

Closed String Vertex Operators

The operators that are inserted inside the Path integral for S-matrix element are of the form,

$$\widehat{V}_{\Psi_j}(k_j^\mu) = \int_M d^2\sigma_j \sqrt{g(\sigma_j)} \widehat{\mathcal{V}}_{\Psi_j}(k_j^\mu, \sigma_j)$$

' $\widehat{\mathcal{V}}_{\Psi_j}(k_j^\mu, \sigma_j)$ ' is the local vertex operator.

Now, we have integrated throughout the manifold to make the operator $\widehat{V}_{\Psi_j}(k_j^\mu)$ diff invariant. But we still have to consider the effect of Weyl transforms. So, overall we need $\widehat{V}_{\Psi_j}(k_j^\mu)$ to be diff $\times$ Weyl invariant. Effectively, it should have Conformal invariance.

For a locally flat metric in conformal gauge, in complex coordinates, it becomes

$$\widehat{V}_{\Psi_j}(k_j^\mu) = \int_M d^2\bar{z}_j \widehat{\mathcal{V}}_{\Psi_j}(k_j^\mu, z_j, \bar{z}_j)$$

Under conformal transform, $z_j \rightarrow f(z_j) \neq \bar{z}_j \rightarrow f^*(\bar{z}_j)$, we know that

$$d^2 z_j \rightarrow f(z_j) f^*(\bar{z}_j) d^2 \bar{z}_j$$

Thus, $\hat{\mathcal{V}}_{\Psi_j}(k_j^\mu, z_j, \bar{z}_j) \rightarrow [f(z_j)]^{-1} [f^*(\bar{z}_j)]^{-1} \hat{\mathcal{V}}_{\Psi_j}(k_j^\mu, z_j, \bar{z}_j)$ is needed. i.e. the local operator has to be a tensor operator of weight $(1, 1)$.

In curved world sheet the restriction is a little more involved, that will be seen later.

Vertex operator for closed String Tachyon in Flat world sheet

Consider the vertex operator given by

$$\hat{\mathcal{V}}_{\Psi_0}(k^\mu) = 2g_c \int_M d^2 \sigma : e^{i k^\mu X_\mu(\sigma)} :$$

where $g_c \rightarrow$ closed string coupling constant due to the presence of Tachyon source string.
 $\rightarrow$ Normalization constant $\sim$ coupling

If we go to flat world sheet & complex coordinates, then, $g_{zz} = g_{\bar{z}\bar{z}}$ & $g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}$

$$\Rightarrow \hat{V}_{\psi_0}(k^\mu) = g_c \int_M d^2z :e^{ik^\mu X_\mu(z, \bar{z})}:$$

Now, the total energy momentum tensor is a sum of T_x^{ab} & T_{ghost}^{ab} . In the flat world sheet, $T^a_a = 0$. Then, as seen in the 2nd chapter, we get the conformal weight of the local operator, $:e^{ik^\mu X_\mu(z, \bar{z})}: \sim (h, \tilde{h}) = \frac{\alpha' k^\mu k_\mu}{4}$

Thus for the insertion $\hat{V}_{\psi_0}(k^\mu)$ to be conformally invariant, we require

$$\frac{\alpha' k^\mu k_\mu}{4} = 1 \Rightarrow k^\mu k_\mu = \frac{4}{\alpha'}$$

But $k^\mu k_\mu = -m^2 = \frac{4}{\alpha'}$. This was exactly the -ve mass state that we got in the light cone quantization, that represented the Ground state of the String. This state represented the Tachyon in $D=26$ for which the total central charge is Zero.

Vertex operator for closed String Massless Particle in Flat world sheet

Here we shall look for the vertex operator that creates the 1st excited state.

Now, $:e^{ik^\mu X_\mu(z, \bar{z})}:$ represents the state with zero oscillator modes $|0; 0; k^\mu\rangle$ on the string as seen before.

In 1st chapter we saw that the modes of X^μ operator acted as ladder operators & gave repⁿ of 1st excited state. And in 2nd chapter we saw that by the state-operator map, the local operators $\partial X^\mu(z, \bar{z})$ represented the operator $\hat{\alpha}_{-1}^\mu$.

Also since no: of left & right moving modes has to be equal as seen in the 1st chapter we need both $\hat{\alpha}_{-1}^\mu$ & $\hat{\tilde{\alpha}}_{-1}^\nu$ to act on the state $|0; 0; k^\mu\rangle$.

$$\left. \begin{aligned} \hat{\alpha}_{-1}^{\mu} &\longrightarrow i \left(\frac{2}{\alpha'} \right)^{1/2} \partial \chi^{\mu} \\ \hat{\tilde{\alpha}}_{-1}^{\nu} &\longrightarrow i \left(\frac{2}{\alpha'} \right)^{1/2} \bar{\partial} \chi^{\nu} \end{aligned} \right\} \begin{array}{l} \text{State} \\ \text{operator} \\ \text{Map} \end{array}$$

So the state $^{\circ} \hat{\alpha}_{-1}^{\mu} \hat{\tilde{\alpha}}_{-1}^{\nu} |0; 0; k^{\mu}\rangle$ can be mapped to the local operator

$$^{\circ} \frac{2}{\alpha'} : \partial \chi^{\mu}(z, \bar{z}) \bar{\partial} \chi^{\nu}(z, \bar{z}) e^{ik^{\mu} \chi_{\mu}(z, \bar{z})} :$$

But we shall introduce a polarization tensor $^{\circ} f_{\mu\nu}$ that determines the type of state along with its polarization. In the 1st chapter we got that if:

$f_{\mu\nu}$ is Traceless & Symmetric $\rightarrow$ Graviton

$f_{\mu\nu}$ is anti-symmetric $\rightarrow B_{\mu\nu}$ field

$f_{\mu\nu} = \eta_{\mu\nu} \rightarrow$ Scalar Dilaton particle

So consider the insertion of vertex operator

$$\widehat{V}_{\Psi_1}(k^\mu) = \frac{2}{\alpha'} g_c \int_M d^2z f_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik^\mu X_\mu} : (z, \bar{z})$$

Now, the weight of such a local operator was found to be,

$$(h, \bar{h}) = \left(\frac{\alpha'}{4} (k_\mu k^\mu) + 1, \frac{\alpha'}{4} (k_\mu k^\mu) + 1 \right)$$

Thus for it to have weights $(1, 1)$ we require $k_\mu k^\mu = 0 \Rightarrow -m^2 = k^\mu k_\mu = 0$. So, 1st excited (oscillator) state gives rise to massless excitations. Also, for it to be a tensor operator we need

$$'k^\mu f_{\mu\nu} = k^\nu f_{\mu\nu} = 0'$$

This represents the transverse polarization condition for the massless excitations.

Closed String Vertex Operators for Arbitrary World Sheet

Now, any Operator that is included in the Polyakov Path integral should always respect the local diff $\times$ Weyl Symmetry.

Let us define a renormalized operator $[\mathcal{F}]_x$

$$[\mathcal{F}]_x = \text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right] \mathcal{F}$$

Here $\mathcal{F}$ is some local operator that depends on χ^μ fields. If $L(\sigma, \sigma')$ is the geodesic distance b/w two points $\sigma \neq \sigma'$, then,

$$\Delta(\sigma, \sigma') = \frac{\alpha'}{2} \ln[L^2(\sigma, \sigma')]$$

For flat metric, $L(\sigma, \sigma') = |\sigma - \sigma'|$, this renormalization reduces to Conformal Normal ordering introduced in 2nd chapter.

This expression of $[\mathcal{F}]_x$ gives the Wick's theorem which instructs us to sum over all ways of contracting pairs of χ^μ in $\mathcal{F}$ & replace the pairs by ' $\Delta(\sigma, \sigma')$ '.

In this definition of $[\mathcal{F}]_x$, there is diff. Symmetry. We can prove this as follows.

Now, under diffeomorphism $\tilde{\sigma} \longrightarrow \tilde{\sigma}(\sigma)$, where $d^2\sigma = J(\tilde{\sigma}) d^2\tilde{\sigma}$ & $J(\tilde{\sigma}) = \det\left(\frac{\partial\sigma}{\partial\tilde{\sigma}}\right)$ is the Jacobian of transformation. We know that,

$$\partial\mathcal{F} = \int_M d^2\sigma \frac{\partial\mathcal{F}}{\partial\chi^\mu(\sigma)} \partial\chi^\mu(\sigma)$$

Put $d^2\sigma = J(\tilde{\sigma}) d^2\tilde{\sigma}$, $\partial\chi^\mu(\sigma) = \partial\tilde{\chi}^\mu(\tilde{\sigma})$

$$\Rightarrow \partial\mathcal{F} = \int_M d^2\tilde{\sigma} J(\tilde{\sigma}) \frac{\partial\mathcal{F}}{\partial\chi^\mu(\sigma)} \partial\tilde{\chi}^\mu(\tilde{\sigma})$$

But $\partial\mathcal{F} = \int_M d^2\tilde{\sigma} \frac{\partial\mathcal{F}}{\partial\tilde{\chi}^\mu(\tilde{\sigma})} \partial\tilde{\chi}^\mu(\tilde{\sigma}) \left. \vphantom{\int_M d^2\tilde{\sigma} \frac{\partial\mathcal{F}}{\partial\tilde{\chi}^\mu(\tilde{\sigma})} \partial\tilde{\chi}^\mu(\tilde{\sigma})} \right\} \text{change of dummy variable}$

Also, $\mathcal{F}$ is diff invariant as it only depends on χ^μ s. Thus,

$$\int_M d^2\sigma \underbrace{\frac{\partial \mathcal{F}}{\partial \chi^\mu(\sigma)} \partial \chi^\mu(\sigma)} = \int_M d^2\tilde{\sigma} \mathcal{T}(\tilde{\sigma}) \frac{\partial \mathcal{F}}{\partial \chi^\mu(\sigma)} \partial \tilde{\chi}^\mu(\tilde{\sigma})$$

$$[\text{replace } \sigma \rightarrow \tilde{\sigma} \ \& \ \partial \chi^\mu(\sigma) = \partial \tilde{\chi}^\mu(\tilde{\sigma})]$$

$$\int_M d^2\tilde{\sigma} \frac{\partial \mathcal{F}}{\partial \tilde{\chi}^\mu(\tilde{\sigma})} \partial \tilde{\chi}^\mu(\tilde{\sigma}) = \int_M d^2\tilde{\sigma} \partial \tilde{\chi}^\mu(\tilde{\sigma}) \frac{\partial (\mathcal{T}(\tilde{\sigma}) \mathcal{F})}{\partial \chi^\mu(\sigma)}$$

$$\rightarrow \frac{\partial \mathcal{F}}{\partial \tilde{\chi}^\mu(\tilde{\sigma})} = \frac{\partial (\mathcal{T}(\tilde{\sigma}) \mathcal{F})}{\partial \chi^\mu(\sigma)} = \mathcal{T}(\tilde{\sigma}) \frac{\partial \mathcal{F}}{\partial \chi^\mu(\sigma)}$$

$$\rightarrow \frac{\partial \mathcal{F}}{\partial \tilde{\chi}^\mu(\tilde{\sigma})} = \mathcal{T}(\tilde{\sigma}) \frac{\partial \mathcal{F}}{\partial \chi^\mu(\sigma)}$$

i.e. under diffeomorphism, as a functional differentiation operator relation,

$$\frac{\partial}{\partial \chi^\mu(\sigma)} \longrightarrow \frac{\partial}{\partial \tilde{\chi}^\mu(\tilde{\sigma})} = \mathcal{T}(\tilde{\sigma}) \frac{\partial}{\partial \chi^\mu(\sigma)}$$

Similarly,

$$\frac{\partial}{\partial \chi^\mu(\sigma)} \longrightarrow \frac{\partial}{\partial \tilde{\chi}^\mu(\tilde{\sigma})} = \mathcal{T}(\tilde{\sigma}) \frac{\partial}{\partial \chi^\mu(\sigma)}$$

The Geodesic distance is a geometric quantity b/w two points, so it is independent of reparameterizations. It is a scalar quantity. Thus $\tilde{\Delta}(\tilde{\sigma}, \tilde{\sigma}') = \Delta(\sigma, \sigma')$

$$\text{Then, } d^2\tilde{\sigma} = \frac{d^2\sigma}{J(\tilde{\sigma})} \quad \& \quad d^2\tilde{\sigma}' = \frac{d^2\sigma'}{J(\tilde{\sigma}')}$$

So now, the operator,

$$\text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right]$$

under diffeomorphism changes as,

$$\text{Exp} \left[\frac{1}{2} \int_M d^2\tilde{\sigma} d^2\tilde{\sigma}' \tilde{\Delta}(\tilde{\sigma}, \tilde{\sigma}') \frac{\partial}{\partial \tilde{\chi}^\mu(\tilde{\sigma})} \frac{\partial}{\partial \tilde{\chi}_\mu(\tilde{\sigma}')} \right]$$

$$= \text{Exp} \left[\frac{1}{2} \int_M \frac{d^2\sigma}{J(\tilde{\sigma})} \frac{d^2\sigma'}{J(\tilde{\sigma}')} \Delta(\sigma, \sigma') J(\tilde{\sigma}) J(\tilde{\sigma}') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right]$$

$$= \text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right]$$

Thus $[\mathcal{F}]_x$ is diff. invariant.

Now, let us check the Weyl invariance of $[\mathcal{F}]_8$ under infinitesimal change

$$\begin{aligned}
 \delta_W [\mathcal{F}]_8 &= \delta_W \left(\underbrace{\text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right]}_{\text{Leibnitz product rule}} \mathcal{F} \right) \\
 &= \left(\text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right] \right) \delta_W \mathcal{F} \\
 &\quad + \underbrace{\delta_W}_{\text{chain rule}} \left(\text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right] \right) \mathcal{F} \\
 &= [\delta_W \mathcal{F}]_8 + \\
 &\quad + \delta_W \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right] \\
 &\quad \hookrightarrow \left(\text{Exp} \left[\frac{1}{2} \int_M d^2\sigma d^2\sigma' \Delta(\sigma, \sigma') \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} \right] \mathcal{F} \right) \\
 &= [\delta_W \mathcal{F}]_8 + \frac{1}{2} \int_M d^2\sigma d^2\sigma' \delta_W [\Delta(\sigma, \sigma')] \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} [\mathcal{F}]_8
 \end{aligned}$$

Geodesic distance is metric dependent. So, when the metric changes, even ' $\Delta(\sigma, \sigma')$ ' changes.

Thus, we can write $\delta_w[\mathcal{F}]_x$ as

$$[\delta_w \mathcal{F}]_x + \frac{1}{2} \int_M d^2\sigma d^2\sigma' \delta_w[\Delta(\sigma, \sigma')] \frac{\partial}{\partial \chi^\mu(\sigma)} \frac{\partial}{\partial \chi_\mu(\sigma')} [\mathcal{F}]_x$$

Now we can check the Weyl variation of different Vertex operators.

Vertex operator for closed String Tachyon in arbitrary world sheet

It is given by,

$$\widehat{V}_{\psi_0}(k^\mu) = 2g_c \int_M d^2\sigma \sqrt{g(\sigma)} [e^{ik^\mu \chi_\mu(\sigma)}]_x$$

Then, its Weyl variation is,

$$\delta_w \widehat{V}_{\psi_0}(k^\mu) = 2g_c \int_M d^2\sigma \delta_w(\sqrt{g(\sigma)} [e^{ik^\mu \chi_\mu(\sigma)}]_x)$$

Under Weyl transformation, $g_{ab} \rightarrow e^{2\omega} g_{ab}$

$$\sqrt{g} \rightarrow e^{2\omega} \sqrt{g} \Rightarrow [\partial_w \sqrt{g(\sigma)}] = 2\partial_w \sqrt{g(\sigma)}$$

$$\begin{aligned} \rightarrow \partial_w \hat{V}_{\psi_0}(k^\mu) &= 2g_c \int_M d^2\sigma (\partial_w \sqrt{g(\sigma)}) [e^{ik^\mu \chi_\mu(\sigma)}]_x \\ &\quad + 2g_c \int_M d^2\sigma \sqrt{g(\sigma)} (\partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_x) \end{aligned}$$

$$= 2g_c \int_M d^2\sigma \sqrt{g(\sigma)} \left[(2\partial_w) [e^{ik^\mu \chi_\mu(\sigma)}]_x + (\partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_x) \right]$$

Now,

$$\partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_x = [\partial_w e^{ik^\mu \chi_\mu(\sigma)}]_x$$

$$+ \frac{1}{2} \int_M d^2\sigma' d^2\sigma'' \partial_w [\Delta(\sigma', \sigma'')] \frac{\partial}{\partial \chi^\mu(\sigma')} \frac{\partial}{\partial \chi_\mu(\sigma'')} [e^{ik^\mu \chi_\mu(\sigma)}]_x$$

We know that $\partial_w e^{ik^\mu \chi_\mu(\sigma)} = 0$

$\Rightarrow \partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_x$ can be written as,

$$= \frac{1}{2} \int_M d^2\sigma' d^2\sigma'' \partial_w [\Delta(\sigma', \sigma'')] \frac{\partial}{\partial \chi^\mu(\sigma')} \frac{\partial}{\partial \chi_\mu(\sigma'')} [e^{ik^\mu \chi_\mu(\sigma)}]_x$$

Now, $\frac{\partial}{\partial \chi^\mu(\sigma')} \frac{\partial}{\partial \chi_\mu(\sigma'')} [e^{ik^\nu \chi_\nu(\sigma)}]_x$

$$= \left[\frac{\partial}{\partial \chi^\mu(\sigma')} \frac{\partial}{\partial \chi_\mu(\sigma'')} e^{ik^\nu \chi_\nu(\sigma)} \right]_x$$

$$= \left[\frac{\partial}{\partial \chi^\mu(\sigma')} \frac{\partial}{\partial \chi_\mu(\sigma'')} e^{ik^\nu \chi_\nu(\sigma)} \right]_x$$

$$= \left[\frac{\partial}{\partial \chi^\mu(\sigma')} e^{ik^\nu \chi_\nu(\sigma)} i k^\nu \frac{\partial \chi_\nu(\sigma)}{\partial \chi_\mu(\sigma'')} \right]_x$$

$$= \left[e^{ik^\nu \chi_\nu(\sigma)} i k_\beta \frac{\partial \chi^\beta(\sigma)}{\partial \chi^\mu(\sigma')} i k^\mu \delta(\sigma - \sigma'') \right]_x$$

$$= \left[e^{ik^\nu \chi_\nu(\sigma)} i k_\mu \delta(\sigma - \sigma') i k^\mu \delta(\sigma - \sigma'') \right]_x$$

$\Rightarrow \frac{\partial}{\partial \chi^\mu(\sigma')} \frac{\partial}{\partial \chi_\mu(\sigma'')} [e^{ik^\nu \chi_\nu(\sigma)}]_x$ is given by

$$= -k_\mu k^\mu \delta(\sigma - \sigma') \delta(\sigma - \sigma'') [e^{ik^\nu \chi_\nu(\sigma)}]_x$$

$\rightarrow \partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma$ can be written as,

$$= \frac{-k_\mu k^\mu}{2} \int_M d^2\sigma' d^2\sigma'' \partial_w [\Delta(\sigma', \sigma'')] \partial(\sigma - \sigma') \partial(\sigma - \sigma'') \times [e^{ik^\nu \chi_\nu(\sigma)}]_\gamma$$

$$\rightarrow \partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma = \frac{-k_\mu k^\mu}{2} \partial_w [\Delta(\sigma, \sigma)] [e^{ik^\nu \chi_\nu(\sigma)}]_\gamma$$

Now, let us find $\partial_w [\Delta(\sigma, \sigma)]$. We know that, $\Delta(\sigma, \sigma') = \frac{\alpha'}{2} \ln[L^2(\sigma, \sigma')]$

Also, we need its weyl variation as $\sigma \rightarrow \sigma'$.

For small distances b/w σ & σ' , in the conformal gauge $g_{ab} = e^{2\omega} \delta_{ab}$, it can be shown that $L(\sigma, \sigma') \approx |\sigma - \sigma'| e^{\omega(\sigma)}$

$$\text{So, } L^2(\sigma, \sigma') \approx (\sigma - \sigma')^2 e^{2\omega(\sigma)}$$

$$\rightarrow \Delta(\sigma, \sigma') \approx \frac{\alpha'}{2} \ln[(\sigma - \sigma')^2 e^{2\omega(\sigma)}]$$

$$\rightarrow \Delta(\sigma, \sigma') \approx \alpha' \omega(\sigma) + \frac{\alpha'}{2} \ln[(\sigma - \sigma')^2]$$

So, Under Weyl transformation, $g_{ab} \rightarrow e^{2\omega} g_{ab}$

for any gauge, $\partial_w[\Delta(\sigma', \sigma'')] \approx \alpha' \partial \omega(\sigma')$

So for $\sigma'' \rightarrow \sigma' = \sigma$, $\partial_w[\Delta(\sigma, \sigma)] = \alpha' \partial \omega(\sigma)$

$$\rightarrow \partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma = \frac{-k_\mu k^\mu}{2} \partial_w [\Delta(\sigma, \sigma)] [e^{ik^\nu \chi_\nu(\sigma)}]_\gamma$$

$$\rightarrow \partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma = -\frac{\alpha'}{2} k_\mu k^\mu \partial \omega(\sigma) [e^{ik^\nu \chi_\nu(\sigma)}]_\gamma$$

$\rightarrow \partial_w \hat{V}_{\psi_0}(k^\mu)$ is given by

$$= 2g_c \int_M d^2\sigma \sqrt{g(\sigma)} \left[(2\partial\omega) [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma + (\partial_w [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma) \right]$$

$$= 2g_c \int_M d^2\sigma \sqrt{g(\sigma)} \left[(2\partial\omega) [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma - \frac{\alpha'}{2} k_\mu k^\mu \partial \omega [e^{ik^\nu \chi_\nu(\sigma)}]_\gamma \right]$$

$$= 2g_c \int_M d^2\sigma \sqrt{g(\sigma)} \left[1 - \frac{\alpha' k_\mu k^\mu}{4} \right] (2\partial\omega(\sigma)) [e^{ik^\mu \chi_\mu(\sigma)}]_\gamma$$

Thus, finally for the Tachyon vertex operator,

$$\hat{V}_{\psi_0}(k^\mu), \text{ its weyl variation is } \delta_\omega \hat{V}_{\psi_0}(k^\mu) \\ = 4g_c \int_M d^2\sigma \sqrt{g(\sigma)} \left[1 - \frac{\alpha' k_\mu k^\mu}{4} \right] \delta\omega(\sigma) \left[e^{ik^\mu X_\mu(\sigma)} \right]_x$$

Thus to have a non-zero weyl variation, it should satisfy

$$\left[1 - \frac{\alpha' k_\mu k^\mu}{4} \right] = 0 \\ \Rightarrow k^\mu k_\mu = \frac{4}{\alpha'}$$

But $k^\mu k_\mu = -m^2 = \frac{4}{\alpha'}$. This was exactly the -ve mass state that we got in the light cone quantization, that represented the Ground state of the String. This state represented the Tachyon in $D=26$ for which the total central charge is Zero.

Massless closed String Vertex operator in arbitrary world sheet

Here we shall look for the vertex operator that creates the 1st excited state in the arbitrary string world sheet.

Now, $[e^{ik^\mu X_\mu(\sigma)}]_x$ represents the state with zero oscillator modes $|0; 0; k^\mu\rangle$ on the string as seen before (Tachyon Vertex).

In 1st chapter we saw that the modes of X^μ operator acted as ladder operators & gave repⁿ of 1st excited state.

And in 2nd chapter for a flat world sheet we saw that by the state-operator map, the local operators ' $\partial_a X^\mu$'s represented the operator ' $\hat{\alpha}_{-1}^\mu$ '. (oscillation mode creation)

But including just one derivative can not give rise to diff-invariant vertex operator. So, our next option is to include two derivatives of X^μ inside the vertex operator. So, it should be of the form,

$$e^{\frac{i}{\alpha'} \int_M d^2\sigma \sqrt{g} A_{\mu\nu}^{ab} [(\partial_a X^\mu)(\partial_b X^\nu) e^{ik^\rho X_\rho}]}$$

where $A_{\mu\nu}^{ab}$ is some structure that can contract with $(\partial_a X^\mu)(\partial_b X^\nu)$ & render it as an invariant under diffeomorphism. Those objects that can contract are ' g^{ab} ' (metric) & the anti-symmetric tensor ' ϵ^{ab} ' given by

$$\epsilon^{ab} = \frac{1}{\sqrt{g}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{g}} \underbrace{\epsilon^{ab}}_{\text{Levi-Civita Symbol}}$$

Consider $\epsilon^{ab} a_{\mu\nu} (\partial_a X^\mu)(\partial_b X^\nu)$ where

$a_{\mu\nu} = a_{\nu\mu}$ is some constant anti-symmetric matrix in μ - ν indices.

$$\epsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu) = \frac{1}{\sqrt{g}} \epsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu)$$

$$d^2\sigma \sqrt{g} \epsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu) = d^2\sigma \epsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu)$$

Under diffeomorphism, $d^2\sigma = J(\sigma') d^2\sigma'$

$$\epsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu) = \epsilon^{ab} \frac{\partial \sigma'^c}{\partial \sigma^a} \frac{\partial \sigma'^d}{\partial \sigma^b} \underbrace{a_{\mu\nu} (\partial_c X^\mu) (\partial_d X^\nu)}_{\text{Non-zero only when } c \neq d}$$

$$= \epsilon^{ab} \frac{\partial \sigma'^1}{\partial \sigma^a} \frac{\partial \sigma'^2}{\partial \sigma^b} a_{\mu\nu} (\partial_1 X^\mu) (\partial_2 X^\nu) + \epsilon^{ab} \frac{\partial \sigma'^2}{\partial \sigma^a} \frac{\partial \sigma'^1}{\partial \sigma^b} a_{\mu\nu} (\partial_2 X^\mu) (\partial_1 X^\nu)$$

$\underbrace{\hspace{10em}}_{\mu \leftrightarrow \nu}$

$$= \epsilon^{ab} \frac{\partial \sigma'^1}{\partial \sigma^a} \frac{\partial \sigma'^2}{\partial \sigma^b} a_{\mu\nu} (\partial_1 X^\mu) (\partial_2 X^\nu) + \epsilon^{ab} \frac{\partial \sigma'^2}{\partial \sigma^a} \frac{\partial \sigma'^1}{\partial \sigma^b} a_{\nu\mu} (\partial_2 X^\nu) (\partial_1 X^\mu)$$

$$= \epsilon^{ab} \frac{\partial \sigma'^1}{\partial \sigma^a} \frac{\partial \sigma'^2}{\partial \sigma^b} a_{\mu\nu} (\partial_1 X^\mu) (\partial_2 X^\nu) \quad (a_{\nu\mu} = -a_{\mu\nu})$$

$$- \epsilon^{ab} \frac{\partial \sigma'^2}{\partial \sigma^a} \frac{\partial \sigma'^1}{\partial \sigma^b} a_{\mu\nu} (\partial_2 X^\nu) (\partial_1 X^\mu)$$

$$= \epsilon^{ab} \left(\frac{\partial \sigma'^1}{\partial \sigma^a} \frac{\partial \sigma'^2}{\partial \sigma^b} - \frac{\partial \sigma'^2}{\partial \sigma^a} \frac{\partial \sigma'^1}{\partial \sigma^b} \right) a_{\mu\nu} (\partial_1 X^\mu) (\partial_2 X^\nu)$$

Now,

$$a_{\mu\nu}(\partial_1 X^\mu)(\partial_2 X^\nu) = \frac{a_{\mu\nu}}{2} \underbrace{\left[(\partial_1 X^\mu)(\partial_2 X^\nu) - (\partial_1 X^\nu)(\partial_2 X^\mu) \right]}$$

only anti-symmetric Part survives

$$\rightarrow a_{\mu\nu}(\partial_1 X^\mu)(\partial_2 X^\nu) = \frac{a_{\mu\nu}}{2} \left[(\partial_1 X^\mu)(\partial_2 X^\nu) - (\partial_2 X^\mu)(\partial_1 X^\nu) \right]$$

$$\rightarrow a_{\mu\nu}(\partial_1 X^\mu)(\partial_2 X^\nu) = \frac{\varepsilon^{cd}}{2} a_{\mu\nu} (\partial_c X^\mu)(\partial_d X^\nu)$$

$$\text{Then, } \varepsilon^{ab} \left(\frac{\partial \sigma'^1}{\partial \sigma^a} \frac{\partial \sigma'^2}{\partial \sigma^b} - \frac{\partial \sigma'^2}{\partial \sigma^a} \frac{\partial \sigma'^1}{\partial \sigma^b} \right)$$

$$= \left(\frac{\partial \sigma'^1}{\partial \sigma^1} \frac{\partial \sigma'^2}{\partial \sigma^2} - \frac{\partial \sigma'^2}{\partial \sigma^2} \frac{\partial \sigma'^1}{\partial \sigma^1} \right) - \left(\frac{\partial \sigma'^1}{\partial \sigma^2} \frac{\partial \sigma'^2}{\partial \sigma^1} - \frac{\partial \sigma'^2}{\partial \sigma^1} \frac{\partial \sigma'^1}{\partial \sigma^2} \right)$$

$$= 2 \left(\frac{\partial \sigma'^1}{\partial \sigma^1} \frac{\partial \sigma'^2}{\partial \sigma^2} - \frac{\partial \sigma'^2}{\partial \sigma^2} \frac{\partial \sigma'^1}{\partial \sigma^1} \right)$$

$$= 2 \underbrace{\left[J(\sigma') \right]^{-1}}_{\text{Inverse Jacobian}}$$

Thus,

$$\varepsilon^{ab} \left(\frac{\partial \sigma'^1}{\partial \sigma^a} \frac{\partial \sigma'^2}{\partial \sigma^b} - \frac{\partial \sigma'^2}{\partial \sigma^a} \frac{\partial \sigma'^1}{\partial \sigma^b} \right) a_{\mu\nu} (\partial_1 X^\mu) (\partial_2 X^\nu)$$

$$= 2 [\mathcal{J}(\sigma')]^{-1} \frac{\varepsilon^{cd}}{2} a_{\mu\nu} (\partial_c X^\mu) (\partial_d X^\nu)$$

$$= [\mathcal{J}(\sigma')]^{-1} \varepsilon^{cd} a_{\mu\nu} (\partial_c X^\mu) (\partial_d X^\nu)$$

So, we get, under diffeomorphism

$$d^2\sigma = \mathcal{J}(\sigma') d^2\sigma'$$

$$\varepsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu) = [\mathcal{J}(\sigma')]^{-1} \varepsilon^{cd} a_{\mu\nu} (\partial_c X^\mu) (\partial_d X^\nu)$$

$\rightarrow d^2\sigma \varepsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu)$ is invariant under diffeomorphism.

Or " $d^2\sigma \sqrt{g} \varepsilon^{ab} a_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu)$ " is invariant under diffeomorphism.

Also " $g^{ab} S_{\mu\nu} (\partial_a X^\mu) (\partial_b X^\nu)$ " where $S_{\mu\nu}$ is some traceless symmetric matrix is a diff invariant combination.

And lastly, we can also include the below particular term " $\frac{g_c}{M} \int d^2\sigma \sqrt{g} \phi R [e^{ik^\rho X_\rho}]_\gamma$ "

in addition to the previous term.

Thus, we can have the vertex operator for 1st excited state to be,

$$\widehat{V}_{\Psi_1}(k^\mu) = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} \left(A_{\mu\nu}^{ab} [(\partial_a X^\mu) (\partial_b X^\nu) e^{ik^\rho X_\rho}]_\gamma + \alpha' \phi R [e^{ik^\rho X_\rho}]_\gamma \right)$$

where $A_{\mu\nu}^{ab} = g^{ab} S_{\mu\nu} + i \epsilon^{ab} a_{\mu\nu}$

This is because of the Euclidean Continuation of time ' σ ' derivatives.

To extend the analysis of Weyl invariance we need to work out the Weyl dependence (3.6.11) of the geodesic distance to higher order. One finds

$$\partial_a \delta_W \Delta(\sigma, \sigma') \Big|_{\sigma'=\sigma} = \frac{1}{2} \alpha' \partial_a \delta \omega(\sigma) , \quad (3.6.15a)$$

$$\partial_a \partial'_b \delta_W \Delta(\sigma, \sigma') \Big|_{\sigma'=\sigma} = \frac{1+\gamma}{2} \alpha' \nabla_a \partial_b \delta \omega(\sigma) , \quad (3.6.15b)$$

$$\nabla_a \partial_b \delta_W \Delta(\sigma, \sigma') \Big|_{\sigma'=\sigma} = -\frac{\gamma}{2} \alpha' \nabla_a \partial_b \delta \omega(\sigma) . \quad (3.6.15c)$$

Here, $\gamma = -\frac{2}{3}$, but we have left it as a parameter for later reference; the third equation can be obtained from the second and the gradient of the first. Using eq. (3.3.5) for the variation of the curvature and eqs. (3.6.7) and (3.6.15) for the variation of the renormalization gives

$$\delta_W V_1 = \frac{g_c}{2} \int d^2 \sigma g^{1/2} \delta \omega \left\{ (g^{ab} S_{\mu\nu} + i\epsilon^{ab} A_{\mu\nu}) [\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X}]_r \right. \\ \left. + \alpha' F R [e^{ik \cdot X}]_r \right\} , \quad (3.6.16)$$

where

$$S_{\mu\nu} = -k^2 s_{\mu\nu} + k_\nu k^\omega s_{\mu\omega} + k_\mu k^\omega s_{\nu\omega} - (1+\gamma) k_\mu k_\nu s^\omega_\omega + 4k_\mu k_\nu \phi , \quad (3.6.17a)$$

$$A_{\mu\nu} = -k^2 a_{\mu\nu} + k_\nu k^\omega a_{\mu\omega} - k_\mu k^\omega a_{\nu\omega} , \quad (3.6.17b)$$

$$F = (\gamma - 1) k^2 \phi + \frac{1}{2} \gamma k^\mu k^\nu s_{\mu\nu} - \frac{1}{4} \gamma (1 + \gamma) k^2 s^\nu_\nu . \quad (3.6.17c)$$

In deriving the Weyl variation (3.6.16) we have integrated by parts and also used the relation

$$[\nabla^2 X^\mu e^{ik \cdot X}]_r = i \frac{\alpha' \gamma}{4} k^\mu R [e^{ik \cdot X}]_r . \quad (3.6.18)$$

The left-hand side would vanish by the naive equation of motion, but here the equation of motion multiplies another operator at the same point. The general principle that holds in this case is that the operator need not vanish but it is not independent — it can be expanded in terms of the other local operators in the theory. In general the coefficients of these operators depend on the renormalization scheme; the result (3.6.18) can be obtained by taking the Weyl variation of each side.

Since $S_{\mu\nu}$, $A_{\mu\nu}$, and F multiply independent operators, the condition for Weyl invariance is

$$S_{\mu\nu} = A_{\mu\nu} = F = 0 . \quad (3.6.19)$$

In order to get an accurate count of the number of operators, we must note that not all V_1 of the form (3.6.14) are independent. Rather, under

$$s_{\mu\nu} \rightarrow s_{\mu\nu} + \xi_\mu k_\nu + k_\mu \xi_\nu , \quad (3.6.20a)$$

$$a_{\mu\nu} \rightarrow a_{\mu\nu} + \zeta_\mu k_\nu - k_\mu \zeta_\nu , \quad (3.6.20b)$$

$$\phi \rightarrow \phi + \frac{\gamma}{2} k \cdot \xi , \quad (3.6.20c)$$

the change in V_1 integrates to zero using the equation of motion (3.6.18). Choose now a vector n^μ satisfying $n \cdot k = 1$ and $n^2 = 0$. A complete set of independent operators is obtained by restricting to

$$n^\mu s_{\mu\nu} = n^\mu a_{\mu\nu} = 0 . \quad (3.6.21)$$

This is $2D$ equations for the $2D$ parameters ξ_μ and ζ_μ ; one can show that one equation and one parameter are trivial, but that eqs. (3.6.21) are otherwise nondegenerate and so define an independent set of vertex operators.

By using condition (3.6.21) and solving first $S_{\mu\nu} n^\mu n^\nu = 0$, then $S_{\mu\nu} n^\mu = A_{\mu\nu} n^\mu = 0$, and finally the full $S_{\mu\nu} = A_{\mu\nu} = F = 0$, one finds that

$$k^2 = 0 , \quad (3.6.22a)$$

$$k^\nu s_{\mu\nu} = k^\mu a_{\mu\nu} = 0 , \quad (3.6.22b)$$

$$\phi = \frac{1 + \gamma}{4} s^\mu{}_\mu . \quad (3.6.22c)$$

$$\gamma = -\frac{2}{3}$$

Again we find a mass-shell condition (3.6.22a), this time corresponding to the first excited states of the closed string. There is also the condition (3.6.22b) that the polarization be transverse to the momentum, as required for a physical polarization of a massless tensor field. In the frame in which

$$k^\mu = (1, 1, 0, 0, \dots, 0) , \quad n^\mu = \frac{1}{2}(-1, 1, 0, 0, \dots, 0) , \quad (3.6.23)$$

conditions (3.6.21) and (3.6.22b) imply that all 0- and 1-components vanish, while eq. (3.6.22c) fixes ϕ . This leaves precisely $(D - 2)^2$ independent operators, the same number of massless closed string states as found in the light-cone gauge.

The condition $n^\mu s_{\mu\nu} = n^\mu a_{\mu\nu} = 0$ is needed to remove zero-norm light-like polarizations. However, the introduction of the vector n^μ spoils manifest Lorentz invariance, and the theory is Lorentz-invariant only because different choices of n^μ are equivalent due to (3.6.20). The equivalence relations (3.6.20) are therefore essential in order for the theory to be Lorentz-invariant with a positive Hilbert space norm. These relations are the signatures of local spacetime symmetries. In particular, ξ_μ is an infinitesimal *spacetime* coordinate transformation, and ζ_μ is the local spacetime symmetry of the antisymmetric tensor field. We now see in the interacting theory the existence of local spacetime symmetries, which we argued in section 1.4 had to be present because of the existence of massless spin-1 and spin-2 fields.

There is a technical point we should mention. Different renormalization schemes assign different names to the renormalized operators. The most common scheme is dimensional regularization; the operators in this scheme are related to those in the scheme above by

$$[e^{ik \cdot X}]_{\text{DR}} = [e^{ik \cdot X}]_{\text{r}} , \quad (3.6.24a)$$

$$[\partial_a X^\mu e^{ik \cdot X}]_{\text{DR}} = [\partial_a X^\mu e^{ik \cdot X}]_{\text{r}} , \quad (3.6.24b)$$

$$[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X}]_{\text{DR}} = [\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X}]_{\text{r}} - \frac{\alpha'}{12} g_{ab} \eta^{\mu\nu} R [e^{ik \cdot X}]_{\text{r}} , \quad (3.6.24c)$$

$$[\nabla_a \partial_b X^\mu e^{ik \cdot X}]_{\text{DR}} = [\nabla_a \partial_b X^\mu e^{ik \cdot X}]_{\text{r}} + i \frac{\alpha'}{12} g_{ab} k^\mu R [e^{ik \cdot X}]_{\text{r}} . \quad (3.6.24d)$$

Comparing with eq. (3.6.18), we see that in dimensional regularization $\gamma = 0$, so the equations of motion hold inside the regulated operator. This is a convenience — many equations simplify. In particular, in the spacetime gauge transformation (3.6.20), ϕ is invariant.

Complete derivation & Proof is
given by Stacy M. Schrons
at the end.

Open String Vertex operators in

Here we need to insert the Vertex operator on the boundary of the world sheet as seen previously.

The tachyon vertex is,

$$\hat{V}_{\psi_0}(k^\mu) = g_0 \int_{\partial \mathcal{M}} ds [e^{ik^\mu X_\mu(\sigma)}]_x$$

We need to figure out the dimension of the boundary operator $:e^{ip \cdot X}:$. It's not the same as for the closed string. The reason is due to presence of the image charge in the propagator (4.57) for a free scalar field on a space with boundary. This propagator appears in the Wick contractions in the OPEs and affects the weights. Let's see why this is the case. Firstly, we look at a single scalar field X ,

$$\begin{aligned} \partial X(z) : e^{ipX(w, \bar{w})} : &= \sum_{n=1}^{\infty} \frac{(ip)^n}{(n-1)!} : X(w, \bar{w})^{n-1} : \left(-\frac{\alpha'}{2} \frac{1}{z-w} - \frac{\alpha'}{2} \frac{1}{z-\bar{w}} \right) + \dots \\ &= -\frac{i\alpha' p}{2} : e^{ipX(w, \bar{w})} : \left(\frac{1}{z-w} + \frac{1}{z-\bar{w}} \right) + \dots \end{aligned}$$

With this result, we can now compute the OPE with T ,

$$T(z) : e^{ipX(w, \bar{w})} : = \frac{\alpha' p^2}{4} : e^{ipX} : \left(\frac{1}{z-w} + \frac{1}{z-\bar{w}} \right)^2 + \dots$$

When the operator $:e^{ipX(w, \bar{w})}:$ is placed on the boundary $w = \bar{w}$, this becomes

$$T(z) : e^{ipX(w, \bar{w})} : = \frac{\alpha' p^2 : e^{ipX(w, \bar{w})} :}{(z-w)^2} + \dots$$

This tells us that the boundary operator $:e^{ip \cdot X}:$ is indeed primary, with weight $\alpha' p^2$.

For the open string, Weyl invariance requires that operators have weight $+1$ in order to cancel the scaling dimension of -1 coming from the boundary integral $\int ds$. So the mass of the open string ground state is

$$M^2 \equiv -p^2 = -\frac{1}{\alpha'}$$

in agreement with the mass of the open string tachyon computed in Section 3.

The vertex operator for the photon is

$$V_{\text{photon}} \sim \int_{\partial\mathcal{M}} ds \, \zeta_a : \partial X^a e^{ip \cdot X} : \quad (5.10)$$

where the index $a = 0, \dots, p$ now runs only over those directions with Neumann boundary conditions that lie parallel to the brane worldvolume. The requirement that this is a primary operator gives $p^a \zeta_a = 0$, while Weyl invariance tells us that $p^2 = 0$. This is the expected behaviour for the momentum and polarization of a photon.

How do we get the image charge for open strings is as follows.

The scalar field $X(z, \bar{z})$ is only defined in the upper-half plane. Suppose we want to implement Neumann boundary conditions. Then the propagator is defined by

$$\langle X(z, \bar{z}) X(w, \bar{w}) \rangle = G(z, \bar{z}; w, \bar{w})$$

which obeys $\partial^2 G = -2\pi\alpha' \delta(z - w, \bar{z} - \bar{w})$ subject to the boundary condition

$$\partial_\sigma G(z, \bar{z}; w, \bar{w})|_{\sigma=0} = 0$$

But we solve problems like this in our electrodynamics courses. A useful way of proceeding is to introduce an “image charge” in the lower-half plane. We now let $X(z, \bar{z})$ vary over the whole complex plane with its dynamics governed by the propagator

$$G(z, \bar{z}; w, \bar{w}) = -\frac{\alpha'}{2} \ln |z - w|^2 - \frac{\alpha'}{2} \ln |z - \bar{w}|^2 \quad (4.57)$$

Much of the remaining discussion of CFTs carries forward with only minor differences. However, there is one point that is simple but worth stressing because it will be of importance later. This concerns the state-operator map. Recall the logic that leads us to this idea: we consider a state at fixed time on the strip and propagate it back to past infinity $\tau \rightarrow -\infty$. After the map to the half-plane, past infinity is again the origin. But now the origin lies on the boundary. We learn that the state-operator map relates states to local operators defined on the boundary.

This fact ensures that theories on a strip have fewer states than those on the cylinder. For example, for a free scalar field, Neumann boundary conditions require $\partial X = \bar{\partial} X$ at $\text{Im} z = 0$. (This follows from the requirement that $\partial_\sigma X = 0$ at $\sigma = 0, \pi$ on the strip). On the cylinder, the operators ∂X and $\bar{\partial} X$ give rise to different states; on the strip they give rise to the same state. This, of course, mirrors what we've seen for the quantization of the open string where boundary conditions mean that we have only half the oscillator modes to play with.

Polchinski's explanation

Open string vertex operators

The extension to the open string is straightforward. We leave the details for the reader as exercise 3.9, giving only some results. The tachyon vertex operator is

$$g_0 \int_{\partial M} ds [e^{ik \cdot X}]_r \quad (3.6.25)$$

and is Weyl-invariant for $k^2 = 1/\alpha'$. The photon vertex operator is

$$-i \frac{g_0}{(2\alpha')^{1/2}} e_\mu \int_{\partial M} ds [\dot{X}^\mu e^{ik \cdot X}]_r, \quad (3.6.26)$$

where the normalization relative to the tachyon is obtained either from the state-operator mapping or unitarity, the i is as discussed for the closed string vertex operator (3.6.14), and the sign is a matter of convention. This operator is Weyl-invariant if $k^2 = 0$ and $k \cdot e = 0$. There is an equivalence $e_\mu \cong e_\mu + \lambda k_\mu$, which is a spacetime gauge transformation. This leaves $D - 2$ transverse polarizations.

3.40 p 105: Eq (3.6.16) The Weyl Transformation for the Massless Vertex Operator for the Polyakov String

This is a subtle and long calculation, so fasten your seatbelts. There were some points that were obscure to me and I posted them on the Physics Stack Exchange. This time my gratitude goes to "Wakabaloola" for helping clarify these issues. At the end of the day it is always gratifying to see how all the details of a calculation conspire to give the final result. This was, if I am permitted a personal note, particularly the case for this calculation.

Some preliminaries first. Under an infinitesimal Weyl transformation

$$\delta_W g_{ab} = 2\delta\omega g_{ab} \quad [3.166]$$

we also have

$$\delta_W g^{ab} = -2\delta\omega g^{ab} \quad [3.167]$$

and

$$\delta_W \sqrt{g} = \frac{1}{2} g^{-1/2} \delta_W g = \frac{1}{2} g^{-1/2} g g^{ab} \delta_W g_{ab} = \frac{1}{2} g^{-1/2} g g^{ab} 2\delta\omega g_{ab} = \sqrt{g} 2\delta\omega \quad [3.168]$$

This implies that

$$\delta_W(\sqrt{g} g^{ab}) = \delta_W(\sqrt{g}) g^{ab} + \sqrt{g} \delta_W g^{ab} = \sqrt{g} 2\delta\omega g^{ab} - 2\sqrt{g} \delta\omega g^{ab} = 0 \quad [3.169]$$

The antisymmetric tensor ϵ^{ab} also transforms non-trivially under a Weyl transformation. Indeed, from $\sqrt{g} \epsilon^{12} = 1$ we have

$$(\delta_W \sqrt{g}) \epsilon^{12} + \sqrt{g} (\delta_W \epsilon^{12}) = 0 \quad \Rightarrow \delta_W \epsilon^{12} = -2\delta\omega \epsilon^{12} \quad [3.170]$$

and thus more generally

$$\delta_W \epsilon^{ab} = -2\delta\omega \epsilon^{ab} \quad [3.171]$$

But we will only need the relation

$$\delta_W(\sqrt{g} \epsilon^{ab}) = 0 \quad [3.172]$$

From (1.2.32) we also know that $\sqrt{g'} R' = \sqrt{g} (R - 2\nabla^2 \delta\omega)$ so that

$$\delta_W(\sqrt{g} R) = -2\sqrt{g} \nabla^2 \delta\omega \quad [3.173]$$

After all these preliminaries, let us now work out Weyl transformation of (3.6.14), using the above and also (3.6.7)

$$\begin{aligned} \delta_W V_1 = & \frac{g_c}{\alpha'} \int d^2\sigma \left\{ \left(\delta_W(\sqrt{g}g^{ab})s_{\mu\nu} + i\delta_W(\sqrt{g}\epsilon^{ab})a_{\mu\nu} \right) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\ & + \alpha' \phi \delta_W(\sqrt{g}R) \left[e^{ik \cdot X(\sigma)} \right]_r \\ & + \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \frac{1}{2} \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\ & \left. + \sqrt{g}\alpha' \phi R \frac{1}{2} \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \end{aligned} \quad [3.174]$$

Recall that $\delta_W e^{ik \cdot X(\sigma)} = \delta_W \partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} = 0$, as there is no metric dependence in these operators. We have just seen that the first line vanishes and so can write

$$\begin{aligned} \delta_W V_1 = & \frac{g_c}{\alpha'} \int d^2\sigma \left\{ -2\alpha' \phi \sqrt{g}(\nabla^2 \delta\omega) \left[e^{ik \cdot X(\sigma)} \right]_r \right. \\ & + \frac{1}{2} \sqrt{g}(g^{ab}s_{\mu\nu} + i\epsilon^{ab}a_{\mu\nu}) \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\ & \left. + \frac{1}{2} \sqrt{g}\alpha' \phi R \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \end{aligned} \quad [3.175]$$

Let us for convenience work out the different lines separately.

We start with the first line

$$\begin{aligned} \mathcal{J}_1 = & -2g_c \phi \int d^2\sigma \sqrt{g}(\nabla^2 \delta\omega) \left[e^{ik \cdot X(\sigma)} \right]_r = -2g_c \phi \int d^2\sigma \sqrt{g}(\nabla_a \partial^a \delta\omega) \left[e^{ik \cdot X(\sigma)} \right]_r \\ = & -2g_c \phi \int d^2\sigma \sqrt{g} \left\{ \nabla_a \left(\partial^a \delta\omega \left[e^{ik \cdot X(\sigma)} \right]_r \right) - \partial^a \delta\omega \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \\ = & -2g_c \phi \int d^2\sigma \partial_a \left(\sqrt{g} \partial^a \delta\omega \left[e^{ik \cdot X(\sigma)} \right]_r \right) + 2g_c \phi \int d^2\sigma \sqrt{g} \partial^a \delta\omega \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r \\ = & +2g_c \phi \int d^2\sigma \sqrt{g} \partial^a \delta\omega \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r \end{aligned} \quad [3.176]$$

where we have used $\partial_a(\sqrt{g}v^a) = \sqrt{g}\nabla_a v^a$ and $\nabla_a \sqrt{g} = 0$. We can now replace the covariant derivative by a partial derivative as it acts on a scalar and partial integrate the other derivative

$$\begin{aligned} \mathcal{J}_1 = & -2g_c \phi \int d^2\sigma \delta\omega \partial^a \left(\sqrt{g} \partial_a \left[e^{ik \cdot X(\sigma)} \right]_r \right) \\ = & -2g_c \phi \int d^2\sigma \delta\omega \sqrt{g} \nabla^a \partial_a \left[e^{ik \cdot X(\sigma)} \right]_r \end{aligned} \quad [3.177]$$

We now have

$$\begin{aligned}
\nabla^a \partial_a e^{ik \cdot X} &= \nabla^a (ik^\mu \partial_a X_\mu) e^{ik \cdot X} \\
&= ik^\mu (\nabla^a \partial_a X_\mu) e^{ik \cdot X} + (ik^\mu \partial_a X_\mu) (ik^\nu \nabla^a X_\nu) e^{ik \cdot X} \\
&= ik^\mu \nabla^2 X_\mu e^{ik \cdot X} - k^\mu k^\nu \partial_a X_\mu \partial^a X_\nu e^{ik \cdot X}
\end{aligned} \tag{3.178}$$

We have freely replaced ∂_a by ∇_a and vice-versa when they are acting on worldsheet scalars. Therefore, using (3.6.18),

$$\begin{aligned}
\left[\nabla^a \partial_a e^{ik \cdot X} \right]_r &= ik^\mu \left[\nabla^2 X_\mu e^{ik \cdot X} \right]_r - k^\mu k^\nu \left[\partial_a X_\mu \partial^a X_\nu e^{ik \cdot X} \right]_r \\
&= ik^\mu \times \left(-i \frac{\alpha'}{6} k_\mu R \left[e^{ik \cdot X(\sigma)} \right]_r \right) - k^\mu k^\nu \left[\partial_a X_\mu \partial^a X_\nu e^{ik \cdot X} \right]_r \\
&= \frac{\alpha'}{6} k^2 R \left[e^{ik \cdot X(\sigma)} \right]_r - k^\mu k^\nu g^{ab} \left[\partial_a X_\mu \partial_b X_\nu e^{ik \cdot X} \right]_r
\end{aligned} \tag{3.179}$$

and thus

$$\begin{aligned}
\mathcal{J}_1 &= -\frac{g_c \alpha'}{3} k^2 \phi \int d^2 \sigma \sqrt{g} \delta \omega R \left[e^{ik \cdot X(\sigma)} \right]_r + 2g_c k^\mu k^\nu \phi \int d^2 \sigma \sqrt{g} \delta \omega g^{ab} \left[\partial_a X_\mu \partial_b X_\nu e^{ik \cdot X} \right]_r \\
&= \frac{g_c}{2} \int d^2 \sigma \sqrt{g} \delta \omega \left\{ -\frac{2\alpha'}{3} k^2 \phi R \left[e^{ik \cdot X(\sigma)} \right]_r + g^{ab} (4k^\mu k^\nu \phi) \left[\partial_a X_\mu \partial_b X_\nu e^{ik \cdot X} \right]_r \right\}
\end{aligned} \tag{3.180}$$

Before we tackle the second, more difficult, line of [3.175], let us do the easier third line

$$\begin{aligned}
\mathcal{J}_3 &= \frac{g_c}{2} \phi \int d^2 \sigma d^2 \sigma' d^2 \sigma'' \sqrt{g} R \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[e^{ik \cdot X(\sigma)} \right]_r \\
&= \frac{g_c}{2} \phi \int d^2 \sigma d^2 \sigma' d^2 \sigma'' \sqrt{g} R \delta_W \Delta(\sigma', \sigma'') (-k^2) \delta^2(\sigma - \sigma') \delta^2(\sigma - \sigma'') \left[e^{ik \cdot X(\sigma)} \right]_r \\
&= -\frac{g_c}{2} \phi k^2 \int d^2 \sigma \sqrt{g} R \delta_W \Delta(\sigma, \sigma) \left[e^{ik \cdot X(\sigma)} \right]_r = -\frac{g_c}{2} \phi k^2 \int d^2 \sigma \sqrt{g} R \alpha' \delta \omega \left[e^{ik \cdot X(\sigma)} \right]_r \\
&= \frac{g_c}{2} \int d^2 \sigma \sqrt{g} \delta \omega (-\alpha' k^2 \phi R) \left[e^{ik \cdot X(\sigma)} \right]_r
\end{aligned} \tag{3.181}$$

Finally, the second line of [3.175], is

$$\mathcal{J}_2 = \frac{g_c}{2\alpha'} \int d^2 \sigma \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \tilde{\mathcal{J}}_2 \tag{3.182}$$

with

$$\tilde{\mathcal{J}}_2 = \int d^2 \sigma' d^2 \sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \tag{3.183}$$

Let us first consider the contribution where the functional derivatives both act on the exponential. This gives

$$\begin{aligned}
\tilde{\mathcal{J}}_{2a} &= \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') (ik^\lambda \delta^2(\sigma' - \sigma)) (ik_\lambda \delta^2(\sigma'' - \sigma)) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\
&= -k^2 \delta_W \Delta(\sigma, \sigma) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\
&= -\alpha' k^2 \delta\omega(\sigma) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r
\end{aligned} \tag{3.184}$$

where we have used (3.6.11). Thus we have a contribution

$$\begin{aligned}
\mathcal{J}_{2a} &= -\frac{g_c}{2} k^2 \int d^2\sigma \sqrt{g} \delta\omega (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\
&= \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega (g^{ab} (-k^2 s_{\mu\nu}) + i\epsilon^{ab} (-k^2 a_{\mu\nu})) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r
\end{aligned} \tag{3.185}$$

Next, take the case where only one of the functional derivatives acts on the exponential. There are four possible combinations

$$\begin{aligned}
\tilde{\mathcal{J}}_{2b} &= \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \left\{ \delta_\lambda^\mu \partial_a \delta^2(\sigma' - \sigma) ik^\lambda \delta^2(\sigma'' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\
&\quad + \delta_\lambda^\nu \partial_b \delta^2(\sigma' - \sigma) ik^\lambda \delta^2(\sigma'' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \\
&\quad + \eta^{\lambda\mu} \partial_a \delta^2(\sigma'' - \sigma) ik_\lambda \delta^2(\sigma' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\
&\quad \left. + \eta^{\lambda\nu} \partial_b \delta^2(\sigma'' - \sigma) ik_\lambda \delta^2(\sigma' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\}
\end{aligned} \tag{3.186}$$

We can already perform one of the integrations:

$$\begin{aligned}
\tilde{\mathcal{J}}_{2b} &= i \int d^2\sigma' \delta_W \Delta(\sigma', \sigma) \left\{ k^\mu \partial_a \delta^2(\sigma' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\
&\quad \left. + k^\nu \partial_b \delta^2(\sigma' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \\
&\quad + i \int d^2\sigma'' \delta_W \Delta(\sigma, \sigma'') \left\{ k^\mu \partial_a \delta^2(\sigma'' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\
&\quad \left. + k^\nu \partial_b \delta^2(\sigma'' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\}
\end{aligned} \tag{3.187}$$

Change the integration variable from σ'' to σ' and use the symmetry $\Delta(\sigma, \sigma') = \Delta(\sigma', \sigma)$

$$\begin{aligned}
 \mathcal{J}_{2b} &= \frac{g_c}{2\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \times 2i\delta_W \Delta(\sigma', \sigma) \\
 &\quad \times \left[k^\mu \partial_a \delta^2(\sigma' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + k^\nu \partial_b \delta^2(\sigma' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right] \\
 &= \frac{ig_c}{\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \delta_W \Delta(\sigma', \sigma) \\
 &\quad \times \left[k^\mu \partial_a \delta^2(\sigma' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + k^\nu \partial_b \delta^2(\sigma' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right] \quad [3.188]
 \end{aligned}$$

We can now perform another partial integration to free up the last delta function. However, it is convenient to first use the chain rule for derivatives of delta functions² in order to change the ∂_a into a ∂'_a :

$$\partial_a \delta^2(\sigma' - \sigma) = -\partial'_a \delta^2(\sigma' - \sigma) \quad [3.189]$$

We perform the chain rule on the derivative of the delta function and partially integrate.

²To see that this is correct, evaluate $I = \int dx dy f(x)g(y)\partial_x \delta(x - y)$ in two ways. First

$$I_1 = - \int dx dy f'(x)g(x)\delta(x - y) = - \int dx f'(x)g(x)$$

Alternatively

$$I_2 = - \int dx dy f(x)g(y)\partial_y \delta(x - y) = \int dx dy f(x)g'(y)\delta(x - y) = \int dx f(x)g'(x)$$

Now

$$I_2 - I_1 = \int dx [f(x)g'(x) + f'(x)g(x)] = \int dx \frac{d}{dx} [f(x)g(x)] = 0$$

which shows that $\partial_x \delta(x - y) = -\partial_y \delta(x - y)$

The ∂'_a and ∂'_b now only act on $\delta_W \Delta(\sigma', \sigma)$. Two minus signs give a plus and so

$$\begin{aligned}
\mathcal{J}_{2b} &= \frac{ig_c}{\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \\
&\quad \times \left\{ \partial'_a \delta_W \Delta(\sigma', \sigma) k^\mu \delta^2(\sigma' - \sigma) \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\
&\quad \left. + \partial'_b \delta_W \Delta(\sigma', \sigma) k^\nu \delta^2(\sigma' - \sigma) \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \\
&= \frac{ig_c}{\alpha'} \int d^2\sigma \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \\
&\quad \times \left\{ \partial'_a \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma} k^\mu \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\
&\quad \left. + \partial'_b \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma} k^\nu \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \tag{3.190}
\end{aligned}$$

Use (3.6.15a), which is, by symmetry, equally valid if we replace ∂_a by ∂'_a

$$\begin{aligned}
\mathcal{J}_{2b} &= \frac{ig_c}{\alpha'} \int d^2\sigma \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \\
&\quad \times \left\{ \frac{1}{2} \alpha' \partial_a \delta\omega k^\mu \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + \frac{1}{2} \alpha' \partial_a \delta\omega k^\nu \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \\
&= \frac{ig_c}{2} \int d^2\sigma \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \\
&\quad \times \left\{ \partial_a \delta\omega k^\mu \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + \partial_a \delta\omega k^\nu \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \tag{3.191}
\end{aligned}$$

Yet another partial integration gives

$$\begin{aligned}
\mathcal{J}_{2b} &= -\frac{ig_c}{2} \int d^2\sigma \delta\omega \left\{ \partial_a \left[\sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right] \right. \\
&\quad \left. + \partial_b \left[\sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\nu \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right] \right\} \tag{3.192}
\end{aligned}$$

We use $\partial_a(\sqrt{g} v^a) = \sqrt{g} \nabla_a v^a$ and the fact that the covariant derivative of g^{ab} , $\sqrt{g}$ and ϵ^{ab}

is zero:

$$\begin{aligned}
\mathcal{J}_{2b} &= -\frac{ig_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \left\{ \nabla_a \left[(g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right] \right. \\
&\quad \left. + \nabla_b \left[(g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\nu \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right] \right\} \\
&= -\frac{ig_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \left\{ (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \nabla_a \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \right. \\
&\quad \left. + (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\nu \nabla_b \left[\partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \quad [3.193]
\end{aligned}$$

Let us now focus on

$$\begin{aligned}
\mathcal{I}_a &= (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \nabla_a \left[\partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r \\
&= (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \left\{ \left[\nabla_a \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + \left[\partial_b X^\nu i k_\lambda \nabla_a X^\lambda e^{ik \cdot X(\sigma)} \right]_r \right\} \quad [3.194]
\end{aligned}$$

We replace ∂_b by ∇_b in the first term and ∇_a by ∂_a in the second term. This is allowed as they both act on worldsheet scalars.

$$\mathcal{I}_a = (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \left\{ \left[\nabla_a \nabla_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + \left[\partial_b X^\nu i k_\lambda \partial_a X^\lambda e^{ik \cdot X(\sigma)} \right]_r \right\} \quad [3.195]$$

Because $\nabla_a \nabla_b = \nabla_b \nabla_a$ when acting on scalars³ the ϵ^{ab} part vanishes with the first term. Thus, also changing dummy variables for the second term,

$$\begin{aligned}
\mathcal{I}_a &= s_{\mu\nu} k^\mu \left[\nabla^2 X^\nu e^{ik \cdot X(\sigma)} \right]_r + (g^{ab} s_{\mu\lambda} + i\epsilon^{ab} a_{\mu\lambda}) i k^\mu k^\lambda \left[\partial_b X^\nu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \\
&= -\frac{i\alpha'}{6} R s_{\mu\nu} k^\mu k^\nu \left[e^{ik \cdot X(\sigma)} \right]_r + (g^{ab} s_{\mu\lambda} + i\epsilon^{ab} a_{\mu\lambda}) i k_\nu k^\lambda \left[\partial_b X^\nu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \quad [3.196]
\end{aligned}$$

Similarly

$$\mathcal{I}_a = -\frac{i\alpha'}{6} R s_{\mu\nu} k^\mu k^\nu \left[e^{ik \cdot X(\sigma)} \right]_r + (g^{ab} s_{\lambda\nu} + i\epsilon^{ab} a_{\lambda\nu}) i k_\mu k^\lambda \left[\partial_b X^\nu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \quad [3.197]$$

3

$$\nabla_a \nabla_b S - \nabla_b \nabla_a S = \partial_a \partial_b S - \Gamma_{ab}^c \partial_c S - \partial_b \partial_a S + \Gamma_{ba}^c \partial_c S = 0$$

as $\Gamma_{ab}^c = \Gamma_{ba}^c$.

and therefore

$$\begin{aligned} \mathcal{J}_{2b} = & \frac{g_c}{2} \int d^2\sigma d\omega \sqrt{g} \left\{ \alpha' R \left(-\frac{1}{3} s_{\mu\nu} k^\mu k^\nu \right) \left[e^{ik \cdot X(\sigma)} \right]_r \right. \\ & \left. + (g^{ab} + i\epsilon^{ab}) \left(s_{\mu\lambda} k_\nu k^\lambda + s_{\lambda\nu} k_\mu k^\lambda \right) \left[\partial_b X^\nu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \end{aligned} \quad [3.198]$$

Let us now consider the case where both functional derivatives act on the $\partial_a X^\mu \partial_b X^\mu$. This gives

$$\begin{aligned} \mathcal{J}_{2c} = & \frac{g_c}{2\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \delta_W \Delta(\sigma', \sigma'') \\ & \times \left[\delta_\lambda^\mu \partial_a \delta^2(\sigma' - \sigma) \eta^{\lambda\nu} \partial_b \delta^2(\sigma'' - \sigma) + \delta_\lambda^\nu \partial_b \delta^2(\sigma' - \sigma) \eta^{\lambda\mu} \partial_a \delta^2(\sigma'' - \sigma) \right] \left[e^{ik \cdot X(\sigma)} \right]_r \end{aligned} \quad [3.199]$$

We interchange integration variables σ' and σ'' in the second term between brackets and use $\Delta(\sigma', \sigma'') = \Delta(\sigma'', \sigma')$

$$\begin{aligned} \mathcal{J}_{2c} = & \frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \delta_W \Delta(\sigma', \sigma'') \\ & \times \eta^{\mu\nu} \partial_a \delta^2(\sigma' - \sigma) \partial_b \delta^2(\sigma'' - \sigma) \left[e^{ik \cdot X(\sigma)} \right]_r \end{aligned} \quad [3.200]$$

We use $a_{\mu\nu} \eta^{\mu\nu} = 0$ and the chain rule for derivatives of the delta function, then perform our first partial integration

$$\begin{aligned} \mathcal{J}_{2c} = & \frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} g^{ab} s_\mu^\mu \delta_W \Delta(\sigma', \sigma'') \partial'_a \delta^2(\sigma' - \sigma) \partial'_b \delta^2(\sigma'' - \sigma) \left[e^{ik \cdot X(\sigma)} \right]_r \\ = & -\frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' d^2\sigma'' \sqrt{g} g^{ab} s_\mu^\mu \partial_b'' \delta_W \Delta(\sigma', \sigma'') \partial'_a \delta^2(\sigma' - \sigma) \delta^2(\sigma'' - \sigma) \left[e^{ik \cdot X(\sigma)} \right]_r \\ = & -\frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} g^{ab} s_\mu^\mu \partial_b \delta_W \Delta(\sigma', \sigma) \partial'_a \delta^2(\sigma' - \sigma) \left[e^{ik \cdot X(\sigma)} \right]_r \end{aligned} \quad [3.201]$$

We follow with the second partial integration

$$\begin{aligned} \mathcal{J}_{2c} = & \frac{g_c}{\alpha'} \int d^2\sigma d^2\sigma' \sqrt{g} g^{ab} s_\mu^\mu \partial'_a \partial_b \delta_W \Delta(\sigma', \sigma) \delta^2(\sigma' - \sigma) \left[e^{ik \cdot X(\sigma)} \right]_r \\ = & \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} g^{ab} s_\mu^\mu \partial'_a \partial_b \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma} \left[e^{ik \cdot X(\sigma)} \right]_r \end{aligned} \quad [3.202]$$

We now use (3.6.15b) and then partially integrate two more times

$$\begin{aligned}
\mathcal{J}_{2c} &= \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} g^{ab} s_\mu^\mu \frac{1}{6} \alpha' \nabla_a \partial_b \delta\omega \left[e^{ik \cdot X(\sigma)} \right]_r = -\frac{g_c}{6} \int d^2\sigma \partial_b \delta\omega \nabla_a \left\{ \sqrt{g} g^{ab} s_\mu^\mu \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \\
&= -\frac{g_c}{6} \int d^2\sigma \partial_b \delta\omega \sqrt{g} g^{ab} s_\mu^\mu \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r = +\frac{g_c}{6} \int d^2\sigma \delta\omega \partial_b \left\{ \sqrt{g} g^{ab} s_\mu^\mu \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \\
&= +\frac{g_c}{6} \int d^2\sigma \delta\omega \sqrt{g} \nabla_b \left\{ g^{ab} s_\mu^\mu \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \\
&= +\frac{g_c}{6} \int d^2\sigma \delta\omega \sqrt{g} g^{ab} s_\mu^\mu \nabla_b \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r
\end{aligned} \tag{3.203}$$

use [3.179] and the fact that $g^{ab} \nabla_b \nabla_a \left[e^{ik \cdot X(\sigma)} \right]_r = \nabla^a \partial_a \left[e^{ik \cdot X} \right]_r$, remember scalars!

$$\begin{aligned}
\mathcal{J}_{2c} &= \frac{g_c}{6} \int d^2\sigma \delta\omega \sqrt{g} s_\lambda^\lambda \left\{ \frac{\alpha'}{6} k^2 R \left[e^{ik \cdot X(\sigma)} \right]_r - k_\mu k_\nu g^{ab} \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X} \right]_r \right\} \\
&= \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \left\{ \alpha' R \left(\frac{1}{18} k^2 s_\lambda^\lambda \right) \left[e^{ik \cdot X(\sigma)} \right]_r + g^{ab} \left(-\frac{1}{3} k_\mu k_\nu s_\lambda^\lambda \right) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X} \right]_r \right\}
\end{aligned} \tag{3.204}$$

We can now bring all the contributions together. They are of the form

$$\delta_W V_1 = \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega \left\{ (g^{ab} s_{\mu\nu} + i\epsilon^{ab} \mathfrak{Q}_{\mu\nu}) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_r + \alpha' R \mathfrak{F} \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \tag{3.205}$$

The contributions form the different parts to $\mathfrak{s}$, $\mathfrak{Q}$ and $\mathfrak{F}$ are summarised in the table below.

	$\mathfrak{s}_{\mu\nu}$	$\mathfrak{Q}_{\mu\nu}$	$\mathfrak{F}$
$\mathcal{J}_1$	$4k_\mu k_\nu \phi$	$-$	$-\frac{2}{3} k^2 \phi$
$\mathcal{J}_{2a}$	$-k^2 s_{\mu\nu}$	$-k^2 a_{\mu\nu}$	$-$
$\mathcal{J}_{2b}$	$k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\nu\lambda}$	$k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda}$	$-\frac{1}{3} k^\mu k^\nu s_{\mu\nu}$
$\mathcal{J}_{2c}$	$-\frac{1}{3} s_\lambda^\lambda k_\mu k_\nu$	$-$	$\frac{1}{18} k^2 s_\lambda^\lambda$
$\mathcal{J}_3$	$-$	$-$	$-k^2 \phi$

Table 3.2: Weyl transformation of the massless vertex operator; contributions to [3.205]

Thus

$$\begin{aligned}
s_{\mu\nu} &= -k^2 s_{\mu\nu} + k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\mu\lambda} - \frac{1}{3} s_\lambda^\lambda k_\mu k_\nu + 4k_\mu k_\nu \phi \\
\mathfrak{Q}_{\mu\nu} &= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda} \\
&= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\mu\lambda} - k^\lambda k_\mu a_{\nu\lambda} \\
\mathfrak{F} &= -\frac{5}{3} k^2 \phi - \frac{1}{3} k^\mu k^\nu s_{\mu\nu} + \frac{1}{18} k^2 s_\lambda^\lambda
\end{aligned} \tag{3.206}$$

We need to compare this with Polchinski's (3.6.17), with $\gamma = -2/3$, i.e.

$$\begin{aligned} S_{\mu\nu} &= -k^2 s_{\mu\nu} + k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\mu\lambda} - \frac{1}{3} s^\lambda_\lambda k_\mu k_\nu + 4k_\mu k_\nu \phi \\ A_{\mu\nu} &= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\mu\lambda} - k^\lambda k_\mu a_{\nu\lambda} \\ F &= -\frac{5}{3} k^2 \phi - \frac{1}{3} k^\mu k^\nu s_{\mu\nu} + \frac{1}{18} k^2 s^\lambda_\lambda \end{aligned} \quad [3.207]$$

Lo and behold! These are identical!

3.41 p 105: Eq (3.6.18) Linking $\nabla^2 X^\mu$ with $k^\mu R$

This is actually a highly non-trivial equation. I haven't found a simple derivation of it, but I will show in a very roundabout way that it is correct. For this I use a derivation provided by "Wakabaloola" on the Physics Stack Exchange, for which I am extremely grateful. As a return gesture, the least I can do is to refer to one of his papers

D. Luest and D. Skliros, Handle Operators in String Theory, arXiv:1912.0155 [hep-th]

which we will denote by [LS] here. The building blocks of most of what follows comes from their section 2. Any errors in my explanation are, of course, solely due to me. It is also worth mentioning that the basis of this work is Polchinski's earlier work published in NBP307 (1988) 61-92.

Because of the length of the calculation, it is useful to summarise what we are going to do. Our first goal is to find transition functions of conformal and Weyl transformations. We will first introduce the concept of holomorphic normal coordinates. These are coordinates in a patch around a point σ_1 of the manifold that are chosen to be "as flat as possible", implying that the connections at that point vanish. This suggests that we introduce the concept of Weyl normal ordering, which is similar to Polchinski's conformal normal ordering but where both points z_1 and z_2 are taken at the same base point. We will then derive the transition function for a change of base points from σ_1 to σ'_1 . This will be achieved in [3.236]. We will then proceed to use this result to work out how a change of coordinates, this time keeping the base point fixed, impacts a local, Weyl normal ordered, operator. I.e we will derive an expression for the derivative of such an operator. This result is achieved in [3.247] and shows that we cannot just bring in the derivative into the normal ordering as there is an additional contribution that depends on the Ricci scalar and its covariant derivatives. As an illustration of this we will show that the formula [3.260] is valid

$$\nabla_a \circ \nabla^a X^\mu e^{ik \cdot X}(\sigma_1)_{\circ z} = \circ \nabla_a X^\mu \nabla^a e^{ik \cdot X}(\sigma_1)_{\circ z} - \frac{i\alpha' k^\mu}{4} R(\sigma_1)_{\circ} e^{ik \cdot X}(\sigma_1)_{\circ z} \quad [3.208]$$

where the $\circ \cdots \circ_z$ denote Weyl normal ordering. This shows how the Ricci scalar can appear in expressions with normal ordering and derivatives.

Next we turn to the case of interest, which is the case of Weyl transformations and derive the transition function for such a transformation. This is given in [3.271]. It is then an easy step to determine how a local operator transforms under a Weyl transformation. This is given in [3.274]. We then derive expression for the propagator and its derivatives in this new scheme. This allows us to focus on requiring Weyl invariance of the vertex operators, giving us a set of equations that lead to the same mass-shell conditions (3.6.22). This then establishes the validity for (3.6.18).

The appendix to this chapter contains a brief introduction to complex structures where we will also derive some results needed in this section. There is a lot of work to be done, so let's get cracking!

HOLOMORPHIC NORMAL COORDINATES AND WEYL NORMAL ORDERING

We have the usual complex coordinates z and $\bar{z}$ on the Riemann surface M under consideration. But let us be very precise. We have a chart on our manifold in which we define holomorphic and anti-holomorphic coordinates, z and $\bar{z}$. We define the complex coordinates in such a way that at a given point σ_1 these complex coordinates vanish. Restricting ourselves to the holomorphic side, as the anti-holomorphic is similar, we denote this set of coordinates by $z_{\sigma_1}(\sigma)$, meaning the holomorphic coordinates based on a chart around the point σ_1 where these coordinates vanish, i.e. where

$$z_{\sigma_1}(\sigma_1) = 0 \quad [3.209]$$

We now define holomorphic normal coordinates as follows. In that local patch around σ_1 go to the conformal gauge $g_{ab} = e^{2\omega} \delta_{ab} = \rho(\sigma) \delta_{ab}$.⁴ We know that in the conformal gauge the metric and the Ricci scalar are given by

$$\begin{aligned} ds^2 &= \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) dz_{\sigma_1} d\bar{z}_{\sigma_1} \\ R &= -4\rho(z_{\sigma_1}, \bar{z}_{\sigma_1})^{-1} \partial_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) \end{aligned} \quad [3.210]$$

Here we have used the fact that $e^{2\omega} = \rho$ and that $\partial_a \partial_a = 4\partial_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}}$. Note that we are explicitly mentioning that these are holomorphic coordinates around the point σ_1 , because soon we will turn our attention to the question what happens in a point nearby σ_1 . We now chose the $z_{\sigma_1}(\sigma)$ in such a way that at $\sigma = \sigma_1$, i.e. where the patch is based and only at that point, the metric is as flat as possible. i.e.

$$\begin{aligned} \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) \Big|_{\sigma=\sigma_1} &= 1 \\ \partial_{z_{\sigma_1}}^n \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) \Big|_{\sigma=\sigma_1} &= 0 \quad \text{for } n \geq 1 \end{aligned} \quad [3.211]$$

⁴That it is always possible to chose such a conformal gauge is explained in the appendix to this chapter, see in particular, the discussion around [3.444].

so that all holomorphic derivatives of ρ vanish at σ_1 . Mixed derivatives (holomorphic plus anti-holomorphic) need not vanish as we cannot generally make the Ricci scalar zero by a choice of coordinates. Holomorphic coordinates satisfying these conditions are called holomorphic normal coordinates,⁵ a term that has been coined by the authors of [LS], which I find very appropriate and hope will stick.

We now define Weyl Normal Ordering by (2.2.7), using $d^2z = 2d^2\sigma$

$$:\mathcal{F}: = \exp \left(\frac{1}{8} \int d^2z_1 d^2z_2 \Delta(z_1, z_2) \frac{\delta}{\delta X^\mu(z_1, \bar{z}_1)} \frac{\delta}{\delta X_\mu(z_2, \bar{z}_2)} \mathcal{F} \right) \Bigg|_{\substack{z_1 = z_{\sigma_1}(\sigma_1) \\ z_2 = z_{\sigma_1}(\sigma_2)}} \quad [3.212]$$

I.e. for an operator based at σ_1 we take the subtractions at the points $z_{\sigma_1}(\sigma_1)$ and $z_{\sigma_1}(\sigma_2)$.

Note that the whole toolkit of CFT can be applied, e.g. the mode expansion of the field X^μ is given by

$$i\partial_{z_{\sigma_1}} X(\sigma) = \sum_{n=-\infty}^{\infty} \frac{\alpha_n^{(z_{\sigma_1})}(\sigma_1)}{z_{\sigma_1}(\sigma)^{n+1}} \quad [3.213]$$

This is heavy on notation, but it emphasises that this is taken in the patch around σ_1 and that the modes depend on that patch and the base point. The same, of course applies for the expansion of the energy-momentum tensor in the Virasoro generators.

THE TRANSITION FUNCTION OF A HOLOMORPHIC COORDINATE

Let us now consider a point nearby σ_1 , say $\sigma'_1 = \sigma_1 + \delta\sigma_1$ and construct a set of holomorphic coordinates $z_{\sigma'_1}$ around that point. Recall again that by construction $z_{\sigma'_1}(\sigma'_1) = 0$. If σ_1 and σ'_1 are close enough that the two holomorphic charts based on them overlap, then the transition function between the two is, by definition of a complex manifold, holomorphic:

$$\begin{aligned} z_{\sigma'_1}(\sigma) &= f_{\sigma'_1\sigma_1}(z_{\sigma_1}(\sigma)) \\ &= z_{\sigma_1}(\sigma) + \delta z_{\sigma_1}(\sigma) \end{aligned} \quad [3.214]$$

This defines $\delta z_{\sigma_1}(\sigma)$, which is a holomorphic function as well, since $f_{\sigma'_1\sigma_1}$ is holomorphic. Our first task will be to determine an equation for this transition function. This is not a straightforward thing to do and we will only achieve this in [3.236].

⁵The fact that we can choose such a set of coordinates that are as flat as possible is not trivial. For a discussion on how to go from a general coordinate system to a holomorphic normal coordinates see the appendix to this chapter and in particular [3.468] which gives an explicit formula for the required transition function from a set of coordinates in the conformal gauge to a set of holomorphic normal coordinates.

The above holomorphic transformation induces a change in metric at a generic point σ that lives in the overlap of the two patches of the form

$$\delta \ln \rho(\sigma) = \nabla_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \nabla_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \quad [3.215]$$

This transformation is derived in the appendix, see [3.452]. Let us now take the $(n-1)^{\text{th}}$ derivative of this, with $n \geq 2$ and evaluate this at the point $\sigma = \sigma_1$. We use the fact that in the conformal gauge the only non-vanishing connections are

$$\begin{aligned} \Gamma_{zz}^z(\sigma) &= \frac{\partial_{z_{\sigma_1}} \rho(\sigma)}{\rho(\sigma)} = \partial_{z_{\sigma_1}} \ln \rho(\sigma) \\ \Gamma_{\bar{z}\bar{z}}^{\bar{z}}(\sigma) &= \frac{\partial_{\bar{z}_{\sigma_1}} \rho(\sigma)}{\rho(\sigma)} = \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \end{aligned} \quad [3.216]$$

We find

$$\begin{aligned} \partial_{z_{\sigma_1}}^{n-1} \delta \ln \rho(\sigma) \Big|_{\sigma=\sigma_1} &= \partial_{z_{\sigma_1}}^{n-1} (\nabla_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \nabla_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma)) \Big|_{\sigma=\sigma_1} \\ &= \partial_{z_{\sigma_1}}^{n-1} \left[\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \partial_{z_{\sigma_1}} \ln \rho(\sigma) \delta z_{\sigma_1}(\sigma) + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \right. \\ &\quad \left. + \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \delta \bar{z}_{\sigma_1}(\sigma) \right] \Big|_{\sigma=\sigma_1} \\ &= \left[\partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma) + \partial_{z_{\sigma_1}}^n \ln \rho(\sigma) \delta z_{\sigma_1}(\sigma) + \partial_{z_{\sigma_1}} \ln \rho(\sigma) \partial_{z_{\sigma_1}}^{n-1} \delta z_{\sigma_1}(\sigma) \right. \\ &\quad \left. + \partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) + \partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \delta \bar{z}_{\sigma_1}(\sigma) + \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \partial_{z_{\sigma_1}}^{n-1} \delta \bar{z}_{\sigma_1}(\sigma) \right] \Big|_{\sigma=\sigma_1} \\ &= \left[\partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma) + (\partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma)) \delta \bar{z}_{\sigma_1}(\sigma) \right] \Big|_{\sigma=\sigma_1} \end{aligned} \quad [3.217]$$

In the last line we have used the fact that we are working in holomorphic normal coordinates such that $\partial_{z_{\sigma_1}}^n \ln \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) \Big|_{\sigma=\sigma_1} = 0$ for $n \geq 1$ and that $\delta \bar{z}_{\sigma_1}$ is holomorphic so that $\partial_{z_{\sigma_1}}^n \delta \bar{z}_{\sigma_1} = 0$ as well. Note that, as we have mentioned earlier, the mixed derivative term $\partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma)$ is not necessarily zero, on the contrary [3.210] links it to the Ricci scalar R . Indeed, note that

$$\partial_{z_{\sigma_1}}^{n-2} [\rho^{-1} \partial_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma)] \Big|_{\sigma=\sigma_1} = \partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma) \Big|_{\sigma=\sigma_1} \quad [3.218]$$

This is easily seen that noting that the derivatives $\partial_{z_{\sigma_1}}^{n-2}$ on the ρ^{-1} are zero when taken in σ_1 by the normal holomorphicity condition. We are then just left with a ρ^{-1} which at the

point σ_1 is equal to one by the same condition. We can thus write

$$\begin{aligned} \left. \frac{\partial^{n-1}}{\partial z_{\sigma_1}} \delta \ln \rho(\sigma) \right|_{\sigma=\sigma_1} &= \partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1) - \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \partial_{z_{\sigma_1}}^{n-2} [-4\rho^{-1} \partial_{z_{\sigma_1}} \partial_{\bar{z}_{\sigma_1}} \ln \rho(\sigma_1)] \\ &= \partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1) - \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \partial_{z_{\sigma_1}}^{n-2} R(\sigma_1) \\ &= \partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1) - \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \nabla_{z_{\sigma_1}}^{n-2} R(\sigma_1) \end{aligned} \quad [3.219]$$

We have here, immediately written the fact that all these quantities are taken at the point σ_1 and in the last line we can replace the ordinary derivatives by the covariant derivatives because as the only non-zero connection $\Gamma_{zz}^z(\sigma) = \partial_{z_{\sigma_1}} \ln \rho(\sigma)$ vanishes at σ_1 .

Requiring that both at σ_1 and at σ'_1 we have holomorphic normal coordinates implies that $\partial_{z_{\sigma_1}}^{n-1} \ln \rho(\sigma_1)$ vanishes and so [3.219] implies that

$$\partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1) = \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \nabla_{z_{\sigma_1}}^{n-2} R(\sigma_1) \quad [3.220]$$

Multiply both sides by $(z_{\sigma_1}(\sigma))^n/n!$ and sum from $n = 2$ to infinity

$$\sum_{n=2}^{\infty} \frac{1}{n!} (z_{\sigma_1}(\sigma))^n \partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1) = \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=2}^{\infty} \frac{1}{n!} (z_{\sigma_1}(\sigma))^n \nabla_{z_{\sigma_1}}^{n-2} R(\sigma_1) \quad [3.221]$$

We can rewrite the LHS as

$$LHS = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1)) (z_{\sigma_1}(\sigma))^n - \delta z_{\sigma_1}(\sigma_1) - (\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)) z_{\sigma_1}(\sigma) \quad [3.222]$$

The sum is just a Taylor expansion. To recognise it, recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n f(a) (x-a)^n \quad [3.223]$$

and Taylor expand $\delta z_{\sigma_1}(\sigma)$ around $z_{\sigma_1}(\sigma_1)$. We thus have $x = z_{\sigma_1}(\sigma)$ and $a = z_{\sigma_1}(\sigma_1)$, with $f \equiv \delta$. It follows that $x - a = z_{\sigma_1}(\sigma) - z_{\sigma_1}(\sigma_1)$. Now recall from [3.209] that the base point is chosen so that $z_{\sigma_1}(\sigma_1) = 0$, albeit it that $\delta z_{\sigma_1}(\sigma_1) = 0$ and its derivatives need not be zero. We thus have $x - a = x = z_{\sigma_1}(\sigma)$ and so

$$\delta z_{\sigma_1}(\sigma) = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma_1)) (z_{\sigma_1}(\sigma))^n \quad [3.224]$$

We therefore have

$$\delta z_{\sigma_1}(\sigma) - \delta z_{\sigma_1}(\sigma_1) - (\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)) z_{\sigma_1}(\sigma) = \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=2}^{\infty} \frac{1}{n!} (z_{\sigma_1}(\sigma))^n \nabla_{z_{\sigma_1}}^{n-2} R(\sigma_1) \quad [3.225]$$

or, equivalently

$$\delta z_{\sigma_1}(\sigma) = \delta z_{\sigma_1}(\sigma_1) + (\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)) z_{\sigma_1}(\sigma) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (z_{\sigma_1}(\sigma))^{n+1} \quad [3.226]$$

where we have changed the summation index $n \rightarrow n+1$. Let us see what we have achieved so far. We have found an expression for $\delta z_{\sigma_1}(\sigma)$ as a power series in $z_{\sigma_1}(\sigma)$ with coefficients that depend only on the base point σ_1 . In detail

$$\delta z_{\sigma_1}(\sigma) = \sum_{n=0}^{\infty} a_n(\sigma_1) (z_{\sigma_1}(\sigma))^n \quad [3.227]$$

with

$$\begin{aligned} a_0 &= \delta z_{\sigma_1}(\sigma_1) \\ a_1 &= \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \\ a_n &= \frac{\delta \bar{z}_{\sigma_1}(\sigma_1) \nabla_{z_{\sigma_1}}^{n-2} R(\sigma_1)}{4(n+1)!} \quad \text{for } n \geq 2 \end{aligned} \quad [3.228]$$

We now add $z_{\sigma_1}(\sigma)$ to both sides. On the left hand side we get $z_{\sigma_1}(\sigma) + \delta z_{\sigma_1}(\sigma)$, which by [3.214] is exactly $z_{\sigma'_1}(\sigma)$. On the RHS this means that the coefficient a_1 changes to $1 + \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)$. Now, to first order in the change δ we can rewrite this as $e^{\text{partial}_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)}$. We can also multiply a_0 and a_n for $n \geq 2$ with this exponential as these coefficients are already of the order $\delta z_{\sigma_1}(\sigma_1)$ and so the difference would just be a second order correction. The upshot of this is that we can write

$$z_{\sigma'_1}(\sigma) = e^{\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)} \sum_{n=0}^{\infty} \tilde{a}_n(\sigma_1) (z_{\sigma_1}(\sigma))^n \quad [3.229]$$

with

$$\begin{aligned} \tilde{a}_0 &= \delta z_{\sigma_1}(\sigma_1) \\ \tilde{a}_1 &= 1 \\ \tilde{a}_n &= \frac{\delta \bar{z}_{\sigma_1}(\sigma_1) \nabla_{z_{\sigma_1}}^{n-2} R(\sigma_1)}{4(n+1)!} \quad \text{for } n \geq 2 \end{aligned} \quad [3.230]$$

or explicitly

$$z_{\sigma'_1}(\sigma) = e^{\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)} \left[\delta z_{\sigma_1}(\sigma_1) + z_{\sigma_1}(\sigma) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (z_{\sigma_1}(\sigma))^{n+1} \right] \quad [3.231]$$

Let us now go back to [3.264] and evaluate it at σ_1 . Since $\rho(\sigma_1) = 1$ by the definition of holomorphic normal coordinates we have $\ln \rho(\sigma_1) = 0$. We furthermore have that all connections vanish at σ_1 , see [3.216]. Therefore

$$\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma_1) = 0 \quad [3.232]$$

For a general complex number $w = x + iy$ we have $w + \bar{w} = 2x = 2\operatorname{Re} w$, thus the above equation says that

$$\operatorname{Re} [\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)] = 0 \quad [3.233]$$

This means that we can write

$$\begin{aligned} \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) &= \operatorname{Re} [\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)] + i \operatorname{Im} [\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)] \\ &= i \operatorname{Im} [\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)] \end{aligned} \quad [3.234]$$

and [3.235] becomes

$$z_{\sigma'_1}(\sigma) = e^{i \operatorname{Im} [\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)]} \left[\delta z_{\sigma_1}(\sigma_1) + z_{\sigma_1}(\sigma) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (z_{\sigma_1}(\sigma))^{n+1} \right] \quad [3.235]$$

The exponential pre-factor is now just a phase factor. We will shove this phase factor under the rug, together with the one that determines the transition functions from coordinates in the conformal gauge to holomorphic normal coordinates, see [3.467] in the appendix to this chapter or for more details **[LS]**.

Our final result is then

$$z_{\sigma'_1}(\sigma) = \delta z_{\sigma_1}(\sigma_1) + z_{\sigma_1}(\sigma) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (z_{\sigma_1}(\sigma))^{n+1} \quad [3.236]$$

This is the holomorphic transition function between two charts with nearby base points σ_1 and σ'_1 . It is holomorphic in σ , any any point in the overlap of the two charts, but due to the appearance of $\delta \bar{z}_{\sigma_1}(\sigma_1)$ it is not holomorphic in the base points σ_1 and σ'_1 .

THE DERIVATIVE OF A NORMAL ORDERED OPERATOR

Using the result [3.236] and the chain rule we can now work out the derivative w.r.t. the base point σ_1 for a given point σ

$$\left. \frac{\partial}{\partial z_{\sigma_1}(\sigma_1)} \right|_{\sigma} = \left. \frac{\partial z_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)} \right|_{\sigma} \frac{\partial}{\partial z_{\sigma_1}(\sigma)} + \left. \frac{\partial \bar{z}_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)} \right|_{\sigma} \frac{\partial}{\partial \bar{z}_{\sigma_1}(\sigma)} \quad [3.237]$$

What does this mean? $\partial z_{\sigma_1}(\sigma)/\partial z_{\sigma_1}(\sigma_1)$ is just the how $z_{\sigma_1}(\sigma)$ varies when we vary the the base point. We can read this off from [3.236]. It is just

$$\begin{aligned} \frac{\partial z_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)} &= \lim_{\delta z_{\sigma_1}(\sigma_1) \rightarrow 0} \frac{z_{\sigma'_1}(\sigma) - z_{\sigma_1}(\sigma)}{\delta z_{\sigma_1}(\sigma_1)} \\ &= \lim_{\delta z_{\sigma_1}(\sigma_1) \rightarrow 0} \frac{\delta z_{\sigma_1}(\sigma_1) + \frac{1}{4} \delta \bar{z}_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{z_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (z_{\sigma_1}(\sigma))^{n+1}}{\delta z_{\sigma_1}(\sigma_1)} \\ &= 1 \end{aligned} \quad [3.238]$$

But we have also a non-vanishing term for the complex conjugate of [3.236]

$$\begin{aligned} \frac{\partial \bar{z}_{\sigma_1}(\sigma)}{\partial z_{\sigma_1}(\sigma_1)} &= \lim_{\delta z_{\sigma_1}(\sigma_1) \rightarrow 0} \frac{\bar{z}_{\sigma'_1}(\sigma) - \bar{z}_{\sigma_1}(\sigma)}{\delta z_{\sigma_1}(\sigma_1)} \\ &= \lim_{\delta z_{\sigma_1}(\sigma_1) \rightarrow 0} \frac{\delta \bar{z}_{\sigma_1}(\sigma_1) + \frac{1}{4} \delta z_{\sigma_1}(\sigma_1) \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (\bar{z}_{\sigma_1}(\sigma))^{n+1}}{\delta z_{\sigma_1}(\sigma_1)} \\ &= + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (\bar{z}_{\sigma_1}(\sigma))^{n+1} \end{aligned} \quad [3.239]$$

and thus

$$\left. \frac{\partial}{\partial z_{\sigma_1}(\sigma_1)} \right|_{\sigma} = \frac{\partial}{\partial z_{\sigma_1}(\sigma)} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} (\bar{z}_{\sigma_1}(\sigma))^{n+1} \frac{\partial}{\partial \bar{z}_{\sigma_1}(\sigma)} \quad [3.240]$$

In this expression we recognise the Virasoro generators

$$L_n^{(z_{\sigma_1})} = -(z_{\sigma_1}(\sigma))^{n+1} \frac{\partial}{\partial z_{\sigma_1}(\sigma)} \quad \text{and} \quad \tilde{L}_n^{(z_{\sigma_1})} = -(\bar{z}_{\sigma_1}(\sigma))^{n+1} \frac{\partial}{\partial \bar{z}_{\sigma_1}(\sigma)} \quad [3.241]$$

We have explicitly added the superscript (z_{σ_1}) to remind ourselves that these are the Virasoro generators for the patch around the base point σ_1 . We can thus write

$$\left. \frac{\partial}{\partial z_{\sigma_1}(\sigma_1)} \right|_{\sigma} = -L_{-1}^{(z_{\sigma_1})} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n^{(\bar{z}_{\sigma_1})} \quad [3.242]$$

This is an expression to the derivative w.r.t. to the variation base point keeping the point σ fixed, i.e. w.r.t. $\delta z_{\sigma_1}(\sigma_1) = (z_{\sigma'_1}(\sigma) - z_{\sigma_1}(\sigma))|_{\sigma=\sigma_1}$. This is a so-called passive variation where we move the frame of reference σ_1 to σ'_1 and keep the coordinate σ fixed. We can obtain the active variation of changing the coordinate but keeping the base point fixed

simply by introducing a minus sign. We thus have that the derivative w.r.t. coordinate for a fixed base point is

$$\partial_{z_{\sigma_1}} = L_{-1}^{(z_{\sigma_1})} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}_{\sigma_1}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n^{(\bar{z}_{\sigma_1})} \quad [3.243]$$

From here on we are working with a fixed base-point σ_1 and so it is superfluous to keep on reminding ourselves that the coordinates z are taking with base point σ_1 . The above equation thus becomes

$$\partial_z = L_{-1} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n \quad [3.244]$$

Let us now see what the corresponding derivative ∂_z does when acting on a local operator that is Weyl normal ordered and is inserted at the base point σ_1 with holomorphic normal coordinates z , viz an operator ${}^{\circ}\mathcal{A}(\sigma_1)_{\circ z}$. We have

$$\partial_z {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} = \left[L_{-1} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n \right] {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} \quad [3.245]$$

We can evaluate $L_{-1} {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z}$ via the OPE of the energy-momentum tensor with the operator ${}^{\circ}\mathcal{A}(\sigma_1)_{\circ z}$

$$L_{-1} {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} = \oint \frac{dz}{2\pi} T(z) \mathcal{A}(w) = {}^{\circ}\partial_z {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} \quad [3.246]$$

and so we just pick up the residue of the simple pole of the simple pole of the OPE which is, as we have seen before by translation invariance ${}^{\circ}\partial_z {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z}$.

We have now achieved our end result for this part. We know how to take the derivative of a normal ordered local operator:

$$\partial_z {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} = {}^{\circ}\partial_z {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n {}^{\circ}\mathcal{A}(\sigma_1)_{\circ z} \quad [3.247]$$

The key point here, and it is worth emphasising it, is that one cannot just take a derivative in a normal ordered product. There is an additional contribution that depends on the anti-holomorphic part of the theory. Note also that these results are very general. They are valid for an arbitrary background and CFT with total central charge zero. These results are also valid off-shell.

AN EXPLICIT EXAMPLE

After all this hard work, let us consider a specific example. We wish to compute

$$\mathfrak{T}^\mu(\sigma_1) = \nabla_a \otimes \nabla^a X^\mu e^{ik \cdot X}(\sigma_1) \otimes_z \quad [3.248]$$

As at the end of the last section, we have not written all the necessary indices, but we are working with holomorphic normal coordinates at base point σ_1 . This means that covariant derivatives $\nabla_z = \nabla_{z_{\sigma_1}}$ are equal to ordinary derivatives and $\rho(\sigma_1) = 1$ with all its derivatives at σ_1 being zero. Replacing the covariant derivative by ordinary derivatives, going to complex coordinates, and using our formula [3.247] for the derivative of a normal ordered operator, we find

$$\begin{aligned} \mathfrak{T}^\mu(\sigma_1) &= 4 \partial_z \otimes \partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \otimes_z \\ &= 4 \otimes \partial_z \left(\partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \right) \otimes_z + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n 4 \otimes \partial_z X^\mu e^{ik \cdot X}(\sigma_1) \otimes_z \end{aligned} \quad [3.249]$$

We now use the OPE to compute

$$\tilde{L}_n \otimes \partial_z X^\mu e^{ik \cdot X}(\sigma_1) \otimes_z = \oint_{\mathcal{C}_z} \frac{dw}{2\pi i} (w-z)^{n+1} \tilde{T}(\bar{w}) \otimes \partial_z X^\mu e^{ik \cdot X}(\sigma_1) \otimes_z \quad [3.250]$$

Let us consider first what we need before we start blindly calculating. Because $n \geq 1$ we need a pole in the OPE $\tilde{T}(\bar{w}) \partial_z X^\mu e^{ik \cdot X}(z)$ of order three or higher to have a non-zero contribution. Here we have not written the normal ordering symbols as these are understood in OPEs. We will also focus further on the holomorphic twin of this expressions, and put conjugates at the end. We thus need the third and higher order poles of

$$T(w) \partial X^\mu e^{ik \cdot X}(z) = -\frac{1}{\alpha'} \partial X^\nu \partial X_\nu(w) \partial X^\mu e^{ik \cdot X}(z) \quad [3.251]$$

The highest order pole we get is when we contract one of the $\partial X^\nu(w)$ with the $\partial X^\mu(z)$ and the other $\partial X^\nu(w)$ with an $X^\sigma(z)$ from the exponential. Using

$$\begin{aligned} X^\mu(w) X^\nu(z) &= -\frac{1}{2} \alpha' \ln |w-z|^2 \\ \partial X^\mu(w) X^\nu(z) &= -\frac{1}{2} \alpha' \frac{1}{w-z} \\ \partial X^\mu(w) \partial X^\nu(z) &= -\frac{1}{2} \alpha' \frac{1}{(w-z)^2} \end{aligned} \quad [3.252]$$

we see that the highest order pole is a third order pole and is given by

$$\begin{aligned} T(w) \partial X^\mu e^{ik \cdot X}(z) &= -\frac{1}{\alpha'} \left[2 \left(-\frac{1}{2} \alpha' \frac{\eta^{\mu\nu}}{(w-z)^2} \right) \left(-\frac{1}{2} \alpha' \frac{ik^\sigma \eta_{\nu\sigma}}{w-z} \right) e^{ik \cdot X}(z) \right] + \dots \\ &= -\frac{ik^\mu \alpha'}{2} \frac{1}{(w-z)^3} e^{ik \cdot X}(z) + \dots \end{aligned} \quad [3.253]$$

Thus

$$\begin{aligned}
L_n \circledast \partial_z X^\mu e^{ik \cdot X}(\sigma_1) \circledast_z &= \oint_{C_z} \frac{dw}{2\pi i} (w-z)^{n+1} \left[-\frac{ik^\mu \alpha'}{2} \frac{1}{(w-z)^3} e^{ik \cdot X}(z) + \dots \right] \\
&= -\frac{i\alpha' k^\mu}{2} \delta_{n,1} \oint_{C_z} \frac{dw}{2\pi i (w-z)} e^{ik \cdot X}(z) \\
&= -\frac{i\alpha' k^\mu}{2} \delta_{n,1} e^{ik \cdot X}(z)
\end{aligned} \tag{3.254}$$

We see that the result is the same for the conjugate twin and therefore

$$\begin{aligned}
\mathfrak{T}^\mu(\sigma_1) &= 4 \circledast \partial_z \left(\partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \right) \circledast_z + \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}}^{n-1} R(\sigma_1)}{(n+1)!} \left(-\frac{i\alpha' k^\mu}{2} \delta_{n,1} e^{ik \cdot X}(z) \right) \\
&= 4 \circledast \partial_z \left(\partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \right) \circledast_z - \frac{i\alpha' k^\mu}{4} R(\sigma_1) \circledast e^{ik \cdot X}(\sigma_1) \circledast_z
\end{aligned} \tag{3.255}$$

In the last line we have once more added the bells and whistles of the normal ordering. Consider now the first term. From [3.246] we have

$$4 \circledast \partial_z \left(\partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \right) \circledast_z = 4 L_{-1} \circledast \partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \circledast_z \tag{3.256}$$

We now need

$$L_{-1} \partial_{\bar{z}} X^\mu e^{ik \cdot X}(z) = \oint_{C_z} \frac{dw}{2\pi i} (w-z)^{-1+1} T(w) \partial_{\bar{z}} X^\mu e^{ik \cdot X}(z) \tag{3.257}$$

The only non-vanishing contribution comes from the single pole in the OPE $T(w) \partial_{\bar{z}} X^\mu e^{ik \cdot X}(z)$, which is easily computed as we know that $e^{ik \cdot X}$ is a primary field, and hence so is $\partial_{\bar{z}} X^\mu e^{ik \cdot X}(z)$ as $T(w) \partial_{\bar{z}} X^\mu(\bar{z})$ has no singular parts in the contour. We can thus write down immediately

$$L_{-1} \partial_{\bar{z}} X^\mu e^{ik \cdot X}(z) = \oint_{C_z} \frac{\partial_{\bar{z}} X^\mu \partial_z e^{ik \cdot X}(w)}{w-z} = \circledast \partial_{\bar{z}} X^\mu \partial_z e^{ik \cdot X}(z) \circledast_z \tag{3.258}$$

where we have added the bells and whistles of normal ordering again at the end. This is actually nothing but a complicated and rigorous way to explain that bring the ∂_z through the $\partial_{\bar{z}} X^\mu$ in a normal ordering, i.e. that $\partial_{\bar{z}} X^\mu$ is anti-holomorphic.

Recall that we are using holomorphic normal coordinates so that we can replace the ordinary derivatives by covariant ones. Thus

$$4 \circledast \partial_z \left(\partial_{\bar{z}} X^\mu e^{ik \cdot X}(\sigma_1) \right) \circledast_z = 4 \circledast \partial_{\bar{z}} X^\mu \partial_z e^{ik \cdot X}(z) \circledast_z = \circledast \nabla_a X^\mu \nabla^a e^{ik \cdot X}(\sigma_1) \circledast_z \tag{3.259}$$

We thus find our final result for $\mathfrak{T}^\mu(\sigma_1)$:

$$\nabla_a \circledast \nabla^a X^\mu e^{ik \cdot X}(\sigma_1) \circledast_z = \circledast \nabla_a X^\mu \nabla^a e^{ik \cdot X}(\sigma_1) \circledast_z + \frac{i\alpha' \gamma k^\mu}{4} R(\sigma_1) \circledast e^{ik \cdot X}(\sigma_1) \circledast_z \tag{3.260}$$

Where we have introduced a parameter γ that is equal to -1 in our case.

WEYL TRANSFORMATIONS IN WEYL NORMAL ORDERING

Let us consider a patch with base point σ_1 holomorphic normal coordinates $z_{\sigma_1}(\sigma)$. We now wish to perform an infinitesimal but general holomorphic change of coordinates

$$z_{\sigma_1}(\sigma) \rightarrow w_{\sigma_1}(\sigma) = z_{\sigma_1}(\sigma) + \delta z_{\sigma_1}(\sigma) = z_{\sigma_1}(\sigma) + \sum_{n=0}^{\infty} \varepsilon_n (z_{\sigma_1}(\sigma))^n \quad [3.261]$$

Note that $\delta z_{\sigma_1}(\sigma)$ has a different meaning than in [3.214]. Here it is a change of coordinates with a fixed base point σ_1 , whereas previously it denoted the impact on the coordinate of a change of base point σ_1 to σ'_1 .

As we are working in holomorphic normal coordinates, the following relations are valid, see [3.210] and [3.211]

$$ds^2 = \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) dz_{\sigma_1} d\bar{z}_{\sigma_1} \quad \text{with} \quad \partial_{z_{\sigma_1}}^n \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) \Big|_{\sigma=\sigma_1} = \delta_{n,0} \quad [3.262]$$

Our aim is to find out how vertex operators transform under Weyl transformations – remember that we have a regularisation scheme, either Pauli-Villars or dimensional regularisation, that is manifestly diffeomorphism invariant and so the survival of the Weyl invariance in the quantum theory is what should concern us – so let us find the coefficient ε_n for a Weyl transformation.

Consider, therefore, the Weyl transformation

$$ds^2 \rightarrow d\hat{s}^2 = e^{\delta\phi(\sigma_1)} \rho(z_{\sigma_1}, \bar{z}_{\sigma_1}) dz_{\sigma_1} d\bar{z}_{\sigma_1} \quad [3.263]$$

In order to satisfy the nitpickers amongst us, let us point out that by $\delta\phi(\sigma_1)$ we actually mean $\delta\phi(z_{\sigma_1}(\sigma), \bar{z}_{\sigma_1}(\sigma))$. From [3.264] we have that, using $\ln \rho = \delta\phi$

$$\delta\phi(\sigma) = \nabla_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma) + \nabla_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma) \quad [3.264]$$

Because both z_{σ_1} and w_{σ_1} are chosen to be holomorphic normal coordinates we have $z_{\sigma_1}(\sigma_1) = w_{\sigma_1}(\sigma_1) = 0$ and hence also $\delta z_{\sigma_1}(\sigma_1) = 0$. But recall that the derivatives of $\delta z_{\sigma_1}(\sigma)$ taken at *sic* are not necessarily zero.

In order to compute $\delta z_{\sigma_1}(\sigma)$ we perform the same trick as before: we take the $(n-1)$ th derivative of $\delta z_{\sigma_1}(\sigma)$ evaluated at $\sigma\sigma_1$, multiply it by $(z_{\sigma_1}(\sigma))^n/n!$ and sum from $n=2$ to infinity

$$\sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \delta\phi(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n = \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \left(\nabla_{z_{\sigma_1}} \delta z_{\sigma_1} + \nabla_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1} \right) (\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n \quad [3.265]$$

We now use [3.216] for the non-vanishing connections and the RHS becomes

$$\begin{aligned}
RHS &= \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \left(\partial_{z_{\sigma_1}} \delta z_{\sigma_1} + (\partial_{z_{\sigma_1}} \ln \rho) \delta z_{\sigma_1} + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1} + (\partial_{\bar{z}_{\sigma_1}} \ln \rho) \delta \bar{z}_{\sigma_1} \right) (\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n \\
&= \sum_{n=2}^{\infty} \frac{1}{n!} \left(\partial_{z_{\sigma_1}}^n \delta z_{\sigma_1} + (\partial_{z_{\sigma_1}}^n \ln \rho) \delta z_{\sigma_1} + (\partial_{z_{\sigma_1}} \ln \rho) \partial_{z_{\sigma_1}}^{n-1} \delta z_{\sigma_1} \right. \\
&\quad \left. + (\partial_{z_{\sigma_1}}^{n-1} \partial_{\bar{z}_{\sigma_1}} \ln \rho) \delta \bar{z}_{\sigma_1} + (\partial_{\bar{z}_{\sigma_1}} \ln \rho) \partial_{z_{\sigma_1}}^{n-1} \delta \bar{z}_{\sigma_1} \right) (\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n \\
&= \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n \tag{3.266}
\end{aligned}$$

Indeed, all but one term survives due to the choice of holomorphic normal coordinates and the fact that $\delta \bar{z}_{\sigma_1}$ is anti-holomorphic. We complete the sum and extract the Taylor expansion to find

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n &= \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^n \delta z_{\sigma_1}(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n - \delta z_{\sigma_1}(\sigma_1) - (\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)) z_{\sigma_1}(\sigma) \\
&= \delta z_{\sigma_1}(\sigma) - (\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)) z_{\sigma_1}(\sigma) \tag{3.267}
\end{aligned}$$

where we have also used $\delta z_{\sigma_1}(\sigma_1) = 0$. We thus have

$$\delta z_{\sigma_1}(\sigma) = (\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)) z_{\sigma_1}(\sigma) + \sum_{n=2}^{\infty} \frac{1}{n!} \partial_{z_{\sigma_1}}^{n-1} \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^n \tag{3.268}$$

We add $z_{\sigma_1}(\sigma)$ to both sides and use the fact that $\delta z_{\sigma_1}(\sigma) + z_{\sigma_1}(\sigma) = w_{\sigma_1}(\sigma)$ to find

$$\begin{aligned}
w_{\sigma_1}(\sigma) &= [1 + \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)] z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^n \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^{n+1} \\
&= e^{\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1)} \left[z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^n \delta \phi(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^{n+1} \right] \tag{3.269}
\end{aligned}$$

The last equation is correct to first order in $\delta \phi$ and thus also δz . The argument of the exponential can be written as

$$\begin{aligned}
\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) &= \operatorname{Re} \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) + i \operatorname{Im} \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \\
&= \frac{1}{2} [\partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) + \partial_{\bar{z}_{\sigma_1}} \delta \bar{z}_{\sigma_1}(\sigma_1)] + i \operatorname{Im} \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \\
&= \frac{1}{2} \delta \phi(\sigma_1) + i \operatorname{Im} \partial_{z_{\sigma_1}} \delta z_{\sigma_1}(\sigma_1) \tag{3.270}
\end{aligned}$$

In the last line we have used [3.264] and the fact that at $\sigma = \sigma_1$ the connections vanish. The imaginary part just gives a phase, which we can, just as before, ignore. We thus find our final result, the transition function for a holomorphic change of coordinates $z_{\sigma_1} \rightarrow w_{\sigma_1}(z_{\sigma_1})$, with fixed base point, corresponding to a Weyl transformation

$$w_{\sigma_1}(\sigma) = e^{\frac{1}{2}\delta\phi(\sigma_1)} \left[z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^n \delta\phi(\sigma) \Big|_{\sigma=\sigma_1} (z_{\sigma_1}(\sigma))^{n+1} \right] \quad [3.271]$$

We can thus write for a Weyl transformation

$$z_{\sigma_1}(\sigma) \rightarrow w_{\sigma_1}(\sigma) = z_{\sigma_1}(\sigma) + \sum_{n=0}^{\infty} \varepsilon_n (z_{\sigma_1}(\sigma))^n \quad [3.272]$$

with

$$\begin{aligned} \varepsilon_0 &= 0 \\ \varepsilon_1 &= \frac{1}{2}\delta\phi \\ \varepsilon_n &= e^{\frac{1}{2}\delta\phi(\sigma_1)} \frac{1}{(n)!} \partial_{z_{\sigma_1}}^{n-1} \delta\phi(\sigma) \Big|_{\sigma=\sigma_1} \quad \text{for } n \geq 2 \end{aligned} \quad [3.273]$$

From standard CFT we know that local operator transforms under a general conformal transformation as

$$\circledast \mathcal{A}^{(w_{\sigma_1})}(\sigma_1)_{\circledast w_{\sigma_1}} = \circledast \mathcal{A}^{(z_{\sigma_1})}(\sigma_1)_{\circledast z_{\sigma_1}} - \sum_{n=0}^{\infty} \left(\varepsilon_n L_n^{(z_{\sigma_1})} + \tilde{\varepsilon}_n \tilde{L}_n^{(z_{\sigma_1})} \right) \circledast \mathcal{A}^{(z_{\sigma_1})}(\sigma_1)_{\circledast z_{\sigma_1}} \quad [3.274]$$

As we have the ε_n from [3.273] and we can evaluate $L_n^{(z_{\sigma_1})} \circledast \mathcal{A}^{(z_{\sigma_1})}(\sigma_1)_{\circledast z_{\sigma_1}}$ by contour operation of the OPE, we thus have an explicit expression of how a local operator changes under a Weyl transformation.

WEYL TRANSFORMATIONS OF THE PROPAGATOR

We are now in a position to really start working on showing that Weyl invariance of the operator V_1 requires the conditions (3.6.16). First we need to work out the Weyl transformations of the propagator $\Delta(\sigma', \sigma)$ and derivatives thereof in our scheme. We have

$$\Delta(\sigma', \sigma) = \frac{\alpha'}{2} \ln |z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)|^2 \quad [3.275]$$

As per our scheme, the geodesic distance is taken with fixed base point. We wish to find the change of this under a Weyl transformation, i.e.

$$\begin{aligned}\delta_W \Delta(\sigma', \sigma) &= \frac{\alpha'}{2} \ln |w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)|^2 - \frac{\alpha'}{2} \ln |z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)|^2 \\ &= \frac{\alpha'}{2} \ln \left| \frac{w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)} \right|^2\end{aligned}\quad [3.276]$$

We have worked out an expression for w_{σ_1} under a Weyl transformation in [3.271] which we write to first order as

$$w_{\sigma_1}(\sigma) = \left(1 + \frac{1}{2}\delta\phi(\sigma_1)\right) z_{\sigma_1}(\sigma) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^n \delta\phi(\sigma_1) (z_{\sigma_1}(\sigma))^{n+1} \quad [3.277]$$

We have written $\delta\phi(\sigma_1)$ but it is, of course, understood that we first take the appropriate number of derivatives of $\delta\phi(\sigma)$ and only afterwards take $\sigma = \sigma_1$. Therefore

$$\begin{aligned}\frac{w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)} &= [z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)]^{-1} \left\{ \left(1 + \frac{1}{2}\delta\phi(\sigma_1)\right) [z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)] \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^n \delta\phi(\sigma_1) [(z_{\sigma_1}(\sigma'))^{n+1} - (z_{\sigma_1}(\sigma))^{n+1}] \right\} \\ &= 1 + \frac{1}{2}\delta\phi(\sigma_1) + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \partial_{z_{\sigma_1}}^n \delta\phi(\sigma_1) \frac{(z_{\sigma_1}(\sigma'))^{n+1} - (z_{\sigma_1}(\sigma))^{n+1}}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)}\end{aligned}\quad [3.278]$$

Let us now look at the fraction in the sum. In the end we will be interested in the limit $\sigma' = \sigma = \sigma_1$. First we have

$$\lim_{x \rightarrow y} \frac{x^{n+1} - y^{n+1}}{x - y} = \lim_{x \rightarrow y} \sum_{k=0}^2 x^k y^{n-k} = (n+1)y^n \quad [3.279]$$

We will thus end up with $n(z_{\sigma_1}(\sigma))^n$, which we will take at $\sigma = \sigma_1$ and hence this vanishes. Therefore at $\sigma' = \sigma = \sigma_1$

$$\ln \frac{w_{\sigma_1}(\sigma') - w_{\sigma_1}(\sigma)}{z_{\sigma_1}(\sigma') - z_{\sigma_1}(\sigma)} = \ln \left(1 + \frac{1}{2}\delta\phi(\sigma_1)\right) = \frac{1}{2}\delta\phi(\sigma_1) \quad [3.280]$$

We have a similar contribution of the conjugate part and thus

$$\delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{2} \times 2 \times \frac{1}{2}\delta\phi(\sigma_1) \quad [3.281]$$

This deserves its standalone formula

$$\delta_W \Delta(\sigma, \sigma') \Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{2} \delta\phi(\sigma_1) \quad [3.282]$$

Recalling that our $\delta\phi$ is equal to the $2\delta\omega$ of Joe, we recover (3.6.11).

Let us now work out $\partial_{z'} \delta_W \Delta(\sigma', \sigma)$ in our scheme. As we will see, this has become pretty easy in this scheme. Using [3.276] and [3.278] we have

$$\begin{aligned} \delta_W \Delta(\sigma', \sigma) &= \frac{\alpha'}{2} \ln \left| 1 + \frac{1}{2} \delta\phi(\sigma_1) + \sum_{n=1}^{\infty} \frac{\partial_{z_{\sigma_1}}^n \delta\phi(\sigma_1)}{(n+1)!} \sum_{k=0}^n (z_{\sigma_1}(\sigma'))^k (z_{\sigma_1}(\sigma))^{n-k} \right|^2 \\ &= \frac{\alpha'}{2} \left[\frac{1}{2} \delta\phi(\sigma_1) + \sum_{n=1}^{\infty} \frac{\partial_{z_{\sigma_1}}^n \delta\phi(\sigma_1)}{(n+1)!} \sum_{k=0}^n (z_{\sigma_1}(\sigma'))^k (z_{\sigma_1}(\sigma))^{n-k} + \text{c.c.} \right] \end{aligned} \quad [3.283]$$

where c.c. stands for complex conjugate. We now need to take the $z' = z_{\sigma_1}(\sigma')$ derivative of this and then put $\sigma' = \sigma = \sigma_1$. There is only one term in the expression that then remains, it is the term with $k = 1$ and $n = 1$ and this gives

$$\partial_{z'} \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{2} \left[\frac{\partial_{z_{\sigma_1}} \delta\phi(\sigma_1)}{2} \right] \quad [3.284]$$

Again, this deserves a standalone formula

$$\partial_{z'} \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma=\sigma_1} = \frac{\alpha'}{4} \partial_z \delta\phi(\sigma_1) \quad [3.285]$$

This is (3.6.15a) with again $\delta\phi = 2\delta\omega$

Let us do one more derivative: $\partial_{z'} \partial_{\bar{z}} \delta_W \Delta(\sigma', \sigma)$. This is pretty simple from [3.283]. There are no mixed z' and $\bar{z}$ terms so that expression vanishes

$$\partial_{z'} \partial_{\bar{z}} \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma=\sigma_1} = 0 \quad [3.286]$$

which is (3.6.15b) for $\gamma = -1$.

WEYL TRANSFORMATIONS OF THE VERTEX OPERATOR V_1

We can now calculate the Weyl Variation of the operator V_1 in the new renormalisation scheme. Recall that we already did this for using the renormalisation in Joe's book. But in order to do that we had to assume (3.6.18) was valid. In our new renormalisation scheme we can calculate this Weyl variation, without having to assume (3.6.18). As we have already done most of the calculation, we will not repeat everything. The reader is referred to the derivation of (3.6.16) on p105 for all details.

We can immediately go to [3.175]

$$\begin{aligned} \delta_W V_1 = & \frac{g_c}{\alpha'} \int d^2\sigma \left\{ -2\alpha' \tilde{\phi} \sqrt{g} (\nabla^2 \delta\omega) \left[e^{ik \cdot X(\sigma)} \right]_w \right. \\ & + \frac{1}{2} \sqrt{g} (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X(\sigma)} \right]_w \\ & \left. + \frac{1}{2} \sqrt{g} \alpha' \tilde{\phi} R \int d^2\sigma' d^2\sigma'' \delta_W \Delta(\sigma', \sigma'') \frac{\delta}{\delta X^\lambda(\sigma')} \frac{\delta}{\delta X_\lambda(\sigma'')} \left[e^{ik \cdot X(\sigma)} \right]_w \right\} \end{aligned} \quad [3.287]$$

We are denoting the renormalisation based on Weyl normal ordering by $[\cdots]_w$. We are taking into account that due to the fact that we are using a different renormalisation scheme, we have assumed a different $\tilde{\phi}$. Once more we take the three lines separately. The first line gives [3.177]

$$\mathcal{J}_1 = -2g_c \tilde{\phi} \int d^2\sigma \delta\omega \sqrt{g} \nabla^a \partial_a \left[e^{ik \cdot X(\sigma)} \right]_w \quad [3.288]$$

We know that we need to be very careful now with derivatives and Weyl normal ordering. Let us first check $\partial_z [e^{ik \cdot X(\sigma)}]_w$. Eq [3.247] tells us how to bring a derivative in a Weyl normal ordered product

$$\partial_z \left[e^{ik \cdot X(\sigma)} \right]_w = \left[\partial_z e^{ik \cdot X(\sigma)} \right]_w + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\nabla_{\bar{z}}^{n-1} R(\sigma_1)}{(n+1)!} \tilde{L}_n \left[e^{ik \cdot X(\sigma)} \right]_w \quad [3.289]$$

Let us first check the second term. We have

$$\begin{aligned} \tilde{L}_n \left[e^{ik \cdot X(\sigma)} \right]_w &= \oint_{C_z} \frac{dw}{2\pi i} (w-z)^{n+1} T(w) e^{ik \cdot X(z)} \\ &= \oint_{C_z} \frac{dw}{2\pi i} (w-z)^{n+1} \left[\frac{(\alpha' k^2/4) e^{ik \cdot X(z)}}{(z-w)^2} + \frac{\partial_z e^{ik \cdot X(z)}}{z-w} \right] \end{aligned} \quad [3.290]$$

The first term of the OPE gives a $(z-w)^{n-1}$ and the second a $(z-w)^n$. For $n \geq 1$ neither of these contributions give a pole and so

$$\tilde{L}_n \left[e^{ik \cdot X(\sigma)} \right]_w = 0 \quad \text{for } n \geq 1 \quad [3.291]$$

and therefore

$$\partial_z \left[e^{ik \cdot X(\sigma)} \right]_w = \left[\partial_z e^{ik \cdot X(\sigma)} \right]_w \quad [3.292]$$

The same holds, of course, for its complex conjugate and hence

$$\begin{aligned}\nabla^a \partial_a \left[e^{ik \cdot X(\sigma)} \right]_{\text{w}} &= \nabla^a \left[\partial_a e^{ik \cdot X(\sigma)} \right]_{\text{w}} = \nabla^a \left[ik_\mu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_{\text{w}} \\ &= ik_\mu \nabla^a \left[\nabla_a X^\mu e^{ik \cdot X(\sigma)} \right]_{\text{w}}\end{aligned}\quad [3.293]$$

But now comes the fruit of all our hard work, because we already calculated this in [3.260]

$$\nabla^a \left[\nabla_a e^{ik \cdot X(\sigma)} \right]_{\text{w}} = ik_\nu \left[\partial_a X^\mu \partial^a X^\nu e^{ik \cdot X(\sigma)} \right]_{\text{w}} + \frac{i\alpha' \gamma k^\mu}{4} R(\sigma_1) \left[e^{ik \cdot X} \right]_{\text{w}} \quad [3.294]$$

We thus find

$$\begin{aligned}\mathcal{J}_1 &= -2g_c \tilde{\phi} \int d^2\sigma \delta\omega \sqrt{g} ik_\mu \left[ik_\nu \left[\partial_a X^\mu \partial^a X^\nu e^{ik \cdot X(\sigma)} \right]_{\text{w}} + \frac{i\alpha' \gamma k^\mu}{4} R(\sigma_1) \left[e^{ik \cdot X} \right]_{\text{w}} \right] \\ &= \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega \left\{ \left(\gamma \alpha' k^2 \tilde{\phi} R \right) \left[e^{ik \cdot X} \right]_{\text{w}} \right. \\ &\quad \left. + \left(4k_\mu k_\nu \tilde{\phi} \right) \left[\partial_a X^\mu \partial^a X^\nu e^{ik \cdot X} \right]_{\text{w}} \right\}\end{aligned}\quad [3.295]$$

Note that this is exactly the same result as in our previous calculation, [3.180] if we set $\gamma = -2/3$ and $\tilde{\phi} = \phi$ as per Joe's book.

The third line is identical to the previous result, as there are no derivatives to mess it up:

$$\mathcal{J}_3 = \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega \left(-\alpha' k^2 \tilde{\phi} R \right) \left[e^{ik \cdot X} \right]_{\text{w}} \quad [3.296]$$

Let us finally focus our attention to the second line. If both functional derivatives act on the exponential we have no derivatives to worry about and we can, once more, just take over the result from our previous calculation, [3.185]

$$\mathcal{J}_{2a} = \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega \left(g^{ab} (-k^2 s_{\mu\nu}) + i\epsilon^{ab} (-k^2 a_{\mu\nu}) \right) \left[\partial_a X^\mu \partial_b X^\nu e^{ik \cdot X} \right]_{\text{w}} \quad [3.297]$$

If one functional derivative acts on the exponential and the other on ∂X , then the previous calculation is valid up to the point where we will apply derivatives on the Weyl normal ordered local operators. This means that we can just take over up to [3.193]

$$\begin{aligned}\mathcal{J}_{2b} &= -\frac{ig_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \left\{ (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\mu \nabla_a \left[\partial_b X^\nu e^{ik \cdot X} \right]_{\text{w}} \right. \\ &\quad \left. + (g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) k^\nu \nabla_b \left[\partial_a X^\mu e^{ik \cdot X} \right]_{\text{w}} \right\}\end{aligned}\quad [3.298]$$

We will consider the symmetric and antisymmetric part separately

$$\mathcal{J}_{2b}^s = -\frac{ig_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \left\{ s_{\mu\nu} k^\mu \nabla^a \left[\nabla_a X^\nu e^{ik \cdot X} \right]_w + s_{\mu\nu} k^\nu \nabla^a \left[\nabla_a X^\mu e^{ik \cdot X} \right]_w \right\} \quad [3.299]$$

We can again use [3.260] and the symmetry of $s_{\mu\nu}$

$$\begin{aligned} \mathcal{J}_{2b}^s &= -ig_c \int d^2\sigma \delta\omega \sqrt{g} s_{\mu\nu} k^\mu \left\{ \left[\nabla_a X^\nu \nabla^a e^{ik \cdot X} \right]_w + \frac{i\alpha' \gamma k^\nu}{4} R \left[e^{ik \cdot X} \right]_w \right\} \\ &= -ig_c \int d^2\sigma \sqrt{g} \delta\omega s_{\mu\nu} k^\mu \left\{ ik_\lambda \left[\partial_a X^\nu \partial^a X^\lambda e^{ik \cdot X} \right]_w + \frac{i\alpha' \gamma k^\nu}{4} R \left[e^{ik \cdot X} \right]_w \right\} \end{aligned} \quad [3.300]$$

Note that we can simplify the first contribution as, ignoring pre-factors,

$$\begin{aligned} s_{\mu\nu} k^\mu k_\lambda \left[\partial_a X^\nu \partial^a X^\lambda e^{ik \cdot X} \right]_w &= g^{ab} s_{\mu\nu} k^\mu k_\lambda \left[\partial_a X^\nu \partial^b X^\lambda e^{ik \cdot X} \right]_w \\ &= \frac{1}{2} g^{ab} \left(s_{\lambda\nu} k^\lambda k_\mu + s_{\mu\lambda} k^\lambda k^\nu \right) \left[\partial_a X^\mu \partial^b X^\nu e^{ik \cdot X} \right]_w \end{aligned} \quad [3.301]$$

We thus find for the symmetric part

$$\begin{aligned} \mathcal{J}_{2b}^s &= \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega \left\{ \alpha' R \left(\frac{\gamma}{2} s_{\mu\nu} k^\mu k^\nu \right) \left[e^{ik \cdot X} \right]_w \right. \\ &\quad \left. + g^{ab} \left(s_{\lambda\nu} k^\lambda k_\mu + s_{\mu\lambda} k^\lambda k^\nu \right) \left[\partial_a X^\mu \partial^b X^\nu e^{ik \cdot X} \right]_w \right\} \end{aligned} \quad [3.302]$$

Let us now focus on the antisymmetric part

$$\mathcal{J}_{2b}^a = -\frac{ig_c}{2} \int d^2\sigma \delta\omega \sqrt{g} i \epsilon^{ab} a_{\mu\nu} \left\{ k^\mu \nabla_a \left[\partial_b X^\nu e^{ik \cdot X} \right]_w + k^\nu \nabla_b \left[\partial_a X^\mu e^{ik \cdot X} \right]_w \right\} \quad [3.303]$$

The only non-vanishing component of ϵ^{ab} is $\epsilon^{z\bar{z}}$ and so we have

$$\mathcal{J}_{2b}^a = -\frac{ig_c}{2} \int d^2\sigma \delta\omega \sqrt{g} i \epsilon^{z\bar{z}} a_{\mu\nu} \left\{ k^\mu \nabla_z \left[\partial_{\bar{z}} X^\nu e^{ik \cdot X} \right]_w + k^\nu \nabla_{\bar{z}} \left[\partial_z X^\mu e^{ik \cdot X} \right]_w \right\} \quad [3.304]$$

We can again just take over [3.260], without a factor of four. Note that the second term in [3.260] will bring down an extra factor of k^μ resulting in a combination $a_{\mu\nu} k^\mu k^\nu$ which is zero by antisymmetry of $a_{\mu\nu}$. So we get

$$\begin{aligned} \mathcal{J}_{2b}^a &= \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} i \epsilon^{z\bar{z}} a_{\mu\nu} \left\{ k^\mu \left[\partial_{\bar{z}} X^\nu \partial_z e^{ik \cdot X} \right]_w + k^\nu \left[\partial_z X^\mu \partial_{\bar{z}} e^{ik \cdot X} \right]_w \right\} \\ &= \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} i \epsilon^{z\bar{z}} a_{\mu\nu} \left\{ k^\mu \left[\partial_{\bar{z}} X^\nu i k_\lambda \partial_z X^\lambda e^{ik \cdot X} \right]_w + k^\nu \left[\partial_z X^\mu i k_\lambda \partial_{\bar{z}} X^\lambda e^{ik \cdot X} \right]_w \right\} \\ &= \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} i \epsilon^{ab} a_{\mu\nu} \left\{ k^\mu \left[\partial_b X^\nu i k_\lambda \partial_a X^\lambda e^{ik \cdot X} \right]_w + k^\nu \left[\partial_a X^\mu i k_\lambda \partial_b X^\lambda e^{ik \cdot X} \right]_w \right\} \\ &= \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} i \epsilon^{ab} \left(s_{\lambda\nu} k^\lambda k_\mu + s_{\mu\lambda} k^\lambda k^\nu \right) \left[\partial_a X^\mu \partial^b X^\nu e^{ik \cdot X} \right]_w \end{aligned} \quad [3.305]$$

Adding the symmetric and antisymmetric part we find

$$\begin{aligned} \mathcal{J}_{2b} = & \frac{g_c}{2} \int d^2\sigma \delta\omega \sqrt{g} \left\{ \alpha' R \left(\frac{\gamma}{2} s_{\mu\nu} k^\mu k^\nu \right) \left[e^{ik \cdot X(\sigma)} \right]_r \right. \\ & \left. + (g^{ab} + i\epsilon^{ab}) \left(s_{\mu\lambda} k_\nu k^\lambda + s_{\lambda\nu} k_\mu k^\lambda \right) \left[\partial_b X^\nu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r \right\} \end{aligned} \quad [3.306]$$

This is the same result [3.198] as in our previous calculation, if we set again $\gamma = -2/3$ there.

Let us finally focus on the term where both functional derivatives act on the $\partial_a X^\mu \partial_b X^\nu$. Once again, we can follow the calculation from the past as long as we don't mess around with derivatives of operators. This means we can go straight to [3.202]

$$\mathcal{J}_{2c} = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} g^{ab} s_\mu^\mu \partial'_a \partial_b \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma} \left[e^{ik \cdot X(\sigma)} \right]_r \quad [3.307]$$

But the only non-vanishing metric components are $g^{z\bar{z}}$ and we have shown in [??] that $\partial_{z'} \partial_{\bar{z}} \delta_W \Delta(\sigma', \sigma) \Big|_{\sigma'=\sigma=\sigma_1} = 0$ and so in Weyl normal ordering we simply have

$$\mathcal{J}_{2c} = 0 \quad [3.308]$$

Bringing the results together, we find We can now bring all the contributions together. They are of the form

$$\delta_W V_1 = \frac{g_c}{2} \int d^2\sigma \sqrt{g} \delta\omega \left\{ (g^{ab} s_{\mu\mu} + i\epsilon^{ab} \mathfrak{Q}_{\mu\nu}) \left[\partial_a X^\mu \partial_a X^\mu e^{ik \cdot X(\sigma)} \right]_r + \alpha' R \mathfrak{F} \left[e^{ik \cdot X(\sigma)} \right]_r \right\} \quad [3.309]$$

The contributions from the different parts to $\mathfrak{s}$, $\mathfrak{Q}$ and $\mathfrak{F}$ are summarised in the table below.

	$\mathfrak{s}_{\mu\nu}$	$\mathfrak{Q}_{\mu\nu}$	$\mathfrak{F}$
$\mathcal{J}_1$	$4k_\mu k_\nu \tilde{\phi}$	—	$\gamma k^2 \tilde{\phi}$
$\mathcal{J}_{2a}$	$-k^2 s_{\mu\nu}$	$-k^2 a_{\mu\nu}$	—
$\mathcal{J}_{2b}$	$k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\nu\lambda}$	$k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda}$	$\frac{\gamma}{2} k^\mu k^\nu s_{\mu\nu}$
$\mathcal{J}_{2c}$	—	—	—
$\mathcal{J}_3$	—	—	$-k^2 \tilde{\phi}$

Table 3.3: Weyl transformation of the massless vertex operator; contributions to [3.205]

Thus

$$\begin{aligned}
s_{\mu\nu} &= -k^2 s_{\mu\nu} + k^\lambda k_\mu s_{\lambda\nu} + k^\lambda k_\mu s_{\mu\lambda} + 4k_\mu k_\nu \tilde{\phi} \\
\mathfrak{a}_{\mu\nu} &= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\lambda\nu} + k^\lambda k_\mu a_{\nu\lambda} \\
&= -k^2 a_{\mu\nu} + k^\lambda k_\mu a_{\mu\lambda} - k^\lambda k_\mu a_{\nu\lambda} \\
\mathfrak{f} &= (\gamma - 1)k^2 \tilde{\phi} + \frac{\gamma}{2} k^\mu k^\nu s_{\mu\nu}
\end{aligned} \tag{3.310}$$

Requiring Weyl invariance of the vertex operator V_1 requires that $s_{\mu\nu} = \mathfrak{a}_{\mu\nu} = \mathfrak{f} = 0$. We can now go through the same reasoning as for (3.6.22), which we briefly repeat here. First we take an n satisfying $n^2 = 0$ and $k \cdot n = 1$. Also $n^\mu s_{\mu\nu} = n^\mu a_{\mu\nu} = 0$. Start by requiring $s_{\mu\nu} n^\mu n^\nu = 0$. This means

$$0 = -k^2 s_{\mu\nu} n^\mu n^\nu + k_\nu k^\omega s_{\mu\omega} n^\mu n^\nu + k_\mu k^\omega s_{\nu\omega} n^\mu n^\nu + 4k_\mu k_\nu n^\mu n^\nu \tilde{\phi} = 4\tilde{\phi} \tag{3.311}$$

so that

$$\tilde{\phi} = 0 \tag{3.312}$$

Now require $s_{\mu\nu} n^\mu = 0$. This gives

$$0 = -k^2 s_{\mu\nu} n^\mu + k_\nu k^\omega s_{\mu\omega} n^\mu + k_\mu k^\omega s_{\nu\omega} n^\mu = (k \cdot n) k^\omega s_{\nu\omega} \tag{3.313}$$

so that

$$k^\mu s_{\mu\nu} = 0 \tag{3.314}$$

Similarly, requiring $A_{\mu\nu} n^\mu = 0$ leads to

$$\begin{aligned}
0 &= -k^2 a_{\mu\nu} n^\mu + k_\nu k^\omega a_{\mu\omega} n^\mu - k_\mu k^\omega a_{\nu\omega} n^\mu \\
&= (k \cdot n) k^\mu a_{\nu\mu}
\end{aligned} \tag{3.315}$$

so that

$$k^\mu a_{\mu\nu} = 0 \tag{3.316}$$

Finally, requiring $s_{\mu\nu} = 0$ gives

$$0 = -k^2 s_{\mu\nu} + k_\nu k^\omega s_{\mu\omega} + k_\mu k^\omega s_{\nu\omega} = -k^2 s_{\mu\nu} \tag{3.317}$$

and so also

$$k^2 = 0 \tag{3.318}$$

Note that $s_{\mu\nu} = 0$ just means that there would be no symmetric part. We now see that $\mathbb{O}_{\mu\nu} = 0$ is also satisfied and so is $\mathbb{F} = 0$. We have exactly the same on-shell conditions as from the renormalisation scheme in Joe's book.

We are now, finally, in a position to argue (3.6.18). And it is now pretty simple. Using the Weyl normal ordering renormalisation scheme we have found the above mass-shell conditions as in Joe's book. Going back through the calculation of the on-shell conditions in Joe's book we see that these can only be satisfied if (3.6.18) is valid. As physical results should be independent of the renormalisation scheme and mass-shell conditions certainly are physical results, this shows that (3.6.18) is indeed valid.

3.42 p 106: Eq (3.6.20) The Independent Parameters of the Massless Vertex Operator, I

Let us first consider the change in $a_{\mu\nu}$. This change will only impact $A_{\mu\nu}$:

$$\begin{aligned} A_{\mu\nu} &\rightarrow A_{\mu\nu} - k^2(\zeta_\mu k_\nu - k_\mu \zeta_\nu) + k_\nu k^\lambda(\zeta_\mu k_\lambda - k_\mu \zeta_\lambda) - k_\mu k^\lambda(\zeta_\nu k_\lambda - k_\nu \zeta_\lambda) \\ &= A_{\mu\nu} - k^2 \zeta_\mu k_\nu + k^2 k_\mu \zeta_\nu + k^2 \zeta_\mu k_\nu - (k \cdot \zeta) k_\mu k_\nu - k^2 k_\mu \zeta_\nu + (k \cdot \zeta) k_\mu k_\nu \\ &= A_{\mu\nu} \end{aligned} \quad [3.319]$$

and so

$$\delta_W V_1(s_{\mu\nu}, a_{\mu\nu} + \zeta_\mu k_\nu - k_\mu \zeta_\nu, \phi) = \delta_W V_1(s_{\mu\nu}, a_{\mu\nu}, \phi) \quad [3.320]$$

Similarly, we find

$$\begin{aligned} S_{\mu\nu} &\rightarrow S_{\mu\nu} - k^2(\xi_\mu k_\nu + k_\mu \xi_\nu) + k_\nu k^\lambda(\xi_\mu k_\lambda + k_\mu \xi_\lambda) + k_\mu k^\lambda(\xi_\nu k_\lambda + k_\nu \xi_\lambda) \\ &\quad - \frac{1}{3} k_\mu k_\nu 2(k \cdot \xi) - \frac{4}{3} k_\mu k_\nu 2(k \cdot \xi) \\ &= S_{\mu\nu} - k^2 \xi_\mu k_\nu - k^2 k_\mu \xi_\nu + k^2 k_\nu \xi_\mu + k_\mu k_\nu (k \cdot \xi) + k^2 k_\mu k_\nu + k_\mu k_\nu (k \cdot \xi) \\ &\quad - 2k_\mu k_\nu (k \cdot \xi) = S_{\mu\nu} \end{aligned} \quad [3.321]$$

We also have

$$\begin{aligned} F &\rightarrow F - \frac{5}{3} k^2 \left(-\frac{1}{3} (k \cdot \xi) \right) - \frac{1}{3} k^\mu k^\nu (\xi_\mu k_\nu + k_\mu \xi_\nu) + \frac{1}{18} k^2 2(k \cdot \xi) \\ &= F + \left[\frac{5}{9} - \frac{2}{3} + \frac{1}{9} \right] k^2 (k \cdot \xi) = F \end{aligned} \quad [3.322]$$

and so

$$\delta_W V_1(s_{\mu\nu} + \xi_\mu k_\nu + k_\mu \xi_\nu, a_{\mu\nu}, \phi - \frac{1}{3} k \cdot \xi) = \delta_W V_1(s_{\mu\nu}, a_{\mu\nu}, \phi) \quad [3.323]$$

3.43 p 106: Eq (3.6.21) The Independent Parameters of the Massless Vertex Operator, II

Let us chose our axes so that $k = (1, 1, 0, \dots, 0)$. It satisfies $k^2 = 0$ as it should. n is then necessarily of the form $n = (-\frac{1}{2}, \frac{1}{2}, \vec{n})$ in order to satisfy $k \cdot n = 1$. Now $n^2 = 0$ implies $\vec{n}^2 = 0$ and so $\vec{n} = 0$. We can thus always chose

$$\begin{aligned} k &= (1, 1, 0, \dots, 0) \\ n &= (-\frac{1}{2}, \frac{1}{2}, 0, \dots, 0) \end{aligned} \quad [3.324]$$

Let us now start from a general $s_{\mu\nu}$ we allow a transformation (3.6.20a) but require that the transformed field satisfies (3.6.61), i.e.

$$\begin{aligned} 0 &= n^\mu (s_{\mu\nu} + \xi_\mu k_\nu + k_\mu \xi_\nu) \\ &= n^\mu s_{\mu\nu} + (n \cdot \xi) k^\nu + \xi^\nu \end{aligned} \quad [3.325]$$

Let us write this out explicitly. For $\nu = 0, 1$ we have

$$\begin{aligned} 0 &= \frac{1}{2}(-s_{00} + s_{10}) + \frac{1}{2}(\xi_0 + \xi_1) + \xi_0 \\ 0 &= \frac{1}{2}(-s_{01} + s_{11}) + \frac{1}{2}(\xi_0 + \xi_1) + \xi_1 \end{aligned} \quad [3.326]$$

This is easily solved for ξ_0 and ξ_1 :

$$\begin{aligned} \xi_0 &= \frac{1}{8}(3s_{00} - 3s_{10} - s_{01} + s_{11}) \\ \xi_1 &= \frac{1}{8}(-s_{00} + s_{10} + 3s_{01} - 3s_{11}) \end{aligned} \quad [3.327]$$

The other equations are, for $\nu = 2, \dots, D-1$

$$0 = \frac{1}{2}(-s_{0\nu} + s_{1\nu}) + \xi_\nu \quad [3.328]$$

which is solved by

$$\xi_\nu = \frac{1}{2}(s_{0\nu} - s_{1\nu}) \quad [3.329]$$

In other words, given an $s_{\mu\nu}$ we can always transform it to a new form that satisfies $n^\mu s_{\mu\nu} = 0$ and still has the same Weyl transformation. A similar argument holds obviously for $a_{\mu\nu}$.

3.44 p 106: Eq (3.6.22) The Independent Parameters of the Massless Vertex Operator, III

Start by requiring $S_{\mu\nu}n^\mu n^\nu = 0$. This means

$$\begin{aligned} 0 &= -k^2 s_{\mu\nu} n^\mu n^\nu + k_\nu k^\omega s_{\mu\omega} n^\mu n^\nu + k_\mu k^\omega s_{\nu\omega} n^\mu n^\nu - (1 + \gamma) k_\mu k_\nu s_\omega^\omega n^\mu n^\nu + 4k_\mu k_\nu n^\mu n^\nu \phi \\ &= [-(1 + \gamma) s_\omega^\omega + 4\phi] k_\mu k_\nu n^\mu n^\nu = [-(1 + \gamma) s_\omega^\omega + 4\phi] (k \cdot n)^2 = -(1 + \gamma) s_\omega^\omega + 4\phi \end{aligned} \quad [3.330]$$

so that

$$\phi = \frac{1 + \gamma}{4} s_\omega^\omega = \frac{1}{12} s_\omega^\omega \quad [3.331]$$

Now require $S_{\mu\nu}n^\mu = 0$. This gives

$$\begin{aligned} 0 &= -k^2 s_{\mu\nu} n^\mu + k_\nu k^\omega s_{\mu\omega} n^\mu + k_\mu k^\omega s_{\nu\omega} n^\mu - (1 + \gamma) k_\mu k_\nu s_\omega^\omega n^\mu + 4k_\mu k_\nu n^\mu \phi \\ &= (k \cdot n) k^\omega s_{\nu\omega} \end{aligned} \quad [3.332]$$

so that

$$k^\mu s_{\mu\nu} = 0 \quad [3.333]$$

Similarly, requiring $A_{\mu\nu}n^\mu = 0$ leads to

$$\begin{aligned} 0 &= -k^2 a_{\mu\nu} n^\mu + k_\nu k^\omega a_{\mu\omega} n^\mu - k_\mu k^\omega a_{\nu\omega} n^\mu \\ &= (k \cdot n) k^\mu a_{\nu\mu} \end{aligned} \quad [3.334]$$

so that

$$k^\mu a_{\mu\nu} = 0 \quad [3.335]$$

Finally, requiring $S_{\mu\nu} = 0$ gives

$$\begin{aligned} 0 &= -k^2 s_{\mu\nu} + k_\nu k^\omega s_{\mu\omega} + k_\mu k^\omega s_{\nu\omega} - (1 + \gamma) k_\mu k_\nu s_\omega^\omega + 4k_\mu k_\nu \phi \\ &= -k^2 s_{\mu\nu} \end{aligned} \quad [3.336]$$

and so also

$$k^2 = 0 \quad [3.337]$$

Note that $s_{\mu\nu} = 0$ just means that there would be no symmetric part. We now see that $A_{\mu\nu} = 0$ is also satisfied and so is $F = 0$.