

Massless Scalar X^μ CFT on a Euclidean Cylinder

Here, we have, the mode expansions for the fields $X^\mu(z, \bar{z})$, $\partial_z X^\mu(z, \bar{z})$ & $\partial_{\bar{z}} X^\mu(z, \bar{z})$

$$X^\mu(z, \bar{z}) = x^\mu - \frac{i\alpha' p^\mu}{2} \ln(|z|^2) + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \left[\frac{\alpha_n^\mu}{z^n} + \frac{\tilde{\alpha}_n^\mu}{\bar{z}^n} \right]$$

where,

holomorphic $\swarrow \partial_z X^\mu(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\alpha_n^\mu}{z^{n+1}}$

anti-holomorphic $\swarrow \partial_{\bar{z}} X^\mu(z, \bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \frac{\tilde{\alpha}_n^\mu}{\bar{z}^{n+1}}$

The Laurent coefficients are the oscillator modes

given by, $\alpha_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \partial_z X^\mu(z, \bar{z})$

$$\tilde{\alpha}_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{d\bar{z}}{2\pi} \bar{z}^n \partial_{\bar{z}} X^\mu(z, \bar{z})$$

' α_n^μ & $\tilde{\alpha}_n^\mu$ are contour independent.'

$$p^\mu = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_0^\mu \text{ is CoM momenta}$$

$$\rightarrow x^\mu = \oint_0^{2\pi} \frac{d\sigma}{2\pi} \chi^\mu(\sigma, 0) \text{ is the Initial CoM location at } \sigma^2 = 0.$$

The canonical commutation relations are,

$$[\hat{\alpha}_n^\mu, \hat{\alpha}_m^\nu] = n \eta^{\mu\nu} \delta_{m, -n}$$

$$[\hat{\tilde{\alpha}}_n^\mu, \hat{\tilde{\alpha}}_m^\nu] = n \eta^{\mu\nu} \delta_{m, -n}$$

$$[\hat{\alpha}_n^\mu, \hat{\tilde{\alpha}}_m^\nu] = 0$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i \eta^{\mu\nu}$$

$$[\hat{p}^\mu, \hat{\alpha}_n^\nu] = 0$$

$$[\hat{p}^\mu, \hat{p}^\nu] = 0$$

$$[\hat{p}^\mu, \hat{\tilde{\alpha}}_n^\nu] = 0$$

Got by relating them to singularity of OPEs

The energy momentum tensor components are,

$$T(z) = T_{zz} = -\frac{1}{\alpha'} (\partial_z \chi^\mu) (\partial_z \chi_\mu)$$

$$\widetilde{T}(\bar{z}) = T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} (\partial_{\bar{z}} \chi^\mu) (\partial_{\bar{z}} \chi_\mu)$$

And the corresponding virasoro generators are,

$$\hat{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu: \eta_{\mu\nu}$$

or

$$\hat{\widetilde{L}}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\hat{\widetilde{\alpha}}_{m-n}^\mu \hat{\widetilde{\alpha}}_n^\nu: \eta_{\mu\nu}$$

where ' : : ' creation - annihilation ordering. The hamiltonian is given by,

$$\hat{H} = \frac{\alpha'}{2} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} (\hat{\alpha}_{-n}^\mu \hat{\alpha}_{\mu n} + \hat{\widetilde{\alpha}}_{-n}^\mu \hat{\widetilde{\alpha}}_{\mu n}) - \frac{D}{12}$$

$$[\hat{H}, \hat{\alpha}_n^\mu] = -n \hat{\alpha}_n^\mu, \quad [\hat{H}, \hat{\alpha}_{-n}^\mu] = n \hat{\alpha}_{-n}^\mu$$

$$[\hat{H}, \hat{\widetilde{\alpha}}_n^\mu] = -n \hat{\widetilde{\alpha}}_n^\mu, \quad [\hat{H}, \hat{\widetilde{\alpha}}_{-n}^\mu] = n \hat{\widetilde{\alpha}}_{-n}^\mu$$

Thus we see that ' $\hat{\alpha}_n^\mu$ ' & ' $\hat{\tilde{\alpha}}_n^\mu$ ' decreases the energy. ' $\hat{\alpha}_{-n}^\mu$ ' & ' $\hat{\tilde{\alpha}}_{-n}^\mu$ ' increases the energy.

$\rightarrow \hat{\alpha}_{-n}^\mu$ & $\hat{\tilde{\alpha}}_{-n}^\mu \rightarrow$ Creation Operator

$\hat{\alpha}_n^\mu$ & $\hat{\tilde{\alpha}}_n^\mu \rightarrow$ Annihilation Operator

But $\hat{P}^\mu$ commutes with $\hat{H}$, $[\hat{H}, \hat{P}^\mu] = 0$
It gives the momentum of CoM. Any eigen state of ' $\hat{P}^\mu$ ' is of the below form

$$' \hat{P}^\mu |k^\mu\rangle = k^\mu |k^\mu\rangle '$$

$\hookrightarrow$ CoM momentum Eigen values

Also define $\hat{N}_n = \frac{\hat{\alpha}_{-n}^\mu \hat{\alpha}_{n\mu}}{n}$ & $\hat{\tilde{N}}_n = \frac{\hat{\tilde{\alpha}}_{-n}^\mu \hat{\tilde{\alpha}}_{n\mu}}{n}$
($n > 0$)

They are number operators which gives the no. of quantum excitations of oscillations of waves at n^{th} mode of the string. There are two possible types of excitations, left moving or the holomorphic sector & other is the right moving or anti-holomorphic sector.

The number operators commute with the Hamiltonian operator. So, we define a general eigen state of Hamiltonian as $|\{N_n\}; \{\tilde{N}_n\}; k^\mu\rangle$ such that

$$\Rightarrow \hat{H}|\{N_n\}; \{\tilde{N}_n\}; k^\mu\rangle = \underbrace{\left[\frac{\alpha'}{2} k^\mu k_\mu + \sum_{n=1}^{\infty} (nN_n + n\tilde{N}_n) - \frac{D}{12} \right]}_{\text{Energy eigen value.}} |\{N_n\}; \{\tilde{N}_n\}; k^\mu\rangle$$

Thus we shall define a general state of the form $|\{0\}; \{0\}; 0\rangle$ which implies the ground state energy to be $-\frac{D}{12}$. But we can renormalize the theory to have $E_g = 0$.

The main characteristic of ground state is it is annihilated by all annihilation operators including $\hat{p}^\mu$ (it is also annihilation op.)

$$\text{i.e. } \hat{\alpha}_n^\mu |\{0\}; \{0\}; 0\rangle = 0 = \hat{\tilde{\alpha}}_n^\mu |\{0\}; \{0\}; 0\rangle, n \geq 0$$

Thus, $|\{0\}; \{0\}; 0\rangle \equiv |0\rangle$ is called Vacuum state.

State operator Isomorphism for Vacuum State

The state annihilated by all ' $\hat{\alpha}_n^\mu$ ' & ' $\hat{\tilde{\alpha}}_n^\mu$ ' where $n > 0$ is the ground state.

$$\text{Now, } \hat{\alpha}_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \partial_z \hat{\chi}^\mu(z, \bar{z})$$

$$\hat{\tilde{\alpha}}_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{d\bar{z}}{2\pi} \bar{z}^n \partial_{\bar{z}} \hat{\chi}^\mu(z, \bar{z})$$

Consider the action of a state ' $|\mathbb{I}\rangle$ ' which is got by the presence of a unit vertex operator $\hat{\mathbb{I}}(0,0)$ at origin.

$$\Rightarrow \hat{\alpha}_n^\mu |\mathbb{I}\rangle = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \underbrace{\partial_z \hat{\chi}^\mu(z, \bar{z}) \hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

This is an operator eqⁿ so we should first consider a state functional that will correspond to unit vertex operator.

Consider the state functional.

$$\chi_f(\sigma) = \chi(x, \sigma)$$

$$\Psi_{\mathbb{I}}[\chi_f(\sigma)] = \int \mathcal{D}[\chi(\sigma)] e^{-S[\chi]} \hat{\mathbb{I}}(0,0), \text{ where}$$

the path integral is evaluated until some radius ' x ' with configuration $\chi_f(\sigma) = \chi(x, \sigma)$.

Now, consider $\hat{\alpha}_n^\mu$ acting on $\Psi_{\mathbb{I}}[\chi_f(\sigma)]$, where

$$\hat{\alpha}_n^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \partial_z \hat{\chi}^\mu(z, \bar{z}), \quad \underline{\underline{|z| = \epsilon < x}}$$

We shall now see it as ' $\hat{\alpha}_n^\mu$ ' acting on the state functional where the path integral has a unit operator inserted in the origin.

$\Rightarrow \hat{\alpha}_n^\mu \Psi_{\mathbb{I}}[\chi_f(\sigma)]$ is given by,

$$\chi_f(\sigma) = \chi(x, \sigma)$$

$$= \sqrt{\frac{2}{\alpha'}} \int \mathcal{D}[\chi(\sigma)] e^{-S[\chi]} \oint_C \frac{dz}{2\pi} z^n \underbrace{\partial_z \hat{\chi}^\mu(z, \bar{z}) \hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

Since ' $\partial_z \hat{\chi}^\mu(z, \bar{z})$ ' & ' $\hat{\mathbb{I}}(0,0)$ ' commute with each other there are no singular terms in OPE b/w them. Thus ' $\partial_z \hat{\chi}^\mu(z, \bar{z}) \hat{\mathbb{I}}(0,0)$ ' is smooth & holomorphic function with no ∞ 's.

This is because we have the action 'S' to depend on ' $\partial \chi^\mu$'s. Whenever the field diverges, action diverges, so this ensures only the smooth functions of ' $\partial_z \hat{\chi}^\mu(z, \bar{z})$ ' contribute.

Thus, if $n \geq 0$, the contour integral vanishes.

$$\Rightarrow \hat{\alpha}_n^\mu \Psi_{\mathbb{I}}[\chi_f(\sigma)] = 0$$

$$\Rightarrow \hat{\alpha}_n^\mu \cdot \underbrace{\hat{\mathbb{I}}(z, \bar{z})}_{\text{Inserted anywhere inside the boundary}} = 0 \Rightarrow \hat{\alpha}_n^\mu \underbrace{|\mathbb{I}, \tau\rangle}_{\text{any radial time}} = 0 \quad (n \geq 0)$$

Thus, the state corresponding to the unit operator is the Vacuum state.

$$\Rightarrow |\mathbb{I}, \tau\rangle \equiv |0\rangle \Rightarrow \text{Vacuum (Zero momenta)}$$

State operator Isomorphism to Excited states

We can repeat the same analysis as before but do it for $\hat{\alpha}_{-n}^\mu$ for $n > 0$. This gives,

$$\hat{\alpha}_{-n}^\mu |0\rangle = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} \overbrace{\partial_z \hat{\chi}^\mu(z, \bar{z}) \hat{\mathbb{I}}(0,0)}^{\text{OPE}} z^n$$

By Cauchy integral formula, we get,

$$\hat{\alpha}_{-n}^\mu |0\rangle = i \sqrt{\frac{2}{\alpha'}} \frac{\partial^n \hat{\chi}^\mu(0,0)}{(n-1)!} \quad \& \quad \text{Similarly,}$$

$$\hat{\tilde{\alpha}}_{-n}^\mu |0\rangle = i \sqrt{\frac{2}{\alpha'}} \frac{\bar{\partial}^n \hat{\chi}^\mu(0,0)}{(n-1)!}$$

$$\Rightarrow \hat{\alpha}_{-n}^\mu |0\rangle = i \sqrt{\frac{2}{\alpha'}} \frac{|\partial^n \hat{\chi}^\mu\rangle}{(n-1)!}, \quad n > 0$$

$$\& \quad \hat{\tilde{\alpha}}_{-n}^\mu |0\rangle = i \sqrt{\frac{2}{\alpha'}} \frac{|\bar{\partial}^n \hat{\chi}^\mu\rangle}{(n-1)!}, \quad n > 0$$

Thus, n^{th} excited state of field χ^μ is got by inserting the vertex operator ' $\partial^n \hat{\chi}^\mu(0,0)$ ' or ' $\bar{\partial}^n \hat{\chi}^\mu(0,0)$ ' inside the Path integral.

Consider the action of ' $\hat{\alpha}_m^\mu$ ' on $|\partial^n \hat{\chi}^\mu\rangle$

$$\hat{\alpha}_m^\nu |\partial^n \hat{\chi}^\mu\rangle = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^m \underbrace{\partial_z \hat{\chi}^\nu(z, \bar{z}) \partial^n \hat{\chi}^\mu(0,0)}_{\text{OPE}} |0\rangle$$

$$\left. \partial_z \hat{\chi}^\nu(z, \bar{z}) \partial_{z'}^n \hat{\chi}^\mu(z', \bar{z}') \right|_{(z', \bar{z}')=(0,0)} = \left. \partial_z \partial_{z'}^n \left(\hat{\chi}^\nu(z, \bar{z}) \hat{\chi}^\mu(0,0) \right) \right|_{(z', \bar{z}')=(0,0)}$$

$$= \left. \partial_z \partial_{z'}^n \left(-\frac{\alpha'}{2} \eta^{\mu\nu} \ln|z - z'|^2 \right) \right|_{(z', \bar{z}')=(0,0)} = -\frac{\alpha'}{2} \eta^{\mu\nu} \left. \partial_{z'}^n \left(\frac{1}{z - z'} \right) \right|_{(z', \bar{z}')=(0,0)}$$

$$\rightarrow \left. \partial_z \hat{\chi}^\nu(z, \bar{z}) \partial^n \hat{\chi}^\mu(0,0) \right|_{(z', \bar{z}')=(0,0)} = -\frac{\alpha'}{2} \eta^{\mu\nu} \frac{n!}{z^{n+1}}$$

$$\rightarrow \hat{\alpha}_m^\nu |\partial^n \hat{\chi}^\mu\rangle = -\eta^{\mu\nu} n! \sqrt{\frac{\alpha'}{2}} \oint_C \frac{dz}{2\pi} \frac{1}{z^{n-m+1}} |0\rangle$$

$$= -i \eta^{\mu\nu} n! \sqrt{\frac{\alpha'}{2}} \delta_{nm} |0\rangle$$

$$\rightarrow \hat{\alpha}_m^\nu \ i \sqrt{\frac{2}{\alpha'}} \frac{|\partial^n \hat{\chi}^\mu\rangle}{n!} = \eta^{\mu\nu} \delta_{nm} |0\rangle$$

Using $\hat{\alpha}_{-n}^{\mu}|0\rangle = i\sqrt{\frac{2}{\alpha'}} \frac{|\partial^n \hat{\chi}^{\mu}\rangle}{(n-1)!}$

$$\rightarrow \hat{\alpha}_m^{\nu} i\sqrt{\frac{2}{\alpha'}} \frac{|\partial^n \hat{\chi}^{\mu}\rangle}{n!} = \frac{\hat{\alpha}_m^{\nu} \hat{\alpha}_{-n}^{\mu}|0\rangle}{n} = \eta^{\mu\nu} \delta_{nm}|0\rangle$$

$$\rightarrow \hat{\alpha}_m^{\nu} \hat{\alpha}_{-n}^{\mu}|0\rangle = n \eta^{\mu\nu} \delta_{nm}|0\rangle$$

Now, $\hat{\alpha}_{-n}^{\mu} \hat{\alpha}_m^{\nu}|0\rangle = 0$

$$\Rightarrow (\hat{\alpha}_m^{\nu} \hat{\alpha}_{-n}^{\mu} - \hat{\alpha}_{-n}^{\mu} \hat{\alpha}_m^{\nu})|0\rangle = n \eta^{\mu\nu} \delta_{nm}|0\rangle$$

$$\Rightarrow [\hat{\alpha}_m^{\nu}, \hat{\alpha}_{-n}^{\mu}]|0\rangle = n \eta^{\mu\nu} \delta_{nm}|0\rangle$$

$$\Rightarrow \underline{[\hat{\alpha}_m^{\nu}, \hat{\alpha}_{-n}^{\mu}] = n \eta^{\mu\nu} \delta_{nm}}$$

This works for any state since every state is built by the action of ' $\hat{\alpha}_{-n}^{\mu}$ ' on $|0\rangle$.

Similarly, we get, $[\hat{\alpha}_m^{\nu}, \hat{\alpha}_{-n}^{\mu}] = n \eta^{\mu\nu} \delta_{nm}$

$$\neq [\hat{\alpha}_m^{\nu}, \hat{\alpha}_{-n}^{\mu}] = 0$$

State-operator Isomorphism to $\hat{\alpha}_{-n}^\mu \neq \hat{\tilde{\alpha}}_{-n}^\mu$ acting on a General state / vertex operator

Consider the action of ' $\hat{\alpha}_{-n}^\mu$ ' with $n > 0$ on a state $|\Theta\rangle$ corresponding to some general vertex operator ' $:\hat{\Theta}(0,0):$ ' which is normal ordered.

$$\begin{aligned}\hat{\alpha}_{-n}^\mu \cdot :\hat{\Theta}(0,0): &= \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^{-n} \partial_z \hat{\chi}^\mu(z, \bar{z}) :\hat{\Theta}(0,0): \\ &= \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^{-n} :\partial_z \hat{\chi}^\mu(z, \bar{z}): :\hat{\Theta}(0,0): \end{aligned}$$

$$\text{Since } \partial_z \hat{\chi}^\nu(z, \bar{z}) :\hat{\Theta}(0,0): = :\partial_z \hat{\chi}^\mu(z, \bar{z}): :\hat{\Theta}(0,0):$$

$$\begin{aligned}\text{Now, } :\partial_z \hat{\chi}^\nu(z, \bar{z}): :\hat{\Theta}(0,0): &= :\partial_z \hat{\chi}^\mu(z, \bar{z}) \hat{\Theta}(0,0): \\ &\quad + \text{Cross Contractions} \end{aligned}$$

$$\begin{aligned}\rightarrow \hat{\alpha}_{-n}^\mu \cdot :\hat{\Theta}(0,0): &= :\hat{\alpha}_{-n}^\mu \cdot \hat{\Theta}(0,0): \\ &\quad + \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} \frac{\left(\text{Cross Contractions} \right)}{z^n} \end{aligned}$$

Now, Cross contraction terms only increase the power of ' z ' in the denominator. Thus we will never get residue since $n \geq 1$.

$$\text{Thus, } \hat{\alpha}_{-n}^{\mu} : \hat{\Theta}(0,0) : = : \hat{\alpha}_{-n}^{\mu} \cdot \hat{\Theta}(0,0) :$$

This can be extended to several oscillators,

$$\begin{aligned} \rightarrow (\hat{\alpha}_{-n_1}^{\mu_1} \hat{\alpha}_{-n_2}^{\mu_2} \hat{\alpha}_{-n_3}^{\mu_3} \dots \hat{\alpha}_{-n_k}^{\mu_k}) : \hat{\Theta}(0,0) : \\ = : (\hat{\alpha}_{-n_1}^{\mu_1} \hat{\alpha}_{-n_2}^{\mu_2} \hat{\alpha}_{-n_3}^{\mu_3} \dots \hat{\alpha}_{-n_k}^{\mu_k}) \cdot \hat{\Theta}(0,0) : \end{aligned}$$

where all ' n_j ' are +ve.

$$\begin{aligned} \Rightarrow (\hat{\alpha}_{-n_1}^{\mu_1} \hat{\alpha}_{-n_2}^{\mu_2} \dots \hat{\alpha}_{-n_k}^{\mu_k}) : \hat{\Theta}(0,0) : \\ = \left(\frac{2}{\alpha'}\right)^{k/2} \oint_{C_1} \oint_{C_2} \dots \oint_{C_k} \frac{dz_1 dz_2 \dots dz_k}{(2\pi)^k} \\ \times \frac{:\partial_{\bar{z}_1} \hat{\chi}^{\mu_1}(z_1, \bar{z}_1) \partial_{\bar{z}_2} \hat{\chi}^{\mu_2}(z_2, \bar{z}_2) \dots \partial_{\bar{z}_k} \hat{\chi}^{\mu_k}(z_k, \bar{z}_k) \hat{\Theta}(0,0):}{z_1^{n_1} z_2^{n_2} \dots z_k^{n_k}} \end{aligned}$$

There are no poles arising from the normal ordered product as $(z_j, \bar{z}_j) \rightarrow (0,0)$ in

$$':\partial_{\bar{z}_1} \hat{\chi}^{\mu_1}(z_1, \bar{z}_1) \partial_{\bar{z}_2} \hat{\chi}^{\mu_2}(z_2, \bar{z}_2) \dots \partial_{\bar{z}_k} \hat{\chi}^{\mu_k}(z_k, \bar{z}_k) \hat{\Theta}(0,0):'$$

Hence,

$$\begin{aligned} \Rightarrow (\hat{\alpha}_{-n_1}^{\mu_1} \hat{\alpha}_{-n_2}^{\mu_2} \dots \hat{\alpha}_{-n_k}^{\mu_k}) : \hat{\Theta}(0,0) : \\ = i^k \left(\frac{2}{\alpha'}\right)^{k/2} \frac{:\partial^{n_1} \hat{\chi}^{\mu_1}(0,0) \partial^{n_2} \hat{\chi}^{\mu_2}(0,0) \dots \partial^{n_k} \hat{\chi}^{\mu_k}(0,0) \hat{\Theta}(0,0):}{(n_1-1)!(n_2-1)! \dots (n_k-1)!} \end{aligned}$$

Thus the state corresponding to some normal ordered operator at origin acted by several ' $\hat{\alpha}$ ' oscillators ' $\hat{\alpha}_{-n_1}^{\mu_1} \hat{\alpha}_{-n_2}^{\mu_2} \dots \hat{\alpha}_{-n_k}^{\mu_k}$ ' is equal to the state corresponding to normal ordered product of

$$: \partial^{n_1} \hat{\chi}^{\mu_1}(0,0) \partial^{n_2} \hat{\chi}^{\mu_2}(0,0) \dots \partial^{n_k} \hat{\chi}^{\mu_k}(0,0) \hat{\Theta}(0,0) :$$

where we replace any

$$\hat{\alpha}_{-n}^{\mu} \longrightarrow i \left(\frac{2}{\alpha'} \right)^{1/2} \frac{\partial^n \hat{\chi}^{\mu}(0,0)}{(n-1)!}$$

Similarly,

$$\hat{\bar{\alpha}}_{-n}^{\mu} \longrightarrow i \left(\frac{2}{\alpha'} \right)^{1/2} \frac{\bar{\partial}^n \hat{\chi}^{\mu}(0,0)}{(n-1)!}$$

$$(n \geq 1)$$

Similarly, if ' $\hat{\chi}^{\mu}$ ' appears to act on a state, we shall replace, $\hat{\chi}^{\mu} \rightarrow \hat{\chi}^{\mu}(0,0)$. Since, in the mode expansion of $\hat{\chi}^{\mu}$, to get it normal ordered, we remove all the singular terms and then take limit $(z, \bar{z}) \rightarrow (0,0)$. This only results in the $(z, \bar{z})$ independent which is $\hat{\chi}^{\mu}$.

Eigen States of $\hat{P}^\mu$

Consider an eigen state of $\hat{P}^\mu$ which is annihilated by all $\hat{\alpha}_n^\mu$ ($n > 0$) and has a definite CoM momenta given by $|\{0\}; \{0\}; k^\mu\rangle$

$$\text{Where } \hat{P}^\mu |\{0\}; \{0\}; k^\mu\rangle = k^\mu |\{0\}; \{0\}; k^\mu\rangle$$

Consider the eigen state of $\hat{x}^\mu$ which is given by $|x^\mu\rangle$ where, $\hat{x}^\mu |x^\mu\rangle = x^\mu |x^\mu\rangle$. Now, this state corresponds to the operator $:\hat{\chi}^\mu(0,0):$.

But consider the commutator of $\hat{x}^\mu$ & $\hat{P}^\nu$ given by, $[\hat{x}^\mu, \hat{P}^\nu] = i\eta^{\mu\nu}$. The translation operator is given by $\hat{U}(a^\mu) = e^{-ia_\mu \hat{P}^\mu}$

$$[\hat{x}^\mu, \hat{U}(a^\nu)] = [\hat{x}^\mu, e^{-ia_\nu \hat{P}^\nu}] = -ia_\nu i\eta^{\mu\nu} e^{-ia^\nu \hat{P}_\nu}$$

$$\Rightarrow [\hat{x}^\mu, \hat{U}(a^\nu)] = a^\mu \hat{U}(a^\mu)$$

$$\begin{aligned} \text{Now, } \hat{x}^\mu \hat{U}(a^\nu) |x^\mu\rangle &= ([\hat{x}^\mu, \hat{U}(a^\nu)] + \hat{U}(a^\nu) \hat{x}^\mu) |x^\mu\rangle \\ &= (a^\mu \hat{U}(a^\nu) + x^\mu \hat{U}(a^\nu)) |x^\mu\rangle \end{aligned}$$

$$\rightarrow \hat{x}^\mu \hat{U}(a^\nu) |x^\mu\rangle = (x^\mu + a^\mu) \hat{U}(a^\nu) |x^\mu\rangle$$

Thus, $\hat{U}(a^\nu) |x^\mu\rangle = |x^\mu + a^\mu\rangle$

Hence ' $\hat{U}(a^\mu) = e^{-ia_\mu \hat{P}^\mu}$ ' is the translation operator and ' $\hat{P}^\mu$ ' generates translations.

Now, consider the action of ' $\hat{U}(a^\mu)$ ' on the eigen state of ' $\hat{P}^\mu$ '

$$\begin{aligned} \hat{U}(a^\mu) |\{0\}; \{0\}; k^\mu\rangle &= e^{-ia_\mu \hat{P}^\mu} |\{0\}; \{0\}; k^\mu\rangle \\ &= e^{-ia_\mu k^\mu} |\{0\}; \{0\}; k^\mu\rangle \end{aligned}$$

$$\rightarrow \underline{\hat{U}(a^\mu) |\{0\}; \{0\}; k^\mu\rangle = e^{-ia_\mu k^\mu} |\{0\}; \{0\}; k^\mu\rangle}$$

Now consider the local vertex operator of the form ' $:e^{ik_\mu \hat{\chi}^\mu(0,0)}:$ '

Now consider the transformation of this vertex operator $\hat{U}(a^\mu) :e^{ik_\mu \hat{\chi}^\mu(0,0)}: \hat{U}^\dagger(a^\mu)$

This form is needed because, $\hat{U}(a^\nu)|x^\mu\rangle = |x^\mu + a^\mu\rangle$
 & under a transformation we need expectation values to remain unchanged. So, if

$$:e^{ik_\mu \hat{\chi}^\mu(0,0)}:_{\text{new}} = \hat{U}(a^\mu):e^{ik_\mu \hat{\chi}^\mu(0,0)}:_{\text{old}} \hat{U}^\dagger(a^\mu)$$

$$\Rightarrow \langle x^\mu + a^\mu | :e^{ik_\mu \hat{\chi}^\mu(0,0)}:_{\text{new}} | x^\mu + a^\mu \rangle$$

$$= \langle x^\mu | \hat{U}^\dagger(a^\mu) \hat{U}(a^\mu) :e^{ik_\mu \hat{\chi}^\mu(0,0)}:_{\text{old}} \hat{U}^\dagger(a^\mu) \hat{U}(a^\mu) | x^\mu \rangle$$

$$= \langle x^\mu | :e^{ik_\mu \hat{\chi}^\mu(0,0)}:_{\text{old}} | x^\mu \rangle \rightarrow \text{Same as new.}$$

Now $:\hat{\chi}^\mu(0,0): \rightarrow \hat{x}^\mu$, & also since

$$\hat{x}^\mu \hat{U}(a^\nu) | x^\mu \rangle = (x^\mu + a^\mu) \hat{U}(a^\nu) | x^\mu \rangle$$

$$\Rightarrow \hat{U}^\dagger(a^\mu) \hat{x}^\mu \hat{U}(a^\nu) | x^\mu \rangle = (x^\mu + a^\mu) | x^\mu \rangle$$

$$\Rightarrow \hat{U}^\dagger(a^\mu) \hat{x}^\mu \hat{U}(a^\nu) = \hat{x}^\mu + a^\mu$$

$$\Rightarrow \underline{\underline{\hat{U}(a^\nu) \hat{x}^\mu \hat{U}^\dagger(a^\mu) = \hat{x}^\mu - a^\mu}}$$

$$\begin{aligned} \rightarrow \hat{U}(a^\mu) : e^{ik_\mu \hat{\chi}^\mu(0,0)} : \hat{U}^\dagger(a^\mu) &= : e^{ik_\mu (\hat{\chi}^\mu(0,0) - a^\mu)} : \\ &= e^{-ia_\mu k^\mu} : e^{ik_\mu \hat{\chi}^\mu(0,0)} : \end{aligned}$$

Thus we see that the state $|\{0\}; \{0\}; k^\mu\rangle$, under translations transform by a phase factor of $e^{-ia_\mu k^\mu}$, i.e.

$$\underline{\hat{U}(a^\mu) |\{0\}; \{0\}; k^\mu\rangle = e^{-ia_\mu k^\mu} |\{0\}; \{0\}; k^\mu\rangle}$$

And the Operator $: e^{ik_\mu \hat{\chi}^\mu(0,0)} :$, under the same translations transform with the same phase factor $e^{-ia_\mu k^\mu}$, i.e.

$$\hat{U}(a^\mu) : e^{ik_\mu \hat{\chi}^\mu(0,0)} : \hat{U}^\dagger(a^\mu) = e^{-ia_\mu k^\mu} : e^{ik_\mu \hat{\chi}^\mu(0,0)} :$$

Thus under state operator isomorphism, the eigen state of $\hat{P}^\mu$ given by $|\{0\}; \{0\}; k^\mu\rangle$ corresponds to the local vertex operator $: e^{ik_\mu \hat{\chi}^\mu(0,0)} :$ at the origin.

Consider $\hat{p}^\mu |\{0\}; \{0\}; k^\mu\rangle \simeq \hat{p}^\mu :e^{ik_\mu \hat{\chi}^\mu(0,0)}:$
 use, $p^\mu = \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu$ & $\alpha_0^\mu = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} \partial_z \chi^\mu(z, \bar{z})$

$$\Rightarrow \hat{p}^\mu |\{0\}; \{0\}; k^\mu\rangle \simeq \sqrt{\frac{2}{\alpha'}} \alpha_0^\mu :e^{ik_\mu \hat{\chi}^\mu(0,0)}:$$

$$= \frac{2}{\alpha'} \oint_C \frac{dz}{2\pi} \underbrace{\partial_z \chi^\mu(z, \bar{z}) :e^{ik_\mu \hat{\chi}^\mu(0,0)}:}_{\text{OPE}}$$

Consider the OPE $\partial_z \chi^\mu(z, \bar{z}) :e^{ik_\mu \hat{\chi}^\mu(0,0)}:$

$$\partial_z \chi^\mu(z, \bar{z}) :e^{ik_\mu \hat{\chi}^\mu(0,0)}: = : \partial_z \chi^\mu(z, \bar{z}) e^{ik_\mu \hat{\chi}^\mu(0,0)} : + \text{Cross Contractions}$$

$$:e^{ik_\nu \chi^\nu(0,0)}: = \sum_{n=0}^{\infty} \frac{i^n}{n!} :[k_\nu \chi^\nu(0,0)]^n:$$

$\rightarrow$

$$\partial_z \chi^\mu(z, \bar{z}) :e^{ik_\nu \hat{\chi}^\nu(0,0)}:$$

$$= \sum_{n=0}^{\infty} \frac{i^n k_\nu n}{n!} \underbrace{\partial_z \chi^\mu(z, \bar{z}) \chi^\nu(0,0)}_{\frac{-\alpha' \eta^{\mu\nu}}{2z}} :[k_\nu \chi^\nu(0,0)]^{n-1}:$$

$$\rightarrow \partial_{\bar{z}} \chi^\mu(z, \bar{z}) : e^{ik_\nu \hat{\chi}^\nu(0,0)} :$$

$$= \frac{-\alpha' i \eta^{\mu\nu} k_\nu}{2\bar{z}} \sum_{n=1}^{\infty} \frac{i^{n-1}}{(n-1)!} : [k_\nu \chi^\nu(0,0)]^{n-1} :$$

$$= \frac{\alpha' k^\mu}{2i\bar{z}} : e^{ik_\nu \hat{\chi}^\nu(0,0)} :$$

$$\Rightarrow \hat{p}^\mu | \{0\}; \{0\}; k^\mu \rangle \simeq \hat{p}^\mu : e^{ik_\mu \hat{\chi}^\mu(0,0)} :$$

$$= \frac{2}{\alpha'} \oint_C \frac{dz}{2\pi} \frac{\alpha' k^\mu}{2i\bar{z}} : e^{ik_\nu \hat{\chi}^\nu(0,0)} :$$

$$= k^\mu : e^{ik_\nu \hat{\chi}^\nu(0,0)} : \underbrace{\oint_C \frac{dz}{2\pi i} \frac{1}{z}}_1$$

$$\Rightarrow \boxed{\hat{p}^\mu : e^{ik_\mu \hat{\chi}^\mu(0,0)} : = k^\mu : e^{ik_\nu \hat{\chi}^\nu(0,0)} :}$$

$$\text{or } \underline{\underline{| \{0\}; \{0\}; k^\mu \rangle \simeq : e^{ik_\nu \hat{\chi}^\nu(0,0)} :}}$$

Path integral formulation to find Vacuum State

Consider the state functional for a single real scalar massless field χ .

$$\chi^{(b)}(\sigma) = \chi(x_b=1, \sigma)$$
$$\Psi_I[\chi^{(b)}(\sigma)] = \int \mathcal{D}[\chi(\sigma)] e^{-S[\chi]} \hat{\mathbb{I}}(0,0), \text{ where}$$

the path integral is evaluated until the unit circle, with boundary $\chi^{(b)}(\sigma) = \chi(x_b=1, \sigma)$.

By conformal mapping b/w cylinder & plane we have for the cylinder $w = \sigma + i\sigma_2$ & $z = e^{-i\omega} \Rightarrow z = \gamma e^{i\theta}$, $\gamma = e^{\sigma_2}$ & $\theta = -\sigma$ in the polar coordinates. So at boundary, we have $z_b = e^{i\theta}$ & $\bar{z} = e^{-i\theta}$.

Let us expand $\chi^{(b)}(\sigma) = \chi(x_b=1, \sigma)$ in terms of Fourier modes as, $\chi^{(b)}(\theta) = \sum_{n=-\infty}^{\infty} \chi_n^{(b)} e^{in\theta}$

Since ' $\chi^{(b)}(\theta)$ ' is real, $(\chi_n^{(b)})^* = \chi_{-n}^{(b)}$

The fourier modes $\chi_n^{(b)}$ are got by

$$\chi_n^{(b)} = \int_0^{2\pi} \frac{d\theta}{2\pi} \chi^{(b)}(\theta) e^{-in\theta}$$

Thus knowing the fourier modes $\{\chi_n^{(b)}\}$, we can know the boundary condition. Hence,

$$\Psi_{\text{I}}[\chi^{(b)}(\sigma)] \equiv \Psi_{\text{I}}[\chi^{(b)}] \equiv \Psi_{\text{I}}[\{\chi_n^{(b)}\}]$$

Now,

$$\Psi_{\text{I}}[\chi^{(b)}] = \int \mathcal{D}[\chi] e^{-\frac{1}{2\pi\alpha'} \int d^2z \partial\chi \bar{\partial}\chi}$$

Here, ' χ ' lies in the interior of the boundary

This is a Gaussian integral in unusual boundary conditions. But we can split the field in the interior by $\chi = \chi_{cl} + \chi'$

where $\chi_{cl} = \chi_0^{(b)} + \sum_{n=1}^{\infty} \left(\chi_n^{(b)} z^n + \chi_{-n}^{(b)} \bar{z}^n \right)$

At the boundary,

$$\begin{aligned}\chi_{cl} &= \chi_o^{(b)} + \sum_{n=1}^{\infty} \left(\chi_n^{(b)} z^n + \chi_{-n}^{(b)} \bar{z}^n \right) \\ &= \chi_o^{(b)} + \sum_{n=1}^{\infty} \left(\chi_n^{(b)} e^{in\theta} + \chi_{-n}^{(b)} e^{-in\theta} \right) \\ &= \sum_{n=-\infty}^{\infty} \chi_n^{(b)} e^{in\theta} \Rightarrow \chi_{cl} = \chi^{(b)}\end{aligned}$$

Thus we see that

$$a) \chi_{cl} = \chi_o^{(b)} + \sum_{n=1}^{\infty} \left(\chi_n^{(b)} z^n + \chi_{-n}^{(b)} \bar{z}^n \right)$$

Satisfies eqⁿ of motion through $|z| \leq 1$
Since,

$$\bar{\partial} \partial \chi_{cl} = \partial \bar{\partial} \chi_{cl} = 0$$

b) At the boundary, $\chi = \chi^{(b)}$ but
also, $\chi_{cl} = \chi^{(b)}$

$$\Rightarrow \underline{\chi' = 0} \text{ at the boundary}$$

Now the action becomes,

$$\begin{aligned} S[\chi] &= \frac{1}{2\pi\alpha'} \int_{|\mathbb{Z}| \leq 1} d^2\mathbb{Z} \partial\chi \bar{\partial}\chi \\ &= \frac{1}{2\pi\alpha'} \int_{|\mathbb{Z}| \leq 1} d^2\mathbb{Z} (\partial\chi_{cl} + \partial\chi') (\bar{\partial}\chi_{cl} + \bar{\partial}\chi') \\ &= \frac{1}{2\pi\alpha'} \int_{|\mathbb{Z}| \leq 1} d^2\mathbb{Z} \left(\partial\chi_{cl} \bar{\partial}\chi_{cl} + \partial\chi' \bar{\partial}\chi' \right. \\ &\quad \left. + \partial\chi_{cl} \bar{\partial}\chi' + \partial\chi' \bar{\partial}\chi_{cl} \right) \end{aligned}$$

If we perform integration by parts in the 3rd & 4th term, due to eqⁿ of motions that ' χ_{cl} ' satisfies, they vanish. Then at the boundary $\chi' = 0$, so 3rd and 4th terms can be completely neglected.

$$\rightarrow S[\chi] = \frac{1}{2\pi\alpha'} \int_{|\mathbb{Z}| \leq 1} d^2\mathbb{Z} (\partial\chi_{cl} \bar{\partial}\chi_{cl} + \partial\chi' \bar{\partial}\chi')$$

$$\Rightarrow \underline{S[\chi] = S[\chi_{cl}] + S[\chi']}$$

Consider $S[\chi_{cl}] = \frac{1}{2\pi\alpha'} \int_{|z| \leq 1} d^2z \partial\chi_{cl} \bar{\partial}\chi_{cl}$

$$= \frac{1}{2\pi\alpha'} \int_{|z| \leq 1} d^2z \left(\sum_{n=1}^{\infty} n \chi_n^{(b)} z^{n-1} \sum_{m=1}^{\infty} m \chi_{-m}^{(b)} \bar{z}^{m-1} \right)$$

$$= \frac{1}{2\pi\alpha'} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nm \chi_n^{(b)} \chi_{-m}^{(b)} \int_{|z| \leq 1} d^2z z^{n-1} \bar{z}^{m-1}$$

Put $z = \gamma e^{i\theta}$ & $\bar{z} = \gamma e^{-i\theta}$

$$|z| \leq 1 \Rightarrow 0 \leq \gamma \leq 1 \quad \& \quad 0 < \theta < 2\pi$$

$$d^2z = 2 du dv, \quad u = \text{Re}(z), \quad v = \text{Im}(z)$$

$$u = \gamma \cos(\theta), \quad v = \gamma \sin(\theta).$$

$$\Rightarrow du dv = \gamma d\gamma d\theta \Rightarrow d^2z = 2\gamma d\gamma d\theta$$

$$\int_{|z| \leq 1} d^2z z^{n-1} \bar{z}^{m-1} = 2 \int_0^1 d\gamma \gamma \int_0^{2\pi} d\theta (\gamma e^{i\theta})^{n-1} (\gamma e^{-i\theta})^{m-1}$$

$$= 2 \underbrace{\int_0^1 d\gamma \gamma^{n+m-1}}_{\frac{1}{n+m}} \underbrace{\int_0^{2\pi} d\theta e^{i(n-m)\theta}}_{2\pi \delta_{nm}}$$

$$\Rightarrow S[\chi_{cl}] = \frac{1}{2\pi\alpha'} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} nm \chi_n^{(b)} \chi_{-m}^{(b)} \frac{4\pi\delta_{nm}}{n+m}$$

$$\Rightarrow S[\chi_{cl}] = \frac{1}{\alpha'} \sum_{n=1}^{\infty} n \chi_n^{(b)} \chi_{-n}^{(b)}$$

Thus $S[\chi_{cl}]$ is a constant and depends only on the boundary conditions & it is independent of the ' χ ' in the interior.

$$\Rightarrow \Psi_{\text{I}}[\chi^{(b)}] = e^{-S[\chi_{cl}]} \int_{\chi^{(b)}} \mathcal{D}[\chi'] e^{-\frac{1}{2\pi\alpha'} \int d^2z \partial\chi' \bar{\partial}\chi'}$$

Thus we have the state functional in terms of ' χ ' that vanish at boundary. Also, the impact of boundary is present only on ' $e^{-S[\chi_{cl}]}$ ' factor.

$$\int_{\chi^{(b)}} \mathcal{D}[\chi'] e^{-\frac{1}{2\pi\alpha'} \int d^2z \partial\chi' \bar{\partial}\chi'} \text{ is independent}$$

of the boundary conditions. Hence a constant with respect to $\chi^{(b)}$.

$$\Rightarrow \boxed{\Psi_{\text{I}}[\{\chi_n^{(b)}\}] = C e^{-\frac{1}{\alpha'} \sum_{n=1}^{\infty} n \chi_n^{(b)} \chi_{-n}^{(b)}}$$

Thus, the ground state is in the form of a gaussian. For each mode $n \geq 1$, we get a mini-gaussian & the state functional is a product of all these mini-Gaussians.

Now we should check if this state functional actually gets annihilated by all $\hat{\alpha}_n$ & $\hat{\tilde{\alpha}}_n$.

Denote $\chi_n^{(b)} \equiv \chi_n$, and define

$$\hat{\alpha}_n = -\frac{in}{\sqrt{2\alpha'}} \chi_{-n} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n}$$

$$\hat{\tilde{\alpha}}_n = -\frac{in}{\sqrt{2\alpha'}} \chi_n - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-n}}$$

We can check these give proper commutator relations when acting of a general functional.

$$\text{We also use, } \frac{\partial}{\partial \chi_n} \chi_m = \delta_{nm}$$

Consider $[\hat{\alpha}_n, \hat{\alpha}_{-m}] \Psi = (\hat{\alpha}_n \hat{\alpha}_{-m} - \hat{\alpha}_{-m} \hat{\alpha}_n) \Psi$

$$= \hat{\alpha}_n \hat{\alpha}_{-m} \Psi - \hat{\alpha}_{-m} \hat{\alpha}_n \Psi$$

$$\hat{\alpha}_n \hat{\alpha}_{-m} \Psi =$$

$$= \left(-\frac{in\chi_{-n}}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \right) \left(\frac{im\chi_m}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-m}} \right) \Psi$$

$$= \left(-\frac{in\chi_{-n}}{\sqrt{2\alpha'}} \left(\frac{im\chi_m}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-m}} \right) \right) \Psi$$

$$- i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \left(\frac{im\chi_m}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-m}} \right) \Psi$$

$$= \frac{nm\chi_{-n}\chi_m}{2\alpha'} \Psi - \frac{n\chi_{-n}}{2} \frac{\partial}{\partial \chi_{-m}} \Psi + \frac{m}{2} \frac{\partial}{\partial \chi_n} \chi_m \Psi$$

$$- \frac{\alpha'}{2} \frac{\partial}{\partial \chi_n} \frac{\partial}{\partial \chi_{-m}} \Psi$$

$$= \frac{nm\chi_{-n}\chi_m}{2\alpha'} \Psi - \frac{n\chi_{-n}}{2} \frac{\partial}{\partial \chi_{-m}} \Psi + \frac{m}{2} \partial_{nm} \Psi$$

$$+ \frac{m}{2} \chi_m \frac{\partial}{\partial \chi_n} \Psi - \frac{\alpha'}{2} \frac{\partial}{\partial \chi_n} \frac{\partial}{\partial \chi_{-m}} \Psi$$

$$\hat{\alpha}_{-m} \hat{\alpha}_n \Psi =$$

$$= \left(\frac{im\chi_m}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-m}} \right) \left(-\frac{in\chi_{-n}}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \right) \Psi$$

$$= \frac{im\chi_m}{\sqrt{2\alpha'}} \left(-\frac{in\chi_{-n}}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \right) \Psi$$

$$- i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-m}} \left(-\frac{in\chi_{-n}}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \right) \Psi$$

$$= \frac{nm}{2\alpha'} \chi_m \chi_{-n} \Psi + \frac{m\chi_m}{2} \frac{\partial}{\partial \chi_n} \Psi - \frac{n}{2} \frac{\partial}{\partial \chi_{-m}} \chi_{-n} \Psi$$

$$- \frac{\alpha'}{2} \frac{\partial}{\partial \chi_{-m}} \frac{\partial}{\partial \chi_n} \Psi$$

$$= \frac{nm\chi_{-n}\chi_m}{2\alpha'} \Psi + \frac{m\chi_m}{2} \frac{\partial}{\partial \chi_n} \Psi - \frac{n}{2} \partial_{nm} \Psi$$

$$- \frac{n}{2} \chi_{-n} \frac{\partial}{\partial \chi_{-m}} \Psi - \frac{\alpha'}{2} \frac{\partial}{\partial \chi_{-m}} \frac{\partial}{\partial \chi_n} \Psi$$

Thus in $\hat{\alpha}_n \hat{\alpha}_{-m} \Psi - \hat{\alpha}_{-m} \hat{\alpha}_n \Psi$ we get only ∂_{nm} term exist others get cancelled.

$$\begin{aligned}\Rightarrow [\hat{\alpha}_n, \hat{\alpha}_{-m}] \Psi &= \hat{\alpha}_n \hat{\alpha}_{-m} \Psi - \hat{\alpha}_{-m} \hat{\alpha}_n \Psi \\ &= \frac{m}{2} \delta_{nm} \Psi + \frac{n}{2} \delta_{nm} \Psi\end{aligned}$$

$\rightarrow$ for any general Ψ , $[\hat{\alpha}_n, \hat{\alpha}_{-m}] \Psi = n \delta_{nm} \Psi$

$$\Rightarrow \underline{[\hat{\alpha}_n, \hat{\alpha}_{-m}] = n \delta_{nm}}$$

In the same way we get,

$$\underline{[\hat{\tilde{\alpha}}_n, \hat{\tilde{\alpha}}_{-m}] = n \delta_{nm}}$$

$$\& \underline{[\hat{\tilde{\alpha}}_n, \hat{\alpha}_{-m}] = 0}$$

Same like how the wave functional is the schrodinger representation of the state, we can say that

$$\hat{\alpha}_n = -\frac{in\chi_{-n}}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \quad \& \quad \hat{\tilde{\alpha}}_n = -\frac{in\chi_n}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_{-n}}$$

form the schrodinger representation of the oscillator modes.

In this representation, we can show that

$$\Psi_{\text{I}}[\{\chi_m\}] = C e^{-\frac{1}{\alpha'} \sum_{n=1}^{\infty} n \chi_n \chi_{-n}}$$

gets annihilated by all $\hat{\alpha}_n \neq \hat{\bar{\alpha}}_n, n \geq 0$.

Consider $\hat{\alpha}_n \Psi_{\text{I}}[\{\chi_m\}]$

$$= \left(-\frac{in\chi_{-n}}{\sqrt{2\alpha'}} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} \right) C e^{-\frac{1}{\alpha'} \sum_{m=1}^{\infty} m \chi_m \chi_{-m}}$$

$$= -\frac{in\chi_{-n}}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial \chi_n} C e^{-\frac{1}{\alpha'} \sum_{m=1}^{\infty} m \chi_m \chi_{-m}}$$

$$= -\frac{in\chi_{-n}}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] - i\sqrt{\frac{\alpha'}{2}} \Psi_{\text{I}}[\{\chi_m\}] \frac{\partial}{\partial \chi_n} \left(-\frac{1}{\alpha'} \sum_{m=1}^{\infty} m \chi_m \chi_{-m} \right)$$

$$= -\frac{in\chi_{-n}}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] + \frac{i}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] \sum_{m=1}^{\infty} m (\chi_{-m} \delta_{nm} + \chi_m \delta_{n,-m})$$

$$\Rightarrow \hat{\alpha}_n \Psi_{\text{I}}[\{\chi_m\}] = -\frac{in\chi_{-n}}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] + \frac{i}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] \sum_{m=1}^{\infty} m (\chi_{-m} \delta_{nm} + \chi_m \delta_{n,-m})$$

For $n > 0$, $\delta_{n,-m}$ is never satisfied,

$$\rightarrow \hat{\alpha}_n \Psi_{\text{I}}[\{\chi_m\}] = -\frac{in\chi_{-n}}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] + \frac{in\chi_{-n}}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}]$$

Thus $\hat{\alpha}_n \Psi_{\text{I}}[\{\chi_m\}] = 0$ for $n \geq 0$.

But for $\hat{\alpha}_{-n}$, $n > 0$ acting on ' Ψ_{I} '

$$\hat{\alpha}_{-n} \Psi_{\text{I}}[\{\chi_m\}] = \frac{in\chi_n}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] + \frac{i}{\sqrt{2\alpha'}} \Psi_{\text{I}}[\{\chi_m\}] \sum_{m=1}^{\infty} m (\underbrace{\chi_{-m} \delta_{-n,m}}_0 + \chi_m \delta_{nm})$$

$$\Rightarrow \underline{\underline{\hat{\alpha}_{-n} \Psi_{\text{I}}[\{\chi_m\}] = i\sqrt{\frac{2}{\alpha'}} n \chi_n \Psi_{\text{I}}[\{\chi_m\}] , n > 0}}$$

$\hookrightarrow 1^{\text{st}}$ Excited state in the n^{th} mode.

Path integral formulation to find 1st excited state

Now, we shall try to find the 1st excited state by showing its correspondance to the local operator " $\partial^m \chi$ " for a single real scalar field. Consider,

$$\chi^{(b)}(\theta) = \chi(x_b=1, \theta)$$

$$\Psi_{\partial^m \chi}[\chi^{(b)}] = \int \mathcal{D}[\chi] e^{-S[\chi]} \partial^m \chi(0,0), \text{ where}$$

the path integral is evaluated until the unit circle, with boundary $\chi^{(b)}(\theta) = \sum_{n=-\infty}^{\infty} \chi_n^{(b)} e^{in\theta}$

We can split the field in the interior by $\chi = \chi_{cl} + \chi'$ where

$$\chi_{cl} = \chi_0^{(b)} + \sum_{n=1}^{\infty} \left(\chi_n^{(b)} z^n + \chi_{-n}^{(b)} \bar{z}^n \right)$$

$$\Rightarrow \bar{\partial} \partial \chi_{cl} = \partial \bar{\partial} \chi_{cl} = 0$$

≠ $\chi' = 0$ at the boundary ($\chi_{cl} = \chi^{(b)}$).

Same like before, the action becomes,

$$S[\chi] = \frac{1}{2\pi\alpha'} \int_{|\mathbb{Z}| \leq 1} d^2\mathbb{Z} (\partial\chi_{cl} \bar{\partial}\chi_{cl} + \partial\chi' \bar{\partial}\chi')$$

$$\Rightarrow \underline{S[\chi] = S[\chi_{cl}] + S[\chi']}$$

We also get $S[\chi_{cl}] = \frac{1}{\alpha'} \sum_{n=1}^{\infty} n \chi_n^{(b)} \chi_{-n}^{(b)}$

Thus, $S[\chi_{cl}]$ is a constant and depends only on the boundary conditions & it is independent of the ' χ ' in the interior.

$$\Psi_{\partial^m \chi}[\{\chi_m\}] = e^{-S[\chi_{cl}]} \int^{\chi^{(b)}} \mathcal{D}[\chi'] e^{-S[\chi']} \partial^m \chi(0,0)$$

Now, $\partial^m \chi = \partial^m \chi_{cl} + \partial^m \chi'$

$$\partial^m \chi_{cl} = \sum_{n=1}^{\infty} \chi_n^{(b)} \partial^m \mathbb{Z}^n = \sum_{n=m}^{\infty} \chi_n^{(b)} \frac{n! \mathbb{Z}^{n-m}}{(n-m)!}$$

$$\Rightarrow \partial^m \chi_{cl}(0,0) = m! \chi_m^{(b)} \quad \text{independent of } \chi'$$

$$\begin{aligned} \Rightarrow \Psi_{\partial^m \chi} [\{\chi_m^{(b)}\}] &= e^{-S[\chi_{cl}]} m! \chi_m^{(b)} \int \mathcal{D}[\chi'] e^{-S[\chi']} \\ &\quad + e^{-S[\chi_{cl}]} \underbrace{\int \mathcal{D}[\chi'] e^{-S[\chi']} \partial^m \chi'(0,0)}_{\text{Constant w.r.t } \chi^{(b)}} \end{aligned}$$

Substitute

$$\Psi_{\mathbb{I}} [\{\chi_m^{(b)}\}] = e^{-S[\chi_{cl}]} \underbrace{\int \mathcal{D}[\chi'] e^{-S[\chi']}}_{\text{Constant w.r.t } \chi^{(b)}}$$

$$\Rightarrow \Psi_{\partial^m \chi} [\{\chi_m^{(b)}\}] = (m! \chi_m^{(b)} + \beta) \Psi_{\mathbb{I}} [\{\chi_m^{(b)}\}]$$

But, by Schrodinger repⁿ of ' $\hat{\alpha}_{-n}$ ', we got,

$$\hat{\alpha}_{-n} \Psi_{\mathbb{I}} [\{\chi_m\}] = i \sqrt{\frac{2}{\alpha'}} n \chi_n \Psi_{\mathbb{I}} [\{\chi_m\}] \quad , n > 0$$

This suggests $\beta = 0$.

We know that in terms of operator eqⁿ,

$$\hat{\alpha}_n = \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \partial_z \hat{\chi}(z, \bar{z})$$

Then, $\hat{\alpha}_n \Psi_{\partial^m \chi}[\chi_m^{(b)}]$ is given by,

$$\begin{aligned} & \chi^{(b)}(\theta) = \chi(x_b=1, \theta) \\ &= \int \mathcal{D}[\chi] e^{-S[\chi]} \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \underbrace{\partial_z \hat{\chi}(z, \bar{z}) \partial^m \chi(0,0)}_{\text{OPE}} \end{aligned}$$

(where $|z| < 1$)

$$\begin{aligned} & \chi^{(b)}(\theta) = \chi(x_b=1, \theta) \\ &= \int \mathcal{D}[\chi] e^{-S[\chi]} \sqrt{\frac{2}{\alpha'}} \oint_C \frac{dz}{2\pi} z^n \underbrace{\partial_z \hat{\chi}(z, \bar{z}) \partial^m \chi(0,0)}_{-\frac{\alpha'}{2} \frac{m!}{z^{m+1}}} \end{aligned}$$

$$\begin{aligned} & \chi^{(b)}(\theta) = \chi(x_b=1, \theta) \\ &= -im! \sqrt{\frac{\alpha'}{2}} \int \mathcal{D}[\chi] e^{-S[\chi]} \underbrace{\oint_C \frac{dz}{2\pi i} \frac{1}{z^{m-n+1}}}_{\delta_{mn}} \end{aligned}$$

$$\Rightarrow \hat{\alpha}_n \Psi_{\partial^m \chi}[\{\chi_m\}] = -im! \sqrt{\frac{\alpha'}{2}} \delta_{mn} \Psi_{\mathbb{I}}[\{\chi_m\}]$$

$$\Rightarrow \hat{\alpha}_n \Psi_{i\sqrt{\frac{2}{\alpha'}} \frac{\partial^m \chi}{(m-1)!}} [\{\chi_m\}] = m \partial_{mn} \Psi_{\mathbb{I}} [\{\chi_m\}]$$

$$\Rightarrow \hat{\alpha}_{-m} \longrightarrow i\sqrt{\frac{2}{\alpha'}} \frac{\partial^m \chi(0,0)}{(m-1)!}$$

Hence, we can set $\beta=0$ &

$$\Rightarrow \Psi_{\partial^m \chi} [\{\chi_n^{(b)}\}] = m! \chi_m^{(b)} \Psi_{\mathbb{I}} [\{\chi_n^{(b)}\}]$$

&

$$\hat{\alpha}_{-n} \Psi_{\mathbb{I}} [\{\chi_m^{(b)}\}] = i\sqrt{\frac{2}{\alpha'}} n \chi_n^{(b)} \Psi_{\mathbb{I}} [\{\chi_m^{(b)}\}], \quad n > 0$$

$$= i\sqrt{\frac{2}{\alpha'}} \frac{n! \chi_n^{(b)} \Psi_{\mathbb{I}} [\{\chi_m^{(b)}\}]}{(n-1)!}$$

Only if $\beta=0$,

$$\hat{\alpha}_{-n} \Psi_{\mathbb{I}} [\{\chi_m^{(b)}\}] = i\sqrt{\frac{2}{\alpha'}} \frac{1}{(n-1)!} \Psi_{\partial^n \chi} [\{\chi_m^{(b)}\}]$$

$$= \Psi_{i\sqrt{\frac{2}{\alpha'}} \frac{\partial^n \chi}{(n-1)!}} [\{\chi_m\}]$$

bc CFT on a Euclidean Cylinder

Here we have the bc-ghost action on the $\mathbb{Z}$ -plane given by

$$S_{bc} = \frac{1}{2\pi} \int d^2z \, \hat{b}(z, \bar{z}) \partial_{\bar{z}} \hat{c}(z, \bar{z})$$

The eqⁿ of motions are given by,

$$\partial_{\bar{z}} \hat{b}(z, \bar{z}) = \partial_{\bar{z}} \hat{c}(z, \bar{z}) = 0$$

Thus, the ghost fields are such that,

$$\begin{aligned} \hat{b}(z, \bar{z}) &\longrightarrow \hat{b}(z) \longrightarrow \text{Holomorphic tensor} \\ \hat{c}(z, \bar{z}) &\longrightarrow \hat{c}(z) \longrightarrow \text{Holomorphic tensor} \end{aligned}$$

Since we have $\hat{b}(z) \neq \hat{c}(z)$ as holomorphic tensor operators with weights $h_b = \lambda \neq h_c = 1 - \lambda$ respectively, we can Laurent expand them as,

$$\hat{b}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{b}_n}{z^{n+\lambda}} \quad \& \quad \hat{c}(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{c}_n}{z^{n+1-\lambda}}$$

$$\left. \begin{aligned} \hat{b}_n &= \oint_{C'} \frac{dz}{2\pi i} z^{n+\lambda-1} \hat{b}(z) \\ \hat{c}_n &= \oint_{C'} \frac{dz}{2\pi i} z^{n-\lambda} \hat{c}(z) \end{aligned} \right\} \begin{array}{l} C' \rightarrow \text{Contour enclosing} \\ \text{the origin} \end{array}$$

Since $\hat{b}(z)$ & $\hat{c}(z)$ are fermionic fields, they anti-commute.

$$\{\hat{b}_n, \hat{c}_m\} = \delta_{n,-m} \quad \& \quad \{\hat{b}_n, \hat{b}_m\} = 0 = \{\hat{c}_n, \hat{c}_m\}$$

We have two primary states $|\uparrow\rangle$ & $|\downarrow\rangle$ such that,

$$\begin{array}{c|c} \hat{c}_0 |\uparrow\rangle = 0 & \hat{b}_0 |\downarrow\rangle = 0 \\ \hat{c}_0 |\downarrow\rangle = |\uparrow\rangle & \hat{b}_0 |\uparrow\rangle = |\downarrow\rangle \end{array}$$

$$\hat{b}_n |\downarrow\rangle = 0 = \hat{c}_n |\downarrow\rangle, \quad n > 0$$

$$\hat{b}_n |\uparrow\rangle = 0 = \hat{c}_n |\uparrow\rangle, \quad n > 0$$

We choose ' $\hat{b}_0$ ' as lowering & ' $\hat{c}_0$ ' as raising operators, this sets, ' $|\downarrow\rangle$ ' as the b-c vacuum.

The energy momentum tensor's holomorphic component is given by,

$$\hat{T}(z) = h_c(\partial_z \hat{b}(z))\hat{c}(z) - h_b \hat{b}(z)(\partial_z \hat{c}(z))$$

where $h_c = \lambda$, $h_b = 1 - \lambda$. The holomorphic central charge is given by ' $1 - 3(2\lambda - 1)^2$ '

The virasoro generators are given by,

$$\hat{L}_m = \sum_{n=-\infty}^{\infty} [m\lambda - n] : \hat{b}_n \hat{c}_{m-n} : + \frac{\lambda[1-\lambda]}{2} \delta_{m,0}$$

If $\lambda > 1$, then the bc-CFT is not unitary & it gives rise to the Ghost states that can have norm-squared to be -ve. And this also makes the central charge to become negative.

State operator Map of Unit Operator

The vacuum state $|0\rangle \equiv |\downarrow\rangle$ is defined as the state which is annihilated by the operators $\hat{b}_0, \hat{b}_n \neq \hat{c}_n$ for $n > 0$.

Consider the state $|\mathbb{I}\rangle$ which is got by the presence of a unit vertex operator $\hat{\mathbb{I}}(0,0)$.

$$\left. \begin{aligned} \hat{b}_n &= \oint_{C'} \frac{dz}{2\pi i} z^{n+\lambda-1} \hat{b}(z) \\ \hat{c}_n &= \oint_{C'} \frac{dz}{2\pi i} z^{n-\lambda} \hat{c}(z) \end{aligned} \right\} \begin{array}{l} C' \rightarrow \text{Contour enclosing} \\ \text{the origin} \end{array}$$

$$\rightarrow \hat{b}_n |\mathbb{I}\rangle = \oint_{C'} \frac{dz}{2\pi i} z^{n+\lambda-1} \underbrace{\hat{b}(z) \hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

$$\text{Similarly, } \hat{c}_n |\mathbb{I}\rangle = \oint_{C'} \frac{dz}{2\pi i} z^{n-\lambda} \underbrace{\hat{c}(z) \hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

This is an operator eqⁿ so they are already inside the path integral weighted by e^{-S} .

Since ' $\hat{I}(0,0)$ ' commutes with both ' $\hat{b}(z)$ ' & ' $\hat{c}(z)$ ', there are no singular terms in the OPEs ' $\hat{b}(z)\hat{I}(0,0)$ ' & ' $\hat{c}(z)\hat{I}(0,0)$ '. So they are holomorphic & divergenceless with no ∞ 's.

Thus in $\hat{b}_n|\mathbb{I}\rangle$, if $n \geq \lambda - 1$ & in $\hat{c}_n|\mathbb{I}\rangle$, if $n > \lambda$, only then they vanish.

$$\hat{b}_n|\mathbb{I}\rangle = 0, \text{ if } n \geq 1 - \lambda$$

$$\& \hat{c}_n|\mathbb{I}\rangle = 0, \text{ if } n \geq \lambda$$

Thus we see that for any value of ' λ ' the state ' $|\mathbb{I}\rangle$ ' does not correspond to the ground / vacuum state.

This is mainly because of the shifts in the exponents of Laurent expansion because of the conformal weights of b & c fields.

Now we shall relate $|\text{II}\rangle$ with the vacuum state. Where, a finite number of creation operators acting on vacuum gives ' $|\text{II}\rangle$ '.

$$\begin{aligned} \text{Since, } \hat{b}_n |\text{II}\rangle &= 0, \text{ if } n \geq 1 - \lambda \approx n > -\lambda \\ &\neq \hat{c}_n |\text{II}\rangle = 0, \text{ if } n \geq \lambda \approx n > \lambda - 1 \end{aligned}$$

The relation can be obtained by seeing how the annihilation conditions shift with ' λ '.

The vacuum state $|0\rangle \equiv |\downarrow\rangle$ is defined as

$$\begin{aligned} \hat{b}_n |0\rangle &= 0, \text{ if } n \geq 0 \\ \hat{c}_n |0\rangle &= 0, \text{ if } n > 0 \end{aligned}$$

Each $\hat{b}_m$ lowers the b -annihilation threshold by '1' when acted on $|0\rangle$.

Each $\hat{c}_m$ lowers the c -annihilation threshold by '1' when acted on $|0\rangle$.

If $\lambda > 1$, $|\text{II}\rangle = \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$

$$\Rightarrow \hat{b}_n (\hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \hat{b}_{-(\lambda-3)} \dots \hat{b}_{-2} \hat{b}_{-1}) |0\rangle = 0$$

when $n = \{1-\lambda, 2-\lambda, 3-\lambda, \dots, -2, -1, 0, 1, \dots, \infty\}$
 because of the property $\hat{b}_m^2 = 0$ if
 $m = \{1-\lambda, 2-\lambda, 3-\lambda, \dots, -2, -1\}$ & for $n \geq 0$
 by definition of vacuum we get it zero.

And then,

$$\hat{c}_n (\hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \hat{b}_{-(\lambda-3)} \dots \hat{b}_{-2} \hat{b}_{-1}) |0\rangle = 0$$

if $n \geq \lambda$ Since all the $\hat{b}_m$ above
 with $m = \{1-\lambda, 2-\lambda, 3-\lambda, \dots, -2, -1\}$ will
 anti-commute with all $\hat{c}_n$ whose $n \geq \lambda$
 & by definition of vacuum, we get
 $\hat{c}_n |0\rangle = 0$ for $n \geq \lambda > 1$.

Thus,

$|\text{II}\rangle = \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \hat{b}_{-(\lambda-3)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$ if $\lambda > 1$.

If $\underline{\lambda = 1}$, $|\text{I}\rangle = |0\rangle$

If $\underline{\lambda < 1}$, $|\text{I}\rangle = \hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-2} \hat{C}_{-1} \hat{C}_0 |0\rangle$

$\Rightarrow \hat{b}_n (\hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-2} \hat{C}_{-1} \hat{C}_0) |0\rangle = 0, n > -\lambda$

Since for $n = \{-\lambda+1, -\lambda+2, \dots, \infty\}$ we get $n > 0$ which makes $\hat{b}_n$ anticommute with all ' $\hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-2} \hat{C}_{-1} \hat{C}_0$ '.

& $\hat{C}_n (\hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-2} \hat{C}_{-1} \hat{C}_0) |0\rangle = 0, n \geq \lambda$

Since for $m = \{\lambda, \lambda+1, \lambda+2, \dots, -2, -1, 0\}$ we get $\hat{C}_m$ to annihilate $|\text{I}\rangle$, since, it results in $\hat{C}_m^2 = 0$ for some ' m '.

And for $n = \{1, 2, 3, \dots, \infty\}$, by defⁿ of vacuum, $\hat{C}_n (\hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-2} \hat{C}_{-1} \hat{C}_0) |0\rangle = 0$

Thus,
$$|\text{I}\rangle = \begin{cases} \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle, & \text{if } \lambda > 1. \\ |0\rangle, & \text{if } \lambda = 1. \\ \hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-1} \hat{C}_0 |0\rangle, & \text{if } \lambda < 1. \end{cases}$$

State operator map for Vacuum State

$$\underline{\text{If } \lambda > 1, |0\rangle = \hat{C}_1 \hat{C}_2 \cdots \hat{C}_{(\lambda-2)} \hat{C}_{(\lambda-1)} |\text{I}\rangle}$$

$$\text{Since, we got, } |\text{I}\rangle = \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$\Rightarrow \hat{C}_1 \hat{C}_2 \cdots \hat{C}_{(\lambda-2)} \hat{C}_{(\lambda-1)} |\text{I}\rangle$$

$$= \hat{C}_1 \hat{C}_2 \cdots \hat{C}_{(\lambda-2)} \hat{C}_{(\lambda-1)} \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$\text{Using } \hat{C}_m \hat{b}_{-m} = 1 - \hat{b}_{-m} \hat{C}_m \\ \text{for } m = \{1, 2, 3, \dots, \lambda-3, \lambda-2, \lambda-1\}$$

& Since $m > 0$, $\hat{C}_m |0\rangle = 0$, we get all pairs to effectively become '1'. Thus,

$$\underline{|0\rangle = \hat{C}_1 \hat{C}_2 \cdots \hat{C}_{(\lambda-2)} \hat{C}_{(\lambda-1)} |\text{I}\rangle \quad \text{for } \lambda > 1}$$

where ' $|\text{I}\rangle$ ' is the state obtained by the unit operator at the origin during the evolution of state in path integrals.

If $\lambda = 1$, $|0\rangle = |I\rangle$

& If $\lambda < 1$, $|0\rangle = \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} |I\rangle$

Since, we got, $|I\rangle = \hat{c}_\lambda \hat{c}_{\lambda+1} \cdots \hat{c}_{-2} \hat{c}_{-1} \hat{c}_0 |0\rangle$

$$\Rightarrow \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} |I\rangle$$

$$= \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} \hat{c}_\lambda \hat{c}_{\lambda+1} \cdots \hat{c}_{-2} \hat{c}_{-1} \hat{c}_0 |0\rangle$$

Using $\hat{b}_m \hat{c}_m = 1 - \hat{c}_m \hat{b}_m$

for $m = \{0, -1, -2, -3, \dots, \lambda+1, \lambda\}$, $\lambda \leq 0$

& Since $\hat{b}_m |0\rangle = 0$ for -ve m , we get all pairs to effectively become '1'.

Thus,

$$\underline{|0\rangle = \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} |I\rangle \text{ for } \lambda < 1}$$

State operator map for $\partial^m \hat{b}(z) \neq \partial^m \hat{c}(z)$

Consider the action of $\hat{b}_{-n}$ & $\hat{c}_{-n}$ on $|\mathbb{I}\rangle$

$$\hat{b}_{-n}|\mathbb{I}\rangle = \oint_{C'} \frac{dz}{2\pi i} z^{-n+\lambda-1} \underbrace{\hat{b}(z)\hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

$$\hat{c}_{-n}|\mathbb{I}\rangle = \oint_{C'} \frac{dz}{2\pi i} z^{-n-\lambda} \underbrace{\hat{c}(z)\hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

Since ' $\hat{\mathbb{I}}(0,0)$ ' commutes with both ' $\hat{b}(z)$ ' & ' $\hat{c}(z)$ ', there are no singular terms in the OPEs ' $\hat{b}(z)\hat{\mathbb{I}}(0,0)$ ' & ' $\hat{c}(z)\hat{\mathbb{I}}(0,0)$ '. So they are holomorphic & divergenceless with no ∞ 's.

Thus $\underbrace{\hat{b}(z)\hat{\mathbb{I}}(0,0)}_{\text{OPE}} \rightarrow :\hat{b}(z):|\mathbb{I}\rangle$
and $\underbrace{\hat{c}(z)\hat{\mathbb{I}}(0,0)}_{\text{OPE}} \rightarrow :\hat{c}(z):|\mathbb{I}\rangle$ } Smooth,
divergenceless
& holomorphic

$$\hat{b}_{-n}|\mathbb{I}\rangle = \oint_{C'} \frac{dz}{2\pi i} \frac{:\hat{b}(z):|\mathbb{I}\rangle}{z^{n-\lambda+1}}$$

$$\text{Thus } \hat{b}_{-n}|\mathbb{I}\rangle = \begin{cases} \frac{\partial^{n-\lambda} \hat{b}(0)|\mathbb{I}\rangle}{(n-\lambda)!} & , n \geq \lambda \\ 0 & , n < \lambda \end{cases}$$

$$\longrightarrow \underline{\underline{(n-\lambda)! \hat{b}_{-n} \longrightarrow \partial^{n-\lambda} \hat{b}(0)}}, n \geq \lambda$$

$$\longrightarrow \boxed{|\partial^m b\rangle \simeq \partial^m \hat{b}(0) \longrightarrow m! \hat{b}_{-m-\lambda}}, m \geq 0$$

$$\text{Similarly, } \hat{c}_{-n}|\mathbb{I}\rangle = \oint_{C'} \frac{dz}{2\pi i} \frac{:\hat{c}(z):|\mathbb{I}\rangle}{z^{n+\lambda}}$$

$$\text{Thus } \hat{c}_{-n}|\mathbb{I}\rangle = \begin{cases} \frac{\partial^{n+\lambda-1} \hat{c}(0)|\mathbb{I}\rangle}{(n+\lambda-1)!} & , n \geq 1-\lambda \\ 0 & , n < 1-\lambda \end{cases}$$

$$\longrightarrow \underline{\underline{(n+\lambda-1)! \hat{c}_{-n} \longrightarrow \partial^{n+\lambda-1} \hat{c}(0)}}, n \geq 1-\lambda$$

$$\longrightarrow \boxed{|\partial^m c\rangle \simeq \partial^m \hat{c}(0) \longrightarrow m! \hat{c}_{\lambda-1-m}}, m \geq 0$$

bc-Ghost Number Operators

The bc-ghost number operator is given by ' $\hat{N}^g$ ' which counts the difference b/w the number of c-type excitations & the b-type excitations.

The ghost number current is given by $\hat{j}_w(w) = -\hat{b}(w)\hat{c}(w)$ in the complex w -coordinates on the cylinder. The charge corresponding to this current in the cylindrical frame is the ghost number operator given by,

$$\hat{N}^g = \oint_0^{2\pi} \frac{d\sigma^1}{2\pi i} : \hat{j}_w(\sigma^1, \sigma^2) :$$

We can relate $\hat{j}_w(w)$ on the cylinder to z -plane by,

$$\hat{j}_z(z) = (\partial_w z)^{-1} \hat{j}_w(w) + q_0 \frac{(\partial_w^2 z)}{(\partial_w z)^2}$$

where

$$\underline{q_0 = \lambda - \frac{1}{2}} \quad (\text{arises due to non-tensorial property})$$

$$\Rightarrow \hat{N}^g = \oint_C \frac{dz}{2\pi i} : \hat{j}_z(z) : - q_0$$

Then the radial frame ghost number charge is given by,

$$\hat{Q}^g = \oint_C \frac{dz}{2\pi i} : j_z(z) : = \hat{N}^g + q_0$$

Using $j_z(z) = b(z)c(z)$ &

$$:\hat{b}(z)\hat{c}(z): - :\hat{b}(z)\hat{c}(z): = \frac{1-\lambda}{z}$$

We get,

$$\hat{N}^g = \hat{c}_0 \hat{b}_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{b}_n - \hat{b}_{-n} \hat{c}_n)$$

$$\Rightarrow \underline{\hat{N}^g |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle} \quad \& \quad \underline{\hat{N}^g |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle}$$

$$|\downarrow\rangle \equiv |0\rangle \quad \& \quad |\uparrow\rangle = \hat{c}_0 |0\rangle$$

$$\Rightarrow \underline{\hat{N}^g |0\rangle = -\frac{1}{2} |0\rangle}$$

The commutation relations b/w $\hat{N}^g$ & mode operators are, $[\hat{N}^g, \hat{b}_m] = -\hat{b}_m$
 $[\hat{N}^g, \hat{c}_m] = \hat{c}_m$

$$\hat{N}^g \hat{b}_m = \hat{b}_m (\hat{N}^g - 1), \quad \forall m$$

$$\hat{N}^g \hat{c}_m = \hat{c}_m (\hat{N}^g + 1), \quad \forall m$$

So any $\hat{b}_m$ decreases the ghost no: by one
 & any $\hat{c}_m$ increases the ghost no: by one

Ghost Number of $|\text{II}\rangle$

Now we shall find the charges of the state corresponding to unit operator.

$$|\text{II}\rangle = \begin{cases} \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle, & \text{if } \lambda > 1. \\ |0\rangle, & \text{if } \lambda = 1. \\ \hat{c}_\lambda \hat{c}_{\lambda+1} \cdots \hat{c}_{-1} \hat{c}_0 |0\rangle, & \text{if } \lambda < 1. \end{cases}$$

If $\lambda > 1$,

$\lambda-1$ no: of $\hat{b}$ operators

$$\hat{N}^g |\text{II}\rangle = \hat{N}^g \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$\begin{aligned}
\hat{N}^g |\mathbb{I}\rangle &= \hat{N}^g \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle \\
&= \hat{b}_{-(\lambda-1)} (\hat{N}^g - 1) \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle \\
&= \hat{b}_{-(\lambda-1)} \hat{N}^g \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle - |\mathbb{I}\rangle \\
&= \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} (\hat{N}^g - 1) \hat{b}_{-(\lambda-3)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle - |\mathbb{I}\rangle \\
&= \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \hat{N}^g \hat{b}_{-(\lambda-3)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle - 2|\mathbb{I}\rangle \\
&= \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \hat{b}_{-(\lambda-3)} \hat{N}^g \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle - 3|\mathbb{I}\rangle
\end{aligned}$$

So, continuing the pattern,

$$\hat{N}^g |\mathbb{I}\rangle = \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} \hat{N}^g |0\rangle - \sum_{i=1}^{\lambda-1} |\mathbb{I}\rangle$$

$$\rightarrow \hat{N}^g |\mathbb{I}\rangle = -\frac{1}{2} |\mathbb{I}\rangle - (\lambda-1) |\mathbb{I}\rangle$$

$$\Rightarrow \underline{\text{If } \lambda > 1, \hat{N}^g |\mathbb{I}\rangle = (\frac{1}{2} - \lambda) |\mathbb{I}\rangle}$$

$$\underline{\text{If } \lambda = 1, \hat{N}^g |\text{II}\rangle = -\frac{1}{2} |\text{II}\rangle}$$

$$\underline{\text{If } \lambda < 1, \hat{N}^g |\text{II}\rangle = \hat{N}^g \underbrace{\hat{c}_\lambda \hat{c}_{\lambda+1} \dots \hat{c}_{-1} \hat{c}_0}_{\lambda+1 \text{ no: of } \hat{c} \text{ operators}} |\text{O}\rangle}$$

$$\Rightarrow \hat{N}^g |\text{II}\rangle = -\frac{1}{2} |\text{II}\rangle + \sum_{i=0}^{\lambda} |\text{II}\rangle$$

$$\Rightarrow \underline{\text{If } \lambda < 1, \hat{N}^g |\text{II}\rangle = \left(\lambda + \frac{1}{2}\right) |\text{II}\rangle}$$

Thus, Even though the unit operator has ghost number Zero, as it commutes with all other operators, the state ' $|\text{II}\rangle$ ' that maps to the unit operator has non zero - Ghost number for any ' λ '.

$$\frac{\langle \text{II} | \hat{N}^g | \text{II} \rangle}{\langle \text{II} | \text{II} \rangle} = \begin{cases} \frac{1}{2} - \lambda & , \lambda > 1 \\ -\frac{1}{2} & , \lambda = 1 \\ \lambda + \frac{1}{2} & , \lambda < 1 \end{cases}$$

Now we shall find the reverse map

If $\lambda > 1$, $|0\rangle = \hat{C}_1 \hat{C}_2 \dots \hat{C}_{(\lambda-2)} \hat{C}_{(\lambda-1)} |\text{II}\rangle$

But, $\hat{C}_n$ corresponds to $\oint_{C'} \frac{dz}{2\pi i} z^{n-\lambda} \hat{C}(z)$

$$\Rightarrow \hat{C}_1 \hat{C}_2 \dots \hat{C}_{(\lambda-2)} \hat{C}_{(\lambda-1)} |\text{II}\rangle$$

$$= \oint_{C'_1} \frac{dz_1}{2\pi i} \oint_{C'_2} \frac{dz_2}{2\pi i} \dots \oint_{C'_{\lambda-1}} \frac{dz_{\lambda-1}}{2\pi i} z_1^{1-\lambda} z_2^{2-\lambda} \dots z_{\lambda-1}^{-1} \\ \times \underbrace{\hat{C}(z_1) \hat{C}(z_2) \dots \hat{C}(z_{\lambda-1}) \hat{\text{II}}(0,0)}_{\text{OPE}}$$

$\Rightarrow$ Ghost no: $-1/2$ state ' $|0\rangle$ ' ground state correspond to the operator

$$\frac{\partial^{(\lambda-2)} \hat{C}(0)}{(\lambda-2)!} \frac{\partial^{\lambda-3} \hat{C}(0)}{(\lambda-3)!} \dots \frac{\partial \hat{C}(0)}{1!} \frac{\hat{C}(0)}{0!} :$$

$$|0\rangle \longrightarrow \frac{\partial^{(\lambda-2)} \hat{C}(0)}{(\lambda-2)!} \frac{\partial^{\lambda-3} \hat{C}(0)}{(\lambda-3)!} \dots \frac{\partial \hat{C}(0)}{1!} \frac{\hat{C}(0)}{0!} :$$

Let us find out which Ghost no: state corresponds to the operators with Ghost no: 1 $\neq$ -1 (i.e. $\hat{C}(z) \neq \hat{b}(z)$)

Consider the state, ' $\hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$ ' whose ghost number is $\frac{3}{2} - \lambda$.

$$\hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle = \hat{C}_{(\lambda-1)} |\text{II}\rangle$$

$$= \oint_C \frac{dz}{2\pi i} z^{(\lambda-1)-\lambda} \underbrace{\hat{C}(z) \hat{\text{II}}(0,0)}_{\text{OPE}}$$

$$= \oint_C \frac{dz}{2\pi i} \frac{:\hat{C}(z):}{z}$$

$\Rightarrow$

$$\boxed{\hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle \longrightarrow \hat{C}(0)}$$

Ghost number $\frac{3}{2} - \lambda$ State

Ghost no:
+1 operator

Similarly let's see, which state maps to Ghost no: '-1' operator

$$|\text{II}\rangle = \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$\Rightarrow \hat{b}_{-\lambda} |\text{II}\rangle = \underbrace{\hat{b}_{-\lambda} \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle}_{\text{Ghost no: } -\lambda - \frac{1}{2} \text{ state}}$$

$$\hat{b}_{-\lambda} |\text{II}\rangle \longrightarrow \oint_{C'} \frac{dz}{2\pi i} z^{-\lambda+\lambda-1} \underbrace{\hat{b}(z) \hat{\text{II}}(0,0)}_{\text{OPE}}$$

$$\hat{b}_{-\lambda} |\text{II}\rangle \longrightarrow \oint_{C'} \frac{dz}{2\pi i} \frac{:\hat{b}(z):}{z} = \hat{b}(0)$$

$$\rightarrow \boxed{\hat{b}_{-\lambda} \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle} \longrightarrow \hat{b}(0)$$

Ghost number $-\lambda - \frac{1}{2}$ state
Ghost no: -1 operator

$$\underline{\text{If } \lambda < 1}, \quad |0\rangle = \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} |\text{II}\rangle$$

$$\text{But, } \hat{b}_n \text{ corresponds to } \oint_{C'} \frac{dz}{2\pi i} \frac{:\hat{b}(z):}{z^{1-n-\lambda}}$$

$$\Rightarrow \hat{b}_n \longrightarrow \frac{\partial^{-\lambda-n} \hat{b}(0)}{(-\lambda-n)!}, \quad n \leq -\lambda$$

$$\Rightarrow |0\rangle = \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} |\text{II}\rangle$$

$$\Rightarrow |0\rangle \longrightarrow \frac{\partial^{-\lambda} \hat{b}(0)}{(-\lambda)!} \frac{\partial^{-\lambda-1} \hat{b}(0)}{(-\lambda-1)!} \cdots \frac{\partial \hat{b}(0)}{1!} \hat{b}(0)$$

Then, let us find out which ghost no: state corresponds to the operators with ghost no: $1 \neq -1$ (i.e. $\hat{c}(z) \neq \hat{b}(z)$), when $\lambda < 1$.

$$|0\rangle = \hat{b}_0 \hat{b}_1 \hat{b}_2 \cdots \hat{b}_{-\lambda-1} \hat{b}_{-\lambda} |\text{II}\rangle$$

$$\Rightarrow \boxed{\hat{c}_{\lambda+1} \cdots \hat{c}_{-1} \hat{c}_0 |0\rangle = \hat{b}_{-\lambda} |\text{II}\rangle \longrightarrow \hat{b}(0)}$$

Ghost number $\lambda - \frac{1}{2}$ state

Ghost no:
-1 operator

$$|\text{II}\rangle = \hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-1} \hat{C}_0 |0\rangle$$

$$\rightarrow \hat{C}_{(\lambda-1)} |\text{II}\rangle = \hat{C}_{(\lambda-1)} \underbrace{\hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-1} \hat{C}_0}_{\text{Ghost number } \lambda + \frac{3}{2} \text{ State}} |0\rangle$$

$\hat{C}_{\lambda-1}$ corresponds to $\hat{C}(0)$ } Ghost number +1 operator

$$\rightarrow \boxed{\hat{C}_{(\lambda-1)} \hat{C}_\lambda \hat{C}_{\lambda+1} \cdots \hat{C}_{-1} \hat{C}_0 |0\rangle} \rightarrow \hat{C}(0)$$

Ghost number $\lambda + \frac{3}{2}$ State
Ghost no: +1 operator

Thus we see a mismatch b/w the Ghost no: of a State & its corresponding operator.
This is because of the Non-tensor property of Ghost Current under conformal transform.

Normal Ordering Constants Using state Operator Maps

The Virasoro generators are given by

$$\hat{L}_m = \oint_C \frac{dz}{2\pi i} z^{m+1} \hat{T}(z)$$

Consider the state corresponding to the unit operator, $|\mathbb{I}\rangle$

$$\Rightarrow \hat{L}_m |\mathbb{I}\rangle \simeq \oint_C \frac{dz}{2\pi i} z^{m+1} \underbrace{\hat{T}(z) \hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

$$\underbrace{\hat{T}(z) \hat{\mathbb{I}}(0,0)}_{\text{OPE}} = :\hat{T}(z): \rightarrow \text{divergenceless \& holomorphic}$$

$$\rightarrow \hat{L}_m |\mathbb{I}\rangle \simeq \oint_C \frac{dz}{2\pi i} z^{m+1} :\hat{T}(z):$$

$$\text{For } m \geq -1, \quad \underline{\hat{L}_m |\mathbb{I}\rangle = 0}$$

$\rightarrow |\mathbb{I}\rangle$ is $SL(2, \mathbb{C})$ invariant.

Consider $\hat{L}_{-m}$, $m > 1$,

$$\hat{L}_{-m}|\mathbb{I}\rangle \simeq \oint_C \frac{dz}{2\pi i} \frac{:\hat{T}(z):}{z^{m-1}} = \partial^{m-2} \hat{T}(0)$$

$$\rightarrow \hat{L}_{-m}|\mathbb{I}\rangle \rightarrow \partial^{m-2} \hat{T}(0)$$

Thus $\hat{T}(0)$ correspond to state $\hat{L}_{-2}|\mathbb{I}\rangle$

$$\text{Similarly, } \hat{L}_m = \oint_C \frac{d\bar{z}}{2\pi i} \bar{z}^{m+1} \hat{\bar{T}}(\bar{z})$$

$$\Rightarrow \hat{L}_m|\mathbb{I}\rangle \simeq \oint_C \frac{d\bar{z}}{2\pi i} \bar{z}^{m+1} \underbrace{\hat{\bar{T}}(\bar{z}) \hat{\mathbb{I}}(0,0)}_{\text{OPE}}$$

$$\underbrace{\hat{\bar{T}}(\bar{z}) \hat{\mathbb{I}}(0,0)}_{\text{OPE}} = :\hat{\bar{T}}(\bar{z}): \rightarrow \text{divergenceless \& anti-holomorphic}$$

$$\rightarrow \hat{L}_m|\mathbb{I}\rangle \simeq \oint_C \frac{d\bar{z}}{2\pi i} \bar{z}^{m+1} :\hat{\bar{T}}(\bar{z}):$$

$$\text{For } m \geq -1, \quad \underline{\hat{L}_m|\mathbb{I}\rangle} = 0$$

$\rightarrow '|\mathbb{I}\rangle'$ is $SL(2, \mathbb{C})$ invariant.

Consider $\hat{L}_{-m}$, $m > 1$,

$$\hat{L}_{-m}|\mathbb{I}\rangle \simeq \oint_C \frac{d\bar{z}}{2\pi i} \frac{:\hat{T}(\bar{z}):}{\bar{z}^{m-1}} = \bar{\partial}^{m-2} \hat{T}(0)$$

$$\rightarrow \hat{L}_{-m}|\mathbb{I}\rangle \longrightarrow \bar{\partial}^{m-2} \hat{T}(0)$$

Thus $\hat{T}(0)$ correspond to state $\hat{L}_{-2}|\mathbb{I}\rangle$

Now, we can find the creation-annihilation normal ordering constants for $\hat{L}_m$ & $\hat{L}_{-m}$ using state operator map.

The Normal ordering constant ' A_m ' for the m^{th} mode Virasoro generator can be got by the difference of $:T(z): - \bar{:}T(z)\bar{:}$, where,

$$A_m = \oint_C \frac{dz}{2\pi i} z^{m+1} [:T(z): - \bar{:}T(z)\bar{:}]$$

$:T(z): \rightarrow$ Conformal normal ordering

$\bar{:}T(z)\bar{:} \rightarrow$ Creation-annihilation normal ordering

$$\text{Now } \oint_C \frac{dz}{2\pi i} z^{m+1} :T(z): \longrightarrow \hat{L}_m |\text{II}\rangle$$

$$\neq \oint_C \frac{dz}{2\pi i} z^{m+1} :T(z): \longrightarrow \hat{L}_m |0\rangle$$

We have chosen vacuum state $|0\rangle$ of the CFT as that state which gets annihilated by all the annihilation operators. So the Creation-annihilation normal ordering puts all creation operators left of annihilation ones. This state ' $|0\rangle$ ' in general need not be $SL(2, \mathbb{C})$ invariant.

But the state $|\text{II}\rangle$ corresponding to the unit operator focuses on the conformal normal ordering. This the state which is actually $SL(2, \mathbb{C})$ invariant.

Thus the Normal ordering constant ' A_m ' is found by the difference b/w the two types of normal ordering.

$$\rightarrow A_m = \frac{\langle \mathbb{I} | \hat{L}_m | \mathbb{I} \rangle}{\langle \mathbb{I} | \mathbb{I} \rangle} - \frac{\langle 0 | \hat{L}_m | 0 \rangle}{\langle 0 | 0 \rangle}$$

Eg: In χ^u CFT, $|\mathbb{I}\rangle = |0\rangle$. So we get $A_m = 0$ for all 'm'.

But for bc-CFT,

$$\hat{L}_m = \sum_{n=-\infty}^{\infty} [m\lambda - n] : \hat{b}_n \hat{c}_{m-n} : + A_m^g \delta_{m,0}$$

Only $\hat{L}_0$ needs a constant. So,

$$\hat{L}_0 = - \sum_{n=-\infty}^{\infty} n : \hat{b}_n \hat{c}_{-n} : + A^g$$

$\hat{L}_0 |\mathbb{I}\rangle = 0$, as $|\mathbb{I}\rangle$ is $SL(2, \mathbb{C})$ invariant

$$\Rightarrow \hat{L}_0 |\mathbb{I}\rangle = - \sum_{n=-\infty}^{\infty} n : \hat{b}_n \hat{c}_{-n} : |\mathbb{I}\rangle - A^g |\mathbb{I}\rangle = 0$$

$$\Rightarrow \underline{A^g |\mathbb{I}\rangle = \sum_{n=-\infty}^{\infty} n : \hat{b}_n \hat{c}_{-n} : |\mathbb{I}\rangle}$$

$$\Rightarrow A^g |II\rangle = - \sum_{n=1}^{\infty} n (\hat{c}_{-n} \hat{b}_n + \hat{b}_{-n} \hat{c}_n) |II\rangle$$

$$|II\rangle = \begin{cases} \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle, & \text{if } \lambda > 1. \\ |0\rangle, & \text{if } \lambda = 1. \\ \hat{c}_{\lambda} \hat{c}_{\lambda+1} \cdots \hat{c}_{-1} \hat{c}_0 |0\rangle, & \text{if } \lambda < 1. \end{cases}$$

If $\lambda > 1$

$$\Rightarrow A^g |II\rangle = - \sum_{n=1}^{\infty} n (\hat{c}_{-n} \hat{b}_n + \hat{b}_{-n} \hat{c}_n) |II\rangle$$

$$= - \sum_{n=1}^{\infty} n (\hat{c}_{-n} \hat{b}_n + \hat{b}_{-n} \hat{c}_n) \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

no
contribution

$$= - \sum_{n=1}^{\infty} n \hat{b}_{-n} \hat{c}_n \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \cdots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

Survives only for

$$n = \{1, 2, 3, \dots, \lambda-1\}$$

$$\Rightarrow A^g |II\rangle = - \sum_{n=1}^{\lambda-1} n (\hat{b}_{-n} \hat{c}_n) \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$\text{Now, } [\hat{b}_{-n} \hat{c}_n, \hat{b}_m] = \hat{b}_{-n} \hat{c}_n \hat{b}_m - \hat{b}_m \hat{b}_{-n} \hat{c}_n$$

$$= \hat{b}_{-n} \delta_{n,-m} - \hat{b}_{-n} \hat{b}_m \hat{c}_n - \hat{b}_m \hat{b}_{-n} \hat{c}_n$$

$$= \hat{b}_{-n} \delta_{n,-m} + \cancel{\hat{b}_m \hat{b}_{-n}} \hat{c}_n - \cancel{\hat{b}_m \hat{b}_{-n}} \hat{c}_n$$

$$[\hat{b}_{-n} \hat{c}_n, \hat{b}_m] = \hat{b}_{-n} \delta_{n,-m}$$

$$\Rightarrow \hat{b}_{-n} \hat{c}_n \hat{b}_m = \hat{b}_{-n} \delta_{n,-m} + \hat{b}_m \hat{b}_{-n} \hat{c}_n$$

$$\Rightarrow A^g |II\rangle = - \sum_{n=1}^{\lambda-1} n (\hat{b}_{-n} \hat{c}_n) \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$= - \sum_{n=1}^{\lambda-1} n (\hat{b}_{-n} \delta_{n,(\lambda-1)} + \hat{b}_{-(\lambda-1)} \hat{b}_{-n} \hat{c}_n) \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$= -(\lambda-1) \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$- \sum_{n=1}^{\lambda-1} n \hat{b}_{-(\lambda-1)} (\hat{b}_{-n} \hat{c}_n \hat{b}_{-(\lambda-2)}) \hat{b}_{-(\lambda-3)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$= -(\lambda-1)|\text{II}\rangle$$

$$- \sum_{n=1}^{\lambda-1} n \hat{b}_{-(\lambda-1)} (\hat{b}_{-n} \hat{c}_{n,(\lambda-2)} + \hat{b}_{-(\lambda-2)} \hat{b}_{-n} \hat{c}_n) \hat{b}_{-(\lambda-3)} \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$= -(\lambda-1)|\text{II}\rangle - (\lambda-2)|\text{II}\rangle$$

$$- \sum_{n=1}^{\lambda-1} n \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} (\hat{b}_{-n} \hat{c}_n \hat{b}_{-(\lambda-3)}) \dots \hat{b}_{-2} \hat{b}_{-1} |0\rangle$$

$$\vdots$$

$$= -(\lambda-1)|\text{II}\rangle - (\lambda-2)|\text{II}\rangle \dots - (1)|\text{II}\rangle$$

$$- \sum_{n=1}^{\lambda-1} n \hat{b}_{-(\lambda-1)} \hat{b}_{-(\lambda-2)} \hat{b}_{-(\lambda-3)} \dots \hat{b}_{-2} \hat{b}_{-1} (\underbrace{\hat{b}_{-n} \hat{c}_n}_0) |0\rangle$$

$$\Rightarrow A^g |\text{II}\rangle = - \sum_{n=1}^{\lambda-1} n |\text{II}\rangle = - \frac{(\lambda-1)\lambda}{2} |\text{II}\rangle$$

$$\Rightarrow \underline{A^g = -\frac{(\lambda-1)\lambda}{2} \text{ if } \lambda > 1}$$

Similarly, $A^g = 0$ if $\lambda = 1$. Since, we would get $|\text{II}\rangle = |0\rangle$ if $\lambda = 1$.

If $\lambda < 1$, $|\text{II}\rangle = \hat{c}_\lambda \hat{c}_{\lambda+1} \cdots \hat{c}_{-2} \hat{c}_{-1} \hat{c}_0 |0\rangle$

$$\& A^g |\text{II}\rangle = - \sum_{n=1}^{\infty} n (\hat{c}_{-n} \hat{b}_n + \underbrace{\hat{b}_{-n} \hat{c}_n}_{\text{No contribution}}) |\text{II}\rangle$$

$$= - \sum_{n=1}^{\infty} n \hat{c}_{-n} \hat{b}_n \hat{c}_\lambda \hat{c}_{\lambda+1} \cdots \hat{c}_{-2} \hat{c}_{-1} \hat{c}_0 |0\rangle$$

(Put $\lambda \rightarrow -|\lambda|$)

$$= - \sum_{n=1}^{\infty} n (\hat{c}_{-n} \hat{b}_n) \hat{c}_{-|\lambda|} \hat{c}_{-|\lambda|+1} \cdots \hat{c}_{-2} \hat{c}_{-1} \hat{c}_0 |0\rangle$$

(only $n = \{|\lambda|, |\lambda|+1, |\lambda|+2, \dots, 2, 1\}$ contribute)

Now, $[\hat{c}_{-n} \hat{b}_n, \hat{c}_m] = \hat{c}_{-n} \hat{b}_n \hat{c}_m - \hat{c}_m \hat{c}_{-n} \hat{b}_n$

$$= \hat{c}_{-n} \delta_{n,-m} - \hat{c}_{-n} \hat{c}_m \hat{b}_n - \hat{c}_m \hat{c}_{-n} \hat{b}_n$$

$$= \hat{c}_{-n} \delta_{n,-m} + \cancel{\hat{c}_m \hat{c}_{-n}} \hat{b}_n - \cancel{\hat{c}_m \hat{c}_{-n}} \hat{b}_n$$

$$\Rightarrow [\hat{c}_{-n} \hat{b}_n, \hat{c}_m] = \hat{c}_{-n} \delta_{n,-m}$$

$$\Rightarrow \hat{c}_{-n} \hat{b}_n \hat{c}_m = \hat{c}_{-n} \delta_{n,-m} + \hat{c}_m \hat{c}_{-n} \hat{b}_n$$

$$\begin{aligned}
 \rightarrow A^g |\text{II}\rangle &= - \sum_{n=1}^{|\lambda|} n (\hat{c}_{-n} \hat{b}_n) \hat{c}_{-|\lambda|} \hat{c}_{-|\lambda|+1} \cdots \hat{c}_{-2} \hat{c}_{-1} \hat{c}_0 |0\rangle \\
 &= - \sum_{n=1}^{|\lambda|} n |\text{II}\rangle = - \frac{|\lambda|(|\lambda|+1)}{2} |\text{II}\rangle
 \end{aligned}$$

Put back $|\lambda| \rightarrow -\lambda$

$$\rightarrow A^g |\text{II}\rangle = - \frac{\lambda(\lambda-1)}{2} |\text{II}\rangle$$

Thus, for any value of ' λ ' we can write

$$\boxed{A^g = - \frac{\lambda(\lambda-1)}{2}}$$

Central Charges & Zero Point Energies

The central charge is present in the expression of hamiltonian as

$$\hat{H} = \hat{L}_0 + \hat{\tilde{L}}_0 - \left(\frac{C + \tilde{C}}{24} \right)$$

Since the identity state annihilates both $\hat{L}_0$ & $\hat{\tilde{L}}_0$, we get

$$\hat{H}|\mathbb{I}\rangle = - \left(\frac{C + \tilde{C}}{24} \right) |\mathbb{I}\rangle$$

In unitary CFTs where $|\mathbb{I}\rangle = |0\rangle$, the energy eigen value corresponds to the Vacuum energy called Casimir Energy or the Zero point energy. It comes from the central charge which is greater than Zero for unitary CFTs.

$$\underline{E_{\text{casimir}} = - \left(\frac{C + \tilde{C}}{24} \right)}$$

This arises due to the finite size of the system of size ' 2π '.

E_{casimir} is inversely proportional to the length. Since, E_{casimir} is in Energy units

$$\& H = - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} (T_{ww} + T_{\bar{w}\bar{w}})$$

$$\rightarrow E_{\text{casimir}} = - \int_0^{2\pi} \frac{d\sigma^1}{2\pi} \left(\frac{C + \tilde{C}}{24} \right)$$

For any other arbitrary ' l ' we get,

$$E_{\text{casimir}} = - \int_0^{2\pi} \frac{d\sigma^1}{l} \left(\frac{C + \tilde{C}}{24} \right)$$

$$\Rightarrow \boxed{E_{\text{casimir}} = - \frac{2\pi}{l} \left(\frac{C + \tilde{C}}{24} \right)}$$

