2}} \simeq \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m \cdot \hat{A}(0,0)}{z^{m+2}}$$

$$\rightarrow \frac{h \hat{A}(0,0)}{z^2} + \frac{(\partial \hat{A})(0,0)}{z} + \text{Non Singular} \simeq \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m \cdot \hat{A}(0,0)}{z^{m+2}}$$

$$\rightarrow \hat{L}_0 \cdot \hat{A}(0,0) = h \hat{A}(0,0), \quad \hat{L}_{-1} \cdot \hat{A}(0,0) = (\partial \hat{A})(0,0)$$

$$\text{For all } m > 0, \quad \hat{L}_m \cdot \hat{A}(0,0) = 0$$

Similarly for the anti-holomorphic part,

$$\rightarrow \hat{\tilde{L}}_0 \cdot \hat{A}(0,0) = \tilde{h} \hat{A}(0,0), \quad \hat{\tilde{L}}_{-1} \cdot \hat{A}(0,0) = (\bar{\partial} \hat{A})(0,0)$$

$$\text{For all } m > 0, \hat{\tilde{L}}_m \cdot \hat{A}(0,0) = 0$$

In terms of states in hilbert space,

$$\rightarrow \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m |A\rangle}{z^{m+2}} = \frac{h |A\rangle}{z^2} + \frac{|\partial A\rangle}{z} + \underbrace{\sum_{m=0}^{\infty} z^m |A^{(m)}\rangle}_{\text{No poles}}$$

$$\rightarrow \hat{L}_0 |A\rangle = h |A\rangle, \quad \hat{L}_{-1} |A\rangle = |\partial A\rangle$$

$$\text{For all } m > 0, \hat{L}_m |A\rangle = 0.$$

Similarly,

$$\hat{\tilde{L}}_0 |A\rangle = \tilde{h} |A\rangle, \quad \hat{\tilde{L}}_{-1} |A\rangle = |\bar{\partial} A\rangle$$

$$\text{For all } m > 0, \hat{\tilde{L}}_m |A\rangle = 0.$$

Thus the Vertex primary operator gives rise to primary states under State-operator map.

Primary States are highest weight States

A state which satisfies $\hat{L}_0|A\rangle = h|A\rangle$,
 $\hat{L}_{-1}|A\rangle = |\partial A\rangle$ & $\hat{L}_m|A\rangle = 0$ ($m > 0$).

Similarly, $\tilde{\hat{L}}_0|A\rangle = \tilde{h}|A\rangle$, $\tilde{\hat{L}}_{-1}|A\rangle = |\bar{\partial} A\rangle$
& $\tilde{\hat{L}}_m|A\rangle = 0$ ($m > 0$) is called as the
highest weight state.

And all other states obtained by the
action of $\hat{L}_{-m}$ or $\tilde{\hat{L}}_{-m}$ on ' $|A\rangle$ ' is called
as descendants. By Virasoro algebra,

$$[\hat{L}_0, \hat{L}_m] = -m \hat{L}_m \Rightarrow \hat{L}_0 \hat{L}_m = \hat{L}_m (\hat{L}_0 - m)$$

$$\& [\tilde{\hat{L}}_0, \tilde{\hat{L}}_m] = -m \tilde{\hat{L}}_m \Rightarrow \tilde{\hat{L}}_0 \tilde{\hat{L}}_m = \tilde{\hat{L}}_m (\tilde{\hat{L}}_0 - m)$$

Thus all $\hat{L}_m$ & $\tilde{\hat{L}}_m$ when $m > 0$, reduce
eigen value of ' $\hat{L}_0$ ' & $\tilde{\hat{L}}_0$. But all $\hat{L}_{-m}$
& $\tilde{\hat{L}}_{-m}$ when $m > 0$, increase the eigen
value of $\hat{L}_0$ & $\tilde{\hat{L}}_0$ respectively.

So a primary state ' $|A\rangle$ ' goes to zero, when $\hat{L}_m$ or $\hat{\bar{L}}_m$ acts on it. So the weight of ' $|A\rangle$ ' is never further reduced by any Virasoro generators ($m > 0$), but it can only be increased by action of $\hat{L}_{-m}$ ($m > 0$). So we call ' $\hat{L}_m$ ' ($m > 0$) as lowering & ' $\hat{L}_{-m}$ ' as raising operators.

If the CFTs we are dealing with have a lower bound on h & $\bar{h}$, then by repeatedly acting lowering operators will always result in a primary state such that further action of any lowering lower operator will annihilate it.

The action of raising operators on a primary state results in descendent states which have higher numerical value of weight. This structure was seen previously in the repⁿ. of Virasoro algebra & Verma module.

Unit Operator & the ground state

For any local operator ' $\hat{\Theta}(0,0)$ ' at origin, we get a corresponding state ' $|\Theta\rangle$ ' on the cylinder at $\sigma^2 = -\infty$. Then, the evolution of the state to some time ' σ^2 ' is got by the path integral where ' $\hat{\Theta}(0,0)$ ' is inserted.

$$\Psi[\phi_f(\sigma), \gamma] = \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{\Theta}(0,0)$$

$\phi_f(\sigma) = \phi(x, \sigma)$

Similarly, if we insert the local operator which is just identity ' $\hat{\mathbb{I}}$ ' at $(0,0)$, then it corresponds to a state ' $|\mathbb{I}\rangle$ ' at $\sigma^2 = -\infty$ on the cylinder. Then, the path integral

$$\int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{\mathbb{I}}(0,0) \left. \vphantom{\int} \right\} \begin{array}{l} \text{evolves the state} \\ \text{to } \sigma^2 \text{ such that} \\ \gamma = e^{\sigma^2} \end{array}$$

$\phi_f(\sigma) = \phi(x, \sigma)$

It doesn't do any disturbance & gives uniform weight.

Let $\phi_f(\sigma) = \phi(x, \sigma)$

$$\Psi_{\mathbb{I}}[\phi_f(\sigma), x] = \int \mathcal{D}[\phi(\sigma)] e^{-S[\phi]} \hat{\mathbb{I}}(0, 0)$$

In terms of evolution operator on the cylinder it can be stated as,

$$\lim_{\sigma_i^2 \rightarrow -\infty} e^{-\hat{H}(\sigma^2 - \sigma_i^2)} |\mathbb{I}\rangle$$

$$= \lim_{\sigma_i^2 \rightarrow -\infty} e^{-\hat{H}(\sigma^2 - \sigma_i^2)} \sum_n |n\rangle \langle n | \mathbb{I} \rangle$$

$$= \lim_{\sigma_i^2 \rightarrow -\infty} \sum_n e^{-E_n(\sigma^2 - \sigma_i^2)} |n\rangle \langle n | \mathbb{I} \rangle$$

$$= \lim_{\sigma_i^2 \rightarrow -\infty} \sum_n \langle n | \mathbb{I} \rangle e^{-E_n \sigma^2} e^{E_n \sigma_i^2} |n\rangle$$

For $\sigma_i^2 \rightarrow -\infty$, ' $e^{E_n \sigma_i^2}$ ' is exponentially suppressed except for the ground state of Hamiltonian whose $E_0 \sim 0$.

$$\Rightarrow \lim_{\sigma_i^2 \rightarrow -\infty} e^{-\hat{H}(\sigma^2 - \sigma_i^2)} |\mathbb{I}\rangle = e^{-E_0 \sigma^2} \langle 0 | \mathbb{I} \rangle |0\rangle$$

Thus projecting any state to infinite past on the euclidean cylinder projects on to Ground State / vacuum. Thus, evolving vacuum state without any disturbance of the local operator or including just the unit operator giving uniform weight to future time results in time evolved ground state / vacuum upto some normalization.

Thus we define $|\mathbb{I}\rangle \equiv |0\rangle \sim \Psi_{\mathbb{I}}[\phi_f(\sigma), \tau]$ as the Vacuum state ignoring normalization.

Consider the OPE of ' $\hat{T}(z)\hat{\mathbb{I}}(0,0)$ ', it has no singular points because they commute.

$$\Rightarrow \hat{T}(z)\hat{\mathbb{I}}(0,0) = \hat{\mathbb{I}}(0,0)\hat{T}(z)$$

So, the OPE has only +ve powers of 'z'.

$$\hat{T}(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m}{z^{m+2}} \Rightarrow \hat{L}_m = \oint_C \frac{dz}{2\pi i} z^{m+1} \hat{T}(z)$$

Now, $\Rightarrow \underbrace{\hat{T}(z)\hat{\mathbb{I}}(0,0)}_{\text{OPE}} = \sum_{m=-\infty}^{\infty} \frac{\hat{L}_m|\mathbb{I}\rangle}{z^{m+2}}$
 has no -ve powers

$\rightarrow$ For all $m \geq -1$ $\hat{L}_m|\mathbb{I}\rangle = 0$

Similarly, $\hat{\tilde{L}}_m|\mathbb{I}\rangle = 0$ for all $m \geq -1$

$\Rightarrow \hat{L}_0|\mathbb{I}\rangle = \hat{\tilde{L}}_0|\mathbb{I}\rangle = 0 \Rightarrow \underline{h_{\mathbb{I}} = \tilde{h}_{\mathbb{I}} = 0}$.

Also, $\hat{L}_{-1}|\mathbb{I}\rangle = \hat{\tilde{L}}_{-1}|\mathbb{I}\rangle = 0 \Rightarrow \underline{\partial\hat{\mathbb{I}} = \bar{\partial}\hat{\mathbb{I}} = 0}$

And $\hat{L}_1|\mathbb{I}\rangle = \hat{\tilde{L}}_1|\mathbb{I}\rangle = 0$.

Thus ground / vacuum state is invariant under full $SL(2, \mathbb{C})$ / mobius transforms.

This is the only state that has this property. Because translation invariance require $\partial\hat{\mathbb{I}} = \bar{\partial}\hat{\mathbb{I}} = 0$, if it doesn't depend on both $z \neq \bar{z}$, then it is unit. No other type of operator has this property.

Hermiticity of Hamiltonian & its density

The hamiltonian density for the euclidean Cylinder is given by

$$\hat{\mathcal{H}} = \hat{T}_{22} = -(\hat{T}_{\omega\omega} + \hat{T}_{\bar{\omega}\bar{\omega}})$$

And the hamiltonian given by $\hat{H} = \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \hat{\mathcal{H}}$ is the generator of σ^2 evolution.

So, in terms of virasoro generators,

$$\hat{T}_{\omega\omega} = - \sum_{m=-\infty}^{\infty} \left(\hat{L}_m - \frac{c\delta_{m0}}{24} \right) e^{im\omega}$$

$$\& \hat{T}_{\bar{\omega}\bar{\omega}} = - \sum_{m=-\infty}^{\infty} \left(\hat{\tilde{L}}_m - \frac{\tilde{c}\delta_{m0}}{24} \right) e^{im\bar{\omega}}$$

$$\rightarrow \hat{\mathcal{H}} = \sum_{m=-\infty}^{\infty} \left(\hat{L}_m e^{im\omega} + \hat{\tilde{L}}_m e^{im\bar{\omega}} \right) - \left(\frac{c + \tilde{c}}{24} \right)$$

The 0^{th} mode term is the hamiltonian $\hat{H}$.

$$\rightarrow \hat{\mathcal{H}} = \hat{H} + \sum_{m=1}^{\infty} \left(\hat{L}_m e^{-m\sigma^2} e^{im\sigma^1} + \hat{L}_{-m} e^{m\sigma^2} e^{-im\sigma^1} \right) + \sum_{m=1}^{\infty} \left(\hat{\tilde{L}}_m e^{m\sigma^2} e^{im\sigma^1} + \hat{\tilde{L}}_{-m} e^{-m\sigma^2} e^{-im\sigma^1} \right)$$

If we now take the hermitian conjugate, then, we need $\hat{H} \neq \hat{\mathcal{H}}$ to be hermitian.

If $L_0^+ = L_0 \neq \tilde{L}_0^+ = \tilde{L}_0 \rightarrow \hat{H}$ is hermitian.

Under Hermitian conjugate,

$$\hat{\mathcal{H}}^+ = \hat{H}^+ + \sum_{m=1}^{\infty} \left(\hat{L}_m^+ e^{-m\sigma^2} e^{-im\sigma^1} + \hat{L}_{-m}^+ e^{m\sigma^2} e^{im\sigma^1} \right) + \sum_{m=1}^{\infty} \left(\hat{\tilde{L}}_m^+ e^{m\sigma^2} e^{-im\sigma^1} + \hat{\tilde{L}}_{-m}^+ e^{-m\sigma^2} e^{im\sigma^1} \right)$$

If $\hat{L}_m^+ = \hat{L}_{-m} \neq \hat{\tilde{L}}_m^+ = \hat{\tilde{L}}_{-m}, \forall m, \neq$

Since $\sigma^2 = i\tau \Rightarrow \sigma^{2+} = -i\tau$ on Lorentzian cylinder, we get,

$$\begin{aligned}\hat{H}^+ &= \hat{H} + \sum_{m=1}^{\infty} \left(\hat{L}_{-m} e^{m\sigma^2} e^{-im\sigma^1} + \hat{L}_m e^{-m\sigma^2} e^{im\sigma^1} \right) \\ &\quad + \sum_{m=1}^{\infty} \left(\hat{\hat{L}}_{-m} e^{-m\sigma^2} e^{im\sigma^1} + \hat{\hat{L}}_m e^{m\sigma^2} e^{-im\sigma^1} \right) \\ &\Rightarrow \hat{H}^+ = \hat{H}\end{aligned}$$

Thus if $\hat{L}_m^+ = \hat{L}_{-m}$ & $\hat{\hat{L}}_m^+ = \hat{\hat{L}}_{-m}$, then we get both $\hat{H}$ and $\hat{H}$ to be hermitian on the Lorentzian Cylinder since we have $\sigma^2 = i\tau \Rightarrow \sigma^{2+} = -i\tau$ on Lorentzian cylinder. But not guaranteed on the Euclidean cylinder.

But the Zeroth-mode of $\hat{H}$ or the ' $\hat{H}$ ' Hamiltonian is hermitian in both Lorentzian & Euclidean cylinder which evolves both ' τ ' & ' σ^2 '. So $L_0^+ = L_0$ & $\hat{\hat{L}}_0^+ = \hat{\hat{L}}_0$ is always true.

Unitary CFTs

A unitary CFT is the one which all the normalizable states have norm squared to be non-negative. There shouldn't be any -ve or imaginary normed states. They also imply, $\hat{L}_m^\dagger = \hat{L}_{-m}$ & $\hat{\tilde{L}}_m^\dagger = \hat{\tilde{L}}_{-m} \quad \forall m$.

This keeps the $\hat{H}$ & $\hat{\tilde{H}}$ Hermitian on Lorentzian cylinder. They are called as Unitary CFTs.

Positive semi-definiteness of h & $\tilde{h}$

This is a strong constraint on unitary CFT which puts a lower bound on weights of any operator which is eigen state of the rotations & dilations.

First consider any primary state $|A\rangle$. It satisfies, $\hat{L}_0|A\rangle = h_A|A\rangle$ & $\hat{\tilde{L}}_0|A\rangle = \tilde{h}_A|A\rangle$, & $\hat{L}_m|A\rangle = 0 \quad \forall m > 0$. Similarly, $\hat{\tilde{L}}_0|A\rangle = \tilde{h}_A|A\rangle$, $\hat{\tilde{L}}_{-1}|A\rangle = |\bar{\partial}A\rangle$ & $\hat{\tilde{L}}_m|A\rangle = 0 \quad \forall m > 0$.

Consider $\|\hat{L}_{-1}|A\rangle\|^2 = \langle A|\hat{L}_{+1}\hat{L}_{-1}|A\rangle$. Now,
 $\langle A|\hat{L}_{-1}\hat{L}_{+1}|A\rangle = 0$. Thus,

$$\begin{aligned}\langle A|[\hat{L}_{+1}, \hat{L}_{-1}]|A\rangle &= \langle A|(\hat{L}_{+1}\hat{L}_{-1} - \hat{L}_{-1}\hat{L}_{+1})|A\rangle \\ &= \langle A|\hat{L}_{+1}\hat{L}_{-1}|A\rangle - \langle A|\hat{L}_{-1}\hat{L}_{+1}|A\rangle \Big|_0 \\ &= \langle A|\hat{L}_{+1}\hat{L}_{-1}|A\rangle = \|\hat{L}_{-1}|A\rangle\|^2\end{aligned}$$

But $[\hat{L}_{+1}, \hat{L}_{-1}] = 2\hat{L}_0$

$$\rightarrow \langle A|[\hat{L}_{+1}, \hat{L}_{-1}]|A\rangle = 2\langle A|\hat{L}_0|A\rangle = 2h$$

Thus for any primary state $|A\rangle$

$$\rightarrow h_A = \frac{\|\hat{L}_{-1}|A\rangle\|^2}{2}, \text{ for unitary CFT,}$$

norm of any state should be 0 or higher.

$$\Rightarrow \underline{h_A \geq 0}.$$

Similarly, we get for the anti-holomorphic part,

$$\tilde{h}_A = \frac{\|\hat{\tilde{L}}_{-1}|A\rangle\|^2}{2} \Rightarrow \underline{\tilde{h}_A \geq 0}.$$

Now, consider descendants of primary states got by acting $\hat{L}_{-m}$'s ($m > 0$) on them. Let, $|\psi\rangle = \hat{L}_{-m_1} \hat{L}_{-m_2} \hat{L}_{-m_3} \dots \hat{L}_{-m_n} |A\rangle$ where all $m_i > 0$.

$$\text{Now, } \hat{L}_0 |\psi\rangle = \left(h_A + \sum_{i=1}^n m_i \right) |\psi\rangle$$

Thus any descendant state has weight greater than ' h_A ' of the primary state it descended from, where $h_A \geq 0$.

If we know the Spectrum of all the primary states of a theory, we get those many irreducible repⁿ of Virasoro Algebra. Then any state can be written as a linear combination of all descendants of all the primary states of the theory.

So, none of the states if it is an eigen state of $\hat{L}_0$ of $\hat{L}_0$ has weight to be -ve. Since all the primary weights are ≥ 0 .

Similarly, the same argument holds for why even $\tilde{h} \geq 0$ for any state which is an eigen state of $\hat{L}_0$.

Consider a state $|\Theta\rangle$ corresponding to some primary operator ' $\hat{\Theta}(z, \bar{z})$ '

If $h_\Theta = 0$, $\Rightarrow \hat{L}_0 |\Theta\rangle = 0$. Since,

$$h_\Theta = \frac{\|\hat{L}_{-1} |\Theta\rangle\|^2}{2} \quad \text{as seen before, we get}$$

$$\|\hat{L}_{-1} |\Theta\rangle\|^2 = 0 \Rightarrow \underline{\hat{L}_{-1} |\Theta\rangle = 0}$$

$$\text{But } \hat{L}_{-1} |\Theta\rangle \simeq \hat{L}_{-1} \cdot \hat{\Theta}(z, \bar{z}) = (\partial \hat{\Theta})(z, \bar{z})$$

$$\Rightarrow (\partial_z \hat{\Theta})(z, \bar{z}) = 0 \Rightarrow \underline{\hat{\Theta} \text{ is anti-holomorphic}}$$

Similarly, if $\tilde{h}_\Theta = 0$, $\Rightarrow \hat{\tilde{L}}_0 |\Theta\rangle = 0$. &

$$\text{Since, } \tilde{h}_\Theta = \frac{\|\hat{\tilde{L}}_{-1} |\Theta\rangle\|^2}{2}, \text{ we get } \|\hat{\tilde{L}}_{-1} |\Theta\rangle\|^2 = 0$$

$$\Rightarrow \underline{\hat{\tilde{L}}_{-1} |\Theta\rangle = 0}$$

$$\text{But } \hat{L}_{-1}|\Theta\rangle \simeq \hat{L}_{-1} \cdot \hat{\Theta}(z, \bar{z}) = (\bar{\partial} \hat{\Theta})(z, \bar{z})$$

$$\Rightarrow (\partial_{\bar{z}} \hat{\Theta})(z, \bar{z}) = 0 \Rightarrow \underline{\hat{\Theta} \text{ is holomorphic}}$$

Thus we get two important results for a unitary CFT. For a primary operator ' $\hat{\Theta}(z, \bar{z})$ ',

If $\tilde{h}_\Theta = 0 \Rightarrow \hat{\Theta} \rightarrow \hat{\Theta}(z)$ is holomorphic.
 $\neq$

If $h_\Theta = 0 \Rightarrow \hat{\Theta} \rightarrow \hat{\Theta}(\bar{z})$ is anti-holomorphic.

The only normalizable primary state which has both weights $h=0$ & $\tilde{h}=0$ is the state $|\mathbb{I}\rangle$ corresponding to the unit operator.

For any primary operator $\hat{\Theta}(z, \bar{z})$, if both $h=0$ & $\tilde{h}=0$, we get $\partial_z \hat{\Theta} = 0 = \partial_{\bar{z}} \hat{\Theta}$.

It implies ' $\hat{\Theta}$ ' is independent of positions. Which makes it a constant or $c\mathbb{I}$ where c is a constant. Hence, proved.

Positive definiteness of central charges

By Virasoro algebra, for $n > 0$,

$$[\hat{L}_n, \hat{L}_{-n}] = 2n\hat{L}_0 + \frac{C}{12}n(n^2-1)$$

$$\& [\hat{\tilde{L}}_n, \hat{\tilde{L}}_{-n}] = 2n\hat{\tilde{L}}_0 + \frac{\tilde{C}}{12}n(n^2-1)$$

For a state corresponding to ground state got by unit operator which will satisfy, $\hat{L}_0|0\rangle = \hat{\tilde{L}}_0|0\rangle = 0$

where $|0\rangle$ is the vacuum at any time, we can write,

$$\frac{C}{12}n(n^2-1) = \langle 0 | [\hat{L}_n, \hat{L}_{-n}] | 0 \rangle = \|\hat{L}_{-n}|0\rangle\|^2$$

For unitary CFT, $\|\hat{L}_{-n}|0\rangle\|^2 > 0$ strictly greater than zero since ' $\hat{L}_{-n}$ ' increases the eigen value of ' $\hat{L}_0$ '. Hence $C > 0$. Similarly we can show that, $\tilde{C} > 0$.

Thus Central charges are always +ve.

X^μ CFT is Unitary for Space-like ' μ '
& non-Unitary for $\mu=0$

Any CFT is unitary, if we define $\hat{L}_m^+ = \hat{L}_{-m}$ & $\hat{\tilde{L}}_m^+ = \hat{\tilde{L}}_{-m}$. But for X^μ CFT, for this to occur, we need

$$\hat{\alpha}_n^{\mu+} = \hat{\alpha}_{-n}^\mu \quad \& \quad \hat{\tilde{\alpha}}_n^{\mu+} = \hat{\tilde{\alpha}}_{-n}^\mu.$$

Where these modes satisfy,

$$[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = n \eta^{\mu\nu} \delta_{m,-n}, \quad [\hat{\tilde{\alpha}}_n^\mu, \hat{\tilde{\alpha}}_m^\nu] = n \eta^{\mu\nu} \delta_{m,-n}$$

Consider a descendent state $\hat{\alpha}_{-n}^\mu |0\rangle$ where $n > 0$. Now, $\|\hat{\alpha}_{-n}^\mu |0\rangle\|^2 = \langle 0 | \hat{\alpha}_n^\mu \hat{\alpha}_{-n}^\mu |0\rangle$

$$= \langle 0 | \hat{\alpha}_n^\mu \hat{\alpha}_{-n}^\mu |0\rangle - \langle 0 | \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\mu |0\rangle$$

$$= \langle 0 | [\hat{\alpha}_n^\mu, \hat{\alpha}_{-n}^\mu] |0\rangle = n \eta^{\mu\mu}$$

$$\Rightarrow \|\hat{\alpha}_{-n}^\mu |0\rangle\|^2 = n \eta^{\mu\mu} \begin{cases} -n, & \mu=0 \\ +n, & \mu=1,2,\dots,D \end{cases}$$

For $\mu=0$, norm² is -ve, so non-unitary.

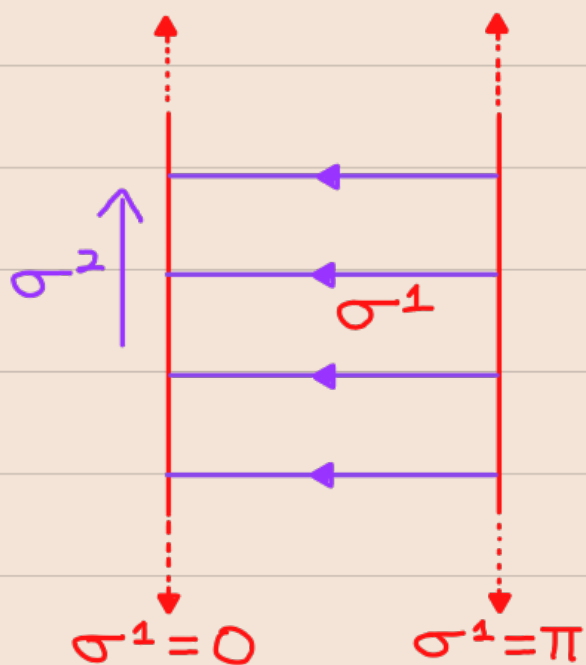
Also in X^μ CFT, the local operator $\hat{X}^\mu(z, \bar{z})$ has weight $(0, 0)$, but still it is not the operator which correspond to Ground state / unit operator. Because the constrain for weights holds only for normalizable states.

But $\hat{X}^\mu|0\rangle$ state is non-normalizable because of infinite range of $\hat{X}^\mu$, so +ve definiteness constraint for weights wont hold for $\hat{X}^\mu$ local operator.

Open String State Operator Map

Consider a CFT defined on an infinitely long strip. The euclidean time coordinate $\sigma^2 = i\tau$ runs from $-\infty$ to $+\infty$. The spatial ' σ^1 ' coordinate takes values b/w 0 to π .

Infinite Strip World Sheet



We define the complex coordinates on the strip as $\omega = \sigma^1 + i\sigma^2$ & $\bar{\omega} = \sigma^1 - i\sigma^2$. So, ω & $\bar{\omega}$ define a point on the world sheet. We do

a conformal map of strip to a plane by defining

$$z = -e^{-i\omega}$$
$$\rightarrow z = -e^{(-i\sigma^1 + \sigma^2)} \rightarrow \underline{z = e^{\sigma^2} e^{i(\pi - \sigma^1)}}$$

$$\rightarrow z = e^{\sigma^2} [-\cos(\sigma^1) + i\sin(\sigma^1)]$$

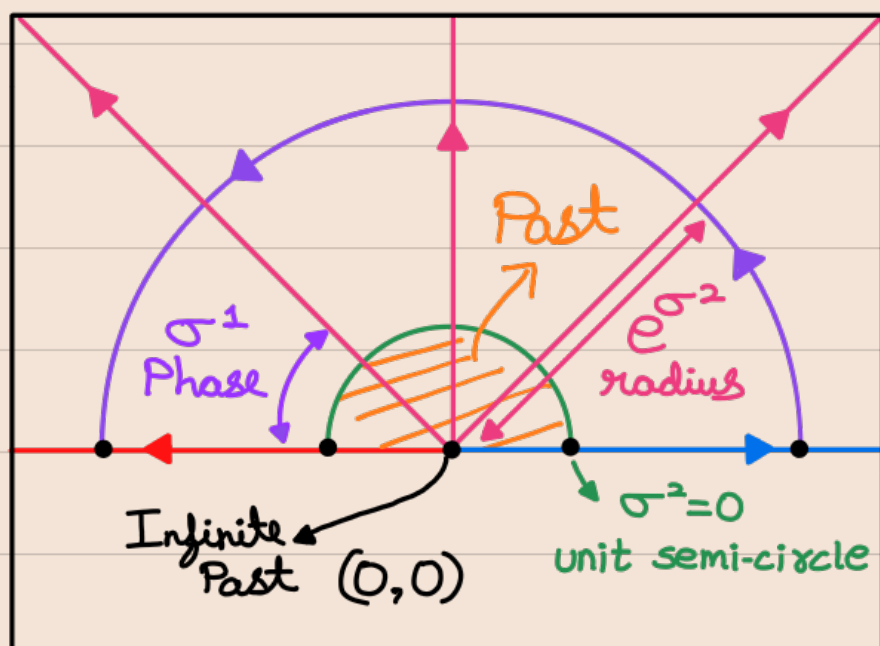
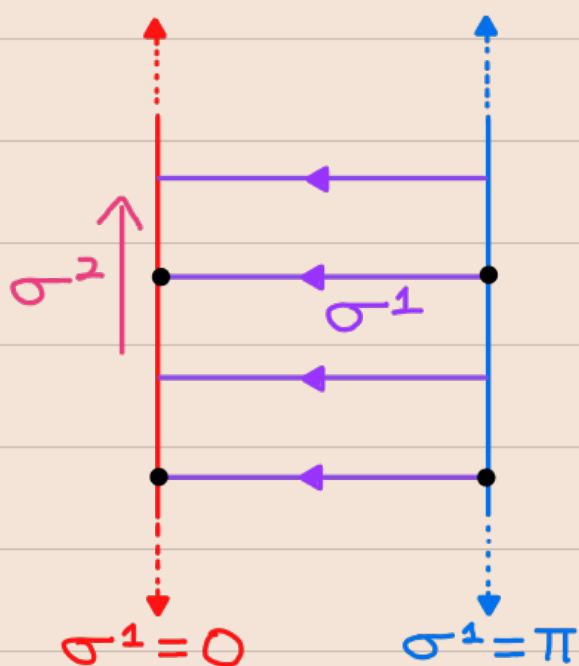
The radial distance in $(z, \bar{z})$ plane relates to the euclidean time by, $\tau = e^{\sigma^2}$. So $\sigma^2 = -\infty$ (Infinite past) is the origin. The Entire Strip is mapped to just the upper half of z -plane.

when $\sigma^1 = 0 \Rightarrow z = -e^{\sigma^2}$, $\text{Im}(z) = 0$. Similarly, if $\sigma^1 = \pi \Rightarrow z = e^{\sigma^2}$, $\text{Im}(z) = 0$.

Thus the entire real line of ' z ' is the boundary of the Strip. The boundary of $\sigma^1 = 0$ maps to $-ve \text{ Re}(z)$ axis and the boundary $\sigma^1 = \pi$ maps to $+ve \text{ Re}(z)$ axis.

Infinite Strip World Sheet

z -plane



Now again here initial state maps to a local operator at the origin. But this time the origin lies on the boundary. Thus,

Every local operator on the boundary $\text{Im}(\bar{z})=0$ ($\sigma^1=0, \pi$) will map to states on the interval (Spatial slice of World Sheet at a particular σ^2 value), which repⁿ the quantum states describing open string along $0 < \sigma^1 < \pi$ at a specified Euclidean time σ^2 .

For open string case, the doubling trick gives one set of holomorphic fields, charges, currents, generators, etc, that are all holomorphic. And thus we can perform closed contour integrals even in $\bar{z}$ -plane.