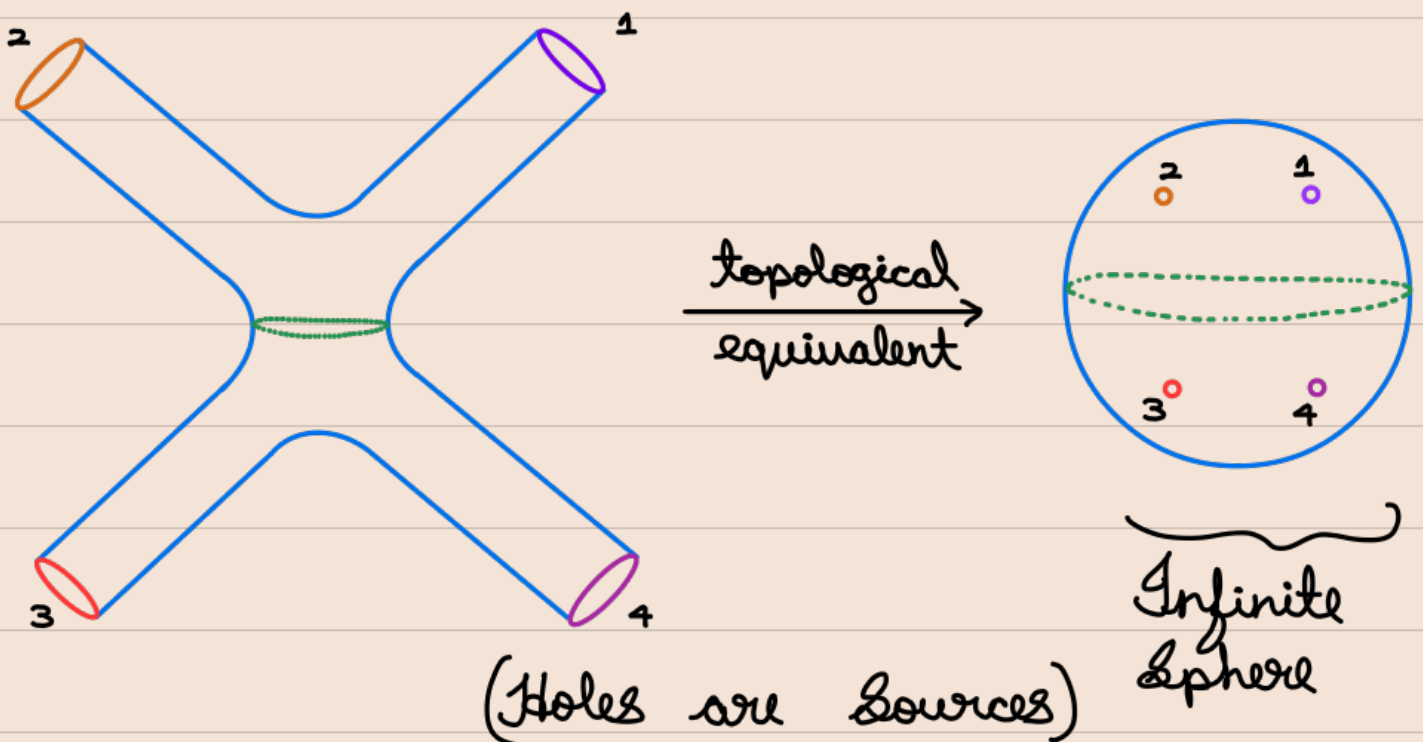


# String S-Matrix

When we want to deal with interactions between the strings we will compute the On-shell Scattering amplitudes called as the elements of String Scattering matrix.

Here the incoming & outgoing strings are taken to infinite time which have Eigen states with definite momenta and Particle representative state. The sources of strings are taken to infinity.

A Scattering amplitude for closed strings



Away from the Scattering process, the strings propagate freely as shown by the incoming & outgoing infinitely long legs.

Each of these legs can be placed in some eigenstate of free string & assigned some space-time momenta ' $p^\mu$ '.

Using state operator map, we know that each of these states at infinity is equivalent to insertion of an appropriate vertex operator on the world sheet. So each source becomes a local disturbance in world sheet.

Then to compute an S-matrix element we use a conformal transformation to bring each of these infinite legs to a finite distance.

Thus the end result is a world sheet with the topology of a sphere dotted with the vertex operators where the legs used to be.

Let,  $w = \sigma^1 + i\sigma^2$  &  $\bar{w} = \sigma^1 - i\sigma^2$  be the world sheet coordinates, where,  $\sigma^1 + 2\pi = \sigma^1$  is the periodic boundary conditions for the closed string. And the  $\sigma^2$  coordinate takes values  $-2\pi t < \sigma^2 < 0$ .

Here, 't' defines the world sheet time for the external source. So,  $t \rightarrow +\infty$  describes a scattering process where one end of the incoming cylinder is at infinite past.

Now, a Cylinder has a conformally equivalent description in terms of the  $z$ -coordinate.

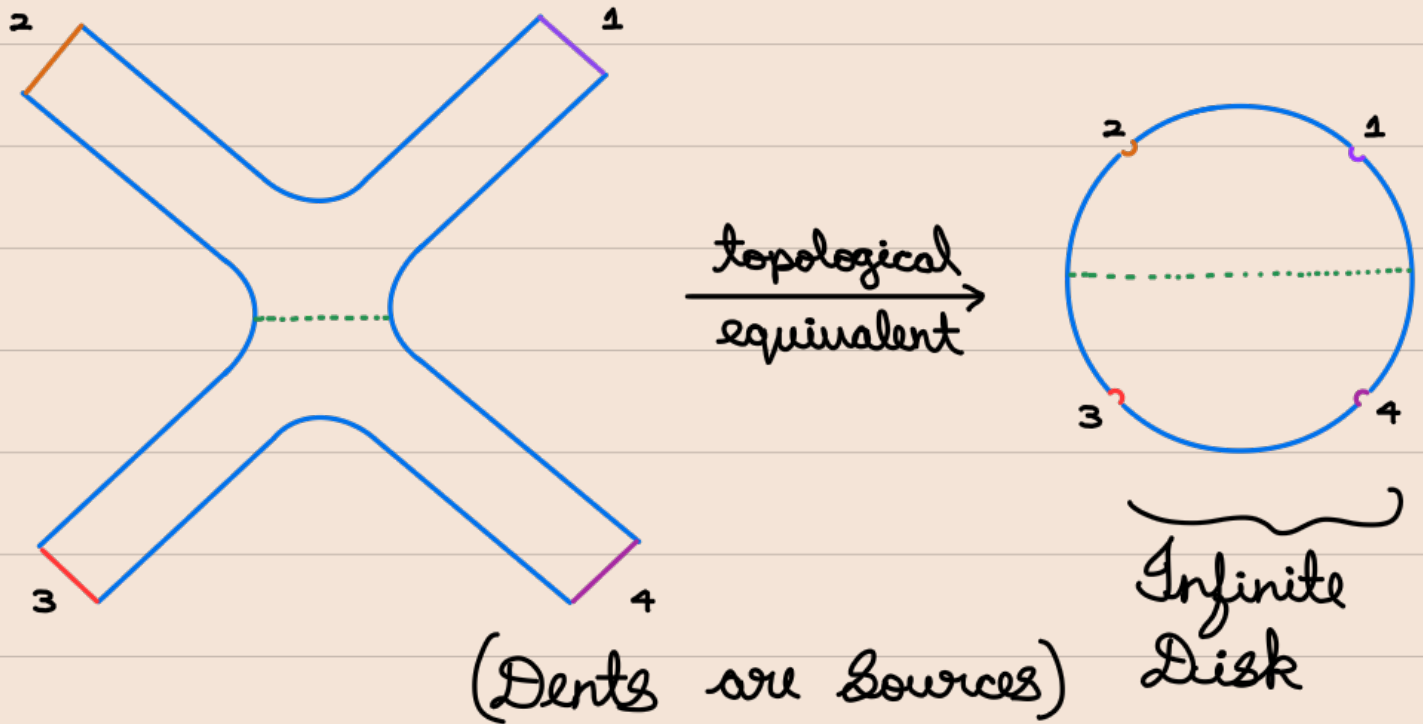
$$z = e^{-i w} \quad \& \quad \bar{z} = e^{-i \bar{w}}$$

$$\Rightarrow z = e^{\sigma^2} e^{-i\sigma^1} \quad \& \quad \bar{z} = e^{\sigma^2} e^{i\sigma^1}$$

Now,  $e^{-2\pi t} \leq |z| \leq 1$  define an annulus for finite 't' & a unit disk for  $t \rightarrow \infty$ .

The external source state of the string is a tiny circle & ultimately a point on the  $z$ -coordinate manifold.

# A Scattering amplitude for open strings



For open string processes we have long strips whose one end reach to infinity. Away from the scattering process, the strings propagate freely as shown by the incoming & outgoing infinitely long legs. Each of these legs can be placed in some eigenstate of free string & assigned some space-time momenta  $p^\mu$ .

The world sheet coordinates are  $W = \sigma^1 + i\sigma^2$  &  $\bar{W} = \sigma^1 - i\sigma^2$ , where  $0 < \sigma^1 < \pi$  &  $\sigma^1 = 0, \pi$  are the boundary. And the  $\sigma^2$  coordinate takes values  $-2\pi \leq \sigma^2 < 0$ .

Here, 't' defines the world sheet time for the external source. So,  $t \rightarrow +\infty$  describes a scattering process where one end of the incoming strip is at infinite past.

Now, a strip has a conformally equivalent description in terms of the  $z$ -coordinate.

$$z = -e^{-i\tau} \quad \& \quad \bar{z} = -e^{-i\bar{\tau}}$$

$$\Rightarrow z = e^{\sigma^2} e^{i(\pi - \sigma^1)} \quad \bar{z} = e^{\sigma^2} e^{-i(\pi - \sigma^1)}$$

The complex phase  $\theta = \pi - \sigma^1$  in the  $z$ -plane takes values b/w  $0 \& \pi$ . So, only the upper half of the  $z$ -plane is accessible. The line  $\sigma^1 = 0 \& \sigma^1 = \pi$  i.e. the  $\text{Re}(z)$  axis are the strip's boundary.

Now,  $e^{-2\pi t} < |z| < 1$  defines a semi-circular ring for finite 't' & a semi unit disk for  $t \rightarrow \infty$ . The external source state of the string is a tiny dent & ultimately a point on the  $z$ -coordinate manifold.

## Sum over Topologies

By state Operator map, each source becomes a local disturbance on the world sheet.

To describe an incoming/outgoing string in some internal state " $\Psi$ ", and a D-Spacetime momenta  $k^\mu$ , there corresponds a local vertex operator " $\hat{\mathcal{V}}_\Psi(k^\mu, \sigma)$ ". The incoming & outgoing states have D-momenta as,

$$k_{in}^\mu = (k_{in}^0, \vec{k}_{in}) = (E, \vec{k}) = k^\mu \hat{\mathcal{V}}_\Psi$$

$$\& \quad k_{out}^\mu = (k_{out}^0, \vec{k}_{out}) = (-E, -\vec{k}) = -k^\mu$$

Since the infinitely long tubes or strips are topologically equivalent to compact surfaces as shown in the diagram previously, where the states at infinite past & future become the point like insertions, we can focus only on compact connected topologies.

In these compact topologies, point like insertions in the interior represents external source for closed string. Similarly the point like insertions in the boundary represents external source for open string.

Then an  $n$ -particle  $S$ -matrix element is,

$$S_{\psi_1, \psi_2, \dots, \psi_n}(k_1^\mu, k_2^\mu, \dots, k_n^\mu) \\ = \sum_{\substack{\text{Compact} \\ \text{Connected} \\ \text{Topologies}}} \int \frac{[D\chi^\mu Dg_{ab}]}{\text{Vol}_{\text{diff}} \times \text{Weyl}} e^{-(S_\chi + \lambda \chi)} \prod_{j=1}^n \hat{V}_{\psi_j}(k_j^\mu)$$

$$\text{Where } \hat{V}_{\psi_j}(k_j^\mu) = \int_M d^2\sigma_j \sqrt{\hat{g}(\sigma_j)} \hat{\mathcal{V}}_{\psi_j}(k_j^\mu, \sigma_j)$$

' $\hat{\mathcal{V}}_{\psi_j}(k_j^\mu, \sigma_j)$ ' is the local vertex operator. It is integrated throughout the manifold to make the operator  $\hat{V}_{\psi_j}(k_j^\mu)$  diff invariant.

## Compact Connected 2D Topologies

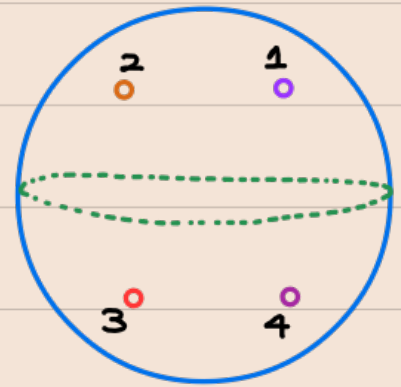
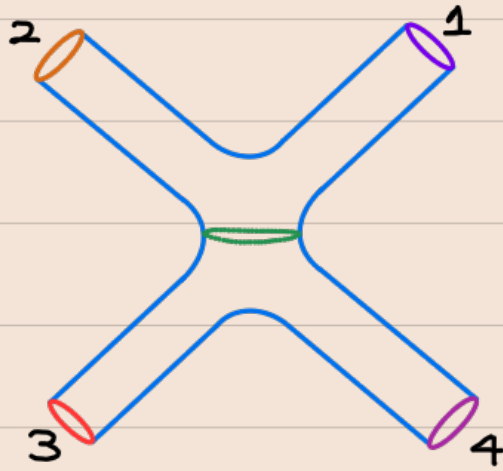
To obtain the connected S-matrix, then, we must sum over all compact, connected topologies for the world-sheet. In two dimensions, the classification of these topologies is well known. Any compact, connected, oriented two-dimensional surface without boundary is topologically equivalent to a sphere with  $g$  handles;  $g$  is known as the *genus* of the surface. In figure 3.10 we show  $g = 0, 1$ , and  $2$ . Boundaries can be added by cutting holes in a closed surface. Any compact, connected, oriented two-dimensional surface is topologically equivalent to a sphere with  $g$  handles and  $b$  holes. For example,  $(g, b) = (0, 1)$  is a disk,  $(0, 2)$  is an annulus, and  $(0, 3)$  is a pair of pants.

To describe unoriented surfaces, it is useful to introduce the cross-cap: cut a hole in the surface and identify diametrically opposite points. In more detail, take a complex coordinate  $z$ , cut out a disk of slightly less than unit radius, and glue the opposite points together by defining pairs of points with  $z' = -1/\bar{z}$  to be equivalent. The resulting surface is unoriented because the gluing is antiholomorphic. Also, unlike the case of a boundary, the cross-cap introduces no edge or other local feature: the edge introduced by the cutting is only the boundary of a coordinate patch. Any compact connected closed surface, oriented or unoriented, can be obtained by adding  $g$  handles and  $c$  cross-caps to the sphere; any compact connected surface can be obtained by adding  $g$  handles,  $b$  boundaries, and  $c$  cross-caps to the sphere. Actually, these descriptions are somewhat redundant: in either the case with boundary or the case without, one obtains every topology exactly once by restricting the sum to let only one of  $g$  and  $c$  be nonzero. For example,  $(g, b, c) = (0, 0, 1)$  is the projective plane,  $(g, b, c) = (0, 1, 1)$  is the Möbius strip,  $(g, b, c) = (0, 0, 2)$  is the Klein bottle, and a torus with cross-cap can be obtained either as  $(g, b, c) = (0, 0, 3)$  or as  $(g, b, c) = (1, 0, 1)$ , trading two cross-caps for a handle. The Euler number is given by

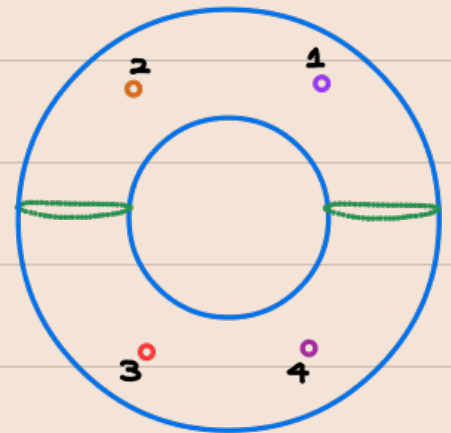
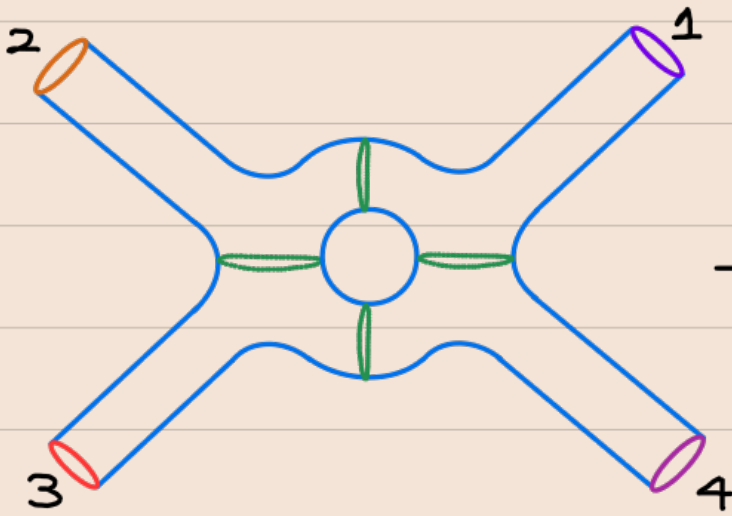
$$\chi = 2 - 2g - b - c. \quad (3.5.6)$$

$g \rightarrow$  no: of Genus,  $b \rightarrow$  no: of boundary (holes)  
 $c \rightarrow$  no: of cross caps

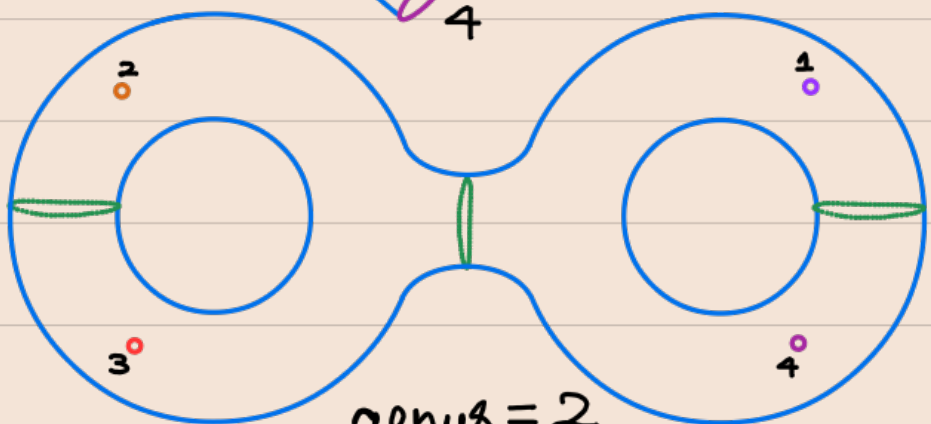
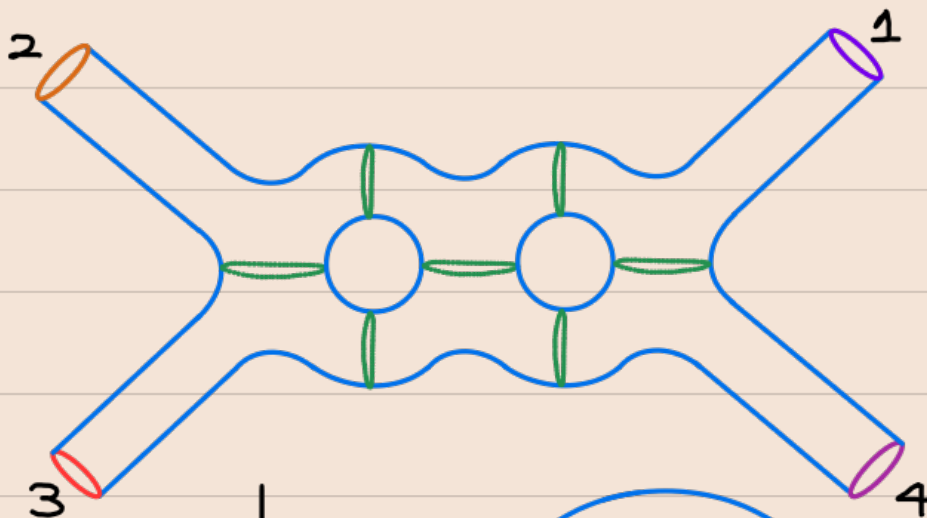
# Boundaryless Compact Topologies: Closed String ( $b=0$ )



genus = 0



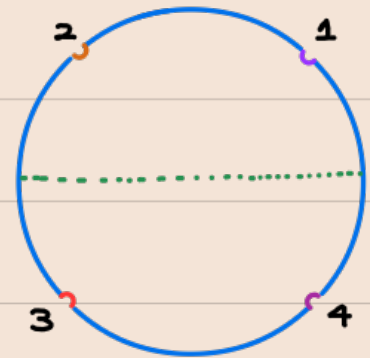
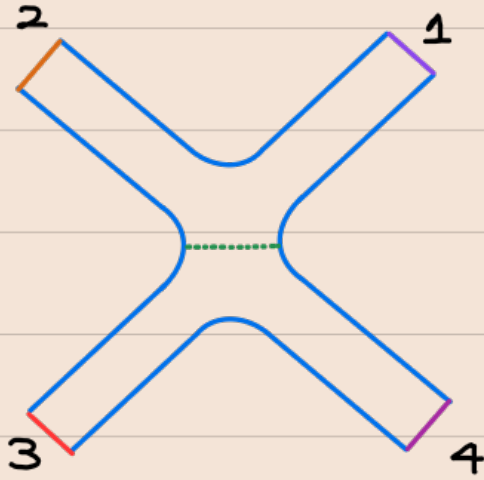
genus = 1



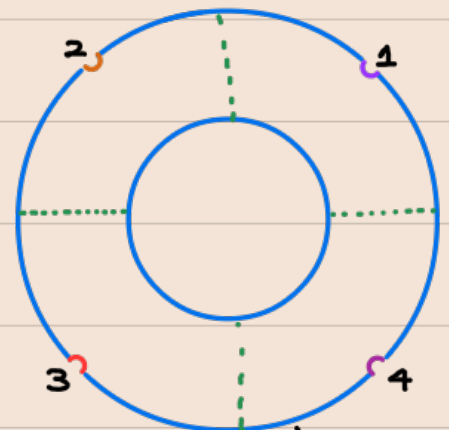
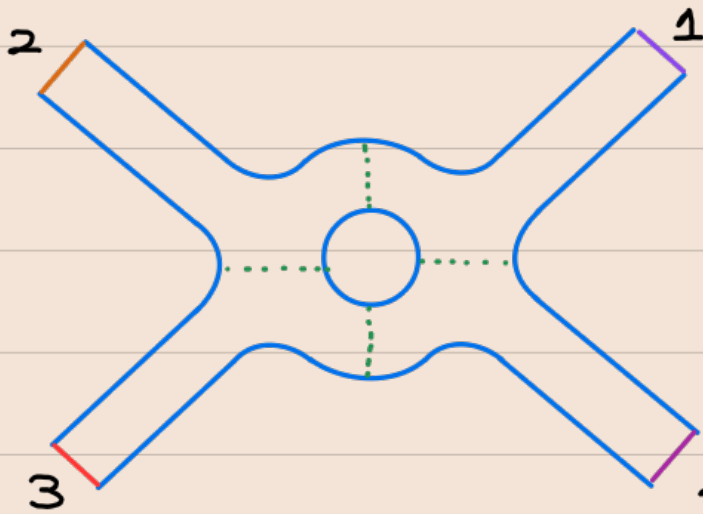
genus = 2

Note that we could have added many more vertex operators which represents more sources

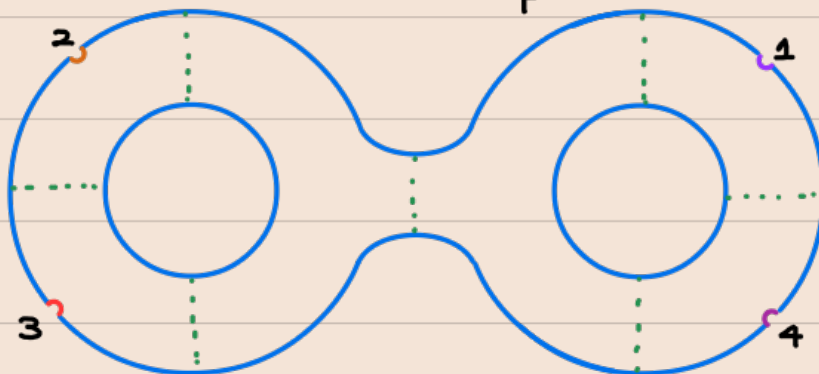
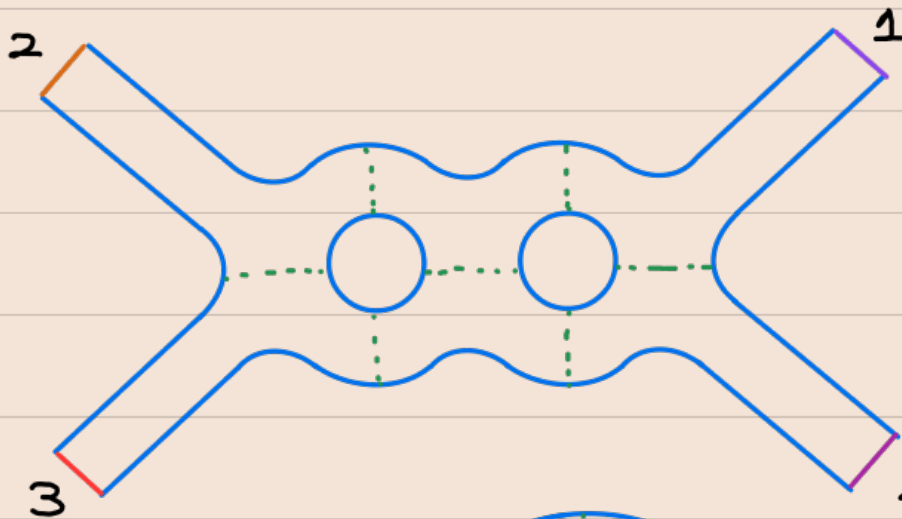
# Compact Topologies with Boundary: Open String ( $g=0, b \neq 0$ )



No: of boundaries = holes = 1



No: of boundaries = holes = 2

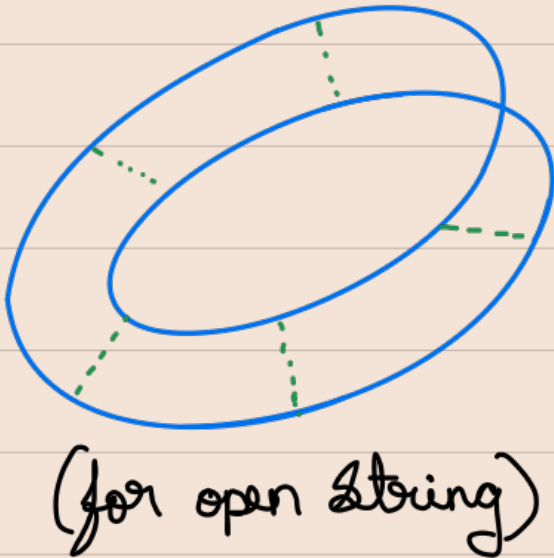


No: of boundaries = holes = 3

Note that we could have added many more vertex operators which represents more sources

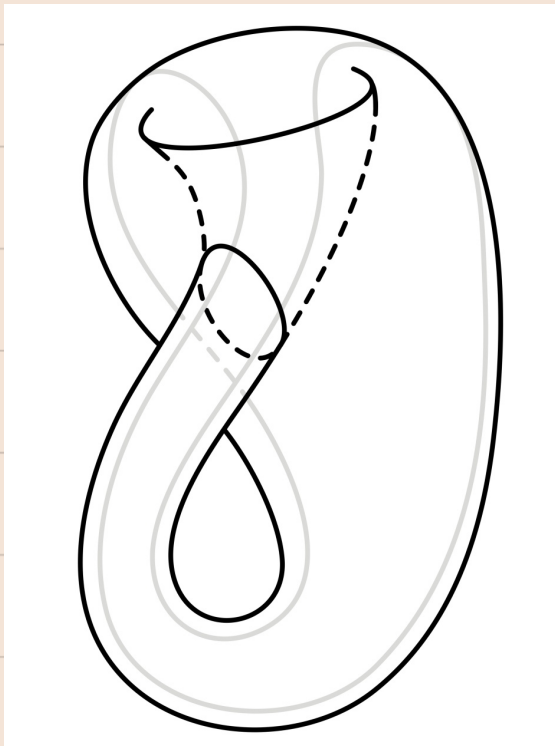
} Topology of a Pant

## Other Exotic Compact Topologies (2D)



→ Mobius Strip

Unoriented Surface  
with no Genus, one  
boundary & one  
Crosscap



→ Klein bottle

Unoriented Surface  
With no boundary  
no genus but  
two Cross Caps.  
(for closed string)

Let us, for convenience summarise the different compact connected 2D topologies, oriented or unoriented. This is done in table 3.3. In that table "o" stands for an oriented surface and "u" for an unoriented surface. The first line, a surface without boundaries, holes or cross-caps is just a sphere with  $g$  handles. The other surfaces are a sphere with extra holes and cross-caps.

| boundaries | orientation | handles | holes | cross-caps |
|------------|-------------|---------|-------|------------|
| —          | o           | $g$     | 0     | 0          |
| ✓          | o           | $g$     | $h$   | 0          |
| —          | u           | $g$     | 0     | $c$        |
| ✓          | u           | $g$     | $h$   | $c$        |

Table 3.1: 2D compact connected surfaces with  $g = \#$  handles,  $h = \#$  holes and  $c = \#$  cross-caps

# Why not Compute Correlation functions

## - David Tong

### 6. String Interactions

So far, despite considerable effort, we've only discussed the free string. We now wish to consider interactions. If we take the analogy with quantum field theory as our guide, then we might be led to think that interactions require us to add various non-linear terms to the action. However, this isn't the case. Any attempt to add extra non-linear terms for the string won't be consistent with our precious gauge symmetries. Instead, rather remarkably, all the information about interacting strings is already contained in the free theory described by the Polyakov action. (Actually, this statement is almost true).

To see that this is at least feasible, try to draw a cartoon picture of two strings interacting. It looks something like the worldsheet shown in the figure. The worldsheet is smooth. In Feynman diagrams in quantum field theory, information about interactions is inserted at vertices, where different lines meet. Here there are no such points. Locally, every part of the diagram looks like a free propagating string. Only globally do we see that the diagram describes interactions.

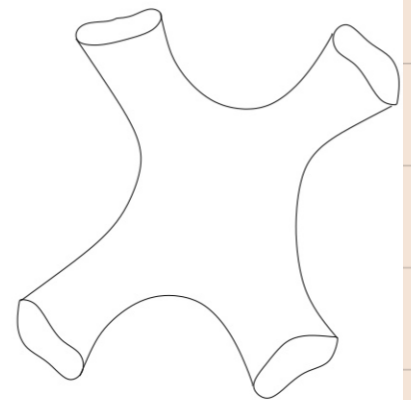


**Figure 31:**

#### 6.1 What to Compute?

If the information about string interactions is already contained in the Polyakov action, let's go ahead and compute something! But what should we compute? One obvious thing to try is the probability for a particular configuration of strings at an early time to evolve into a new configuration at some later time. For example, we could try to compute the amplitude associated to the diagram above, stipulating fixed curves for the string ends.

No one knows how to do this. Moreover, there are words that we can drape around this failure that suggests this isn't really a sensible thing to compute. I'll now try to explain these words. Let's start by returning to the familiar framework of quantum field theory in a fixed background. There the basic objects that we can compute are correlation functions,

$$\langle \phi(x_1) \dots \phi(x_n) \rangle \quad (6.1)$$

After a Fourier transform, these describe Feynman diagrams in which the external legs carry arbitrary momenta. For this reason, they are referred to as *off-shell*. To get the scattering amplitudes, we simply need to put the external legs on-shell (and perform a few other little tricks captured in the LSZ reduction formula).

The discussion above needs amendment if we turn on gravity. Gravity is a gauge theory and the gauge symmetries are diffeomorphisms. In a gauge theory, only gauge invariant observables make sense. But the correlation function (6.1) is not gauge invariant because its value changes under a diffeomorphism which maps the points  $x_i$  to another point. This emphasizes an important fact: there are no local off-shell gauge invariant observables in a theory of gravity.

There is another way to say this. We know, by causality, that space-like separated operators should commute in a quantum field theory. But in gravity the question of whether operators are space-like separated becomes a dynamical issue and the causal structure can fluctuate due to quantum effects. This provides another reason why we are unable to define local gauge invariant observables in any theory of quantum gravity.

Let's now return to string theory. Computing the evolution of string configurations for a finite time is analogous to computing off-shell correlation functions in QFT. But string theory is a theory of gravity so such things probably don't make sense. For this reason, we retreat from attempting to compute correlation functions, back to the S-matrix.

## Why no off shell-Amplitudes? - Polchinski

Observe that a naive attempt to define off-shell amplitudes, by taking  $k^\mu$  off the mass shell, is inconsistent with the local world-sheet symmetries. In position space this means that a local probe of the string world-sheet, given by inserting

$$\delta^D(X(\sigma) - x_0) = \int \frac{d^D k}{(2\pi)^D} \exp[ik \cdot (X(\sigma) - x_0)] \quad (3.6.12)$$

into the path integral, is inconsistent because it involves all momenta.

From several points of view this is not surprising. First, the argument given above that we could use pointlike sources (vertex operators) for string states required a limiting process that essentially restricts us to on-shell questions. Second, string theory contains gravity, and in general relativity simple off-shell amplitudes do not exist. This is because we have to specify the location of the probe, but the coordinates are unphysical and coordinate-invariant off-shell quantities are much more complicated. Third, in string theory we do not have the freedom to introduce additional fields to measure local observables (analogous to the way electroweak processes are used to probe strongly interacting systems): we must use the strings themselves, or other objects that we will discuss (D-branes and solitons) that are intrinsic to the theory.

One can define off-shell amplitudes, at least in perturbation theory, if one fixes a gauge (light-cone gauge being simplest for this) and uses stringy sources. The S-matrix formula (3.5.5) can then be derived in a way more akin to the reduction formula from quantum field theory, where it is possible to define finite-time transition amplitudes and then take the infinite-time limit. Incidentally, the reader who is trying to relate the present discussion to field theory should note that the path integral expression (3.5.5) is not analogous to a Green's function but will in fact have the properties of an S-matrix element. The analogous object in field theory is the Green's function with its external propagators amputated and its external momenta restricted to the mass shell.

The discussion of the local probe (3.6.12) also shows the problem with a contact interaction between strings. The simplest form for such an interaction would be

$$\int_M d^2\sigma g(\sigma)^{1/2} \int_M d^2\sigma' g(\sigma')^{1/2} \delta^D(X(\sigma) - X(\sigma')) , \quad (3.6.13)$$

which is nonvanishing whenever the world-sheet self-intersects. However, the delta function involves all momenta, as in eq. (3.6.12), so this is not Weyl-invariant. The structure of string theory is quite rigid: strings can couple to strings only in the way we have described at the beginning of this chapter.