

Normalized Polyakov Path Integral

There is an enormous overcounting in the Path integral given by

$$\int [\mathcal{D}\chi^\mu \mathcal{D}g_{ab}] e^{-S} = \int [\mathcal{D}\chi^\mu \mathcal{D}g_{ab}] e^{-S_x} e^{-\lambda \tilde{\chi}}$$

where $S_x = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} g^{ab} (\partial_a \chi^\mu) (\partial_b \chi_\mu)$

is the Polyakov action. And $\tilde{\chi} = \chi - \frac{1}{4} n_{\text{corner}}$

where $\chi = \frac{1}{4\pi} \int_M d^2\sigma \sqrt{g} R + \frac{1}{2\pi} \int_{\partial M} ds k$ is

the Euler characteristic.

This over-counting is a result of the gauge-symmetries present in the Polyakov action namely the diffeomorphism $\times$ Weyl Symmetries.

Two different configuration (X^μ, g_{ab}) and (X'^μ, g'_{ab}) that are related by diffeomorphism $\times$ Weyl symmetries represent the same physical configurations. So, we are over-counting all possible configurations in the path-integral measure $[DX^\mu Dg_{ab}]$. Thus we need to divide by the $\text{Vol}_{\text{diff} \times \text{Weyl}}$ gauge group to remove the over counting. This implies that we should be integrating over only the physically distinct configurations, not related by Weyl or diffeomorphism symmetry transformations.

$$\Rightarrow Z = \frac{\int [DX^\mu Dg_{ab}] e^{-S}}{\text{Vol}_{\text{diff} \times \text{Weyl}}} \quad \text{is appropriate.}$$

The over-counting can be avoided by Gauge-fixing, integrating over a slice that cuts through each gauge equivalent metrics once & then obtain correct measure on the slice by Faddeev-Popov method.

Fiducial Metric & Residual gauge Symmetry

The world sheet metric, being symmetric, has three independent components, so there are those many gauge functions. They are given by local scale of the metric (related to Weyl invariance) and two world sheet coordinates (related to diffeomorphism).

Now, these gauge symmetries allow us to put the metric into any form of our choosing, this chosen metric is called as the Fiducial metric ' $\hat{g}_{ab}$ '. This represents our choice of gauge fixing. So, setting a $\hat{g}_{ab}$ fixes the gauge we are working on.

Now, we can also show that any metric is locally equivalent to flat metric, where we can use Weyl & diffeomorphism symmetry to do the fixing.

Consider g_{ab} as a world sheet metric. If we Weyl transform the metric to g'_{ab} , we know that both $g'_{ab} \neq g_{ab}$ represent the same configuration due to the Weyl Symmetry in the action.

$$\rightarrow g'_{ab}(\sigma^a) = e^{2\phi(\sigma^a)} g_{ab}(\sigma^a)$$

Then the Ricci Scalars (R) of these metrics are related by,

$$\sqrt{g'} R' = \sqrt{g} (R - 2\nabla^2 \phi)$$

where R is built out of g_{ab} & R' is built out of g'_{ab} . So, if we choose $\phi(\sigma^a)$ such that it satisfies ' $R - 2\nabla^2 \phi = 0$ ' for ϕ which can always be solved atleast locally.

Then we get $R' = 0$. So, then the world sheet described by the Weyl transformed metric g'_{ab} should describe a flat metric (if ' $R - 2\nabla^2 \phi = 0$ ' is solvable globally) or locally flat metric (if ' $R - 2\nabla^2 \phi = 0$ ' is solvable only locally).

This is true because in two dimensions,

$$R'_{abcd} = \frac{R'}{2} [g'_{ac} g'_{bd} - g'_{ad} g'_{bc}]$$

↓
(Riemann Tensor)

So if $R' = 0$, then $R'_{abcd} = 0$. If all Riemann tensor components are zero, then it implies that the world sheet described by the corresponding metric is flat (or locally flat).

Thus g'_{ab} represents flat metric or locally flat smooth manifold.

Note: Only those world sheets that are topologically equivalent to sphere or a infinite cylinder can be transformed to a flat metric. But metrics that have holes (like a torus) can not be transformed to a flat metric globally, but only locally.

Now, provided we do a weyl transform we get locally flat world sheet, then we can do diffeomorphism transforms & reach a flat metric γ_{ab} (locally).

So we can set $\hat{g}_{ab} = \gamma_{ab}$, this is known as flat / Unit gauge condition.

Sometimes, if we want to consider the effect of diff group alone, we reach something called conformal gauge. We can use up two diffeomorphism degree of freedom to reach a unit metric upto a weyl rescaling.

Thus in conformal gauge one can choose

$$\underline{\underline{\hat{g}_{ab} = e^{2\omega(\sigma^a)} \gamma_{ab}}}$$

Conformal invariance as residual gauge symmetry

Even after fixing a particular gauge, say unit-flat gauge, we have used up the three gauge degrees of freedom of the metric.

In flat gauge, let us introduce complex coordinates $z = \sigma^1 + i\sigma^2$ & $\bar{z} = \sigma^1 - i\sigma^2$ then the flat metric takes the form

$$g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$$

Then the line element is $ds^2 = dz d\bar{z}$

Now, under conformal transformation, if

$z \rightarrow f(z) = z'$ & $\bar{z} \rightarrow f^*(\bar{z}) = \bar{z}'$, then

$$dz' = (\partial_z f) dz \quad \& \quad d\bar{z}' = (\partial_{\bar{z}} f^*) d\bar{z}$$

$$\Rightarrow (ds')^2 = |\partial_z f(z)|^2 dz d\bar{z}$$

$$\Rightarrow (ds')^2 = |\partial_z f(z)|^2 (ds)^2$$

This is a diffeomorphism transformation.

Now, combine a Weyl transformation.

$$(ds'')^2 = e^{2\omega} (ds')^2 = e^{2\omega} |\partial_z f(z)|^2 (ds)^2$$

If we choose ' ω ' such that

$$e^{2\omega} |\partial_z f(z)|^2 = 1 \quad \text{or} \quad \omega = -\ln |\partial_z f(z)|$$

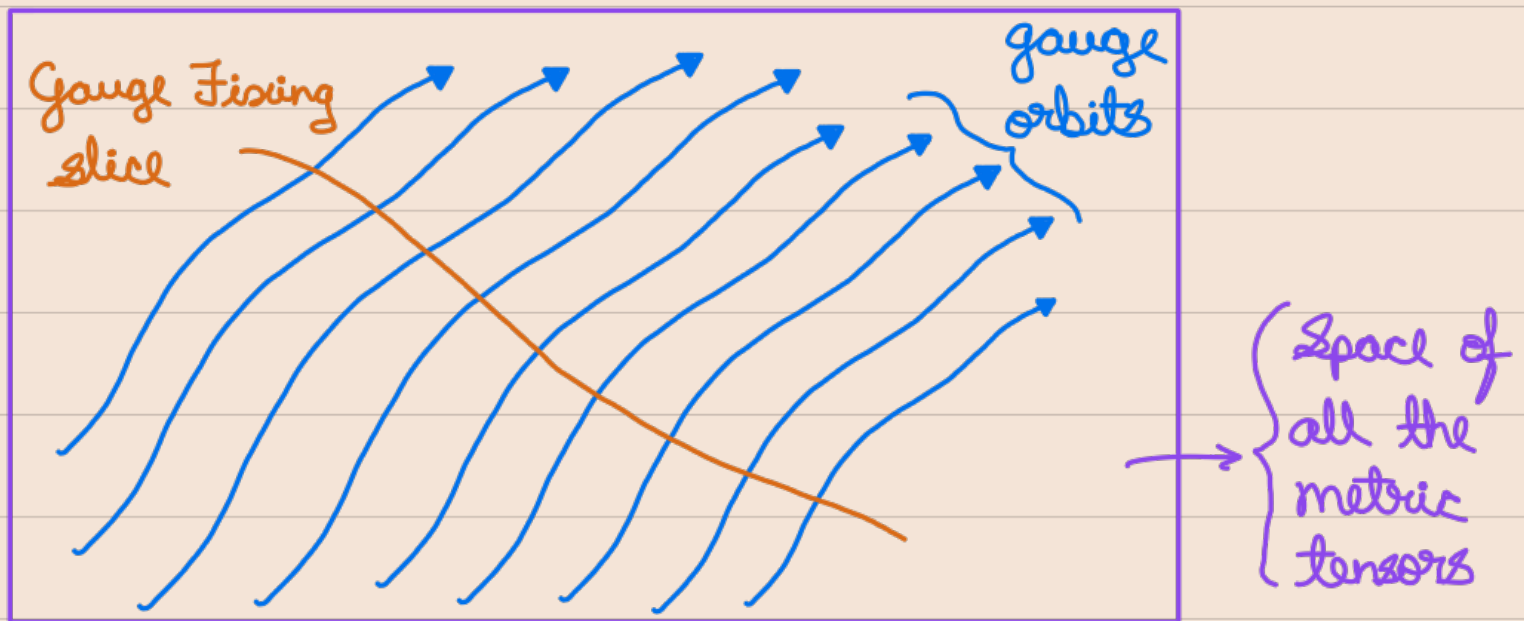
Then the combined diff $\times$ weyl invariance still fixes the metric invariant.

$$\text{i.e.} \quad (ds)^2 \xrightarrow{\text{diff} \times \text{weyl}} (ds)^2$$

- The full diffeomorphism group $\text{Diff}(\Sigma)$ is infinite-dimensional. The holomorphic maps satisfying $\bar{\partial}f = 0$ are solutions of a strong PDE constraint. On a compact Riemann surface the space of holomorphic automorphisms is typically finite-dimensional (e.g. for the sphere it is the 6-real-parameter Möbius group $PSL(2, \mathbb{C})$; for genus $g \geq 2$ it is finite and often trivial). Even if locally one can write many holomorphic functions, the holomorphic subset sits in a much lower dimensional manifold than Diff . In functional-measure terms the probability of selecting a random diffeo that is holomorphic is zero. That is what Polchinski means by "measure zero."
- Concretely: if you parametrize reparameterizations by infinitely many Fourier modes, the holomorphic condition imposes infinitely many constraints, reducing the functional dimension drastically (in compact case down to a finite number). Hence negligible (measure zero) in path-integral measure.

Faddeev - Popov Procedure

The Faddeev-Popov procedure provides a way to split our Integration over physically distinct configurations (not related by any $\text{diff} \times \text{Weyl}$ transformations) and those over gauge orbits.



A particular gauge orbit is the set of all gauge field configurations (metric tensors) that are connected to each other by $\text{diff} \times \text{Weyl}$ transformations. So, $\text{diff} \times \text{Weyl}$ Inequivalent metric tensors belong to different gauge orbits.

A gauge fixing slice is a sub-space of the configuration space of all metric tensors which intersects every possible orbit once & only once.

Doing a path integral over all the possible metric tensors & then dividing by the volume of the $\text{diff} \times \text{Weyl}$ gauge group is same as integrating over just the gauge fixing slice which intersects all orbits once & only once. This way we can avoid over counting.

So, in the process of converting the path integral over all the possible metric tensors to a path integral over just the gauge fixing slice, we need to account for the Jacobian of transformation.

We shall parameterize different gauge orbits by a parameter ' E '. So, for every ' E ', there is a representative metric tensor $\hat{g}_{ab}(E)$ that is the fiducial metric for the particular orbit.

Let ' ζ ' represent the combined diff $\times$ Weyl transformations. So, ζ maps $g_{ab} \rightarrow g_{ab}^{\zeta}$

$$\zeta: g_{ab} \rightarrow g_{ab}^{\zeta}(\sigma) = e^{2\omega(\sigma)} \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b} g_{cd}(\sigma)$$

where $\sigma \equiv \sigma^a = (\sigma^1, \sigma^2)$ are old coordinates
& $\sigma' \equiv \sigma'^a = (\sigma'^1, \sigma'^2)$ are reparametrized ones.

So, for each ' ζ ' we have a representative metric tensor $\hat{g}_{ab}(\zeta)$ & an orbit given by the map ' ζ ' which takes $\hat{g}_{ab}(\zeta)$ to $\hat{g}_{ab}^{\zeta}(\zeta)$ in the same orbit.

In the path integral, we need to only consider ' $\hat{g}_{ab}(\zeta)$ ' or the gauge fixing slice.
& not entire space of ' g_{ab} 's.

So in this process we need to account for the jacobian of transformation given by the Faddeev-Popov determinant.

Faddeev - Popov Determinant

To find the relevant Jacobian we shall use the defⁿ of Faddeev - Popov determinant ' $\Delta_{FP}(g_{ab})$ ' which is defined by the eqⁿ,

$$[\Delta_{FP}(g_{ab})]^{-1} = \int dE_s d\zeta_s \delta[g_{ab} - \hat{g}_{ab}^{\zeta_s}(E_s)]$$

dE_s & $d\zeta_s$ are gauge invariant measures.

The dirac - delta is actually a functional that which imposes the condition $g_{ab} = \hat{g}_{ab}^{\zeta_s}(E_s)$.

We need an ideal $\hat{g}_{ab}^{\zeta_s}(E_s)$ such that any metric tensor ' g_{ab} ' can be got by $\hat{g}_{ab}$ for some orbit ' E_s ' & any diff x Weyl transform ' ζ_s '. Only then $\Delta_{FP}(g_{ab})$ is defined & meaningful.

It can be proved that the Faddeev - Popov determinant $\Delta_{FP}(g_{ab})$ is gauge invariant.

$$i.e. \quad \Delta_{FP}(g_{ab}^{\zeta_s}) = \Delta_{FP}(g_{ab}) \quad \text{for any } \zeta_s.$$

Proof: Consider the defⁿ of $[\Delta_{FP}(g_{ab}^{\zeta'})]^{-1}$

$$[\Delta_{FP}(g_{ab}^{\zeta'})]^{-1} = \int d\xi_g d\zeta_g \delta[g_{ab}^{\zeta'} - \hat{g}_{ab}^{\zeta}(\xi_g)]$$

The dirac delta forces $g_{ab}^{\zeta'} = \hat{g}_{ab}^{\zeta}(\xi_g)$ or it is equivalent to forcing $g_{ab} = \hat{g}_{ab}^{\zeta \cdot (\zeta')^{-1}}(\xi_g)$

Since the delta function is gauge invariant,

$$\rightarrow [\Delta_{FP}(g_{ab}^{\zeta'})]^{-1} = \int d\xi_g d\zeta_g \delta[g_{ab} - \hat{g}_{ab}^{\zeta \cdot (\zeta')^{-1}}(\xi_g)]$$

Denote $\zeta \cdot (\zeta')^{-1} = \zeta''$ & since the measure is gauge invariant, $d\zeta_g = d\zeta''$

$$\rightarrow [\Delta_{FP}(g_{ab}^{\zeta'})]^{-1} = \underbrace{\int d\xi_g d\zeta'' \delta[g_{ab} - \hat{g}_{ab}^{\zeta''}(\xi_g)]}_{\text{def}^n \text{ of } [\Delta_{FP}(g_{ab})]^{-1}}$$

$$\Rightarrow \underline{\Delta_{FP}(g_{ab})} = \underline{\Delta_{FP}(g_{ab}^{\zeta})} \text{ for any } \zeta_g$$

Thus $\Delta_{FP}(g_{ab})$ is gauge invariant.

Evaluating the Normalized Path Integral

So, we shall insert

$$1 = \Delta_{FP}(g_{ab}) \int d\xi_3 d\zeta_3 \delta[g_{ab} - \hat{g}_{ab}^{\zeta_3}(\xi_3)]$$

inside the path integral measure.

$$Z = \frac{\int [D\chi^\mu Dg_{ab}]}{\text{Vol}_{\text{diff}} \times \text{Weyl}} (1) e^{-S[\chi^\mu, g_{ab}]}$$

$$= \frac{\int [D\chi^\mu Dg_{ab}]}{\text{Vol}_{\text{diff}} \times \text{Weyl}} \Delta_{FP}(g_{ab}) \int d\xi_3 d\zeta_3 \delta[g_{ab} - \hat{g}_{ab}^{\zeta_3}(\xi_3)] e^{-S[\chi^\mu, g_{ab}]}$$

$$= \int d\xi_3 d\zeta_3 \int [D\chi^\mu] \underbrace{\frac{\int [Dg_{ab}] \delta[g_{ab} - \hat{g}_{ab}^{\zeta_3}(\xi_3)] \Delta_{FP}(g_{ab}) e^{-S[\chi^\mu, g_{ab}]}]{\text{Vol}_{\text{diff}} \times \text{Weyl}}}_{\text{Integrate over } g_{ab}}$$

(this replaces $g_{ab} \rightarrow \hat{g}_{ab}^{\zeta_3}(\xi_3) \neq \chi^\mu \xrightarrow[\text{variable}]{\text{dummy}} \chi^{\mu\zeta_3}$)

$$= \frac{\int d\xi_3 d\zeta_3}{\text{Vol}_{\text{diff}} \times \text{Weyl}} \int [D\chi^{\mu\zeta_3}] \Delta_{FP}[\hat{g}_{ab}^{\zeta_3}(\xi_3)] e^{-S[\chi^{\mu\zeta_3}, \hat{g}_{ab}^{\zeta_3}(\xi_3)]}$$

Now, $\Delta_{FP}[\hat{g}_{ab}^{\zeta}(\xi)] \neq S[\chi^{\mu\zeta}, \hat{g}_{ab}^{\zeta}(\xi)]$ is gauge invariant.

So replace $\Delta_{FP}[\hat{g}_{ab}^{\zeta}(\xi)] \rightarrow \Delta_{FP}[\hat{g}_{ab}(\xi)]$ and $S[\chi^{\mu\zeta}, \hat{g}_{ab}^{\zeta}(\xi)] \rightarrow S[\chi^{\mu}, \hat{g}_{ab}(\xi)]$.

$$\rightarrow Z[\hat{g}_{ab}] = \frac{\int d\xi d\zeta}{\text{Vol}_{\text{diff} \times \text{Weyl}}} \int [D\chi^{\mu}] \Delta_{FP}[\hat{g}_{ab}(\xi)] e^{-S[\chi^{\mu}, \hat{g}_{ab}(\xi)]}$$

Now, nothing inside the integral depends on the choice of ' ζ '. But since the choice of all possible ' ζ ' will cover the entire $\text{diff} \times \text{Weyl}$ equivalent configurations,

$$\rightarrow \int d\zeta = \text{Vol}_{\text{diff} \times \text{Weyl}}$$

$$\rightarrow Z[\hat{g}_{ab}] = \int d\xi \int [D\chi^{\mu}] \Delta_{FP}[\hat{g}_{ab}(\xi)] e^{-S[\chi^{\mu}, \hat{g}_{ab}(\xi)]}$$

Thus integration over all possible metric tensor is replaced by integration over only the gauge fixing slice. But the appropriate Jacobian is the Faddeev-Popov determinant.

To find $\delta g_{ab}(\sigma^a) = g_{ab}^{\zeta}(\sigma^a) - g_{ab}(\sigma^a)$

We shall find the above expression for infinitesimal diff x weyl transform, i.e., where ζ is close to Identity which ensures $\delta g_{ab}(\sigma^a)$ is infinitesimal.

Now $\zeta: g_{ab} \rightarrow g_{ab}^{\zeta}(\sigma^i)$ where $\sigma^{i1} = \sigma^{i1}(\sigma^1, \sigma^2)$ & $\sigma^{i2} = \sigma^{i2}(\sigma^1, \sigma^2)$ are new coordinates.

$$\text{So, } g_{ab}^{\zeta}(\sigma^i) = e^{2\omega(\sigma)} \frac{\partial \sigma^c}{\partial \sigma^{i1}} \frac{\partial \sigma^d}{\partial \sigma^{i2}} g_{cd}(\sigma)$$

$$\text{where } e^{2\omega(\sigma)} \sim e^{2\delta\omega(\sigma)} = 1 + 2\delta\omega(\sigma)$$

$$\& \sigma^{i1} = \sigma^1 + \delta\sigma^1 \rightarrow \frac{\partial \sigma^a}{\partial \sigma^{i1}} = \partial_a^1 - \partial_a(\delta\sigma^1)$$

So, upto 1st order,

$$\begin{aligned} g_{ab}^{\zeta}(\sigma^i) &= [1 + 2\delta\omega(\sigma)] [\partial_a^c - \partial_a(\delta\sigma^c)] [\partial_b^d - \partial_b(\delta\sigma^d)] g_{cd}(\sigma) \\ &= \partial_a^c \partial_b^d g_{cd}(\sigma) + 2\delta\omega(\sigma) \partial_a^c \partial_b^d g_{cd}(\sigma) \\ &\quad - \partial_a(\delta\sigma^c) \partial_b^d g_{cd}(\sigma) - \partial_a^c \partial_b(\delta\sigma^d) g_{cd}(\sigma) \end{aligned}$$

$$\begin{aligned} \rightarrow g_{ab}^{\zeta}(\sigma') &= g_{ab}(\sigma) + 2\partial\omega(\sigma)g_{ab}(\sigma) \\ &\quad - \partial_a(\partial\sigma^c) \underbrace{g_{cb}(\sigma)}_{=g_{bc}(\sigma)} - \partial_b(\partial\sigma^d) \underbrace{g_{ad}(\sigma)}_{d \rightarrow c} \end{aligned}$$

$$\begin{aligned} \rightarrow g_{ab}^{\zeta}(\sigma') &= g_{ab}(\sigma) + 2\partial\omega(\sigma)g_{ab}(\sigma) \\ &\quad - \partial_a(\partial\sigma^c)g_{bc}(\sigma) - \partial_b(\partial\sigma^c)g_{ac}(\sigma) \end{aligned}$$

$$\begin{aligned} \text{Now } g_{ab}^{\zeta}(\sigma) &= g_{ab}^{\zeta}(\sigma' - \partial\sigma^c) \\ &= g_{ab}^{\zeta}(\sigma') - \partial\sigma^c \partial_c g_{ab}^{\zeta}(\sigma') \end{aligned}$$

$$\rightarrow \partial g_{ab}(\sigma) = g_{ab}^{\zeta}(\sigma) - g_{ab}(\sigma)$$

$$\partial g_{ab}(\sigma) = g_{ab}^{\zeta}(\sigma') - \partial\sigma^c \partial_c g_{ab}^{\zeta}(\sigma') - g_{ab}(\sigma)$$

$$\partial\sigma^c \partial_c g_{ab}^{\zeta}(\sigma') = \partial\sigma^c \partial_c g_{ab}(\sigma) \text{ upto 1st order}$$

$$\rightarrow \partial g_{ab}(\sigma) = g_{ab}^{\zeta}(\sigma') - g_{ab}(\sigma) - \partial\sigma^c \partial_c g_{ab}(\sigma)$$

$$\begin{aligned} \rightarrow \partial g_{ab}(\sigma) &= 2\partial\omega(\sigma)g_{ab}(\sigma) - \partial_a(\partial\sigma^c)g_{bc}(\sigma) \\ &\quad - \partial_b(\partial\sigma^c)g_{ac}(\sigma) - \partial\sigma^c \partial_c g_{ab}(\sigma) \end{aligned}$$

Every quantity is a function of ' σ '. So, no need to explicitly mention it.

$$\text{Now, } \delta \sigma^c = g^{cd} \delta \sigma_d$$

$$\begin{aligned} \Rightarrow \partial_a (\delta \sigma^c) &= \partial_a (g^{cd} \delta \sigma_d) \\ &= \delta \sigma_d (\partial_a g^{cd}) + g^{cd} (\partial_a \delta \sigma_d) \end{aligned}$$

$$\begin{aligned} \rightarrow \delta g_{ab} &= 2\delta \omega g_{ab} - \delta \sigma_d (\partial_a g^{cd}) g_{bc} \\ &\quad - g^{cd} g_{bc} (\partial_a \delta \sigma_d) - \delta \sigma_d (\partial_b g^{cd}) g_{ac} \\ &\quad - g^{cd} g_{ac} (\partial_b \delta \sigma_d) - g^{cd} \delta \sigma_d \partial_c g_{ab} \end{aligned}$$

$$\begin{aligned} \rightarrow \delta g_{ab} &= 2\delta \omega g_{ab} - \delta_b^d (\partial_a \delta \sigma_d) - \delta_a^d (\partial_b \delta \sigma_d) \\ &\quad - \delta \sigma_d (\partial_a g^{cd}) g_{bc} - \delta \sigma_d (\partial_b g^{cd}) g_{ac} \\ &\quad - \delta \sigma_d g^{cd} \partial_c g_{ab} \end{aligned}$$

$$\begin{aligned} \rightarrow \delta g_{ab} &= 2\delta \omega g_{ab} - (\partial_a \delta \sigma_b) - (\partial_b \delta \sigma_a) \\ &\quad - \delta \sigma_d \left[(\partial_a g^{cd}) g_{bc} + (\partial_b g^{cd}) g_{ac} + g^{cd} \partial_c g_{ab} \right] \end{aligned}$$

$$\text{Since } g^{cd} g_{bc} = \delta_b^d \rightarrow \partial_a (g^{cd} g_{bc}) = 0$$

$$\text{So, } (\partial_a g^{cd}) g_{bc} = -g^{cd} (\partial_a g_{bc})$$

Using this relation,

$$\rightarrow \delta g_{ab} = 2\delta\omega g_{ab} - (\partial_a \delta\sigma_b) - (\partial_b \delta\sigma_a) + \delta\sigma_d \left[g^{cd} (\partial_a g_{bc}) + g^{cd} (\partial_b g_{ac}) - g^{cd} \partial_c g_{ab} \right]$$

$$\rightarrow \delta g_{ab} = 2\delta\omega g_{ab} - (\partial_a \delta\sigma_b) - (\partial_b \delta\sigma_a) + \delta\sigma_d \underbrace{g^{cd} \left[(\partial_a g_{bc}) + (\partial_b g_{ac}) - \partial_c g_{ab} \right]}_{2\Gamma_{ab}^d}$$

where $\Gamma_{ab}^d = \frac{g^{cd}}{2} \left[(\partial_a g_{bc}) + (\partial_b g_{ac}) - \partial_c g_{ab} \right]$ is the christoffel symbol.

$$\rightarrow \delta g_{ab} = 2\delta\omega g_{ab} - (\partial_a \delta\sigma_b) - (\partial_b \delta\sigma_a) + 2\delta\sigma_d \Gamma_{ab}^d$$

Now $\nabla_a \delta\sigma_b = \partial_a \delta\sigma_b - \Gamma_{ab}^d \delta\sigma_d$ is the covariant derivative.

$$\text{So, } (\partial_a \delta\sigma_b) = \nabla_a \delta\sigma_b + \Gamma_{ab}^d \delta\sigma_d$$

$$(\partial_b \delta\sigma_a) = \nabla_b \delta\sigma_a + \Gamma_{ba}^d \delta\sigma_d$$

Since $\Gamma_{ab}^d = \Gamma_{ba}^d$, we get,

$$\delta g_{ab} = 2\delta\omega g_{ab} - \nabla_a \delta\sigma_b - \cancel{\Gamma_{ab}^d \delta\sigma_d} - \nabla_b \delta\sigma_a - \cancel{\Gamma_{ba}^d \delta\sigma_d} + 2\cancel{\delta\sigma_d \Gamma_{ab}^d}$$

$$\Rightarrow \boxed{\delta g_{ab} = 2\delta\omega g_{ab} - \nabla_a \delta\sigma_b - \nabla_b \delta\sigma_a}$$

We shall define a differential operator ' P_1 ' that takes any vector to a traceless symmetric rank-2 tensor as,

$$(P_1 v)_{ab} = \frac{1}{2} [\nabla_a v_b + \nabla_b v_a - g_{ab} \nabla_c v^c]$$

$$\text{So } g^{db} (P_1 v)_{ab} = \frac{1}{2} [\nabla_a v^d + \nabla^d v_a - \delta_a^d \nabla_c v^c]$$

$$\begin{aligned} \Rightarrow \underbrace{g^{ab} (P_1 v)_{ab}}_{\text{trace}} &= \frac{1}{2} [\nabla_a v^a + \nabla^a v_a - \delta_a^a \nabla_c v^c] \\ &= \frac{1}{2} (2 - \delta_a^a) \nabla_a v^a \end{aligned}$$

Thus for 2-d world sheet, $\delta_a^a = 2 \Rightarrow \underline{\underline{g^{ab} (P_1 v)_{ab} = 0}}$

$$\Rightarrow \delta g_{ab} = 2\delta\omega g_{ab} - 2(P_1 \delta\sigma)_{ab} - g_{ab} (\nabla_c \delta\sigma^c)$$

$$\Rightarrow \boxed{\delta g_{ab} = [2\delta\omega - (\nabla_c \delta\sigma^c)] g_{ab} - 2(P_1 \delta\sigma)_{ab}}$$

Evaluation of Faddeev-Popov determinant

$$\underbrace{\left(\Delta_{FP}[\hat{g}_{ab}(\xi_s)]\right)^{-1}} = \int d\xi'_s d\zeta_s \delta[\hat{g}_{ab}(\xi_s) - \hat{g}_{ab}^{\zeta_s}(\xi'_s)]$$

It is a function of ' ξ_s ' $\rightarrow$ So it tells which equivalence class (orbit) we are evaluating on.

Now $\delta[\hat{g}_{ab}(\xi_s) - \hat{g}_{ab}^{\zeta_s}(\xi'_s)]$ is non zero only when $\xi'_s \rightarrow \xi_s$ and $\zeta_s \rightarrow 1$, i.e. when we are on the same orbit & when the diff $\times$ Weyl transform tends to identity.

So, we can evaluate the integral when ' ζ_s ' is close to Identity & is on the same orbit ' ξ_s '. So we should find

$$\delta \hat{g}_{ab}(\sigma^a) = \hat{g}_{ab}^{\zeta_s}(\sigma^a) - \hat{g}_{ab}(\sigma^a) \text{ for the same } \xi_s \text{ orbit. If } \zeta_s \rightarrow 1, \text{ then } \delta \hat{g}_{ab}(\sigma^a) \text{ is infinitesimal. So, } \hat{g}_{ab}^{\zeta_s}(\sigma) = e^{2\omega} \frac{\partial \sigma^c}{\partial \sigma^{1a}} \frac{\partial \sigma^d}{\partial \sigma^{1b}} \hat{g}_{cd}(\sigma)$$

where $e^{2\omega} \sim 1 + 2\delta\omega$ & $\sigma^{1a} = \sigma^a + \delta\sigma^a$.

As seen before, we get $\delta \hat{g}_{ab}(\xi) = \hat{g}_{ab}(\xi) - \hat{g}_{ab}^{\mathcal{L}}(\xi)$ as,

$$\delta \hat{g}_{ab}(\xi) = [2\delta\omega - (\hat{\nabla}_c^{\xi} \delta\sigma^c)] \hat{g}_{ab}(\xi) - 2(\hat{P}_1^{\xi} \delta\sigma)_{ab}$$

$$(\hat{P}_1^{\xi} \delta\sigma)_{ab} = \frac{1}{2} [\hat{\nabla}_a^{\xi} \delta\sigma_b + \hat{\nabla}_b^{\xi} \delta\sigma_a - \hat{g}_{ab}(\xi) \hat{\nabla}_c^{\xi} \delta\sigma^c]$$

where a hat '^' on the quantities tell us that we are calculating the objects with respect to the fiducial metric ' $\hat{g}_{ab}(\xi)$ '.

$$\hat{\nabla}_a^{\xi} \delta\sigma_b = \partial_a \delta\sigma_b - \hat{\Gamma}_{ab}^{c\xi} \delta\sigma_c$$

where $\hat{\Gamma}_{ab}^{d\xi} = \frac{\hat{g}^{cd}(\xi)}{2} [\partial_a \hat{g}_{bc}(\xi) + \partial_b \hat{g}_{ac}(\xi) - \partial_c \hat{g}_{ab}(\xi)]$

Thus near the identity $\mathcal{L}_g \rightarrow 1 \neq$ as $\xi'_g \rightarrow \xi_g$

$$(\Delta_{FP}[\hat{g}_{ab}(\xi)])^{-1} = \int d\mathcal{L}_g \delta[\hat{g}_{ab}(\xi) - \hat{g}_{ab}^{\mathcal{L}}(\xi'_g)]$$

$$\rightarrow (\Delta_{FP}[\hat{g}_{ab}(\xi)])^{-1} = \int d\mathcal{L}_g \delta[-\delta \hat{g}_{ab}(\xi)]$$

Also $d\mathcal{L}_g \sim [2\delta\omega] \underbrace{[2\delta\sigma_a]}_{\text{gauge lie algebra measure}}$

$\downarrow$
 gauge lie group measure

$$\Rightarrow (\Delta_{FP}[\hat{g}_{ab}(\xi)])^{-1} = \int [D\omega][D\sigma_a] \delta \left(\left[(\hat{\nabla}_c^\xi \partial \sigma^c) - 2\omega \right] \hat{g}_{ab}(\xi) + 2(\hat{P}_1^\xi \partial \sigma)_{ab} \right)$$

Now we shall write the path integral representation of the delta functional.

Same like how $\delta(x) = \int \frac{dp}{2\pi} e^{ipx}$ we can write,

$$\delta[x_{ab}] = \int \frac{[Dp^{ab}]}{2\pi} \exp \left(i \int d^2\sigma \sqrt{\hat{g}} p^{ab}(\sigma) x_{ab}(\sigma) \right)$$

Here x_{ab} & p^{ab} are rank-2 tensor functions.

Let $B^{ab} = \frac{p^{ab}}{2\pi}$. Thus,

$$\rightarrow \delta[x_{ab}] = \int [DB^{ab}] \exp \left(2\pi i \int d^2\sigma \sqrt{\hat{g}} B^{ab}(\sigma) x_{ab}(\sigma) \right)$$

Now replace $x_{ab} \rightarrow \left[(\hat{\nabla}_c^\xi \partial \sigma^c) - 2\omega \right] \hat{g}_{ab}(\xi) + 2(\hat{P}_1^\xi \partial \sigma)_{ab}$

$$(\Delta_{FP}[\hat{g}_{ab}(\xi)])^{-1} = \int [D\omega][D\sigma_a][DB^{ab}]$$

$$\times e^{\left[2\pi i \int d^2\sigma \sqrt{\hat{g}} \left(B^{ab} \hat{g}_{ab} \left[(\hat{\nabla}_c^\xi \partial \sigma^c) - 2\omega \right] + 2 B^{ab} (\hat{P}_1^\xi \partial \sigma)_{ab} \right) \right]}$$

Now, since $[(\hat{\nabla}_c^\xi \partial \sigma^c) - 2\partial \omega] \hat{g}_{ab}(\xi) + 2(\hat{P}_1^\xi \partial \sigma)_{ab}$ is symmetric & traceless 2-tensor, the delta functional forces even β^{ab} to be a symmetric & traceless 2-tensor. This can be proven by doing the ' $\partial \omega$ ' integral. Consider,

$\int [\partial \partial \omega] \exp[-4\pi i \int d^2\sigma \sqrt{g} \beta^{ab} \hat{g}_{ab} \partial \omega]$ this is the functional repⁿ of $\partial[-2\beta^{ab} \hat{g}_{ab}]$.

$$\rightarrow \partial[-2\beta^{ab} \hat{g}_{ab}] = \int [\partial \partial \omega] \exp[-4\pi i \int d^2\sigma \sqrt{g} \beta^{ab} \hat{g}_{ab} \partial \omega]$$

This restricts $\beta^{ab} \hat{g}_{ab} = 0 \rightarrow \beta^{ab}$ is traceless w.r.t the fiducial gauge-fixed metric tensor.

Thus, at any specified ' ξ ' $(\Delta_{FP}[\hat{g}_{ab}])^{-1}$ is,

$$\int [\partial \partial \omega] [\partial \beta^{ab}] \partial[-2\beta^{ab} \hat{g}_{ab}]$$

$$\times e^{\left[2\pi i \int d^2\sigma \sqrt{g} \left(\underbrace{\beta^{ab} \hat{g}_{ab} (\hat{\nabla}_c \partial \sigma^c)}_{\downarrow} + 2\beta^{ab} (\hat{P}_1^\xi \partial \sigma)_{ab} \right) \right]}$$

$\rightarrow \beta^{ab} \hat{g}_{ab}$ term doesn't contribute.

Ignoring normalization constants, we get,

$$\left(\Delta_{FP}[\hat{g}_{ab}^{\xi}]\right)^{-1} = \int [D\sigma_a][DB^{ab}] e^{\left[4\pi i \int d^2\sigma \sqrt{\hat{g}^{\xi}} B^{ab} (\hat{P}_1^{\xi} \sigma)_{ab}\right]}$$

Since, $(\hat{P}_1^{\xi} \sigma)_{ab}$ is symmetric & traceless, it forces even B^{ab} to be symmetric. Thus in the path integral, $[DB^{ab}]$ represents all possible rank 2 traceless symmetric tensors.

Now we can relate the reciprocal of path integral of commuting fields (σ^a) & B^{ab} to the path integral of anti-commuting fields that are Grassmann variables.

The logic is as follows. The path-integral for $\left(\Delta_{FP}[\hat{g}_{ab}^{\xi}]\right)^{-1}$ is quadratic in B^{ab} & σ_b . So, it can be thought as the calculation of the determinant of the operator ' $\hat{P}_1^{\xi}$ '. i.e. it computes the inverse determinant of projection of ' $\hat{P}_1^{\xi}$ ' onto symmetric & traceless rank-2 tensors.

But the same determinant of ' $\hat{\Delta}^\xi$ ' can be found by the path integral of grassmann anti-commuting fields b_{ab} & c^a which are also quadratic in them. The relation is,

$$\int [D\partial\sigma_a][D\beta^{ab}] e^{[4\pi i \int d^2\sigma \sqrt{\hat{g}}^\xi \beta^{ab} (\hat{P}_1^\xi \partial\sigma)_{ab}]} \\ = (\Delta_{FP}[\hat{g}_{ab}^\xi])^{-1} = \left(\int [D b_{ab}][D c^a] e^{[2 \int d^2\sigma \sqrt{\hat{g}}^\xi b_{ab} (\hat{P}_1^\xi c)^{ab}] } \right)^{-1}$$

$c^a \rightarrow$ Vector, $b_{ab} \rightarrow$ Symmetric & traceless rank-2 tensor

Thus the Faddeev-Popov determinant is,

$$\Delta_{FP}[\hat{g}_{ab}^\xi] = \int [D b_{ab}][D c^a] e^{[2 \int d^2\sigma \sqrt{\hat{g}}^\xi b_{ab} (\hat{P}_1^\xi c)^{ab}]}$$

where b & c are anti-commuting grassmann fields which are also called as Ghost fields. Ignoring normalizing constants, we can write,

$$\Delta_{FP}[\hat{g}_{ab}^\xi] = \int [D b_{ab}][D c^a] e^{-S_{ghost}[\hat{g}_{ab}^\xi, b_{ab}, c^a]}$$

where $S_{\text{ghost}}[\hat{g}_{ab}^{\xi}, b_{ab}, c^b] = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}^{\xi}} b_{ab} (\hat{P}_1^{\xi} c)^{ab}$

is the ghost field action. The ghost path integral results in a quantity proportional to $\det(\hat{P}_1)$. That is why ' Δ_{FP} ' is called as a 'determinant'. Also, we can show that, as

$$(\hat{P}_1 c)^{ab} = \frac{1}{2} [\hat{\nabla}^a c^b + \hat{\nabla}^b c^a - \hat{g}^{ab}(\xi) \hat{\nabla}_d c^d]$$

$$\Rightarrow b_{ab} (\hat{P}_1 c)^{ab} = b_{ab} \hat{\nabla}^a c^b$$

Inserting this S_{ghost} in the Polyakov Path integral, we get,

$$Z[\hat{g}_{ab}] = \int d\xi [\mathcal{D}\chi^{\mu}] [\mathcal{D}b_{ab}] [\mathcal{D}c^a] e^{-(\hat{S}_{\text{Poly}} + \hat{S}_{\text{ghost}})}$$

So, the consequence of gauge fixing is that the auxiliary fields that we constructed now have the same status as dynamical fields ' χ^{μ} '. These ghost fields c & b are not physical & are non-observables, but they are just Mathematical constructs.

Conformal Gauge & bc CFT

In conformal gauge, $\hat{g}_{ab} = e^{2\phi(\sigma)} \delta_{ab}$,
 $\rightarrow \sqrt{\det(\hat{g})} = e^{2\phi(\sigma)}$

Note that we are only interested in the local theory, so for any orbit $\mathcal{E}_S$, the above works as all manifolds are locally flat.

$$S_{\text{ghost}}[\hat{g}_{ab}, b_{ab}, c^b] = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ab} (\hat{P}_1^{\mathcal{E}} c)^{ab}$$

$$\rightarrow S_{\text{ghost}}[\hat{g}_{ab}, b_{ab}, c^b] = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ab} \hat{\nabla}^a c^b$$

Let us define complex variables,

$$Z = \sigma^1 + i\sigma^2$$

$$\bar{Z} = \sigma^1 - i\sigma^2$$

$$\Rightarrow \partial \equiv \partial_Z = \frac{1}{2} (\partial_1 - i\partial_2)$$

$$\& \bar{\partial} \equiv \partial_{\bar{Z}} = \frac{1}{2} (\partial_1 + i\partial_2)$$

$$\text{The measure is } d^2\sigma = \frac{dZ d\bar{Z}}{2}$$

The metric tensor is $\hat{g}_{ab} = e^{2\phi(\sigma)} \delta_{ab}$
 In complex coordinates conformal gauge

$$\hat{g} = e^{2\phi(z, \bar{z})} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} g_{z\bar{z}} & g_{\bar{z}z} \\ g_{\bar{z}z} & g_{z\bar{z}} \end{bmatrix}$$

†

$$\hat{g}^{-1} = e^{-2\phi(z, \bar{z})} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} g^{z\bar{z}} & g^{z\bar{z}} \\ g^{\bar{z}z} & g^{\bar{z}z} \end{bmatrix}$$

Now, $b_{ab} \hat{\nabla}^a c^b = b_{ab} \hat{g}^{ad} \hat{\nabla}_d c^b$, since b_{ab} is traceless, † since it is symmetric the only non-zero components in the complex coordinates are $b_{z\bar{z}}$ † $b_{\bar{z}z}$. Thus

$$\begin{aligned} b_{ab} \hat{\nabla}^a c^b &= b_{ab} \hat{g}^{ad} \hat{\nabla}_d c^b = b_{z\bar{z}} \hat{g}^{z\bar{z}} \hat{\nabla}_{\bar{z}} c^b \\ &\quad + b_{\bar{z}z} \hat{g}^{\bar{z}z} \hat{\nabla}_z c^b \\ &= 2e^{-2\phi} (b_{z\bar{z}} \hat{\nabla}_{\bar{z}} c^b + b_{\bar{z}z} \hat{\nabla}_z c^b) \end{aligned}$$

$$\rightarrow \underline{b_{ab} \hat{\nabla}^a c^b = 2e^{-2\phi} (b_{z\bar{z}} \hat{\nabla}_{\bar{z}} c^z + b_{\bar{z}z} \hat{\nabla}_z c^{\bar{z}})}$$

$$\text{Now, } \hat{\nabla}_{\bar{z}} c^z = \partial_{\bar{z}} c^z + \hat{\Gamma}_{\bar{z}z}^z c^z + \hat{\Gamma}_{\bar{z}\bar{z}}^z c^{\bar{z}}$$

$$\& \hat{\nabla}_z c^{\bar{z}} = \partial_z c^{\bar{z}} + \hat{\Gamma}_{zz}^{\bar{z}} c^z + \hat{\Gamma}_{z\bar{z}}^{\bar{z}} c^{\bar{z}}$$

$$\hat{\Gamma}_{ab}^c = \frac{g^{cd}}{2} \left[\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab} \right]$$

$$\bullet \hat{\Gamma}_{\bar{z}z}^z = \frac{g^{z\bar{d}}}{2} \left[\partial_{\bar{z}} g_{z\bar{d}} + \partial_z g_{\bar{z}\bar{d}} - \partial_{\bar{d}} g_{\bar{z}z} \right]$$

$$= \frac{g^{z\bar{z}}}{2} \left[\cancel{\partial_{\bar{z}} g_{z\bar{z}}} + \underbrace{\partial_z g_{\bar{z}\bar{z}}}_0 - \cancel{\partial_{\bar{z}} g_{\bar{z}z}} \right]$$

$d = \bar{z}$
remains

$$\Rightarrow \underline{\hat{\Gamma}_{\bar{z}z}^z = 0 = \hat{\Gamma}_{z\bar{z}}^z}$$

$$\bullet \hat{\Gamma}_{\bar{z}\bar{z}}^z = \frac{g^{z\bar{d}}}{2} \left[\partial_{\bar{z}} g_{\bar{z}\bar{d}} + \partial_{\bar{z}} g_{\bar{z}\bar{d}} - \underbrace{\partial_{\bar{d}} g_{\bar{z}\bar{z}}}_0 \right]$$

$$= \frac{g^{z\bar{z}}}{2} \left[\underbrace{\partial_{\bar{z}} g_{\bar{z}\bar{z}}}_0 + \underbrace{\partial_{\bar{z}} g_{\bar{z}\bar{z}}}_0 - \underbrace{\partial_{\bar{z}} g_{\bar{z}\bar{z}}}_0 \right]$$

$$\Rightarrow \underline{\hat{\Gamma}_{\bar{z}\bar{z}}^z = 0}$$

- $$\hat{\Gamma}_{\bar{z}\bar{z}}^{\bar{z}} = \frac{g^{\bar{z}d}}{2} \left[\partial_{\bar{z}} g_{\bar{z}d} + \partial_{\bar{z}} g_{\bar{z}d} - \underbrace{\partial_d g_{\bar{z}\bar{z}}}_0 \right]$$

$$= \frac{g^{\bar{z}z}}{2} \left[\partial_{\bar{z}} \underbrace{g_{\bar{z}z}}_0 + \partial_{\bar{z}} \underbrace{g_{\bar{z}z}}_0 - \partial_d \underbrace{g_{\bar{z}\bar{z}}}_0 \right]$$

$$\Rightarrow \underline{\hat{\Gamma}_{\bar{z}\bar{z}}^{\bar{z}} = 0}$$

- $$\hat{\Gamma}_{\bar{z}\bar{z}}^{\bar{z}} = \frac{g^{\bar{z}d}}{2} \left[\partial_{\bar{z}} g_{\bar{z}d} + \partial_{\bar{z}} g_{\bar{z}d} - \partial_d g_{\bar{z}\bar{z}} \right]$$

$$= \frac{g^{\bar{z}z}}{2} \left[\cancel{\partial_{\bar{z}} g_{\bar{z}z}} + \partial_{\bar{z}} \underbrace{g_{\bar{z}z}}_0 - \cancel{\partial_{\bar{z}} g_{\bar{z}\bar{z}}} \right]$$

$$\Rightarrow \underline{\hat{\Gamma}_{\bar{z}\bar{z}}^{\bar{z}} = 0 = \hat{\Gamma}_{\bar{z}\bar{z}}^{\bar{z}}}$$

- $$\hat{\Gamma}_{\bar{z}\bar{z}}^z = \frac{g^{z\bar{d}}}{2} \left[\partial_{\bar{z}} g_{\bar{z}d} + \partial_{\bar{z}} g_{\bar{z}d} - \underbrace{\partial_d g_{\bar{z}\bar{z}}}_0 \right] = g^{z\bar{z}} \partial_{\bar{z}} g_{\bar{z}\bar{z}}$$

$$\Rightarrow \boxed{\hat{\Gamma}_{\bar{z}\bar{z}}^z = \partial_{\bar{z}} \phi(z, \bar{z})}$$

The only non zero metric connections
 $\nearrow$ connections

- $$\text{Similarly, } \boxed{\hat{\Gamma}_{\bar{z}\bar{z}}^{\bar{z}} = \partial_{\bar{z}} \phi(z, \bar{z})}$$

Thus, $\hat{\nabla}_{\bar{z}} c^z = \partial_{\bar{z}} c^z$ & $\hat{\nabla}_z c^{\bar{z}} = \partial_z c^{\bar{z}}$

But $\hat{\nabla}_z c^z = \partial_z c^z + c^z \partial_z \phi(z, \bar{z})$

& $\hat{\nabla}_{\bar{z}} c^{\bar{z}} = \partial_{\bar{z}} c^{\bar{z}} + c^{\bar{z}} \partial_{\bar{z}} \phi(z, \bar{z})$

So, we get,

$b_{ab} \hat{\nabla}^a c^b = 2e^{-2\phi} (b_{z\bar{z}} \partial_{\bar{z}} c^z + b_{\bar{z}z} \partial_z c^{\bar{z}})$

Thus, in conformal gauge,

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ab} \hat{\nabla}^a c^b$$

$$\rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z (b_{z\bar{z}} \partial_{\bar{z}} c^z + b_{\bar{z}z} \partial_z c^{\bar{z}})$$

The ghost action factorizes into two free theories and also independent of Weyl factor in conformal gauge. Thus this ghost action is Weyl Invariant. And also it has conformal Symmetry under $z \rightarrow f(z)$ & $\bar{z} \rightarrow f^*(\bar{z})$, which will be proven later.

S_{ghost} in conformal gauge in σ^a coordinates

$$S_{\text{ghost}}[\hat{g}_{ab}, b_{ab}, c^b] = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ab} \hat{\nabla}^a c^b$$

In conformal gauge, $\hat{g}_{ab} = e^{2\phi} \delta_{ab}$

$$\hat{g}^{ab} = e^{-2\phi} \delta^{ab} \quad \& \quad \sqrt{\hat{g}} = e^{2\phi}$$

$$b_{ab} \hat{\nabla}^a c^b = b_{ab} g^{ad} \hat{\nabla}_d c^b$$

$$= b_{ab} e^{-2\phi} \delta^{ad} (\partial_d c^b + \Gamma_{df}^b c^f)$$

$$\Rightarrow b_{ab} \hat{\nabla}^a c^b = e^{-2\phi} b_{ab} (\partial^a c^b + \delta^{ad} \Gamma_{df}^b c^f)$$

$$\Gamma_{df}^b = \frac{g^{bl}}{2} [\partial_d g_{fl} + \partial_f g_{dl} - \partial_l g_{df}]$$

$$= \delta^{bl} [\partial_{fl} \phi + \partial_{dl} \phi - \partial_{df} \phi]$$

$$\Rightarrow \Gamma_{df}^b = \delta_f^b \partial_d \phi + \delta_d^b \partial_f \phi - \delta_{df} \delta^{bl} \partial_l \phi$$

$$\Rightarrow \Gamma_{df}^b c^f = c^f \delta_f^b \partial_d \phi + c^f \delta_d^b \partial_f \phi - c^f \delta_{df} \delta^{bl} \partial_l \phi$$

$$\Rightarrow \Gamma_{df}^b c^f = c^b \partial_d \phi + c^f \partial_d^b \partial_f \phi - c_d \partial^b \phi$$

$$\Rightarrow \partial^{ad} \Gamma_{df}^b c^f = c^b \partial^a \phi + c^f \partial^{ab} \partial_f \phi - c^a \partial^b \phi$$

$$\begin{aligned} \Rightarrow b_{ab} \partial^{ad} \Gamma_{df}^b c^f &= b_{ab} c^b \partial^a \phi + c^f b_{ab} \partial^{ab} \partial_f \phi \\ &\quad - b_{ab} c^a \partial^b \phi \end{aligned}$$

$$\begin{aligned} \Rightarrow e^{-2\phi} b_{ab} \partial^{ad} \Gamma_{df}^b c^f &= e^{-2\phi} b_{ab} c^b \partial^a \phi \\ &\quad + c^f b_{ab} \hat{g}^{ab} \partial_f \phi \\ &\quad - e^{-2\phi} b_{ab} c^a \partial^b \phi \end{aligned}$$

$$\text{But } b_{ab} \hat{g}^{ab} = 0.$$

$$\Rightarrow b_{ab} \partial^{ad} \Gamma_{df}^b c^f = b_{ab} c^b \partial^a \phi - \underbrace{b_{ab} c^a \partial^b \phi}_{=b_{ba}}$$

$$= b_{ab} c^b \partial^a \phi - b_{ba} c^a \partial^b \phi \}_{a \leftrightarrow b}$$

$$= b_{ab} \underbrace{c^b \partial^a \phi - c^b \partial^a \phi}_0$$

$$\Rightarrow b_{ab} \partial^{ad} \Gamma_{df}^b c^f = 0 // \text{ in conformal gauge}$$

$$\Rightarrow \underline{\underline{b_{ab} \hat{\nabla}^a c^b = e^{-2\phi} b_{ab} \partial^a c^b}}$$

$$\text{Thus, } S_{\text{ghost}}[\hat{g}_{ab}, b_{ab}, c^b] = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ab} \hat{\nabla}^a c^b$$

$$= \frac{1}{2\pi} \int d^2\sigma e^{2\phi} e^{-2\phi} b_{ab} \partial^a c^b$$

$$\rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma b_{ab} \partial^a c^b$$

Weyl neutral & non-neutral components

This also implies that b_{ab} & c^a are Weyl Neutral/Invariant. Since the action is Weyl invariant, under $\hat{g}_{ab} \rightarrow e^{2\omega} \hat{g}_{ab}$ it is unchanged. But this is not a coordinate change, thus $b_{zz} \rightarrow b_{zz}$, $b_{\bar{z}\bar{z}} \rightarrow b_{\bar{z}\bar{z}}$, $c^z \rightarrow c^z$ & $c^{\bar{z}} \rightarrow c^{\bar{z}}$.

Consider b^{ab} components.

$$\hat{b}^{ab} = \hat{g}^{ac} \hat{g}^{bd} \hat{b}_{cd} \Rightarrow b^{zz} = g^{z\bar{z}} g^{z\bar{z}} b_{\bar{z}\bar{z}}$$

$$\rightarrow b^{zz} = 4 e^{-4\phi} b_{\bar{z}\bar{z}}$$

Then under Weyl rescaling, $\hat{g}_{ab} \rightarrow e^{2\omega} \hat{g}_{ab}$
 $\Rightarrow \phi \rightarrow \phi + \omega$

$$\Rightarrow b^{z\bar{z}} = 4e^{-4\phi} b_{\bar{z}\bar{z}} \longrightarrow e^{-4\omega} 4e^{-4\phi} b_{\bar{z}\bar{z}}$$

$$\Rightarrow \underbrace{b^{z\bar{z}} \longrightarrow e^{-4\omega} b^{z\bar{z}}}_{\text{Not Weyl Invariant}} \text{ under Weyl rescaling}$$

$$\text{Similarly, } b^{\bar{z}\bar{z}} \xrightarrow{\text{Weyl rescaling}} \underbrace{e^{-4\omega} b^{\bar{z}\bar{z}}}_{\text{Not Weyl Invariant}}$$

For C_a components.

$$C_a = g_{ab} C^b, C_z = g_{z\bar{z}} C^{\bar{z}} = e^{2\phi} C^{\bar{z}}$$

$$\text{Under Weyl rescaling, } C_z \longrightarrow e^{2\omega} C_z$$

$$\text{Similarly } C_{\bar{z}} \longrightarrow e^{2\omega} C_{\bar{z}} \quad \underbrace{\hspace{1cm}}_{\text{Not Weyl Invariant}}$$

Conformal Weights of b & c fields

Under $z \rightarrow z' = f(z)$ & $\bar{z} \rightarrow \bar{z}' = f^*(\bar{z})$,
Since ' b ' is a rank-2 covariant tensor,
using tensorial properties of ' b ' we get,

$$b_{z'z'} = \frac{\partial z}{\partial z'} \frac{\partial z}{\partial z'} b_{zz} + \underbrace{\frac{\partial \bar{z}}{\partial z'} \frac{\partial \bar{z}}{\partial z'}}_{=0} b_{\bar{z}\bar{z}}$$

(noting $b_{z\bar{z}} = b_{\bar{z}z} = 0$) Zero in conformal transform

$$\rightarrow \underline{b_{z'z'} = (\partial f)^{-2} b_{zz}} \quad \left. \vphantom{\underline{b_{z'z'} = (\partial f)^{-2} b_{zz}}} \right\} \text{Under Conformal transform}$$

$$\text{Similarly, } \underline{b_{\bar{z}'\bar{z}'} = (\bar{\partial} f^*)^{-2} b_{\bar{z}\bar{z}}}$$

Thus the conformal weight of b_{zz} is $(2,0)$
and the conformal weight of $b_{\bar{z}\bar{z}}$ is $(0,2)$

$$\text{Then, } c^{z'} = \frac{\partial z'}{\partial z} c^z + \underbrace{\frac{\partial z'}{\partial \bar{z}}}_{=0} c^{\bar{z}}$$

$$\rightarrow c^{z'} = (\partial f)^1 c^z. \text{ Similarly, } c^{\bar{z}'} = (\bar{\partial} f^*)^1 c^{\bar{z}}$$

Thus the conformal weight of c^z is $(-1,0)$
and the conformal weight of $c^{\bar{z}}$ is $(0,-1)$

Conformal Symmetry of ghost action

This action has Conformal Symmetry that is a residual symmetry of combined diff $\times$ Weyl transformation which doesn't affect the gauge fixing condition & also keeps the action invariant. Here, $z \rightarrow z' = f(z)$ & $\bar{z} \rightarrow \bar{z}' = f^*(\bar{z})$ & the Weyl transformation is $e^{2\omega} = |\partial_z f(z)|^{-2}$.

The line element is $ds^2 = dz d\bar{z}$. Under conformal transformation,

$$dz' = (\partial_z f) dz \quad \& \quad d\bar{z}' = (\partial_{\bar{z}} f^*) d\bar{z}$$

$$ds^2 = dz d\bar{z} = (\partial_z f)^{-1} (\partial_{\bar{z}} f^*)^{-1} dz' d\bar{z}'$$

$$\rightarrow ds^2 = |\partial_z f(z)|^{-2} (ds')^2$$

$$\text{Under Weyl, } (ds')^2 \rightarrow (ds'')^2 = e^{2\omega} (ds')^2$$

$$\rightarrow ds^2 = |\partial_z f(z)|^{-2} e^{-2\omega} (ds'')^2$$

$$\rightarrow (ds'')^2 = |\partial_z f(z)|^2 e^{2\omega} ds^2$$

So if $e^{2\omega} |\partial_z f(z)|^2 = 1$,

i.e. $e^{2\omega} = |\partial_z f(z)|^{-2} \rightarrow$ if $\omega = -\ln |\partial_z f(z)|$
then

$$(ds'')^2 = ds^2 \rightarrow \text{Metric tensor is invariant.}$$

Now, let's check the invariance of Ghost action under Conformal transformation.

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}})$$

Under $z \rightarrow z' = f(z) \quad \& \quad \bar{z} \rightarrow \bar{z}' = f^*(\bar{z})$,

$$d^2z = \text{area element} = dz d\bar{z} = |(\partial f)|^{-2} d^2z'$$

$$\rightarrow d^2z' = |(\partial f)|^2 d^2z$$

$$b_{z'z'} \partial_{\bar{z}'} c^{z'} = (\partial f)^{-2} b_{zz} (\bar{\partial} f^*)^{-1} (\partial f)^1 \partial_{\bar{z}} c^z$$

$$\Rightarrow b_{z'z'} \partial_{\bar{z}'} c^{z'} = (\partial f)^{-1} (\bar{\partial} f^*)^{-1} b_{zz} \partial_{\bar{z}} c^z$$

$$\Rightarrow b_{z' \bar{z}'} \partial_{\bar{z}'} c^{z'} = |(\partial f)|^{-2} b_{z \bar{z}} \partial_{\bar{z}} c^z$$

$$\text{Similarly, } b_{\bar{z}' z'} \partial_{z'} c^{\bar{z}'} = |(\partial f)|^{-2} b_{\bar{z} z} \partial_z c^{\bar{z}}$$

$$\rightarrow d^2 z' b_{z' \bar{z}'} \partial_{\bar{z}'} c^{z'} = |(\partial f)|^2 d^2 z |(\partial f)|^{-2} b_{z \bar{z}} \partial_{\bar{z}} c^z$$

$$\rightarrow \underline{d^2 z' b_{z' \bar{z}'} \partial_{\bar{z}'} c^{z'} = d^2 z b_{z \bar{z}} \partial_{\bar{z}} c^z}$$

$$\text{Similarly, } \underline{d^2 \bar{z}' b_{\bar{z}' z'} \partial_{z'} c^{\bar{z}'} = d^2 \bar{z} b_{\bar{z} z} \partial_z c^{\bar{z}}}$$

Thus the ghost action is invariant under the conformal transformation.

We can denote $b_{z \bar{z}} \equiv b(z, \bar{z})$, $c^z \equiv c(z, \bar{z})$
 $\& b_{\bar{z} z} \equiv \tilde{b}(z, \bar{z})$, $c^{\bar{z}} \equiv \tilde{c}(z, \bar{z})$

where $b(z, \bar{z})$, $c(z, \bar{z})$, $\tilde{b}(z, \bar{z})$ & $\tilde{c}(z, \bar{z})$ are Grassmann fields which have the conformal weights $(2, 0)$, $(-1, 0)$, $(0, 2)$ & $(0, -1)$ respectively. This is the $bc + \tilde{b}\tilde{c}$ CFT.

$$S_{\text{ghost}} = S_{bc} + S_{\tilde{b}\tilde{c}}$$

$$\text{where } S_{bc} = \frac{1}{2\pi} \int d^2z \, b(z, \bar{z}) \partial_{\bar{z}} c(z, \bar{z})$$

$$\& S_{\tilde{b}\tilde{c}} = \frac{1}{2\pi} \int d^2z \, \tilde{b}(z, \bar{z}) \partial_z \tilde{c}(z, \bar{z})$$

The eqⁿ of motion fixes b & c fields as holomorphic functions and the $\tilde{b}$ & $\tilde{c}$ fields as anti-holomorphic functions.

For the closed string case, where the worldsheet has no end point boundary, we have two independent CFTs.

Now all results that we got in the previous chapter for closed string S_{ghost} CFT can be used & applied here.

But with $\lambda = 2$ & $\tilde{\lambda} = 2$

Open String boundary Conditions

In open string case, we have a patch of world sheet with end points of σ^1 as the boundary (line of constant σ^1).

Here, path integral should be thought of as integral over fields that depend on a fixed coordinate region.

Thus under the diffeomorphism the boundary should map to itself. So ' $\partial\sigma^a$ ' should be tangential & no normal component should be present. $\Rightarrow n_a \partial\sigma^a = 0$, where ' n_a ' is the normal to the boundary.

This corresponds to $n_a c^a = 0$. So, c^a is proportional to tangent vector ' $\dot{z}^a$ '. Thus the boundary condition of c^a is,

$$\underline{n_a c^a = 0} \rightarrow \text{at boundary}$$

The equations of motion for S_{ghost} action is got by its variation. The boundary term gives boundary conditions for b_{ab} . We previously got,

$$\rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int_M d^2\sigma b_{ab} \partial^a c^b$$

$$\underbrace{\delta^{(c)} S_{\text{ghost}}}_{\text{w.r.t 'c'}} = \frac{1}{2\pi} \int_M d^2\sigma b_{ab} \partial^a (\delta c^b)$$

Integrate by parts

$$= \int_{\partial M} ds \delta c^b (n^a b_{ab}) - \frac{1}{2\pi} \int_M d^2\sigma \delta c^b (\partial^a b_{ab})$$

at boundary $\delta c^b = t^b \Rightarrow$ tangent

Thus for boundary term to vanish we need

$$\underline{\underline{n^a t^b b_{ab} = 0}} \rightarrow \text{at boundary}$$

This is the boundary condition for b_{ab} .

Energy Momentum Tensor

The covariant ghost action is,

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d$$

$$\rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} b_{ad} \hat{g}^{af} \left(\partial_f c^d + \hat{\Gamma}_{fh}^d c^h \right)$$

The energy momentum tensor is got by varying the Action with respect to $\hat{g}_{ab}$.

$$\text{i.e. } \hat{T}^{mn} = - \frac{4\pi}{\sqrt{\hat{g}}} \frac{\partial S_{\text{ghost}}}{\partial \hat{g}_{mn}}$$

hat '^' indicates that the action is varied with respect to the gauge fixed metric.

We fix 'b_{ad}' & 'c^d' as constants. Also we shall use, $\partial \sqrt{\hat{g}} = -\frac{1}{2} \sqrt{\hat{g}} \hat{g}_{mn} \partial \hat{g}^{mn}$,

$$\partial \hat{\Gamma}_{fh}^d = \frac{1}{2} \hat{g}^{dm} \left(\nabla_f \partial \hat{g}_{mh} + \nabla_h \partial \hat{g}_{mf} - \nabla_m \partial \hat{g}_{fh} \right)$$

$$\& \partial \hat{g}_{mn} = -\hat{g}_{mh} \hat{g}_{nf} \partial \hat{g}^{hf}$$

$$\delta S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma \left[(\partial \sqrt{\hat{g}}) b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d \right. \\ \left. + \sqrt{\hat{g}} b_{ad} \hat{\nabla}_f c^d (\partial \hat{g}^{af}) \right]_{\substack{a \rightarrow m \\ f \rightarrow n}} \\ \left. + \sqrt{\hat{g}} b_{ad} \hat{g}^{af} (\partial \hat{\Gamma}_{fh}^d c^h) \right]$$

$$= \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \left[\left(-\frac{1}{2} \hat{g}_{mn} \partial \hat{g}^{mn} \right) b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d \right. \\ \left. + (\partial \hat{g}^{mn}) b_{md} \hat{\nabla}_n c^d + b_{ad} \hat{g}^{af} c^h \partial \hat{\Gamma}_{fh}^d \right]$$

Since $\partial \hat{g}^{mn}$ is symmetric, in the 2nd term

$$(\partial \hat{g}^{mn}) b_{md} \hat{\nabla}_n c^d = (\partial \hat{g}^{mn}) \underbrace{(b_{md} \hat{\nabla}_n c^d + b_{nd} \hat{\nabla}_m c^d)}_2$$

Thus,

$$\delta S_{\text{ghost}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \left[(\partial \hat{g}^{mn}) (b_{md} \hat{\nabla}_n c^d + b_{nd} \hat{\nabla}_m c^d) \right. \\ \left. - (\partial \hat{g}^{mn}) \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d \right. \\ \left. + 2 b_{ad} \hat{g}^{af} c^h (\partial \hat{\Gamma}_{fh}^d) \right]$$

Consider the last term,

$$\partial \hat{\Gamma}_{fh}^d = \frac{1}{2} \hat{g}^{dm} (\nabla_f \partial \hat{g}_{mh} + \nabla_h \partial \hat{g}_{mf} - \nabla_m \partial \hat{g}_{fh})$$

$$2b_{ad} \hat{g}^{af} c^h \partial \hat{\Gamma}_{fh}^d$$

$$= b_{ad} \hat{g}^{af} c^h \hat{g}^{dm} (\nabla_f \partial \hat{g}_{mh} + \nabla_h \partial \hat{g}_{mf} - \nabla_m \partial \hat{g}_{fh})$$

$$= b^{fm} c^h (\nabla_f \partial \hat{g}_{mh} + \nabla_h \partial \hat{g}_{mf} - \nabla_m \partial \hat{g}_{fh})$$

$$= b^{fm} (\underbrace{\nabla_f \partial \hat{g}_{mh} - \nabla_m \partial \hat{g}_{fh}}_{\text{Zero since } b^{fm} = b^{mf}}) c^h + b^{fm} c^h \nabla_h \partial \hat{g}_{mf}$$

$$\rightarrow 2b_{ad} \hat{g}^{af} c^h \partial \hat{\Gamma}_{fh}^d = b^{fm} c^h \nabla_h (\partial \hat{g}_{mf})$$

$$\frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} (2b_{ad} \hat{g}^{af} c^h \partial \hat{\Gamma}_{fh}^d)$$

$$= \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} b^{fm} c^h \nabla_h (\partial \hat{g}_{mf})$$

Use integration by parts

$$\int d^2\sigma \sqrt{\hat{g}} \omega^{fmh} \nabla_f \partial \hat{g}_{mh} = - \int d^2\sigma \sqrt{\hat{g}} \partial \hat{g}_{mh} \nabla_f \omega^{fmh}$$

(This is valid because of metric compatibility)

$$\begin{aligned}
&\Rightarrow \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} (2 b_{ad} \hat{g}^{af} c^h \partial \hat{\Gamma}_{fh}^d) \\
&= \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} b^{fm} c^h \nabla_h (\partial \hat{g}_{mf}) \\
&= \frac{-1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \underbrace{\partial \hat{g}_{mf}}_{f \rightarrow n} \nabla_h (b^{fm} c^h) \\
&= -\frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} (\partial \hat{g}_{mn}) \nabla_h (b^{mn} c^h) \\
&\quad \text{use } \partial \hat{g}_{mn} = -\hat{g}_{ml} \hat{g}_{nf} \partial \hat{g}^{lf} \\
&= \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \hat{g}_{ml} \hat{g}_{nf} \partial \hat{g}^{lf} \nabla_h (b^{mn} c^h) \\
&= \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \underbrace{\partial \hat{g}^{lf}}_{l \rightarrow m, f \rightarrow n} \nabla_h (b_{lf} c^h) \\
&= \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \partial \hat{g}^{mn} \left[(\nabla_h b_{mn}) c^h + \underbrace{b_{mn} (\nabla_h c^h)}_{\text{traceless condition} \left\{ \begin{array}{l} \text{doesn't} \\ \text{contribute} \end{array} \right.}} \right]
\end{aligned}$$

Thus we finally get,

$$\delta S_{\text{ghost}} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \left[(\partial \hat{g}^{mn}) (b_{md} \hat{\nabla}_n c^d + b_{nd} \hat{\nabla}_m c^d) - (\partial \hat{g}^{mn}) \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d + \partial \hat{g}^{mn} (\nabla_h b_{mn}) c^h \right]$$

$$\text{Now, } \frac{\delta S_{\text{ghost}}}{\delta \hat{g}^{mn}} = \frac{1}{4\pi} \sqrt{\hat{g}} \left[b_{md} \hat{\nabla}_n c^d + b_{nd} \hat{\nabla}_m c^d - \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d + \nabla_h (b_{mn}) c^h \right]$$

$$\hat{T}_{mn} = -\frac{4\pi}{\sqrt{\hat{g}}} \frac{\delta}{\delta \hat{g}^{mn}} S_{\text{ghost}}$$

$$\rightarrow \hat{T}_{mn} = \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d - b_{md} \hat{\nabla}_n c^d - b_{nd} \hat{\nabla}_m c^d - (\nabla_d b_{mn}) c^d$$

$$\rightarrow \boxed{\hat{T}_{mn} = \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d - b_{md} \hat{\nabla}_n c^d - b_{nd} \hat{\nabla}_m c^d - (\nabla_d b_{mn}) c^d}$$

$$\begin{aligned} \text{Now } \hat{g}^{mn} \hat{T}_{mn} &= \hat{g}^{mn} \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d \\ &\quad - \hat{g}^{mn} b_{md} \hat{\nabla}_n c^d - \hat{g}^{mn} b_{nd} \hat{\nabla}_m c^d \\ &\quad - \underbrace{\hat{g}^{mn} (\nabla_d b_{mn}) c^d}_0 \end{aligned}$$

Vanishes due
to metric compatibility

Thus we get,

$$\hat{g}^{mn} \hat{T}_{mn} = \delta_m^m b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d - b_{md} \hat{\nabla}^m c^d - b_{nd} \hat{\nabla}^n c^d$$

Now for 2d world sheet $\delta_m^m = 2$

$$\Rightarrow \hat{g}^{mn} \hat{T}_{mn} = 2 b_{ad} \hat{\nabla}^a c^d - \underbrace{b_{md} \hat{\nabla}^m c^d + b_{nd} \hat{\nabla}^n c^d}_{\text{All cancels}}$$

Thus for the bc theory in any gauge condition, the energy momentum tensor is traceless, i.e. $\hat{T}_m^m = 0$

We shall simplify 2nd, 3rd & 4th term

$$\begin{aligned} & b_{md} \hat{\nabla}_n c^d + b_{nd} \hat{\nabla}_m c^d + (\nabla_d b_{mn}) c^d \\ &= b_{md} \partial_n c^d + b_{md} \Gamma_{nh}^d c^h \\ & \quad + b_{nd} \partial_m c^d + b_{nd} \Gamma_{mh}^d c^h \\ & \quad + (\partial_d b_{mn}) c^d - \underbrace{\Gamma_{dm}^h b_{hn} c^d}_{h \leftrightarrow d} - \underbrace{\Gamma_{dn}^h b_{mh} c^d}_{h \leftrightarrow d} \end{aligned}$$

$$\begin{aligned}
&= b_{md} \partial_n c^d + b_{md} \overset{d}{\Gamma}_{nh}^d c^h + b_{nd} \partial_m c^d + b_{nd} \overset{d}{\Gamma}_{mh}^d c^h \\
&\quad + (\partial_d b_{mn}) c^d - b_{dn} \overset{d}{\Gamma}_{hm}^d c^h - b_{md} \overset{d}{\Gamma}_{hn}^d c^h
\end{aligned}$$

$$= b_{md} \partial_n c^d + b_{nd} \partial_m c^d + (\partial_d b_{mn}) c^d$$

Thus, in any gauge condition,

$$\begin{aligned}
\hat{T}_{mn} &= \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d - b_{md} \partial_n c^d \\
&\quad - b_{nd} \partial_m c^d - (\partial_d b_{mn}) c^d
\end{aligned}$$

Energy Momentum tensor in Conformal Gauge

In conformal gauge, $\hat{g}_{ab} = e^{2\phi} \delta_{ab}$

$$\hat{g}^{ab} = e^{-2\phi} \delta^{ab} \quad \& \quad \sqrt{\hat{g}} = e^{2\phi}$$

$$\hat{\nabla}_f c^d = \partial_f c^d + \overset{d}{\Gamma}_{fh}^d c^h$$

$$\overset{d}{\Gamma}_{hf}^d = \delta_f^d \partial_h \phi + \delta_h^d \partial_f \phi - \hat{g}_{hf} \hat{g}^{dl} \partial_l \phi$$

1st term:

$$\hat{\nabla}_f c^d = \partial_f c^d + \Gamma_{fh}^d c^h$$

$$\begin{aligned} \rightarrow \hat{g}^{af} \hat{\nabla}_f c^d &= \hat{g}^{af} \partial_f c^d + \hat{g}^{af} \Gamma_{fh}^d c^h \\ &= \hat{g}^{af} \partial_f c^d + \hat{g}^{af} \partial_f^d c^h \partial_h \phi \\ &\quad + \hat{g}^{af} \partial_h^d c^h \partial_f \phi - \hat{g}^{af} \hat{g}_{hf} \hat{g}^{dl} c^h \partial_l \phi \end{aligned}$$

$$\begin{aligned} \rightarrow \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d &= \hat{g}_{mn} b_{ad} \hat{g}^{af} \partial_f c^d \\ &\quad + \hat{g}_{mn} b_{ad} \hat{g}^{af} \partial_f^d c^h \partial_h \phi \\ &\quad + \hat{g}_{mn} b_{ad} \hat{g}^{af} \partial_h^d c^h \partial_f \phi \\ &\quad - \hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{g}_{hf} \hat{g}^{dl} c^h \partial_l \phi \end{aligned}$$

$$= \hat{g}_{mn} b_{ad} \hat{g}^{af} \partial_f c^d + \hat{g}_{mn} \cancel{b_{ad}} \hat{g}^{ad} c^h \partial_h \phi$$

$$+ \hat{g}_{mn} \cancel{b_{ah}} \hat{g}^{af} c^h \partial_f \phi - \hat{g}_{mn} \cancel{b_{ha}} \hat{g}^{af} c^h \partial_f \phi$$

$$\rightarrow \underline{\hat{g}_{mn} b_{ad} \hat{g}^{af} \hat{\nabla}_f c^d = \hat{g}_{mn} \hat{g}^{af} b_{ad} \partial_f c^d}$$

Thus in conformal gauge,

$$\hat{T}_{mn} = \hat{g}_{mn} \hat{g}^{af} b_{ad} \partial_f c^d - b_{md} \partial_n c^d - b_{nd} \partial_m c^d - (\partial_d b_{mn}) c^d$$

The energy momentum tensor components are weyl invariant in conformal gauge. Since there is no explicit weyl factor dependence ($\hat{g}_{mn} \hat{g}^{af} = \gamma_{mn} \gamma^{af}$).

But it is traceless in any gauge. So the trace is weyl invariant.

Energy momentum tensor in Complex coordinates

In conformal gauge,

$$\hat{g} = e^{2\phi(z, \bar{z})} \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} g_{zz} & g_{z\bar{z}} \\ g_{\bar{z}z} & g_{\bar{z}\bar{z}} \end{bmatrix}$$

$$\hat{g}^{-1} = e^{-2\phi(z, \bar{z})} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} g^{z\bar{z}} & g^{z\bar{z}} \\ g^{\bar{z}z} & g^{\bar{z}\bar{z}} \end{bmatrix}$$

$$b_{z\bar{z}} = b_{\bar{z}z} = 0 \quad (\text{Traceless condition})$$

$$\hat{T}_{mn} = \hat{g}_{mn} \hat{g}^{af} b_{ad} \partial_f c^d - b_{md} \partial_n c^d - b_{nd} \partial_m c^d - (\partial_d b_{mn}) c^d$$

Now,

$$\hat{g}^{af} b_{ad} \partial_f c^d = \hat{g}^{zf} b_{zd} \partial_f c^d + \hat{g}^{\bar{z}f} b_{\bar{z}d} \partial_f c^d$$

$$\Rightarrow \hat{g}^{af} b_{ad} \partial_f c^d = \hat{g}^{z\bar{z}} b_{z\bar{z}} \partial_{\bar{z}} c^z + \hat{g}^{\bar{z}z} b_{\bar{z}z} \partial_z c^{\bar{z}}$$

$$T_{z\bar{z}} = \hat{g}_{z\bar{z}} \hat{g}^{af} b_{ad} \partial_f c^d - 2b_{zd} \partial_{\bar{z}} c^d - (\partial_d b_{z\bar{z}}) c^d$$

$$= -2b_{z\bar{z}} \partial_{\bar{z}} c^z - (\partial_z b_{z\bar{z}}) c^z - (\partial_{\bar{z}} b_{z\bar{z}}) c^{\bar{z}}$$

If the eqⁿ of motion is obeyed, $\partial_{\bar{z}} b_{z\bar{z}} = 0$

Thus,

$$\underline{T_{zz} = -2b_{zz} \partial_z c^z - (\partial_z b_{zz}) c^z}$$

Now, if eqⁿ of motion is obeyed,

$$b_{zz} \equiv b(z, \bar{z}) = b(z), \quad c^z \equiv c(z, \bar{z}) = c(z)$$

$$\Rightarrow T_{zz} = T(z) \rightarrow \text{holomorphic.}$$

$$\text{Thus, } T(z) = -2b(z) [\partial_z c(z)] - [\partial_z b(z)] c(z)$$

$$\Rightarrow \boxed{T(z) = 2[\partial_z c(z)] b(z) - [\partial_z b(z)] c(z)}$$

Similarly,

$$\hat{T}_{\bar{z}\bar{z}} = \underbrace{\hat{g}_{\bar{z}\bar{z}} \hat{g}^{af}}_0 b_{ad} \partial_f c^d - 2b_{\bar{z}d} \partial_{\bar{z}} c^d - (\partial_d b_{\bar{z}\bar{z}}) c^d$$

$$= -2b_{\bar{z}\bar{z}} \partial_{\bar{z}} c^{\bar{z}} - \underbrace{(\partial_{\bar{z}} b_{\bar{z}\bar{z}}) c^z}_0 - (\partial_{\bar{z}} b_{\bar{z}\bar{z}}) c^{\bar{z}}$$

0 by eqⁿ of motion

$$\Rightarrow T_{\bar{z}\bar{z}} = -2b_{\bar{z}\bar{z}} \partial_{\bar{z}} c^{\bar{z}} - (\partial_{\bar{z}} b_{\bar{z}\bar{z}}) c^{\bar{z}}$$

Now, if eqⁿ of motion is obeyed,

$$b_{\bar{z}\bar{z}} \equiv \tilde{b}(z, \bar{z}) = \tilde{b}(\bar{z}) \quad , \quad c^{\bar{z}} \equiv \tilde{c}(z, \bar{z}) = \tilde{c}(\bar{z})$$

$$\Rightarrow T_{z\bar{z}} = \tilde{T}(\bar{z}) \rightarrow \text{anti-holomorphic}$$

$$\rightarrow \tilde{T}(\bar{z}) = -2\tilde{b}(\bar{z})[\partial_{\bar{z}}\tilde{c}(\bar{z})] - [\partial_{\bar{z}}\tilde{b}(\bar{z})]\tilde{c}(\bar{z})$$

$$\rightarrow \boxed{\tilde{T}(\bar{z}) = 2[\partial_{\bar{z}}\tilde{c}(\bar{z})]\tilde{b}(\bar{z}) - [\partial_{\bar{z}}\tilde{b}(\bar{z})]\tilde{c}(\bar{z})}$$

Gross components,

$$\hat{T}_{z\bar{z}} = \hat{g}_{z\bar{z}} \hat{g}^{af} b_{ad} \partial_f c^d - b_{zd} \partial_{\bar{z}} c^d - b_{\bar{z}d} \partial_z c^d - (\partial_d b_{z\bar{z}}) c^d$$

$$= \overbrace{\hat{g}_{z\bar{z}} \hat{g}^{z\bar{z}}}^1 b_{z\bar{z}} \partial_{\bar{z}} c^z + \overbrace{\hat{g}_{z\bar{z}} \hat{g}^{\bar{z}z}}^1 b_{\bar{z}z} \partial_z c^{\bar{z}} - b_{z\bar{z}} \partial_{\bar{z}} c^z - b_{\bar{z}z} \partial_z c^{\bar{z}}$$

$$= b_{z\bar{z}} \partial_{\bar{z}} c^z + b_{\bar{z}z} \partial_z c^{\bar{z}} - b_{z\bar{z}} \partial_{\bar{z}} c^z - b_{\bar{z}z} \partial_z c^{\bar{z}}$$

$$\text{Thus, } \underline{\underline{\hat{T}_{z\bar{z}}}} = 0 \quad (\text{even off shell / on shell})$$

Thus we have derived the energy momentum tensor components for a $bc \neq \tilde{b}\tilde{c}$ ghost CFT which matches with the $\lambda=2$ $\neq$ $\tilde{\lambda}=2$ $bc + \tilde{b}\tilde{c}$ CFT that we studied in the previous chapter.

This results in the OPE of the ghost fields

$$b(z)c(w) \sim \frac{1}{z-w}, \quad c(z)b(w) \sim \frac{1}{z-w}$$

$$\tilde{c}(\bar{z})\tilde{b}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}, \quad \tilde{b}(\bar{z})\tilde{c}(\bar{w}) \sim \frac{1}{\bar{z}-\bar{w}}$$

The OPE of these fields with the energy momentum tensor shows that these fields are primary operators w.r.t their conformal transformation properties with weights as follows,

$$(h_b, \tilde{h}_b) = (2, 0), \quad (h_c, \tilde{h}_c) = (-1, 0),$$

$$(h_{\tilde{b}}, \tilde{h}_{\tilde{b}}) = (0, 2) \neq (h_c, \tilde{h}_c) = (0, -1)$$

The TT OPEs give the conformal central charges as $c_{bc} = -26$ &
 $\tilde{c}_{\tilde{b}\tilde{c}} = -26$

$T(\mathbb{Z})$ has weight $(h_T, \tilde{h}_T) = (2, 0)$

$\tilde{T}(\bar{\mathbb{Z}})$ has weight $(h_{\tilde{T}}, \tilde{h}_{\tilde{T}}) = (0, 2)$