

Presence of Anomalies when Quantizing a Gauge theory

In Gauge theories, an Anomaly is a Quantum outcome / effect which breaks the Gauge Symmetry that was present in the classical version of that theory.

In the classical Polyakov path integral the gauge symmetries present are:

- a) Diffeomorphisms (reparameterizations)
- b) Weyl Invariance (local rescaling of metric)

But in quantizing any gauge theory we do Gauge fixing (choose a representative metric) & regularize UV divergences (introduce the counter term / regulator fields). This could break gauge symmetry in the quantum level, if the necessary regulator term which cancel divergences doesn't respect the classical gauge symmetry. So we need to check this.

To cancel the UV divergences we need to add a Pauli-Villars regulator with a massive regulator field of the form

$$S_{\text{reg}} = \int d^2\sigma \sqrt{\hat{g}} \left[\hat{g}^{ab} (\partial_a \gamma^\nu) (\partial_b \gamma_\nu) + \mu^2 \gamma^\nu \gamma_\nu \right]$$

Here the kinetic term of the regulator will cancel the UV divergences and at low momenta the mass is taken to be very large so the regulator's contribution will be exponentially suppressed.

Thus this regulator cancels UV divergence at large momenta & also leaves the IR (low momenta) physics intact.

So the regulator field respects diff & the Poincaré invariance even in quantum level. But the mass term breaks Weyl symmetry. As it introduces length scale for the system & thus system is not scale invariant now.

Thus we need to check the Weyl invariance by hand. So let us check if this holds as an operator equation.

Since Weyl Invariance can be thought of repeated infinitesimal Weyl transform, let us check what happens as a metric is transformed infinitesimally.

Now, we have the Path integral after gauge fixing as,

$$Z[\hat{g}_{ab}] = \int d\mathcal{E} [\mathcal{D}\chi^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}]}$$

Now, in the presence of arbitrary insertions we get operator eqⁿ of the form,

$$\begin{aligned} \langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle_{\hat{g}} &= \\ &= \int d\mathcal{E} [\mathcal{D}\chi^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}]} \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \end{aligned}$$

Now even if $\hat{g}_{ab} \rightarrow \hat{g}_{ab}^{\xi}$ we need the above eqⁿ to remain invariant. i.e.

$$\langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}^{\xi}}$$

Let us check what happens to $\langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}$ as the metric changes infinitesimally.

$$\begin{aligned} \delta \langle \theta_1 \dots \theta_n \rangle_{\xi} &= \\ &= \int d\xi [\partial \chi^\mu \partial b_{ab} \partial c^a] e^{-\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}]} (-\delta \hat{S}) \theta_1 \dots \theta_n \\ &\rightarrow \delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\xi} = \langle (-\delta \hat{S}) \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} \end{aligned}$$

We know that the hilbert's definition of Energy momentum tensor is,

$$\hat{T}_{ab} = -\frac{4\pi}{\sqrt{\hat{g}}} \frac{\delta \hat{S}}{\delta \hat{g}^{ab}}$$

So, when the metric changes infinitesimally.

$$(-\delta \hat{S}) = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \hat{T}_{ab} \delta \hat{g}^{ab}$$

Thus, under $\delta \hat{g}_{ab}(\sigma^a) = \hat{g}_{ab}^{\zeta}(\sigma^a) - \hat{g}_{ab}(\sigma^a)$,
 where ' ζ ' is very close to identity, we get,

$$\delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\zeta} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \langle \delta \hat{g}^{ab} \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}$$

Now if ' ζ ' is infinitesimal diff transform,
 where $\sigma^a \rightarrow \sigma^a + \delta \sigma^a$

$$\delta g_{ab} = -\nabla_a \delta \sigma_b - \nabla_b \delta \sigma_a, \quad \delta g^{ab} = \nabla^a \delta \sigma^b + \nabla^b \delta \sigma^a$$

Then, we get $\delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\text{diff}}$

$$= \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \langle (\nabla^a \delta \sigma^b + \nabla^b \delta \sigma^a) \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}$$

Since $\hat{T}_{ab}$ is symmetric,

$$= \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \langle (\nabla^a \delta \sigma^b) \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}$$

Integrate by parts such that boundary term vanishes,

$$= \frac{-1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \langle \nabla^a \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} \delta \sigma^b$$

Thus,

$$\delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\text{diff}} = \frac{-1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \langle \nabla^a \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} \delta \sigma^b$$

Now, if $\langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}^{\text{diff}}$ under diff transform, then,

$$\delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\text{diff}} = 0$$

$$\rightarrow \frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \langle \nabla^a \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} \delta \sigma^b = 0$$

Since $\delta \sigma^b$ is arbitrary, $\langle \nabla^a \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = 0$

Thus the energy-momentum tensor is covariantly conserved if diff invariance is present as an operator statement.

Conversely, if the energy momentum tensor is covariantly conserved, i.e., $\langle \nabla^a \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = 0$
Then, $\langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}^{\text{diff}} \rightarrow \text{has diff invariance}$

Now, if ' ζ ' is infinitesimal Weyl transform,
Then,

$$\delta g_{ab} = 2\delta\omega g_{ab} \Rightarrow \delta g^{ab} = -2\delta\omega g^{ab}$$

Then, we get $\delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\text{Weyl}}$

$$= \frac{-1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \delta\omega \langle g^{ab} \hat{T}_{ab} \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}$$

Thus we get,

$$\delta \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\text{Weyl}} = \frac{-1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \delta\omega \langle \hat{T}^a_a \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}}$$

So if $\langle \hat{T}^a_a \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = 0$ for any $\delta\omega$,
then there is Weyl Symmetry in quantum
operator level $\rightarrow \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}^{\text{Weyl}}}$

The converse of the above statement is also
true. i.e. if Weyl Symmetry holds even
in the quantum level, i.e. if

$$\langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = \langle \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}^{\text{Weyl}}}$$
$$\rightarrow \langle \hat{T}^a_a \theta_1 \theta_2 \dots \theta_n \rangle_{\hat{g}} = 0$$

Now, since we have a diff & Poincare invariant regulator, we have trace $\hat{T}^a_a$ also to obey these symmetries.

Since we couldn't find a Weyl invariant regulator, at the quantum level, tracelessness of energy-momentum tensor might not be satisfied. $\rightarrow \langle \hat{T}^a_a \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle_g \stackrel{?}{=} 0$.

But in previous chapter in the flat world sheet metric, we got that the Polyakov & Ghost action had conformal symmetry even in quantum level. i.e., the energy-momentum tensor was conserved & traceless as operator equations. Thus, in flat space metric, even the quantum theory obeyed Weyl symmetry.

Thus in arbitrary curved world sheet the trace of energy momentum tensor has to be proportional to something which is zero in flat world sheet & non-zero in curved one.

Since,

- Trace is diff & poincare invariant
- Trace depends on curvature of world sheet
- Trace is zero in the flat world sheet
- Trace is non-zero in curved world sheet

For a two dimensional world sheet, the appropriate candidate is some polynomial of Ricci Scalar without the constant term.

$$\Rightarrow \text{Trace}(\hat{T}) = a_1 \hat{R} + a_2 \hat{R}^2 + a_3 \hat{R}^3 + \dots$$

So, if world sheet is flat then $\hat{R}_{\text{flat}} = 0$,
 $\implies \text{Trace}(\hat{T})_{\text{flat}} = 0$.

But any arbitrary power of Ricci Scalar is not allowed. It is fixed by dimensional analysis. We know that,

$$\delta \hat{S} = \frac{-1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \hat{T}_{ab} \delta \hat{g}^{ab}$$

Now ' $\hat{S}$ ', ' $\hat{g}^{ab}$ ' & ' $\hat{g}_{ab}$ ' are dimensionless.

Then $\underline{\underline{[\hat{T}_{ab}] = [\sigma]^{-2}}} \rightarrow [\text{Trace}(\hat{T})] = [\sigma]^{-2}$

Now, $\partial_c \hat{g}_{ab} = \frac{\partial \hat{g}_{ab}}{\partial \sigma^c} \rightarrow [\partial_c \hat{g}_{ab}] = [\sigma]^{-1}$

But, $\hat{\Gamma}_{bc}^a = \frac{\hat{g}^{ad}}{2} (\partial_b \hat{g}_{cd} + \partial_c \hat{g}_{bd} - \partial_d \hat{g}_{bc})$
 $\rightarrow [\hat{\Gamma}_{bc}^a] = [\sigma]^{-1}$

The Riemann tensor is,

$$\hat{R}_{bcd}^a = \partial_c \hat{\Gamma}_{bd}^a - \partial_d \hat{\Gamma}_{bc}^a + \hat{\Gamma}_{hc}^a \hat{\Gamma}_{bd}^h - \hat{\Gamma}_{hd}^a \hat{\Gamma}_{bc}^h$$

$$\rightarrow [\hat{R}_{bcd}^a] = [\sigma]^{-2}$$

The Ricci tensor is, $\hat{R}_{bd} = \hat{g}^{ch} \hat{g}_{ha} \hat{R}_{bcd}^a = \hat{R}_{bcd}^c$

And the Ricci Scalar is $\hat{R} = \hat{g}^{bd} \hat{R}_{bd}$

Thus $\underline{\underline{\hat{R} = \hat{g}^{bd} \hat{g}^{ch} \hat{g}_{ha} \hat{R}_{bcd}^a}}$

$\rightarrow \underline{\underline{[\hat{R}] = [\sigma]^{-2}}}$. Thus, $\boxed{\text{Trace}(\hat{T}) \propto \hat{R}}$

This should hold as an operator eqⁿ. So the Proportionality constant should be dimensionless w.r.t world sheet.

$\rightarrow \boxed{\text{Trace}(\hat{T}) = \hat{a}_1 \hat{R}}$

Calculation of Weyl Anomaly

Method-1: Comparing Weyl Transforms

Consider the energy momentum tensor in Euclidean coordinates

$$\hat{T}_{ab} = \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & \hat{T}_{22} \end{bmatrix}$$

The trace is, $\text{Trace}(\hat{T}) = \hat{g}^{ab} \hat{T}_{ab} = a_1 \hat{R}$

$$\rightarrow \boxed{\hat{T}_{ab} = \frac{\hat{g}_{ab} a_1 \hat{R}}{2}}$$

Only then, $\hat{T}_{ab} \hat{g}^{ab} = a_1 \hat{R}$, we get this since $\hat{g}^{ab} \hat{g}_{ab} = 2$

In conformal gauge, $\hat{g}_{ab} = e^{2\phi(\sigma^a)} \delta_{ab}$
So, in complex $(z, \bar{z})$ coordinates,

$$\hat{g}_{z\bar{z}} = \hat{g}_{\bar{z}z} = \frac{1}{2} e^{2\phi(z, \bar{z})} \quad \& \quad \hat{g}_{zz} = \hat{g}_{\bar{z}\bar{z}} = 0$$

$$\rightarrow \boxed{\hat{T}_{z\bar{z}} = \hat{g}_{z\bar{z}} \frac{a_1 \hat{R}}{2} = \hat{T}_{\bar{z}z}}$$

Now, $\hat{\nabla}^{\bar{z}} \hat{T}_{\bar{z}z} = \hat{g}_{\bar{z}z} \hat{\nabla}^{\bar{z}} \frac{a_1 \hat{R}}{2}$

Since R is a world sheet scalar

$$\hat{\nabla}^{\bar{z}} \hat{R} = \partial^{\bar{z}} \hat{R} = \hat{g}^{\bar{z}z} \partial_z \hat{R} + \underbrace{\hat{g}^{\bar{z}\bar{z}} \partial_{\bar{z}} \hat{R}}_0$$

$$\Rightarrow \hat{\nabla}^{\bar{z}} T_{\bar{z}z} = \frac{a_1}{2} \underbrace{\hat{g}_{\bar{z}z} \hat{g}^{\bar{z}z}}_1 \partial_z \hat{R}$$

$$\Rightarrow \hat{\nabla}^{\bar{z}} \hat{T}_{\bar{z}z} = \frac{a_1}{2} \partial_z \hat{R}$$

Similarly, $\hat{\nabla}^z \hat{T}_{z\bar{z}} = \frac{a_1}{2} \partial_{\bar{z}} \hat{R}$

But since ' $\hat{T}$ ' is conserved, $\hat{\nabla}^a \hat{T}_{ab} = 0$

$$\rightarrow \hat{\nabla}^z \hat{T}_{zz} + \hat{\nabla}^{\bar{z}} \hat{T}_{\bar{z}z} = 0$$

$$\neq \hat{\nabla}^z \hat{T}_{z\bar{z}} + \hat{\nabla}^{\bar{z}} \hat{T}_{\bar{z}\bar{z}} = 0$$

Thus we get,

$$\hat{\nabla}^z \hat{T}_{zz} = -\frac{a_1}{2} \partial_z \hat{R} \quad , \quad \hat{\nabla}^{\bar{z}} \hat{T}_{\bar{z}\bar{z}} = -\frac{a_1}{2} \partial_{\bar{z}} \hat{R}$$

Now, we shall do weyl transformation of the above equations about flat world sheet.

We know that, under $\hat{g}_{ab} \rightarrow e^{2\omega} \hat{g}_{ab}$, the Ricci scalar transforms as,

$$\sqrt{\hat{g}'} \hat{R}' = \sqrt{\hat{g}} (\hat{R} - 2\hat{\nabla}^2 \omega)$$

For infinitesimal weyl transform,

$$\hat{g}_{ab} \rightarrow \hat{g}'_{ab} = e^{2\delta\omega} \simeq (1 + 2\delta\omega) \hat{g}_{ab}$$

$$\rightarrow \delta_\omega \hat{g}_{ab} = 2\delta\omega \hat{g}_{ab}$$

Now,

$$\delta_\omega \hat{R} = \hat{R}' - \hat{R} = \frac{\sqrt{\hat{g}}}{\sqrt{\hat{g}'}} (\hat{R} - 2\hat{\nabla}^2 \delta\omega) - \hat{R}$$

$$\frac{\sqrt{\hat{g}}}{\sqrt{\hat{g}'}} = \frac{\sqrt{\hat{g}}}{e^{2\delta\omega} \sqrt{\hat{g}}} = e^{-2\delta\omega} \simeq (1 - 2\delta\omega)$$

$$\begin{aligned}\rightarrow \partial_\omega \hat{R} &= (1 - 2\partial\omega)(R - 2\hat{\nabla}^2 \partial\omega) - \hat{R} \\ &= \hat{R} - 2\partial\omega \hat{R} - 2\hat{\nabla}^2 \partial\omega - \hat{R}\end{aligned}$$

$$\rightarrow \underline{\underline{\partial_\omega \hat{R} = -2\partial\omega \hat{R} - 2\hat{\nabla}^2 \partial\omega}}$$

In conformal gauge, $\hat{g}_{ab} = e^{2\phi} \partial_{ab}$
This is like a Weyl transform of unit gauge.

$$\Rightarrow \sqrt{\hat{g}} \hat{R} = \sqrt{\partial_{ab}} (\hat{R}_{(unit)} - 2\nabla_{(unit)}^2 \phi)$$

$$\sqrt{\hat{g}} = e^{2\phi} \quad \& \quad \text{in unit gauge,}$$

$$\hat{R}_{(unit)} = 0 \quad \& \quad \sqrt{\partial_{ab}} = 1 \quad \& \quad \nabla_{(unit)}^2 = \partial^{ab} \partial_a \partial_b$$

$$\Rightarrow \underline{\underline{\hat{R} = -2e^{-2\phi} \partial^{ab} \partial_a \partial_b \phi = -2\hat{g}^{ab} \partial_a \partial_b \phi}}$$

$$\& \quad \hat{\nabla}^2 = \hat{g}^{ab} \partial_a \partial_b. \text{ Thus in conformal gauge,}$$

$$\partial_\omega \hat{R} = -2\partial\omega (-2\hat{g}^{ab} \partial_a \partial_b \phi) - 2\hat{g}^{ab} \partial_a \partial_b \partial\omega$$

$$\rightarrow \partial_\omega \hat{R} = 4\partial\omega \hat{g}^{ab} \partial_a \partial_b \phi - 2\hat{g}^{ab} \partial_a \partial_b \partial\omega$$

Expanding this about flat world sheet is like putting $\phi \rightarrow 0$,

$$\rightarrow \partial_w \hat{R} = -2\partial^{ab} \partial_a \partial_b \partial \omega$$

In complex coordinates,

$$\partial_w \hat{R} = -2\hat{g}^{z\bar{z}} \partial_z \partial_{\bar{z}} \partial \omega - 2\hat{g}^{\bar{z}z} \partial_{\bar{z}} \partial_z \partial \omega$$

$$\rightarrow \partial_w \hat{R} = -8\partial_z \partial_{\bar{z}} \partial \omega$$

$$\rightarrow \underline{-\frac{a_1}{2} \partial_z (\partial_w \hat{R}) = 4a_1 \partial_z^2 \partial_{\bar{z}} \partial \omega}$$

$$\& \underline{-\frac{a_1}{2} \partial_{\bar{z}} (\partial_w \hat{R}) = 4a_1 \partial_z \partial_{\bar{z}}^2 \partial \omega}$$

Now we shall find the weyl transformation of $\hat{T}_{zz}$ & $\hat{T}_{\bar{z}\bar{z}}$ expanding it around the flat world sheet.

If we are initially at flat world sheet then any conformal transformation to the energy momentum tensor can be found.

Initially in flat world sheet the trace vanishes
 So, $T_{z\bar{z}} = T_{\bar{z}z} = 0$, & $\nabla_{\text{flat}} \rightarrow \partial$. By the
 conservation of T ,

$$\rightarrow \partial^z T_{zz} + \overbrace{\partial^{\bar{z}} T_{\bar{z}z}}^0 = 0$$

$$\underbrace{\partial^z T_{z\bar{z}}}_0 + \partial^{\bar{z}} T_{\bar{z}\bar{z}} = 0$$

$$\Rightarrow \partial^z T_{zz} = 0 = \partial^{\bar{z}} T_{\bar{z}\bar{z}} \quad \left. \vphantom{\begin{matrix} \partial^z T_{zz} = 0 \\ \partial_{\bar{z}} T_{z\bar{z}} = 0 \end{matrix}} \right\} \begin{matrix} T_{zz} \rightarrow T_{zz}(z) \\ T_{\bar{z}\bar{z}} \rightarrow T_{\bar{z}\bar{z}}(\bar{z}) \end{matrix}$$

$$\Rightarrow \underline{\partial_{\bar{z}} T_{z\bar{z}} = 0 = \partial_z T_{\bar{z}z}}$$

Under Conformal transform,

$$\frac{\delta_c T_{zz}}{\epsilon} = -\frac{c}{12} \partial_z^3 \mathcal{V}(z) - 2T_{zz} \partial_z \mathcal{V}(z) - \mathcal{V}(z) \partial_z T_{zz}$$

$$\frac{\delta_c T_{\bar{z}\bar{z}}}{\epsilon} = -\frac{\tilde{c}}{12} \partial_{\bar{z}}^3 \mathcal{V}^*(\bar{z}) - 2T_{\bar{z}\bar{z}} \partial_{\bar{z}} \mathcal{V}^*(\bar{z}) - \mathcal{V}^*(\bar{z}) \partial_{\bar{z}} T_{\bar{z}\bar{z}}$$

But any conformal transformation is a
 combination of Weyl & diff transforms.

Consider the diffeomorphism

$$z \rightarrow z' = z + \varepsilon v(z) \quad \neq \quad \bar{z} \rightarrow \bar{z}' = \bar{z} + \varepsilon v^*(\bar{z})$$

$$\text{So, } T_{z'z'}(z') = \frac{\partial z}{\partial z'} \frac{\partial z}{\partial \bar{z}'} T_{z\bar{z}}(z) \quad \left(\begin{array}{c} \text{other terms} \\ \text{vanish} \end{array} \right)$$

$$= \frac{1}{[1 + \varepsilon \partial_z v(z)]^2} T_{z\bar{z}} = [1 - 2\varepsilon \partial_z v(z)] T_{z\bar{z}}$$

$$\Rightarrow T_{z'z'}(z') = [1 - 2\varepsilon \partial_z v(z)] T_{z\bar{z}}(z)$$

$$\begin{aligned} \text{But } T_{z'z'}(z') &= T_{z'z'}(z + \varepsilon v(z)) \\ &= T_{z'z'}(z) + \varepsilon v(z) \partial_z T_{z'z'}(z) \end{aligned}$$

$$\text{Now, } \varepsilon T_{z'z'}(z) \sim \varepsilon [1 - 2\varepsilon \partial_z v(z)] T_{z\bar{z}}(z)$$

$\underbrace{\hspace{10em}}_{\text{Vanish } \varepsilon^2 \sim 0}$

$$\Rightarrow T_{z'z'}(z') = T_{z'z'}(z) + \varepsilon v(z) \partial_z T_{z\bar{z}}(z)$$

$$\rightarrow [1 - 2\varepsilon \partial_z v(z)] T_{z\bar{z}}(z)$$

$$= T_{z'z'}(z) + \varepsilon v(z) \partial_z T_{z\bar{z}}(z)$$

$$\rightarrow T_{z\bar{z}}(z) - 2\varepsilon T_{z\bar{z}}(z) \partial_{\bar{z}} v(z)$$

$$= T_{z'\bar{z}'}(z) + \varepsilon v(z) \partial_{\bar{z}} T_{z\bar{z}}(z)$$

$$\rightarrow T_{z\bar{z}}(z) - T_{z'\bar{z}'}(z) = 2\varepsilon T_{z\bar{z}}(z) \partial_{\bar{z}} v(z)$$

$$+ \varepsilon v(z) \partial_{\bar{z}} T_{z\bar{z}}(z)$$

$$\Rightarrow \frac{-\partial_{\text{diff}} T_{z\bar{z}}(z)}{\varepsilon} = 2 T_{z\bar{z}}(z) \partial_{\bar{z}} v(z) + v(z) \partial_{\bar{z}} T_{z\bar{z}}(z)$$

Thus “ $-2 T_{z\bar{z}}(z) \partial_{\bar{z}} v(z) - v(z) \partial_{\bar{z}} T_{z\bar{z}}(z)$ ” part in the conformal transformation of $T_{z\bar{z}}$ is the diffeomorphism.

$$\Rightarrow \frac{\partial_c T_{z\bar{z}}(z)}{\varepsilon} = -\frac{c}{12} \partial_{\bar{z}}^3 v(z) + \partial_{\text{diff}} T_{z\bar{z}}(z)$$

Similarly, “ $-2 T_{\bar{z}\bar{z}} \partial_{\bar{z}} v^*(\bar{z}) - v^*(\bar{z}) \partial_{\bar{z}} T_{\bar{z}\bar{z}}$ ” part in the conformal transformation of $T_{\bar{z}\bar{z}}$ is the diffeomorphism.

$$\Rightarrow \frac{\partial_c T_{\bar{z}\bar{z}}(\bar{z})}{\varepsilon} = -\frac{\tilde{c}}{12} \partial_{\bar{z}}^3 v^*(\bar{z}) + \partial_{\text{diff}} T_{\bar{z}\bar{z}}(\bar{z})$$

Now we should see how metric changes under this conformal transform.

$$(ds')^2 = (dz')(d\bar{z}) = \partial_z [z + \varepsilon \varphi(z)] dz \\ \times \partial_{\bar{z}} [\bar{z} + \varepsilon \varphi^*(\bar{z})] d\bar{z}$$

$$\rightarrow (ds')^2 = [1 + \varepsilon \partial_z \varphi(z)] [1 + \varepsilon \partial_{\bar{z}} \varphi^*(\bar{z})] (ds)^2$$

Thus conformal transformation changes the metric by the scaling factor

$$e^{2\omega} = [1 + \varepsilon \partial_z \varphi(z)] [1 + \varepsilon \partial_{\bar{z}} \varphi^*(\bar{z})]$$

$$\rightarrow 1 + 2\omega = [1 + \varepsilon \partial_z \varphi(z)] [1 + \varepsilon \partial_{\bar{z}} \varphi^*(\bar{z})]$$

$$\rightarrow 1 + 2\omega = 1 + \varepsilon \partial_z \varphi(z) + \varepsilon \partial_{\bar{z}} \varphi^*(\bar{z})$$

$$\Rightarrow 2\omega = \varepsilon \partial_z \varphi(z) + \varepsilon \partial_{\bar{z}} \varphi^*(\bar{z})$$

$$\rightarrow \partial_z^2 (2\omega) = \varepsilon \partial_z^3 \varphi(z)$$

$$\nrightarrow \partial_{\bar{z}}^2 (2\omega) = \varepsilon \partial_{\bar{z}}^3 \varphi^*(\bar{z})$$

$$\rightarrow \frac{c}{12} \partial_{\bar{z}}^3 \varphi(z) = \frac{c}{12} \frac{\partial_{\bar{z}}^2 (2\partial\omega)}{\epsilon}$$

$$\& \frac{\tilde{c}}{12} \partial_{\bar{\bar{z}}}^3 \varphi^*(\bar{z}) = \frac{\tilde{c}}{12} \frac{\partial_{\bar{\bar{z}}}^2 (2\partial\omega)}{\epsilon}$$

Use these relations in $\partial_c T_{zz}$ & $\partial_c T_{\bar{z}\bar{z}}$

$$\rightarrow \frac{\partial_c T_{zz}(z)}{\epsilon} = -\frac{c}{6} \frac{\partial_{\bar{z}}^2 \partial\omega}{\epsilon} + \partial_{\text{diff}} T_{zz}(z)$$

$$\& \frac{\partial_c T_{\bar{z}\bar{z}}(\bar{z})}{\epsilon} = -\frac{\tilde{c}}{6} \frac{\partial_{\bar{\bar{z}}}^2 \partial\omega}{\epsilon} + \partial_{\text{diff}} T_{\bar{z}\bar{z}}(\bar{z})$$

$$\begin{aligned} \Rightarrow \partial_w T_{zz} &= -\frac{c}{6} \partial_{\bar{z}}^2 \partial\omega \\ \& \partial_w T_{\bar{z}\bar{z}} &= -\frac{\tilde{c}}{6} \partial_{\bar{\bar{z}}}^2 \partial\omega \end{aligned} \left. \vphantom{\begin{aligned} \Rightarrow \partial_w T_{zz} &= -\frac{c}{6} \partial_{\bar{z}}^2 \partial\omega \\ \& \partial_w T_{\bar{z}\bar{z}} &= -\frac{\tilde{c}}{6} \partial_{\bar{\bar{z}}}^2 \partial\omega \end{aligned}} \right\} \text{Weyl transform}$$

Now,

$$\partial^z \partial_w T_{zz} = -\frac{c}{6} \partial^z \partial_{\bar{z}}^2 \partial\omega = -\frac{c}{3} \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial\omega$$

$$\partial^{\bar{z}} \partial_w T_{\bar{z}\bar{z}} = -\frac{\tilde{c}}{6} \partial^{\bar{z}} \partial_{\bar{\bar{z}}}^2 \partial\omega = -\frac{\tilde{c}}{3} \partial_{\bar{z}} \partial_{\bar{\bar{z}}}^2 \partial\omega$$

$$(\text{used } \partial^z = 2\partial_{\bar{z}} \quad \& \quad \partial^{\bar{z}} = 2\partial_{\bar{\bar{z}}})$$

$$\text{But } \partial^{\bar{z}} T_{z\bar{z}} = -\frac{a_1}{2} \partial_{\bar{z}} R$$

$$\Rightarrow \partial^{\bar{z}} \partial_w T_{z\bar{z}} = -\frac{a_1}{2} \partial_{\bar{z}} \partial_w R$$

We previously got

$$-\frac{a_1}{2} \partial_{\bar{z}} (\partial_w R) = 4a_1 \partial_{\bar{z}}^2 \partial_{\bar{z}} \partial \omega$$

$$\Rightarrow \partial^{\bar{z}} \partial_w T_{z\bar{z}} = -\frac{c}{3} \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial \omega$$

↙ compare

$$= -\frac{a_1}{2} \partial_{\bar{z}} (\partial_w R) = 4a_1 \partial_{\bar{z}}^2 \partial_{\bar{z}} \partial \omega$$

$$\Rightarrow -\frac{c}{3} \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial \omega = 4a_1 \partial_{\bar{z}}^2 \partial_{\bar{z}} \partial \omega$$

$$\Rightarrow \boxed{a_1 = -\frac{c}{12}}$$

Now, for the anti-holomorphic part,

$$\partial^{\bar{z}} \partial_w T_{\bar{z}\bar{z}} = -\frac{\bar{c}}{3} \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial \omega$$

$$\text{But } \partial^{\bar{z}} T_{\bar{z}\bar{z}} = -\frac{a_1}{2} \partial_{\bar{z}} R$$

$$\Rightarrow \partial^{\bar{z}} \partial_w T_{\bar{z}\bar{z}} = -\frac{a_1}{2} \partial_{\bar{z}} \partial_w R$$

In the exact same way, using,

$$\underline{-\frac{a_1}{2} \partial_{\bar{z}} (\partial_w R) = 4a_1 \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial_w}$$

$$\rightarrow -\frac{\tilde{c}}{3} \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial_w = 4a_1 \partial_{\bar{z}} \partial_{\bar{z}}^2 \partial_w$$

$$\Rightarrow \boxed{a_1 = -\frac{\tilde{c}}{12}}$$

Thus we finally get as an operator relation,

$$\boxed{T_{\text{trace}}(T) = -\frac{C}{12} R = -\frac{\tilde{c}}{12} R}$$

So if we wish to have consistent CFT theories in curved background we need $C = \tilde{c}$ and both have to be set to Zero.

$C \neq 0 \ \& \ \tilde{c} \neq 0$ leads to Weyl Anomaly

$C \neq \tilde{c}$ leads to Gravitational Anomaly

Method-2: Through Path Integral

The gauge fixed path integral is given by,

$$Z[\hat{g}_{ab}] = \int d\mathcal{E}_g [\mathcal{D}x^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-(\hat{S}[x^\mu, b_{ab}, c^a, \hat{g}_{ab}])}$$

We are in fact interested only on Weyl transformations in any orbit ' $\mathcal{E}_g$ ' where all are locally similar, thus we shall focus on Weyl transforms in a particular orbit.

$$Z[\hat{g}_{ab}, \mathcal{E}_g] = \int [\mathcal{D}x^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-(\hat{S}[x^\mu, b_{ab}, c^a, \hat{g}_{ab}])}$$

Now, we do the Weyl variation of $Z[\hat{g}_{ab}, \mathcal{E}_g]$.

$$\partial_w Z[\hat{g}_{ab}, \mathcal{E}_g] = \int [\mathcal{D}x^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-(\hat{S}[x^\mu, b_{ab}, c^a, \hat{g}_{ab}])} (-\partial_w \hat{S})$$

$$\text{where, } (-\partial_w \hat{S}) = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \hat{T}_{ab} \partial_w \hat{g}^{ab}$$

$$\& \partial_w g_{ab} = 2\partial\omega g_{ab} \Rightarrow \partial_w g^{ab} = -2\partial\omega g^{ab}$$

$$\text{Then, } (-\partial_\omega \hat{S}) = -\frac{1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \partial\omega \hat{T}_{ab} \hat{g}^{ab}$$

$$\& \text{Trace}(\hat{T}) = \hat{T}_{ab} \hat{g}^{ab} = a_1 \hat{R}$$

Now, $\hat{R}$ is independent of $[\partial X^\mu \partial b_{ab} \partial C^a]$, it depends only on the gauge fixed metric $\& ' \sigma '$:

$$\rightarrow \partial_\omega Z[\hat{g}_{ab}, \xi_g] = -Z[\hat{g}_{ab}, \xi_g] \frac{a_1}{2\pi} \int d^2\sigma \sqrt{\hat{g}} \partial\omega \hat{R}$$

In conformal Gauge, $\hat{g}_{ab} = e^{2\omega} \partial_{ab}$

$$\sqrt{\hat{g}} = e^{2\omega} \& R = -2e^{-2\omega} \partial^{ab} \partial_a \partial_b \omega$$

So, we are actually looking for the change ' ∂_ω ' got by $2\omega \rightarrow 2\omega + 2\delta\omega$ transformation.

$$\rightarrow \partial_\omega Z[e^{2\omega} \partial_{ab}] = \underbrace{Z[e^{2\omega} \partial_{ab}]}_{\text{denotes Conformal gauge}} \frac{a_1}{\pi} \int d^2\sigma \partial\omega \underbrace{\partial^{ab} \partial_a \partial_b \omega}_{\substack{\text{(do int. by} \\ \text{Parts)}}}$$

$$\rightarrow \boxed{\frac{\partial_\omega Z[e^{2\omega} \partial_{ab}]}{Z[e^{2\omega} \partial_{ab}]} = -\frac{a_1}{2\pi} \int d^2\sigma (2\delta\omega) \partial^{ab} (\partial_a \omega) (\partial_b \omega)}$$

Now we can functional integrate the above expression, from the limit $2\omega=0$ which denotes flat metric gauge to $2\omega=2\phi$ that denotes some conformal gauge. Thus,

$$Z[e^{2\phi}\gamma_{ab}] = Z[\gamma_{ab}] \exp\left(-\frac{a_1}{2\pi} \int d^2\sigma \gamma^{ab} (\partial_a \phi) (\partial_b \phi)\right)$$

This can be verified by taking a functional derivative w.r.t 'φ' & setting $\phi \rightarrow \omega$ & we get back the functional differential equation.

So, now we got how the path integral 'Z' depends on the conformal Gauge metric in a particular orbit. But locally every metric is equivalent to conformal gauge through diffeomorphism. Thus we should be able to write ' $Z[\hat{g}_{ab}]$ ' for any metric in a diffeomorphism invariant way such that when in conformal gauge we get back $Z[e^{2\phi}\gamma_{ab}]$ expression as above.

Such a diff invariant functional is given by,

$$Z[\hat{g}_{ab}, \xi_g] = Z[g_{ab}, \xi_g] e^{F[\hat{g}_{ab}]}$$

$$F[\hat{g}_{ab}] = \frac{\alpha_1}{8\pi} \int d^2\sigma d^2\sigma' \sqrt{\hat{g}}(\sigma) \hat{R}(\sigma) \hat{G}_1(\sigma, \sigma') \sqrt{\hat{g}}(\sigma') \hat{R}(\sigma')$$

where the functional 'F' is diff invariant.

The primed coordinate σ' & unprimed one ' σ ' just represents different world sheet locations.

And $\hat{G}_1(\sigma, \sigma')$ is the scalar green's function satisfying, $\sqrt{\hat{g}}(\sigma) \hat{\nabla}^a \hat{\nabla}_a \hat{G}_1(\sigma, \sigma') = \delta^{(2)}(\sigma - \sigma')$

It can be shown that in conformal gauge, the above expression goes to the familiar form.

If $\hat{g}_{ab}(\sigma) = e^{2\omega(\sigma)} g_{ab}$, then $\sqrt{\hat{g}}(\sigma) = e^{2\omega(\sigma)}$

$$\& R(\sigma) = -2 e^{-2\omega(\sigma)} g^{ab} \partial_a \partial_b \omega(\sigma)$$

$$\hat{R}(\sigma') = -2 e^{-2\omega(\sigma')} g^{ab} \partial_a \partial_b \omega(\sigma')$$

$$\partial_a \partial_b \omega(\sigma) = \frac{\partial}{\partial \sigma^a} \frac{\partial \omega(\sigma)}{\partial \sigma^b}, \partial_{a'} \partial_{b'} \omega(\sigma') = \frac{\partial}{\partial \sigma'^a} \frac{\partial \omega(\sigma')}{\partial \sigma'^b}$$

Now,

$$\begin{aligned}
 F[\hat{g}_{ab}] &= \frac{a_1}{8\pi} \int d^2\sigma d^2\sigma' \sqrt{\hat{g}}(\sigma) \hat{R}(\sigma) \hat{G}_1(\sigma, \sigma') \sqrt{\hat{g}}(\sigma') \hat{R}(\sigma') \\
 &= \frac{a_1}{8\pi} \int d^2\sigma d^2\sigma' e^{2\omega(\sigma)} (-2) e^{-2\omega(\sigma)} \partial^{ab} \partial_a \partial_b \omega(\sigma) \hat{G}_1(\sigma, \sigma') \\
 &\quad \times e^{2\omega(\sigma')} (-2) e^{-2\omega(\sigma')} \partial^{c'd'} \partial_c \partial_{d'} \omega(\sigma') \\
 &= \frac{a_1}{2\pi} \int d^2\sigma d^2\sigma' \partial^{ab} \underbrace{[\partial_a \partial_b \omega(\sigma) \hat{G}_1(\sigma, \sigma')]}_{\text{(Integrate by parts twice)}} \partial^{c'd'} \partial_c \partial_{d'} \omega(\sigma')
 \end{aligned}$$

$$= \frac{a_1}{2\pi} \int d^2\sigma d^2\sigma' \omega(\sigma) \partial^{ab} \partial_b \partial_a \hat{G}_1(\sigma, \sigma') \partial^{c'd'} \partial_c \partial_{d'} \omega(\sigma')$$

Now, $\sqrt{\hat{g}}(\sigma) \hat{\nabla}_a \hat{\nabla}^a \hat{G}_1(\sigma, \sigma') = \partial^{(2)}(\sigma - \sigma')$

$$\Rightarrow \sqrt{\hat{g}}(\sigma) \left[\frac{1}{\sqrt{\hat{g}}(\sigma)} \partial_a (\sqrt{\hat{g}}(\sigma) \partial^a \hat{G}_1(\sigma, \sigma')) \right] = \partial^{(2)}(\sigma - \sigma')$$

$$\Rightarrow \partial_a (\sqrt{\hat{g}}(\sigma) \hat{g}^{ab}(\sigma) \partial_b \hat{G}_1(\sigma, \sigma')) = \partial^{(2)}(\sigma - \sigma')$$

In conformal gauge $\sqrt{\hat{g}}(\sigma) \hat{g}^{ab}(\sigma) = \partial^{ab}$

$$\Rightarrow \partial^{ab} \partial_b \partial_a \hat{G}_1(\sigma, \sigma') = \partial^{(2)}(\sigma - \sigma')$$

$\Rightarrow F[\hat{g}_{ab}]$ in conformal gauge is

$$= \frac{a_1}{2\pi} \int d^2\sigma d^2\sigma' \omega(\sigma) \partial^{ab} \partial_b \partial_a \hat{G}(\sigma, \sigma') \partial^{c'd'} \partial_c \partial_{d'} \omega(\sigma')$$

$$= \frac{a_1}{2\pi} \int d^2\sigma d^2\sigma' \omega(\sigma) \partial^{(2)}(\sigma - \sigma') \partial^{c'd'} \partial_c \partial_{d'} \omega(\sigma')$$

(do ' σ' ' integration)

$$\Rightarrow F[\hat{g}_{ab}] = \frac{a_1}{2\pi} \int d^2\sigma' \omega(\sigma') \underbrace{\partial^{c'd'} \partial_c \partial_{d'} \omega(\sigma')}_{\substack{\sigma' \rightarrow \sigma \text{ dummy} \\ \text{variable}}}$$

replace $c \rightarrow a$ & $d \rightarrow b$

$$\Rightarrow F[\hat{g}_{ab}] = \frac{a_1}{2\pi} \int d^2\sigma \omega(\sigma) \partial^{ab} \partial_b \partial_a \omega(\sigma)$$

Integrate by parts

$$\Rightarrow F[\hat{g}_{ab}] = -\frac{a_1}{2\pi} \int d^2\sigma \partial^{ab} \partial_a \omega(\sigma) \partial_b \omega(\sigma)$$

Thus, in conformal gauge,

$$Z[e^{2\omega} \hat{g}_{ab}] = Z[\hat{g}_{ab}] \exp\left(-\frac{a_1}{2\pi} \int d^2\sigma \partial^{ab} \partial_a \omega(\sigma) \partial_b \omega(\sigma)\right)$$

So, we recovered the previous expression.

Hence, for any metric that is diff equivalent to conformal gauge we can write

$$Z[\hat{g}_{ab}, \mathcal{E}_g] = Z[g_{ab}, \mathcal{E}_g] e^{F[\hat{g}_{ab}]}, \text{ where,}$$

$$F[\hat{g}_{ab}] = \frac{a_1}{8\pi} \int d^2\sigma d^2\sigma' \sqrt{\hat{g}(\sigma)} \hat{R}(\sigma) \hat{G}_1(\sigma, \sigma') \sqrt{\hat{g}(\sigma')} \hat{R}(\sigma')$$

Thus the path integral is completely determined by the functional ' $F[\hat{g}_{ab}]$ ', it is this which causes the Weyl anomaly.

This can also be extended to the expectation value of arbitrary local operators inserted inside the path integral.

$$\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle_{\hat{g}, \mathcal{E}_g} = \langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle_{g, \mathcal{E}_g} e^{F[\hat{g}_{ab}]}$$

where, $\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n \rangle_{\hat{g}, \mathcal{E}_g}$ is given by,

$$= \int [dx^\mu db_{ab} dc^a] e^{-\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}]} \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n$$

Expanding $F[\hat{g}_{ab}]$ near flat metric

Let $\hat{g}_{ab} = \delta_{ab} + \epsilon h_{ab}$, $\epsilon \sim$ infinitesimal
where ' h_{ab} ' is the perturbation (Symmetric)

In complex coordinates, it becomes

$$\begin{bmatrix} \hat{g}_{zz} & \hat{g}_{z\bar{z}} \\ \hat{g}_{\bar{z}z} & \hat{g}_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} + \epsilon \begin{bmatrix} h_{zz} & h_{z\bar{z}} \\ h_{\bar{z}z} & h_{\bar{z}\bar{z}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{g}_{zz} & \hat{g}_{z\bar{z}} \\ \hat{g}_{\bar{z}z} & \hat{g}_{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} \epsilon h_{zz} & 1/2 + \epsilon h_{z\bar{z}} \\ 1/2 + \epsilon h_{\bar{z}z} & \epsilon h_{\bar{z}\bar{z}} \end{bmatrix}$$

Then the inverse metric is,

$$\begin{bmatrix} \hat{g}^{zz} & \hat{g}^{z\bar{z}} \\ \hat{g}^{\bar{z}z} & \hat{g}^{\bar{z}\bar{z}} \end{bmatrix} = \begin{bmatrix} -4\epsilon h_{\bar{z}\bar{z}} & 2 - 4\epsilon h_{z\bar{z}} \\ 2 - 4\epsilon h_{\bar{z}z} & -4\epsilon h_{zz} \end{bmatrix} + \mathcal{O}(\epsilon^2)$$

Correct upto 1st order in ' ϵ '.

Then $\sqrt{\hat{g}} = 1 + 2\epsilon h_{z\bar{z}} + \mathcal{O}(\epsilon^2)$

The Ricci scalar is given by,

$$\hat{R} = 4\epsilon \bar{\partial}^2 h_{z\bar{z}} + 4\epsilon \partial^2 h_{\bar{z}\bar{z}} - 8\epsilon \partial \bar{\partial} h_{z\bar{z}} + \mathcal{O}(\epsilon^2)$$

The Green's function is the solution of

$$\rightarrow \sqrt{\hat{g}}(\sigma) \hat{\nabla}_a \hat{\nabla}^a \hat{G}_1(\sigma, \sigma') = \delta^{(2)}(\sigma - \sigma')$$

$$\rightarrow \partial_a (\sqrt{\hat{g}}(\sigma) \hat{g}^{ab}(\sigma) \partial_b \hat{G}_1(\sigma, \sigma')) = \delta^{(2)}(\sigma - \sigma')$$

Now, in complex coordinates,

$$\delta^{(2)}(\sigma - \sigma') = 2 \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\begin{aligned} \Rightarrow 2 \delta^{(2)}(z - z', \bar{z} - \bar{z}') &= \partial \left((1 + 2\epsilon h_{z\bar{z}}) \hat{g}^{z\bar{z}} \partial \hat{G}_1(\sigma, \sigma') \right) \\ &\quad + \bar{\partial} \left((1 + 2\epsilon h_{z\bar{z}}) \hat{g}^{\bar{z}z} \bar{\partial} \hat{G}_1(\sigma, \sigma') \right) \\ &\quad + 2 \partial \left((1 + 2\epsilon h_{z\bar{z}}) \hat{g}^{\bar{z}z} \bar{\partial} \hat{G}_1(\sigma, \sigma') \right) \end{aligned}$$

$$\Rightarrow \mathcal{G}^{(2)}(z - z', \bar{z} - \bar{z}') =$$

$$= -4\epsilon \partial \left((1 + 2\epsilon h_{z\bar{z}}) h_{\bar{z}\bar{z}} \partial \hat{G}_1(\sigma, \sigma') \right)$$

$$- 4\epsilon \bar{\partial} \left((1 + 2\epsilon h_{z\bar{z}}) h_{z\bar{z}} \bar{\partial} \hat{G}_1(\sigma, \sigma') \right)$$

$$+ 2 \partial \left((1 + 2\epsilon h_{z\bar{z}}) (2 - 4\epsilon h_{z\bar{z}}) \bar{\partial} \hat{G}_1(\sigma, \sigma') \right)$$

$$= -4\epsilon \partial (h_{\bar{z}\bar{z}} \partial \hat{G}_1(\sigma, \sigma')) - 4\epsilon \bar{\partial} (h_{z\bar{z}} \bar{\partial} \hat{G}_1(\sigma, \sigma')) \\ + 4 \partial \bar{\partial} \hat{G}_1(\sigma, \sigma')$$

$$\Rightarrow \mathcal{G}^{(2)}(z - z', \bar{z} - \bar{z}') = 2 \partial \bar{\partial} \hat{G}_1(\sigma, \sigma')$$

$$- 2\epsilon \left[\partial (h_{\bar{z}\bar{z}} \partial \hat{G}_1(\sigma, \sigma')) + \bar{\partial} (h_{z\bar{z}} \bar{\partial} \hat{G}_1(\sigma, \sigma')) \right]$$

$$\longrightarrow \mathcal{G}^{(2)}(z - z', \bar{z} - \bar{z}') = 2 \partial \bar{\partial} \hat{G}_1(\sigma, \sigma') + \mathcal{O}(\epsilon^2)$$

We know that

$$\partial \bar{\partial} \ln |z - z'|^2 = 2\pi \mathcal{G}^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\longrightarrow \hat{G}_1(\sigma, \sigma') = \frac{1}{4\pi} \ln |z - z'|^2 + \mathcal{O}(\epsilon)$$

Now, Substituting all these in 'F'

$$F[\hat{g}_{ab}] = \frac{a_1}{8\pi} \int d^2\sigma d^2\sigma' \sqrt{\hat{g}(\sigma)} \hat{R}(\sigma) \hat{G}_1(\sigma, \sigma') \sqrt{\hat{g}(\sigma')} \hat{R}(\sigma')$$

$$d^2\sigma = \frac{d^2z}{2} \quad \& \quad d^2\sigma' = \frac{d^2z'}{2}$$

$$\sqrt{\hat{g}(\sigma)} = 1 + 2\epsilon h_{z\bar{z}} \quad , \quad \sqrt{\hat{g}(\sigma')} = 1 + 2\epsilon h_{z'\bar{z}'}$$

$$\hat{R}(\sigma) = 4\epsilon \bar{\partial}^2 h_{z\bar{z}} + 4\epsilon \partial^2 h_{\bar{z}\bar{z}} - 8\epsilon \partial \bar{\partial} h_{z\bar{z}} + \mathcal{O}(\epsilon^2)$$

$$\hat{R}(\sigma') = 4\epsilon \bar{\partial}'^2 h_{z'\bar{z}'} + 4\epsilon \partial'^2 h_{\bar{z}'\bar{z}'} - 8\epsilon \partial' \bar{\partial}' h_{z'\bar{z}'} + \mathcal{O}(\epsilon^2)$$

$$\& \hat{G}_1(\sigma, \sigma') = \frac{1}{4\pi} \ln |z - z'|^2 + \mathcal{O}(\epsilon)$$

Since $\hat{R}(\sigma) \propto \epsilon$, $\mathcal{O}(\epsilon)$ terms in $\hat{G}_1$ will not contribute. Also only 0^{th} order term in $\sqrt{\hat{g}(\sigma)} \sim 1$ will contribute. Thus,

$$F[g+h] \sim \frac{a_1}{8\pi} \int \frac{d^2z d^2z'}{4} \hat{R}(\sigma) \frac{1}{4\pi} \ln |z - z'|^2 \hat{R}(\sigma')$$

$$\begin{aligned}
F[\beta+h] &\sim \frac{a_1}{8\pi^2} \int d^2z d^2z' \ln|z-z'|^2 \\
&\quad \times (\bar{\partial}^2 h_{zz} + \partial^2 h_{\bar{z}\bar{z}} - 2\partial\bar{\partial} h_{z\bar{z}}) \\
&\quad \times (\bar{\partial}'^2 h_{z'z'} + \partial'^2 h_{\bar{z}'\bar{z}'} - 2\partial'\bar{\partial}' h_{z'\bar{z}'}) \\
&= \frac{a_1}{8\pi^2} \int d^2z d^2z' \ln|z-z'|^2 \\
&\quad \times (\bar{\partial}^2 h_{zz} + \partial^2 h_{\bar{z}\bar{z}} - 2\partial\bar{\partial} h_{z\bar{z}}) \\
&\quad \times (\bar{\partial}'^2 h_{z'z'} + \partial'^2 h_{\bar{z}'\bar{z}'} - 2\partial'\bar{\partial}' h_{z'\bar{z}'})
\end{aligned}$$

Now, note that we wish to compare the resulting equation to the expectation of the $T_{zz} T_{z'z'}$ OPE & then get the value of T^a_a . This relates T^a_a to central charge 'c'.

$$\text{Now, } \partial_w Z[\hat{g}_{ab}, \xi_g] = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \partial \hat{g}^{ab} \langle \hat{T}_{ab} \rangle_{\hat{g}, \xi_g}$$

To extract ' $T_{zz} T_{z'z'}$ ' OPE, we need to have $(\partial \hat{g}^{ab})_{\text{complex}} \sim h^{zz}$ only.

$$h^{zz} \sim 4h_{\bar{z}\bar{z}}$$

Thus we wish to only calculate terms only upto 2nd order of ' $h_{\bar{z}\bar{z}}$ ' & set h_{zz} & $h_{z\bar{z}}$ to zero.

But we could also compare the $T_{\bar{z}\bar{z}} T_{\bar{z}'\bar{z}'}$ OPE & then get the value of T_a^a . This relates T_a^a to central charge ' $\tilde{c}$ '.

For now, if we focus on ' $h_{\bar{z}\bar{z}}$ ', where we set h_{zz} & $h_{z\bar{z}}$ to zero.

Then $\sqrt{\hat{g}} = 1 + \mathcal{O}(\epsilon^3)$, $\hat{R} = 4\epsilon \partial^2 h_{\bar{z}\bar{z}} + \mathcal{O}(\epsilon^3)$

$$\& \hat{G}_1(\sigma, \sigma') = \frac{1}{4\pi} \ln |z - z'|^2 + \mathcal{O}(\epsilon^3)$$

Thus, upto 2nd order in ' $h_{\bar{z}\bar{z}}$ ' & ' $h_{\bar{z}'\bar{z}'}$ ',

$$F[\mathcal{G}+h] = \frac{a_1}{8\pi^2} \int d^2z d^2z' \ln |z - z'|^2 (\partial^2 h_{\bar{z}\bar{z}}) (\partial'^2 h_{\bar{z}'\bar{z}'})$$

Integrate by parts (no boundary)

Then, after 4 times integration by parts,

$$F[\mathcal{Z}+h] = \frac{a_1}{8\pi^2} \int d^2\bar{z} d^2\bar{z}' h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'} \partial_{\bar{z}}^2 \partial_{\bar{z}'}^2 (\ln|z - z'|^2)$$

$$\begin{aligned} \partial_{\bar{z}}^2 \partial_{\bar{z}'}^2 (\ln|z - z'|^2) &= \partial_{\bar{z}}^2 \partial_{\bar{z}'}^2 \ln(z - z') \\ &= \partial_{\bar{z}'}^2 \partial_{\bar{z}} \frac{1}{(z - z')} = \partial_{\bar{z}'}^2 \frac{-1}{(z - z')^2} \\ &= \partial_{\bar{z}'} \frac{2}{(z - z')^3} = \frac{-6}{(z - z')^4} \end{aligned}$$

Thus upto 2nd order we get,

$$\underbrace{F[\mathcal{Z}+h]}_{h_{\bar{z}\bar{z}} \neq 0, h_{z\bar{z}} = h_{z\bar{z}} = 0} = \frac{-3a_1}{4\pi^2} \int d^2\bar{z} d^2\bar{z}' \frac{h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'}}{(z - z')^4}$$

$$\text{Since, } \mathcal{Z}[\mathcal{Z}+h] = \mathcal{Z}[\mathcal{Z}] e^{F[\mathcal{Z}+h]}$$

$$\begin{aligned} \Rightarrow \underbrace{\ln \left[\frac{\mathcal{Z}[\mathcal{Z}+h]}{\mathcal{Z}[\mathcal{Z}]} \right]}_{h_{\bar{z}\bar{z}} \neq 0, h_{z\bar{z}} = h_{z\bar{z}} = 0} &\sim \frac{-3a_1}{4\pi^2} \int d^2\bar{z} d^2\bar{z}' \frac{h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'}}{(z - z')^4} \end{aligned}$$

Similarly, had we focused on h_{zz} instead,

$$\rightarrow \underbrace{F[\beta+h]} = \frac{-3a_1}{4\pi^2} \int d^2z d^2z' \frac{h_{zz} h_{z'z'}}{(\bar{z} - \bar{z}')^4}$$

$$h_{zz} \neq 0, h_{\bar{z}\bar{z}} = h_{z\bar{z}} = 0$$

$$\rightarrow \underbrace{\ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right]} \sim \frac{-3a_1}{4\pi^2} \int d^2z d^2z' \frac{h_{zz} h_{z'z'}}{(\bar{z} - \bar{z}')^4}$$

$$h_{zz} \neq 0, h_{\bar{z}\bar{z}} = h_{z\bar{z}} = 0$$

Extracting TT OPEs

By perturbation in the metric,

$$Z[\beta+h] = \int [Dx^\mu D b_{ab} D c^a] e^{-\hat{S}[x^\mu, b_{ab}, c^a, \beta+h]}$$

$$\text{Now, } \delta \hat{S} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \hat{T}_{ab} \delta \hat{g}^{ab} \equiv \frac{1}{4\pi} \hat{T} h$$

$$(\delta \hat{g}_{ab} = h_{ab})$$

$$\left[\sqrt{\hat{g}} = 1 + 2\epsilon h_{z\bar{z}} + \mathcal{O}(\epsilon^2) \right]$$

$$(\hat{T}_{ab} \delta \hat{g}^{ab} = -\hat{T}^{ab} \delta \hat{g}_{ab})$$

$$\hat{S}[\chi^\mu, b_{ab}, c^a, \beta+h] = \hat{S}[\chi^\mu, b_{ab}, c^a, \beta] + \frac{1}{4\pi} \hat{T} h$$

$$\rightarrow Z[\beta+h] = \int [\mathcal{D}\chi^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-\hat{S}[\chi^\mu, b_{ab}, c^a, \beta]} \times e^{-\frac{1}{4\pi} \hat{T} h}$$

$$\rightarrow Z[\beta+h] = \left\langle e^{-\frac{1}{4\pi} \hat{T} h} \right\rangle_\beta$$

$$= \left\langle 1 - \frac{1}{4\pi} \hat{T} h + \frac{1}{32\pi^2} (\hat{T} h)^2 \right\rangle_\beta$$

$$= Z[\beta] - \frac{1}{4\pi} \langle \hat{T} h \rangle_\beta + \frac{1}{32\pi^2} \langle (\hat{T} h)^2 \rangle_\beta$$

$$\rightarrow \frac{Z[\beta+h]}{Z[\beta]} = 1 - \frac{1}{4\pi} \frac{\langle \hat{T} h \rangle_\beta}{Z[\beta]} + \frac{1}{32\pi^2} \frac{\langle (\hat{T} h)^2 \rangle_\beta}{Z[\beta]}$$

Neglecting $(Z[\beta])^{-2}$ orders,

$$\ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right] \simeq -\frac{1}{4\pi} \frac{\langle \hat{T} h \rangle_\beta}{Z[\beta]} + \frac{1}{32\pi^2} \frac{\langle (\hat{T} h)^2 \rangle_\beta}{Z[\beta]}$$

Consider the 1st term.

$$\langle \hat{T} h \rangle_g = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \langle h^{ab} \hat{T}_{ab} \rangle_g$$

Upto 1st order, (in complex)

$$\langle \hat{T} h \rangle_g = \frac{1}{8\pi} \int d^2z \left[\langle h^{zz} \hat{T}_{zz} \rangle_g + 2 \langle h^{z\bar{z}} \hat{T}_{z\bar{z}} \rangle_g + \langle h^{\bar{z}\bar{z}} \hat{T}_{\bar{z}\bar{z}} \rangle_g \right]$$

$$\left. \begin{array}{l} \hat{T}_{zz} \text{ has spin } +2 \\ \hat{T}_{\bar{z}\bar{z}} \text{ has spin } -2 \\ \hat{T}_{z\bar{z}} \text{ has spin } 0 \end{array} \right\} \begin{array}{l} \text{Because of rotational} \\ \text{invariance,} \\ \int d^2z \langle \hat{\Theta} \rangle_g = 0 \end{array}$$

If ' $\hat{\Theta}$ ' has Non-zero spin.

$$\Rightarrow \langle \hat{T} h \rangle_g = \frac{1}{4\pi} \int d^2z \langle h^{z\bar{z}} \hat{T}_{z\bar{z}} \rangle_g$$

$$h^{z\bar{z}} \sim \hat{g}^{z\bar{z}} \hat{g}^{\bar{z}z} h_{\bar{z}z} \sim 4 h_{\bar{z}z} \text{ (Upto 1st order)}$$

$$\Rightarrow \langle \hat{T} h \rangle_g = \frac{1}{\pi} \int d^2z \langle h_{z\bar{z}} \hat{T}_{z\bar{z}} \rangle_g$$

For the other term

$$\langle (\hat{T}h)^2 \rangle_g = \int d^2\sigma d^2\sigma' \underbrace{\sqrt{\hat{g}} \sqrt{\hat{g}'}}_1 h^{ab} h^{a'b'} \overbrace{\langle \hat{T}_{ab} \hat{T}_{a'b'} \rangle_g}^{\text{OPE}}$$

$$\rightarrow \langle (\hat{T}h)^2 \rangle_g = \int d^2\sigma d^2\sigma' h^{ab} h^{a'b'} \langle \hat{T}_{ab} \hat{T}_{a'b'} \rangle_g$$

Thus,

$$\begin{aligned} \ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right] &\simeq \frac{1}{32\pi^2} \int d^2\sigma d^2\sigma' \frac{h^{ab} h^{a'b'}}{Z[\beta]} \langle \hat{T}_{ab} \hat{T}_{a'b'} \rangle_g \\ &\quad - \frac{1}{\pi} \int d^2z \frac{\langle h_{z\bar{z}} \hat{T}_{z\bar{z}} \rangle_g}{Z[\beta]} \end{aligned}$$

For $h_{\bar{z}\bar{z}} \neq 0, h_{zz} = h_{z\bar{z}} = 0,$

$$\hat{T}_{ab} h^{ab} = h^{\bar{z}\bar{z}} \hat{T}_{\bar{z}\bar{z}} = 4h_{\bar{z}\bar{z}} \hat{T}_{z\bar{z}} \quad (\text{upto 1st order})$$

Thus,

$$\ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right] \simeq \frac{1}{32\pi^2} \int \frac{d^2z d^2z'}{4} \frac{4^2 h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'}}{Z[\beta]} \langle \hat{T}_{z\bar{z}} \hat{T}_{z'\bar{z}'} \rangle_g$$

$\rightarrow \ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right]$ can be written as,

$$= \frac{1}{8\pi^2 Z[\beta]} \int d^2z d^2z' h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'} \langle \hat{T}_{z\bar{z}} \hat{T}_{z'\bar{z}'} \rangle_\beta$$

$\langle \hat{T}_{z\bar{z}} \hat{T}_{z'\bar{z}'} \rangle_\beta = \text{OPE in usual flat space CFT}$

$$\langle \hat{T}_{z\bar{z}} \hat{T}_{z'\bar{z}'} \rangle_\beta \sim \frac{\langle C/2 \rangle_\beta}{(z-z')^4} + \frac{2\langle \hat{T}_{z\bar{z}} \rangle_\beta}{(z-z')^2} + \frac{\langle \partial_z T_{z\bar{z}} \rangle_\beta}{z-z'}$$

$$\langle C/2 \rangle_\beta = C/2 Z[\beta]$$

$$\begin{aligned} \rightarrow \ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right] &= \frac{C/2 Z[\beta]}{8\pi^2 Z[\beta]} \int d^2z d^2z' \frac{h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'}}{(z-z')^4} \\ &+ \frac{2}{8\pi^2 Z[\beta]} \int d^2z d^2z' h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'} \frac{\langle \hat{T}_{z\bar{z}} \rangle_\beta}{(z-z')^2} \\ &+ \frac{1}{8\pi^2 Z[\beta]} \int d^2z d^2z' h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'} \frac{\langle \partial_z T_{z\bar{z}} \rangle_\beta}{z-z'} \end{aligned}$$

$$\rightarrow \ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right] \sim \frac{c}{16\pi^2} \int d^2z d^2z' \frac{h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'}}{(z-z')^4}$$

$$h_{\bar{z}\bar{z}} \neq 0, h_{zz} = h_{z\bar{z}} = 0 \quad \left(\begin{array}{l} \text{We need this coefficient} \\ \text{as it matches with} \\ \text{the below.} \end{array} \right)$$

Previously we got,

$$\ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right] \sim \frac{-3a_1}{4\pi^2} \int d^2z d^2z' \frac{h_{\bar{z}\bar{z}} h_{\bar{z}'\bar{z}'}}{(z-z')^4}$$

$$h_{\bar{z}\bar{z}} \neq 0, h_{zz} = h_{z\bar{z}} = 0$$

Thus by comparing coefficients both side

$$\frac{c}{16} = \frac{-3a_1}{4}$$

$$\Rightarrow \boxed{a_1 = \frac{-c}{12}}$$

For $h_{zz} \neq 0, h_{\bar{z}\bar{z}} = h_{z\bar{z}} = 0,$

$$\hat{T}_{ab} h^{ab} = h^{\bar{z}\bar{z}} \hat{T}_{\bar{z}\bar{z}} = 4 h_{zz} \hat{T}_{\bar{z}\bar{z}} \quad (\text{upto 1st order})$$

$$\ln \left[\frac{Z[\partial+h]}{Z[\partial]} \right] \simeq \frac{1}{32\pi^2} \int \frac{d^2z d^2z'}{4} \frac{4^2 h_{zz} h_{z'z'}}{Z[\partial]} \langle \hat{T}_{\bar{z}\bar{z}} \hat{T}_{\bar{z}'\bar{z}'} \rangle_{\partial}$$

$$= \frac{1}{8\pi^2 Z[\partial]} \int d^2z d^2z' h_{zz} h_{z'z'} \langle \hat{T}_{\bar{z}\bar{z}} \hat{T}_{\bar{z}'\bar{z}'} \rangle_{\partial}$$

$$\langle \hat{T}_{\bar{z}\bar{z}} \hat{T}_{\bar{z}'\bar{z}'} \rangle_{\partial} = \text{OPE in usual flat space CFT}$$

$$\langle \hat{T}_{\bar{z}\bar{z}} \hat{T}_{\bar{z}'\bar{z}'} \rangle_{\partial} \sim \frac{\langle \tilde{C}/2 \rangle_{\partial}}{(\bar{z} - \bar{z}')^4} + \frac{2\langle \hat{T}_{\bar{z}\bar{z}} \rangle_{\partial}}{(\bar{z} - \bar{z}')^2} + \frac{\langle \partial_{\bar{z}} T_{\bar{z}\bar{z}} \rangle_{\partial}}{\bar{z} - \bar{z}'}$$

$$\langle C/2 \rangle_{\partial} = \tilde{C}/2 Z[\partial]$$

$$\rightarrow \ln \left[\frac{Z[\partial+h]}{Z[\partial]} \right] \simeq \frac{\tilde{C}/2 Z[\partial]}{8\pi^2 Z[\partial]} \int d^2z d^2z' \frac{h_{zz} h_{z'z'}}{(\bar{z} - \bar{z}')^4}$$

$h_{zz} \neq 0, h_{\bar{z}\bar{z}} = h_{z\bar{z}} = 0$ (We need only this term as it matches with the previous result)

$$\rightarrow \underbrace{\ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right]} \sim \frac{\tilde{c}}{16\pi^2} \int d^2z d^2z' \frac{h_{zz} h_{z'z'}}{(\bar{z} - \bar{z}')^4}$$

$$h_{zz} \neq 0, h_{\bar{z}\bar{z}} = h_{z\bar{z}} = 0$$

Previously we got,

$$\underbrace{\ln \left[\frac{Z[\beta+h]}{Z[\beta]} \right]} \sim \frac{-3a_1}{4\pi^2} \int d^2z d^2z' \frac{h_{zz} h_{z'z'}}{(\bar{z} - \bar{z}')^4}$$

$$h_{zz} \neq 0, h_{\bar{z}\bar{z}} = h_{z\bar{z}} = 0$$

Thus by comparing coefficients both side

$$\frac{\tilde{c}}{16} = \frac{-3a_1}{4}$$

$$\Rightarrow \boxed{a_1 = \frac{-\tilde{c}}{12}}$$

And we also need $c = \tilde{c}$.

Method-3: Through TT OPEs

Here we shall derive the contact term b/w the $T_{z\bar{z}}$ & $T_{z'\bar{z}'}$ operators.

We know that in flat gauge,

$$\partial^z \hat{T}_{z\bar{z}} + \partial^{\bar{z}} \hat{T}_{\bar{z}z} = 0 \rightarrow \partial_{\bar{z}} \hat{T}_{z\bar{z}} + \partial_z \hat{T}_{\bar{z}z} = 0$$

$$\text{So } \partial_z \hat{T}_{z\bar{z}} = -\partial_{\bar{z}} \hat{T}_{\bar{z}z}, \quad \partial_{z'} \hat{T}_{z'\bar{z}'} = -\partial_{\bar{z}'} \hat{T}_{\bar{z}'z'}$$

Then the OPE for

$$\begin{aligned} (\partial_z \hat{T}_{z\bar{z}})(\partial_{z'} \hat{T}_{z'\bar{z}'}) &= (\partial_{\bar{z}} \hat{T}_{\bar{z}z})(\partial_{\bar{z}'} \hat{T}_{\bar{z}'z'}) = \partial_{\bar{z}} \partial_{\bar{z}'} \left(\hat{T}_{z\bar{z}} \hat{T}_{z'\bar{z}'} \right) \quad \text{OPE} \\ &= c/2 \partial_{\bar{z}} \partial_{\bar{z}'} \left[\frac{1}{(z - z')^4} \right] + 2 \partial_{\bar{z}} \partial_{\bar{z}'} \left[\frac{\hat{T}_{z\bar{z}}}{(z - z')^4} \right] \\ &\quad + \partial_{\bar{z}} \partial_{\bar{z}'} \left[\frac{\partial_z T_{z\bar{z}}}{z - z'} \right] \end{aligned}$$

(2nd and 3rd term don't contribute)

$$\text{Now, } \partial_{\bar{z}} \partial_{\bar{z}'} \left[\frac{1}{(z - z')^4} \right] = \frac{\partial_{\bar{z}} \partial_{\bar{z}'} \partial_z^2 \partial_{z'}}{6} \left[\frac{1}{(z - z')} \right]$$

$$= \frac{\partial_z^2 \partial_{z'} \partial_{\bar{z}'} \partial_{\bar{z}} (\ln |z - \bar{z}|^2)}{6}$$

We know,

$$\partial_{\bar{z}} \partial_z \ln |z - \bar{z}|^2 = 2\pi \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\partial_{\bar{z}} \partial_{\bar{z}'} \left[\frac{1}{(z - z')^4} \right] = \frac{\pi}{3} \partial_z^2 \partial_{z'} \partial_{\bar{z}'} \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\Rightarrow (\partial_{\bar{z}} \hat{T}_{z\bar{z}}) (\partial_{z'} \hat{T}_{z'\bar{z}'}) \sim \frac{\pi c}{6} \partial_z^2 \partial_{z'} \partial_{\bar{z}'} \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

We neglected other terms as they won't contribute to the calculation of Weyl Anomaly. The below term is the one that is responsible for Weyl Anomaly.

$$\Rightarrow \hat{T}_{z\bar{z}} \hat{T}_{z'\bar{z}'} \sim \frac{\pi c}{6} \partial_z \partial_{\bar{z}'} \delta^{(2)}(z - z', \bar{z} - \bar{z}')$$

This is called as the contact term.

Now, we know that $\langle \hat{T}^a_a \rangle_g = 0$ in flat gauge, but if we do an infinitesimal Weyl transform, then

$$\langle \hat{T}^a_a \rangle_g = \int [D\chi^\mu D b_{ab} D c^a] e^{-\left(\hat{S}[\chi^\mu, b_{ab}, c^a, g]\right)} \hat{T}^a_a$$

$$\partial_w \langle \hat{T}^a_a \rangle_g = \frac{1}{4\pi} \int d^2\sigma' \sqrt{\hat{g}'} \partial\omega \langle \hat{T}^a_a \partial_w \hat{g}^{b'c'} \hat{T}_{b'c'} \rangle_g$$

$$\partial_w \hat{g}^{bc} = -2\omega \hat{g}^{bc}, \quad (\sqrt{\hat{g}'} \partial\omega) \sim \partial\omega$$

Thus we get Weyl transform of $\langle \hat{T}^a_a(\sigma) \rangle_g$

$$\partial_w \langle \hat{T}^a_a(\sigma) \rangle_g = \frac{-1}{2\pi} \int d^2\sigma' \partial\omega(\sigma') \langle \hat{T}^a_a(\sigma) \hat{T}^b_b(\sigma') \rangle_g$$

$$\langle \hat{T}^a_a(\sigma) \hat{T}^b_b(\sigma') \rangle_g = \langle \partial^{ca} T_{ca}(\sigma) \partial^{db} T_{db}(\sigma') \rangle_g$$

In complex coordinates

$$\langle \partial^{ca} T_{ca}(\sigma) \partial^{db} T_{db}(\sigma') \rangle_g =$$

$$\langle (g^{z\bar{z}} T_{z\bar{z}} + g^{\bar{z}z} T_{\bar{z}z}) (g^{z'\bar{z}'} T_{z'\bar{z}'} + g^{\bar{z}'z'} T_{\bar{z}'z'}) \rangle_g$$

$$\rightarrow \langle \hat{T}_a^a(\sigma) \hat{T}_b^b(\sigma') \rangle_g = 16 \langle T_{z\bar{z}} T_{z'\bar{z}'} \rangle_g$$

$$\text{But } \langle T_{z\bar{z}} T_{z'\bar{z}'} \rangle_g \sim \frac{\pi c}{6} \partial_z \partial_{\bar{z}'} \partial^{(2)}(z - z', \bar{z} - \bar{z}')$$

$$\langle \hat{T}_a^a(\sigma) \hat{T}_b^b(\sigma') \rangle_g = \frac{8\pi c}{3} \partial_z \partial_{\bar{z}'} \partial^{(2)}(z - z', \bar{z} - \bar{z}')$$

Now,

$$8 \partial_z \partial_{\bar{z}'} \partial^{(2)}(z - z', \bar{z} - \bar{z}') = - \partial^a \partial_a \partial^{(2)}(\sigma - \sigma')$$

$$\rightarrow \langle \hat{T}_a^a(\sigma) \hat{T}_b^b(\sigma') \rangle_g = - \frac{\pi c}{3} \partial^a \partial_a \partial^{(2)}(\sigma - \sigma')$$

Finally,

$$\delta_w \langle \hat{T}_a^a(\sigma) \rangle_g = \frac{c}{6} \int d^2\sigma' \delta\omega(\sigma') \partial^a \partial_a \partial^{(2)}(\sigma - \sigma')$$

Int. by parts twice

$$\delta_w \langle \hat{T}_a^a(\sigma) \rangle_g = \frac{c}{6} \int d^2\sigma' \partial^a \partial_a \delta\omega(\sigma') \partial^{(2)}(\sigma - \sigma')$$

$$\Rightarrow \boxed{\delta_w \langle \hat{T}_a^a(\sigma) \rangle_g = \frac{c}{6} \partial^a \partial_a \delta\omega(\sigma)}$$

But the Ricci Scalar under the small weyl transform is,

$$\begin{aligned} (\partial_w \hat{R}) &= -8 \partial_z \partial_{\bar{z}} \partial \omega \\ \text{or } (\partial_w \hat{R}) &= -2 \partial^a \partial_a \partial \omega \end{aligned} \quad \left. \vphantom{\begin{aligned} (\partial_w \hat{R}) &= -8 \partial_z \partial_{\bar{z}} \partial \omega \\ (\partial_w \hat{R}) &= -2 \partial^a \partial_a \partial \omega \end{aligned}} \right\} \text{flat gauge.}$$

$$\Rightarrow \partial_w \langle \hat{T}^a_a(\sigma) \rangle_\partial = \frac{-c}{6} \frac{(\partial_w \hat{R})}{2}$$

$$\Rightarrow \partial_w \langle \hat{T}^a_a(\sigma) \rangle_\partial = \frac{-c}{12} \partial_w \hat{R}$$

$$\Rightarrow \boxed{\langle \hat{T}^a_a \rangle_{\hat{g}} = \frac{-c}{12} \hat{R}}$$

In exact same way we also get,

$$\boxed{\langle \hat{T}^a_a \rangle_{\hat{g}} = \frac{-\tilde{c}}{12} \hat{R}}$$

Critical Dimensions

In Polyakov Path integral after Gauge fixing, we got a new Ghost action Path integral as the Jacobian.

The bc Ghosts have $\lambda=2$, $\tilde{\lambda}=2$ whose central charge is $C_g = -26$ & $\tilde{C}_g = -26$. But we also know that the D-Scalar X^μ theory has central charge $C_x = D$ & $\tilde{C}_x = D$.

Thus total central charge is

$$C = C_x + C_g \quad \& \quad \tilde{C} = \tilde{C}_x + \tilde{C}_g$$

$$\Rightarrow \underline{C = D - 26} \quad \& \quad \underline{\tilde{C} = D - 26}$$

So, to have a Weyl invariant quantum theory we need $C = \tilde{C} = 0$. Thus the critical Dimension for X^μ theory should be,

$$\boxed{D_{\text{critical}} = 26}$$

Inclusion of Boundaries

All the previous calculations were for World Sheets without end-point boundaries.

But now we would like to see how the boundary terms contribute to expectation for the trace of energy momentum tensor.

We can write a diff. invariant possibility for the trace

$$\text{Trace}(\hat{T}) = a_0 + a_1 \hat{R}$$

Earlier in chapter '2' we set $a_0 = 0$ by choice (neglecting normal ordering).

But even if $a_0 \neq 0$, we can add a Cosmological counter term to set

$$\text{Trace}(\hat{T}) \propto \hat{R}$$

Removing cosmological constant

Consider $S_{ct} = b_0 \int_M d^2\sigma \sqrt{\hat{g}}$ } cosmological term

Adding S_{ct} to the action breaks Weyl invariance but the diff-invariance is still obeyed. This changes the energy momentum tensor as follows.

$$\underbrace{\hat{T}^{ab}(\sigma)}_{\text{new } T^{ab}} = \underbrace{\frac{4\pi}{\sqrt{\hat{g}}} \frac{\partial S}{\partial \hat{g}_{ab}}}_{\text{old } \hat{T}^{ab}} + \underbrace{\frac{4\pi}{\sqrt{\hat{g}}} \frac{\partial S_{ct}}{\partial \hat{g}_{ab}}}_{T_{ct}^{ab}}$$

$$\Rightarrow \hat{T}_{\text{new}}^{ab}(\sigma) = \hat{T}_{\text{old}}^{ab}(\sigma) + \frac{4\pi b_0}{\sqrt{\hat{g}(\sigma)}} \int_M d^2\sigma' \frac{\partial \sqrt{\hat{g}(\sigma')}}{\partial \hat{g}_{ab}(\sigma)}$$

$$\text{But } \partial \sqrt{\hat{g}} = \frac{1}{2} \sqrt{\hat{g}} \hat{g}^{mn} \partial \hat{g}_{mn},$$

$$\rightarrow \frac{\partial \sqrt{\hat{g}(\sigma')}}{\partial \hat{g}_{ab}(\sigma)} = \frac{1}{2} \sqrt{\hat{g}(\sigma')} \hat{g}^{mn}(\sigma') \underbrace{\frac{\partial \hat{g}_{mn}(\sigma')}{\partial \hat{g}_{ab}(\sigma)}}_{\partial_m^a \partial_n^b \delta^{(2)}(\sigma - \sigma')}$$

$$\Rightarrow \hat{T}_{\text{new}}^{ab}(\sigma) = \hat{T}_{\text{old}}^{ab}(\sigma)$$

$$+ \frac{4\pi b_0}{\sqrt{\hat{g}(\sigma)}} \int_M d^2\sigma' \frac{1}{2} \sqrt{\hat{g}(\sigma')} \hat{g}^{ab}(\sigma') \delta^{(2)}(\sigma - \sigma')$$

$$\Rightarrow \underline{\hat{T}_{\text{new}}^{ab}(\sigma) = \hat{T}_{\text{old}}^{ab}(\sigma) + 2\pi b_0 \hat{g}^{ab}(\sigma)}$$

$$\text{Then } \hat{g}_{ab} \hat{T}_{\text{new}}^{ab} = \hat{g}_{ab} \hat{T}_{\text{old}}^{ab} + 2\pi b_0 \underbrace{\hat{g}_{ab} \hat{g}^{ab}}_2$$

$$\rightarrow \text{Trace}(\hat{T}_{\text{new}}) = \text{Trace}(\hat{T}_{\text{old}}) + 4\pi b_0$$

$$\text{Trace}(\hat{T}_{\text{old}}) = a_0 + a_1 \hat{R}$$

$$\rightarrow \text{Trace}(\hat{T}_{\text{new}}) = a_1 \hat{R} + a_0 + 4\pi b_0$$

$$\text{If we set } b_0 = -\frac{a_0}{4\pi}$$

$$\text{Then, } \underline{\underline{\text{Trace}(\hat{T}_{\text{new}}) = a_1 \hat{R}}}$$

Thus we have implicitly done this procedure in previous chapter of flat gauge CFT.

But no such local counter term can remove the constant a_1 . Thus if it is non-zero, then there is Weyl anomaly.

Variation of Path integral with boundary

The Euler characteristic in the action is

$$\hat{\chi} = \frac{1}{4\pi} \int_M d^2\sigma \sqrt{\hat{g}} \hat{R} + \frac{1}{2\pi} \int_{\partial M} ds \hat{k}$$

where ' $\hat{R}$ ' is the Ricci scalar of the gauge fixed metric, $\hat{k} = t^a n_b \hat{\nabla}_a t^b$ is the geodesic curvature of the boundary & $ds = \sqrt{\hat{g}_{11}} d\sigma^1$ for the σ^1 boundary & $ds = \sqrt{\hat{g}_{22}} d\sigma^2$ is the euclidean time along the σ^2 boundary. t^a is the tangent unit vector & n^a is a unit outward normal vector orthogonal to t^a at the boundary.

$\hat{\chi}$ is Weyl and diff invariant. So, its variation is not included in the previous calculations of world sheet without boundary.

In the presence of boundary let us see how the path integral changes.

$$Z[\hat{g}_{ab}] = \int d\mathcal{E}_g [\mathcal{D}\chi^\mu \mathcal{D}b_{ab} \mathcal{D}c^a] e^{-\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}]}$$

$$\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}] = \underbrace{\hat{S}_\chi[\chi^\mu, \hat{g}]}_{\text{Polyakov}} + \underbrace{\hat{S}_g[b_{ab}, c^a, \hat{g}]}_{\text{Ghost}} + \lambda \underbrace{\hat{\chi}[\hat{g}]}_{\text{Euler no.}}$$

$$\text{Now } \partial \hat{S} = \partial \hat{S}_\chi + \partial \hat{S}_g + \lambda \partial \hat{\chi}[\hat{g}]$$

If $\partial \hat{S}$ is due to the change of metric, then, $\partial \hat{S}_\chi + \partial \hat{S}_g$ gives the definition of energy momentum tensor.

$$\partial \hat{S}_\chi + \partial \hat{S}_g = \frac{-1}{4\pi} \int_M d^2\sigma \sqrt{\hat{g}} \hat{T}_{ab} \partial \hat{g}^{ab}$$

Specifically for Weyl transform

$$\partial_w \hat{g}^{ab} = -2 \hat{g}^{ab} \partial \omega$$

$$\rightarrow \partial_w \hat{S}_\chi + \partial_w \hat{S}_g = \frac{1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \partial \omega \hat{g}^{ab} \hat{T}_{ab}$$

$$\hat{g}_{ab} \hat{T}^{ab} = a_0 + a_1 \hat{R}$$

$$\rightarrow \partial_w \hat{S}_x + \partial_w \hat{S}_g = \frac{1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \partial\omega (a_0 + a_1 \hat{R})$$

Now, consider the Euler characteristic,

$$\hat{\chi} = \frac{1}{4\pi} \int_M d^2\sigma \sqrt{\hat{g}} \hat{R} + \frac{1}{2\pi} \int_{\partial M} ds \hat{k}$$

In 1st chapter we saw, if $\hat{g}^{ab} \rightarrow e^{2\omega} \hat{g}^{ab}$

$$\int_M d^2\sigma \sqrt{\hat{g}} \hat{R} \rightarrow \int_M d^2\sigma \sqrt{\hat{g}} \hat{R} - 2 \int_{\partial M} ds (n^a \partial_a \omega)$$

$$\begin{aligned} k &\rightarrow (k + n^a \partial_a \omega) e^{-\omega} \sim k(1 - \partial\omega) + n^a \partial_a \partial\omega \\ ds &\rightarrow e^{\omega} ds \sim (1 + \partial\omega) ds \xrightarrow{\text{Infinitesimal}} \\ k ds &\rightarrow (k + n^a \partial_a \omega) ds \sim k ds + ds n^a \partial_a \partial\omega \end{aligned}$$

Thus in effective ' $\hat{\chi}$ ' is Weyl Invariant.
But still we could keep these changes & see if we can cancel them by some counter terms.

So a most general $\partial_w \hat{S}_x + \partial_w \hat{S}_g + \lambda \partial_w \hat{\chi}$ under infinitesimal Weyl transform is,

$$\frac{1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \partial\omega(a_0 + a_1 \hat{R}) \left\} \text{from } \hat{T}^a_a \right.$$

some cosmological const.

$$+ \frac{1}{2\pi} \int_{\partial M} ds \widetilde{a_2 \partial\omega + a_3 k \partial\omega + a_4 n^a \partial_a \partial\omega}$$

by boundary variation in $\hat{\chi}$

$$\rightarrow \partial_w \hat{S}_x + \partial_w \hat{S}_g + \lambda \partial_w \hat{\chi}$$

$$= \frac{1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \partial\omega(a_0 + a_1 \hat{R}) + \frac{1}{2\pi} \int_{\partial M} ds (a_2 + a_3 k + a_4 n^a \partial_a) \partial\omega$$

Now the Partition function changes as

$$\partial_w Z[\hat{g}_{ab}, \xi_g] = \int [dx^\mu db_{ab} dc^a] e^{-(\hat{S}[\chi^\mu, b_{ab}, c^a, \hat{g}_{ab}])} (-\partial_w \hat{S})$$

$$\text{Here } \hat{S} = S_x + \hat{S}_g + \lambda \hat{\chi}$$

Thus the most general variation is,

$$\begin{aligned} \frac{\delta_w Z[\hat{g}_{ab}, \xi_g]}{Z[\hat{g}_{ab}, \xi_g]} &= \delta_w [\ln(Z[\hat{g}_{ab}, \xi_g])] \\ &= -\frac{1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \delta\omega (a_0 + a_1 \hat{R}) \\ &\quad - \frac{1}{2\pi} \int_{\partial M} ds (a_2 + a_3 K + a_4 n^a \partial_a) \delta\omega \end{aligned}$$

For a closed string the boundary term will not appear & we should deal with 1st term only. For now we deal with both.

The possible counter terms are

$$S_{ct} = b_0 \int_M d^2\sigma \sqrt{\hat{g}} + \int_{\partial M} ds (b_2 + b_4 K)$$

b_0 removes a_0 , b_2 removes a_2 and b_4 removes a_4 .

Then Consider the Weyl transform of S_{ct}

$$\sqrt{\hat{g}} \rightarrow e^{2\omega} \sqrt{\hat{g}} \sim (1+2\partial\omega) \sqrt{\hat{g}}$$

$$R \rightarrow (R + n^a \partial_a \omega) e^{-\omega} \sim R(1-\partial\omega) + n^a \partial_a \partial\omega$$

$$ds \rightarrow e^{\omega} ds \sim (1+\partial\omega) ds$$

$$\begin{aligned} \text{Then } S_{ct} &\rightarrow b_0 \int_M d^2\sigma e^{2\omega} \sqrt{\hat{g}} \\ &\quad + \int_{\partial M} ds e^{\omega} [b_2 + b_4 (R + n^a \partial_a \omega) e^{-\omega}] \end{aligned}$$

(For $\omega \rightarrow \partial\omega$)

$$\sim b_0 \int_M d^2\sigma \sqrt{\hat{g}} + 2b_0 \int_M d^2\sigma \sqrt{\hat{g}} \partial\omega$$

$$+ \int_{\partial M} ds b_2 + \int_{\partial M} ds b_2 \partial\omega + b_4 \int_{\partial M} ds (R + n^a \partial_a \partial\omega)$$

$$\rightarrow \delta_\omega S_{ct} = 2b_0 \int_M d^2\sigma \sqrt{\hat{g}} \partial\omega + \int_{\partial M} ds (b_2 + b_4 n^a \partial_a) \partial\omega$$

Then, because of the counter term contribution in path integral, $\delta_w [\ln(Z[\hat{g}_{ab}, \xi_s])]$ in the presence of path integral is,

$$\begin{aligned} \delta_w [\ln(Z[\hat{g}_{ab}, \xi_s])] &= -\frac{1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \delta\omega (a_0 + a_1 \hat{R}) \\ &\quad - \frac{1}{2\pi} \int_{\partial M} ds (a_2 + a_3 k + a_4 n^a \partial_a) \delta\omega \\ &\quad + 2b_0 \int_M d^2\sigma \sqrt{\hat{g}} \delta\omega + \int_{\partial M} ds (b_2 + b_4 n^a \partial_a) \delta\omega \end{aligned}$$

If $b_0 = -\frac{a_0}{4\pi}$, $b_2 = -\frac{a_2}{2\pi}$, $b_4 = -\frac{a_4}{2\pi}$, then

$$\begin{aligned} \delta_w [\ln(Z[\hat{g}_{ab}, \xi_s])] &= -\frac{a_1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \delta\omega \hat{R} \\ &\quad - \frac{a_3}{2\pi} \int_{\partial M} ds k \delta\omega \end{aligned}$$

Thus it is a_1 & a_3 that are the Potential anomalies for Open String. For closed String a_3 term doesn't even exist.

Wess - Zumino Consistency Condition

This puts a strong constrain to the structure of possible anomalies.

Consider $\delta_{w_2} \delta_{w_1} [\ln(Z[g_{ab}, \xi_s])]$

$$= -\frac{a_1}{2\pi} \delta_{w_2} \int_M d^2\sigma \delta\omega_1 \sqrt{\hat{g}} \hat{R} - \frac{a_3}{2\pi} \int_{\partial M} \delta_{w_2}(ds k) \delta\omega_1$$

after including counter terms

As seen previously,

$$\delta_{w_2} \int_M d^2\sigma \sqrt{\hat{g}} \hat{R} = -2 \int_{\partial M} ds n^a \partial_a \delta\omega_2$$

$$\delta_{w_2}(ds k) = ds n^a \partial_a \delta\omega_2$$

$$\begin{aligned} &= \frac{a_1}{\pi} \int_M d^2\sigma \delta\omega_1 \sqrt{\hat{g}} \hat{\nabla}^2 \delta\omega_2 + \frac{a_3}{2\pi} \int_{\partial M} ds \delta\omega_1 \delta\omega_2 k \\ &\quad - \frac{a_3}{2\pi} \int_{\partial M} ds \delta\omega_1 n^a \partial_a \delta\omega_2 \end{aligned}$$

Wess - Zumino Consistency Condition

This puts a strong constrain to the structure of possible anomalies.

Consider $\delta_{w_2} \delta_{w_1} [\ln(Z[\hat{g}_{ab}, \hat{E}_g])]$

$$= -\frac{a_1}{2\pi} \int_M d^2\sigma \delta\omega_1 (\delta_{w_2} \sqrt{\hat{g}} \hat{R}) - \frac{a_3}{2\pi} \int_{\partial M} \delta_{w_2}(dsR) \delta\omega_1$$

after including counter terms

$$\delta_{w_2}(\hat{R}) = -2\delta\omega_2 \hat{R} - 2\hat{\nabla}^2 \delta\omega_2$$

$$\delta_{w_2}(\sqrt{\hat{g}}) = 2\delta\omega_2 \sqrt{\hat{g}}$$

$$\delta_{w_2}(dsR) = ds n^a \partial_a \delta\omega_2$$

$$= -\frac{a_1}{2\pi} \int_M d^2\sigma \delta\omega_1 \left[\cancel{2\delta\omega_2 \sqrt{\hat{g}}} \hat{R} \right. \\ \left. - \sqrt{\hat{g}} (\cancel{2\delta\omega_2 \hat{R}} + 2\hat{\nabla}^2 \delta\omega_2) \right] \\ + \frac{a_3}{2\pi} \int_{\partial M} ds \delta\omega_1 n^a \partial_a \delta\omega_2$$

$$= \frac{a_1}{\pi} \int_M d^2\sigma \, \partial\omega_1 \sqrt{\hat{g}} \, \hat{\nabla}^2 \partial\omega_2 - \frac{a_3}{2\pi} \int_{\partial M} ds \, \partial\omega_1 n^a \partial_a \partial\omega_2$$

$$= \frac{a_1}{\pi} \int_M d^2\sigma \, \partial\omega_1 \sqrt{\hat{g}} \, \hat{\nabla}^a \hat{\nabla}_a \partial\omega_2 - \frac{a_3}{2\pi} \int_{\partial M} ds \, \partial\omega_1 n^a \partial_a \partial\omega_2$$

Int. by parts

$$= \frac{a_1}{\pi} \int_{\partial M} ds \, \partial\omega_1 \sqrt{\hat{g}} \, n^a \hat{\nabla}_a \partial\omega_2 \longrightarrow \text{Boundary term}$$

$$- \frac{a_1}{\pi} \int_M d^2\sigma \, \sqrt{\hat{g}} \, (\hat{\nabla}^a \partial\omega_1) (\hat{\nabla}_a \partial\omega_2)$$

$$- \frac{a_3}{2\pi} \int_{\partial M} ds \, \partial\omega_1 n^a \partial_a \partial\omega_2$$

$$\hat{\nabla}_a \partial\omega_1 = \partial_a \partial\omega_1$$

$$\hat{\nabla}_a \partial\omega_2 = \partial_a \partial\omega_2$$

$$\longrightarrow \partial_{\omega_2} \partial_{\omega_1} [\ln(Z[\hat{g}_{ab}, \xi_3])] \text{ is}$$

$$= -\frac{a_1}{\pi} \int_M d^2\sigma \, \sqrt{\hat{g}} \, (\partial_a \partial\omega_1) (\partial_a \partial\omega_2)$$

$$+ \frac{2a_1 - a_3}{2\pi} \int_{\partial M} ds \, \partial\omega_1 n^a \partial_a \partial\omega_2$$

Now since order of $\delta_{w_2} \delta_{w_1} [\ln(Z[\hat{g}_{ab}, \xi])]$ should be equal to $\delta_{w_1} \delta_{w_2} [\ln(Z[\hat{g}_{ab}, \xi])]$.

Only those terms symmetric in δw_1 & δw_2 can exist. Thus it puts a hard constraint of $\boxed{2a_1 - a_3 = 0}$

Thus, we finally get Weyl Anomaly eqⁿ,

$$\delta_w [\ln(Z[\hat{g}_{ab}, \xi])] = -\frac{a_1}{2\pi} \int_M d^2\sigma \sqrt{\hat{g}} \delta w \hat{R} - \frac{a_1}{\pi} \int_{\partial M} ds K \delta w$$

Thus even in the case of open strings there is only one constant ' a_1 ' because of which Weyl anomaly exists.

Since as $\partial M \rightarrow 0$ we need $a_1^{\text{open}} \rightarrow a_1^{\text{closed}}$ we will get even in this case,

$$\boxed{a_1 = -\frac{C}{12}}$$

No, any new anomaly because of boundary.

Thus, Weyl anomaly is only because of the physics in the interior of world sheet.