Point Particle in Flat Space-time

The action for a point particle in flat space-time is, $S_{\text{Particle}} = \frac{1}{2}\int d\tau \left[\mathcal{E}(\tau)^{-1}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu - \mathcal{E}(\tau)m^2\right]$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots 1)_{D\times D}$ is the minkowski tensor describing a D-dim flat space-time. $\mathcal{E}(\tau)$ is an auxillary 'Einbein' field. We also need $\mathcal{E}(\tau)$ to be invariant under reparametorization of $\tau \longrightarrow \tilde{\tau}$. i.e.

If $\tau \longrightarrow \tilde{\tau}(\tau)$, then $d\tau = \frac{\partial\tau}{\partial\tilde{\tau}}d\tilde{\tau}$ &

$$\dot{X}^\mu\dot{X}^\nu = \partial_\tau X^\mu(\tau)\partial_\tau X^\nu(\tau) = \left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^{-2}\partial_{\tilde{\tau}}\tilde{X}^\mu(\tilde{\tau})\partial_{\tilde{\tau}}\tilde{X}^\nu(\tilde{\tau})$$

$$\longrightarrow \dot{X}^\mu\dot{X}^\nu = \left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^{-2}\dot{\tilde{X}}^\mu\dot{\tilde{X}}^\nu \;\&\; \text{If } \mathcal{E}(\tau) = \frac{\partial\tilde{\tau}}{\partial\tau}\tilde{\mathcal{E}}(\tilde{\tau}),$$

then,

$$d\tau\,\mathcal{E}(\tau)^{-1}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = \frac{\partial\tau}{\partial\tilde{\tau}}d\tilde{\tau}\,\frac{\partial\tau}{\partial\tilde{\tau}}\,\tilde{\mathcal{E}}(\tilde{\tau})^{-1}\left(\frac{\partial\tilde{\tau}}{\partial\tau}\right)^{-2}\dot{\tilde{X}}^\mu\dot{\tilde{X}}^\nu$$

Thus $d\tau\,\mathcal{E}(\tau)^{-1}\eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = d\tilde{\tau}\,\tilde{\mathcal{E}}(\tilde{\tau})^{-1}\dot{\tilde{X}}^\mu\dot{\tilde{X}}^\nu$

& $d\tau\,\mathcal{E}(\tau) = \frac{\partial\tau}{\partial\tilde{\tau}} d\tilde{\tau}\,\frac{\partial\tilde{\tau}}{\partial\tau}\tilde{\mathcal{E}}(\tilde{\tau}) = d\tilde{\tau}\,\tilde{\mathcal{E}}(\tilde{\tau})$

So, $S_{Particle}$ is invariant under $\tau \longrightarrow \tilde{\tau}(\tau)$

'$\mathcal{E}(\tau)$' is actually related to the world-line metric '$\gamma_{\tau\tau}(\tau)$' as '$\underline{\mathcal{E}(\tau) = \sqrt{-\gamma_{\tau\tau}(\tau)}}$'.
↳ Lorentzian

The eq^n of motion for X^μ is,

$$\frac{dP^\mu}{d\tau} = 0 \text{ , where } P^\mu = \mathcal{E}^{-1}\dot{X}^\mu$$

$$\Rightarrow \underline{\ddot{X}^\mu = \mathcal{E}^{-1}\dot{\mathcal{E}}\dot{X}^\mu}$$

& Eq^n of motion for $\mathcal{E}(\tau)$ is

$$\mathcal{E}^{-2}\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu} + m^2 = 0$$

$$\rightarrow \mathcal{E}^2 = \frac{-\dot{X}^\mu\dot{X}^\nu\eta_{\mu\nu}}{m^2}$$

Thus the $\mathcal{E}(\tau)$ is completely fixed.

$$\rightarrow P^\mu = \mathcal{E}^{-1}\dot{X}^\mu = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^\rho\dot{X}^\sigma\eta_{\rho\sigma}}}$$

$$\longrightarrow \underline{p^\mu p^\nu \eta_{\mu\nu} = -m^2} \quad \text{(On mass-shell)}$$

Now, for massless particles,

$$S = \frac{1}{2}\int d\tau\, e^{-1}\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}$$

& the eqⁿ of motion for x^μ is,

$$\frac{dp^\mu}{d\tau} = 0 \text{ , where } p^\mu = e^{-1}\dot{x}^\mu$$

Eqⁿ of motion for $e(\tau)$: $e^{-2}\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} = 0$

$$\longrightarrow \underline{p^\mu p^\nu \eta_{\mu\nu} = 0}$$ → On-mass shell Conditions for the massless particles.

Thus

$$S_{\text{Particle}} = \frac{1}{2}\int d\tau \left[e(\tau)^{-1}\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - e(\tau)m^2\right]$$

describes a relativistic free particle, both massive & massless.

Point Particle in Curved Space-time

Here we just replace the minkowski metric $\eta_{\mu\nu}$ with space time dependent space-time metric tensor $G_{\mu\nu}(X)$ that has the info. on the space-time curvature.

$$\longrightarrow S_{Particle} = \frac{1}{2}\int d\tau \left[\mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} - \mathcal{E}(\tau)\, m^2\right]$$

where $G_{\mu\nu}(X) \longrightarrow G_{\mu\nu}(X^{\rho}(\tau))$

Now, if we find the eq^{n} of motion for $\mathcal{E}(\tau)$, we get,

$$-\frac{1}{2}\mathcal{E}(\tau)^{-2} G_{\mu\nu}(X)\dot{X}^{\mu}\dot{X}^{\nu} - m^2 = 0$$

$$\longrightarrow \mathcal{E}(\tau)^2 = \frac{-G_{\mu\nu}(X)\dot{X}^{\mu}\dot{X}^{\nu}}{m^2}$$

Eq^{n} of motion for X^{μ} is

$$\frac{d}{d\tau}\left[\mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{X}^{\nu}\right] = \frac{1}{2}\mathcal{E}(\tau)^{-1}\dot{X}^{\nu}\dot{X}^{\rho}\frac{\partial}{\partial X^{\mu}} G_{\nu\rho}$$

Now, by chain rule,

$$\frac{d}{d\tau}\left[\mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{x}^{\nu}\right] = -\mathcal{E}(\tau)^{-2} G_{\mu\nu}\dot{x}^{\nu}\frac{d\mathcal{E}(\tau)}{d\tau}$$

$$+\mathcal{E}(\tau)^{-1}\frac{dG_{\mu\nu}}{d\tau}\dot{x}^{\nu} + \mathcal{E}(\tau)^{-1} G_{\mu\nu}\ddot{x}^{\nu}$$

$$\rightarrow \mathcal{E}(\tau)^{-1} G_{\mu\nu}\ddot{x}^{\nu} + \mathcal{E}(\tau)^{-1}\dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial G_{\mu\nu}}{\partial x^{\rho}}$$

$$-\mathcal{E}(\tau)^{-2} G_{\mu\nu}\dot{x}^{\nu}\frac{d\mathcal{E}(\tau)}{d\tau}$$

$$= \frac{1}{2}\mathcal{E}(\tau)^{-1}\dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial}{\partial x^{\mu}} G_{\nu\rho}$$

$$\rightarrow G_{\mu\nu}\ddot{x}^{\nu} + \dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial G_{\mu\nu}}{\partial x^{\rho}}$$

$$-\mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{x}^{\nu}\frac{d\mathcal{E}(\tau)}{d\tau} = \frac{1}{2}\dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial}{\partial x^{\mu}} G_{\nu\rho}$$

$$\rightarrow G_{\mu\nu}\ddot{x}^{\nu} + \dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial G_{\mu\nu}}{\partial x^{\rho}} - \frac{1}{2}\dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial}{\partial x^{\mu}} G_{\nu\rho}$$

$$= \mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{x}^{\nu}\frac{d\mathcal{E}(\tau)}{d\tau}$$

Now,

$$\Gamma^{\rho}_{\mu\nu} = \frac{G^{\rho\alpha}}{2}\left[\frac{\partial G_{\alpha\nu}}{\partial x^{\mu}} + \frac{\partial G_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial G_{\mu\nu}}{\partial x^{\alpha}}\right]$$

$$\rightarrow G_{\rho\beta}\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left[\frac{\partial G_{\beta\nu}}{\partial x^{\mu}} + \frac{\partial G_{\mu\beta}}{\partial x^{\nu}} - \frac{\partial G_{\mu\nu}}{\partial x^{\beta}}\right]$$

$$\rightarrow \dot{x}^{\mu}\dot{x}^{\nu} G_{\rho\beta}\Gamma^{\rho}_{\mu\nu}$$

$$= \frac{1}{2}\left[\underbrace{\dot{x}^{\mu}\dot{x}^{\nu}\frac{\partial G_{\beta\nu}}{\partial x^{\mu}} + \dot{x}^{\mu}\dot{x}^{\nu}\frac{\partial G_{\mu\beta}}{\partial x^{\nu}}}_{\text{Combine}} - \dot{x}^{\mu}\dot{x}^{\nu}\frac{\partial G_{\mu\nu}}{\partial x^{\beta}}\right]$$

$$\rightarrow \dot{x}^{\mu}\dot{x}^{\nu} G_{\rho\beta}\Gamma^{\rho}_{\mu\nu} = \dot{x}^{\nu}\dot{x}^{\mu}\frac{\partial G_{\beta\nu}}{\partial x^{\mu}} - \frac{1}{2}\dot{x}^{\mu}\dot{x}^{\nu}\frac{\partial G_{\mu\nu}}{\partial x^{\beta}}$$

Use above eqⁿ in

$$G_{\mu\nu}\ddot{x}^{\nu} + \dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial G_{\mu\nu}}{\partial x^{\rho}} - \frac{1}{2}\dot{x}^{\nu}\dot{x}^{\rho}\frac{\partial}{\partial x^{\mu}}G_{\nu\rho}$$

$$= \mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{x}^{\nu}\frac{d\mathcal{E}(\tau)}{d\tau}$$

$$\Rightarrow G_{\mu\nu}\ddot{\chi}^{\nu} + \dot{\chi}^{\nu}\dot{\chi}^{\rho}\, G_{\mu\beta}\Gamma^{\beta}_{\nu\rho}$$

$$= \mathcal{E}(\tau)^{-1} G_{\mu\nu}\dot{\chi}^{\nu}\frac{d\mathcal{E}(\tau)}{d\tau}$$

$$\Rightarrow \ddot{\chi}_{\mu} + \dot{\chi}^{\nu}\dot{\chi}^{\rho}\, G_{\mu\beta}\Gamma^{\beta}_{\nu\rho} = \dot{\chi}_{\mu}\mathcal{E}(\tau)^{-1}\frac{d\mathcal{E}(\tau)}{d\tau}$$

Multiply by $G^{\mu\alpha}$ & then replace $\alpha \to \mu$

$$\Rightarrow \ddot{\chi}^{\mu} + \dot{\chi}^{\nu}\dot{\chi}^{\rho}\Gamma^{\mu}_{\nu\rho} = \dot{\chi}^{\mu}\mathcal{E}(\tau)^{-1}\frac{d\mathcal{E}(\tau)}{d\tau}$$

If $\tau \to$ Proper time then

$$\mathcal{E}(\tau)^2 = \frac{-G_{\mu\nu}(\chi)\dot{\chi}^{\mu}\dot{\chi}^{\nu}}{m^2} = \frac{-\dot{\chi}^{\mu}\dot{\chi}_{\mu}}{m^2}$$

But $\dot{\chi}^{\mu}\dot{\chi}_{\mu} = -c^2$ $\}$ $c =$ Speed of light

Then, $\mathcal{E}(\tau) = \frac{c}{m} \Rightarrow \frac{d\mathcal{E}(\tau)}{d\tau} = 0$

Thus we get the Geodesic eqn

$$\boxed{\ddot{\chi}^{\mu} + \dot{\chi}^{\nu}\dot{\chi}^{\rho}\Gamma^{\mu}_{\nu\rho} = 0}$$

Polyakov action in Curved Space-time

Here we will just replace the Minkowski metric by the curved space-time metric, in the Polyakov action for arbitrary world sheet manifold.

$$S_G = \frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, g^{ab}\, G_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu)$$

But we previously saw that a Graviton in itself a state of string, got by the massless excitation of the closed string. So we should think of a Curved Space-time background as a Coherent state of Gravitons in the background of Minkowski Space-time.

To justify this, consider a Space-time which is close to being flat minkowski.

i.e. $$G_{\mu\nu}(X) = \eta_{\mu\nu} + H_{\mu\nu}(X)$$

$$\Rightarrow S_G = \frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, g^{ab}\, G_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu)$$

$$= \frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, g^{ab}\,(\eta_{\mu\nu} + H_{\mu\nu}(X))(\partial_a X^\mu)(\partial_b X^\nu)$$

$$= \underbrace{\frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, g^{ab}\, \eta_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu)}_{S_X} + \frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, g^{ab} H_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu)$$

$$\Rightarrow S_G = S_X + \int_M \frac{d^2\sigma \sqrt{g}}{4\pi\alpha'} g^{ab} H_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu)$$

$$\Rightarrow S_G = S_X + S_H$$

where,

$$S_H = \int_M \frac{d^2\sigma \sqrt{g}}{4\pi\alpha'} g^{ab} H_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu)$$

And S_X is Polyakov action in the flat Minkowski Space-time.

Then the Polyakov Path integral becomes,

$$Z_C = \int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{Vol_{diff \times Weyl}} e^{-S_X[X^\mu, g_{ab}]} e^{-S_H[X^\mu, g_{ab}]}$$

$$= \int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{Vol_{diff \times Weyl}} e^{-S_X[X^\mu, g_{ab}]} \underbrace{(1 - S_H)}_{\text{Upto } 1^{st} \text{ order}}$$

$$= \int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{Vol_{diff \times Weyl}} e^{-S_X[X^\mu, g_{ab}]} \Big\} Z_X$$

$$- \int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{Vol_{diff \times Weyl}} e^{-S_X[X^\mu, g_{ab}]} S_H$$

$$\rightarrow Z_G = Z_X - \int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{Vol_{diff \times Weyl}} e^{-S_X[X^\mu, g_{ab}]} S_H$$

where Z_X is the Polyakov action in flat Minkowski Space-time background.

If $H_{\mu\nu}(X) = -4\pi g_c e^{ik_\rho X^\rho} \underbrace{\mathcal{E}_{\mu\nu}}_{\text{Symmetric traceless}}$

Then, $Z_G - Z_X$ is given by,

$$-\int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{\text{Vol}_{\text{diff}\times\text{Weyl}}} e^{-S_X[X^\mu, g_{ab}]}$$

$$\times \int_M \frac{d^2\sigma \sqrt{g}}{4\pi\alpha'} g^{ab} (-4\pi g_c e^{ik_\rho X^\rho} \mathcal{E}_{\mu\nu})(\partial_a X^\mu)(\partial_b X^\nu)$$

$\longrightarrow$ $Z_G - Z_X$ is given by

$$\int \frac{[\mathcal{D}X^\mu \mathcal{D}g_{ab}]}{\text{Vol}_{\text{diff}\times\text{Weyl}}} e^{-S_X[X^\mu, g_{ab}]}$$

$$\times \frac{g_c}{\alpha'} \int_M d^2\sigma \sqrt{g} \mathcal{E}_{\mu\nu} e^{ik_\rho X^\rho} g^{ab} (\partial_a X^\mu)(\partial_b X^\nu)$$

Vertex operator for Graviton excitation

Thus we get a slightly curved metric to relate somehow to the graviton's vertex operator. So, the curved metric Polyakov action is got by exponentiating the vertex operator corresponding to Graviton. This is like generating a coherent state of Graviton excitations.

Generatization to Other Background Fields

Instead of having just Graviton background we shall consider adding even the anti-symmetric tensor field background and the Dilaton background. Then the action is of the form called as Non-Linear Sigma model.

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\left[\mathcal{A}^{ab}_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu) + \alpha' R\Phi(X)\right]$$

where $$\mathcal{A}^{ab}_{\mu\nu}(X) = \left[g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\right]$$

$G_{\mu\nu}(X) \longrightarrow$ Graviton field

$B_{\mu\nu}(X) \longrightarrow$ Anti-Symmetric tensor field

$\Phi(X) \longrightarrow$ Dilaton field

$\epsilon^{ab} = \frac{1}{\sqrt{g}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{\varepsilon^{ab}}{\sqrt{g}}$, $g^{ab} \longrightarrow$ World sheet Metric tensor

$R \longrightarrow$ Ricci Tensor built by world sheet metric

We have a Space-time gauge invariance provided that the quantities $G_{\mu\nu}(X)$ & $B_{\mu\nu}(X)$ behave as rank-2 tensors under $X^{\mu} \longrightarrow X'^{\mu} = X'^{\mu}(X)$ transform and $\Phi(X)$ transforms as a scalar.

Also if $\mathcal{B}B_{\mu\nu}(X) = \frac{\partial \zeta_\nu}{\partial X^\mu} - \frac{\partial \zeta_\mu}{\partial X^\nu}$, then

$$\epsilon^{ab}\,\mathcal{B}B_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu) = \epsilon^{ab}\left[\frac{\partial \zeta_\nu}{\partial X^\mu} - \frac{\partial \zeta_\mu}{\partial X^\nu}\right](\partial_a X^\mu)(\partial_b X^\nu)$$

$$= \epsilon^{ab}\frac{\partial \zeta_\nu}{\partial X^\mu}(\partial_a X^\mu)(\partial_b X^\nu) - \underbrace{\epsilon^{ab}\frac{\partial \zeta_\mu}{\partial X^\nu}(\partial_a X^\mu)(\partial_b X^\nu)}_{\mu \longleftrightarrow \nu}$$

$$\Longrightarrow \epsilon^{ab}\partial B_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu) =$$

$$= \epsilon^{ab}\frac{\partial \zeta_{3\nu}}{\partial X^\mu}(\partial_a X^\mu)(\partial_b X^\nu) - \epsilon^{ab}\frac{\partial \zeta_{3\nu}}{\partial X^\mu}(\partial_a X^\nu)(\partial_b X^\mu)$$

$$(\epsilon^{ab} = -\epsilon^{ba})$$

$$= \epsilon^{ab}\frac{\partial \zeta_{3\nu}}{\partial X^\mu}(\partial_a X^\mu)(\partial_b X^\nu) + \underbrace{\epsilon^{ba}\frac{\partial \zeta_{3\nu}}{\partial X^\mu}(\partial_a X^\nu)(\partial_b X^\mu)}_{a \leftrightarrow b}$$

$$\Longrightarrow \epsilon^{ab}\partial B_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu) = 2\epsilon^{ab}\frac{\partial \zeta_{3\nu}}{\partial X^\mu}(\partial_a X^\mu)(\partial_b X^\nu)$$

Then, $\int_M d^2\sigma \sqrt{g}\, \epsilon^{ab}\partial B_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu) =$

$$= 2\int_M d^2\sigma\, \varepsilon^{ab}\frac{\partial \zeta_{3\nu}}{\partial X^\mu}(\partial_a X^\mu)(\partial_b X^\nu)$$

Use $(\partial_a X^\mu)\frac{\partial \zeta_{3\nu}}{\partial X^\mu} = \partial_a[\zeta_{3\nu}(X)]$

$$= 2\int_M d^2\sigma\, \varepsilon^{ab}(\partial_b X^\nu)\partial_a[\zeta_{3\nu}(X)]$$

$$= 2\int_M d^2\sigma \left[\partial_a[\varepsilon^{ab}(\partial_b X^\nu)\zeta_{3\nu}(X)] - \zeta_{3\nu}(X)\varepsilon^{ab}\partial_a\partial_b X^\nu\right]$$

Now $\varepsilon^{ab}\underbrace{\partial_a\partial_b X^\nu}_{\text{Symmetric in } a,b} = 0$

$$\longrightarrow \int_M d^2\sigma \sqrt{g}\, \epsilon^{ab}\, \delta B_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu) =$$

$$= 2\int_M d^2\sigma\, \partial_a\left[\varepsilon^{ab}(\partial_b X^\nu)\zeta_\nu(X)\right]$$

Thus, if $B_{\mu\nu} \longrightarrow B_{\mu\nu} + \delta B_{\mu\nu}$, where

$\delta B_{\mu\nu}(X) = \dfrac{\partial \zeta_\nu}{\partial X^\mu} - \dfrac{\partial \zeta_\mu}{\partial X^\nu}$, we get

$$S_\sigma \longrightarrow S_\sigma + \frac{1}{4\pi\alpha'}\int_M d^2\sigma \sqrt{g}\, \epsilon^{ab}\, \delta B_{\mu\nu}(\partial_a X^\mu)(\partial_b X^\nu)$$

$$\rightarrow S_\sigma \longrightarrow S_\sigma + \frac{1}{2\pi\alpha'}\int_M d^2\sigma\, \partial_a\left[\varepsilon^{ab}(\partial_b X^\nu)\zeta_\nu(X)\right]$$

Thus only a total derivative term gets added to the Lagrangian density which keeps the action invariant.

Non-Linear Sigma Model

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\left[G^{ab}_{\mu\nu}(X)(\partial_a X^\mu)(\partial_b X^\nu) + \alpha' R\Phi(X)\right]$$

The theory thus depends only on gauge-invariant objects built out of the metric and other fields. One can think of the fields X^μ as being coordinates on a manifold. This is called the *target space*, because the X^μ define an embedding,

$$\text{world-sheet} \;\rightarrow\; \text{target}\,. \tag{3.7.9}$$

In string theory, the target space is spacetime itself. A field theory such as (3.7.2), in which the kinetic term is field-dependent and so field space is effectively a curved manifold, is known for historic reasons as a *nonlinear sigma model*. Nonlinear sigma models have had many applications in particle physics and quantum field theory. For example, the neutral and charged pion fields can be regarded to good approximation as coordinates on the group manifold $SU(2)$.

The nonlinear sigma model action is no longer quadratic in X^μ, so the path integral is now an *interacting* two-dimensional quantum field theory. Expand the path integral around the classical solution $X^\mu(\sigma) = x_0^\mu$ for any chosen point x_0^μ. With $X^\mu(\sigma) = x_0^\mu + Y^\mu(\sigma)$,

$$\begin{aligned} G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = \Big[&G_{\mu\nu}(x_0) + G_{\mu\nu,\omega}(x_0)Y^\omega \\ &+ \tfrac{1}{2}G_{\mu\nu,\omega\rho}(x_0)Y^\omega Y^\rho + \ldots\Big]\partial_a Y^\mu \partial_b Y^\nu\,. \end{aligned} \tag{3.7.10}$$

The first term in this expansion is a quadratic kinetic term for the field Y^μ. The next term is a cubic interaction, and so forth. The coupling constants in the expansion, $G_{\mu\nu,\omega}(x_0)$ and so on, involve derivatives of the metric at x_0. In a target space with characteristic radius of curvature R_c, derivatives of the metric are of order R_c^{-1}. The effective dimensionless coupling in the theory is therefore $\alpha'^{1/2}R_c^{-1}$. If the radius of curvature R_c is much greater than the characteristic length scale of the string, then this coupling is small and perturbation theory in the two-dimensional field theory is a useful tool.

Notice that in this same regime we can employ another tool as well. Since the characteristic length scale is long compared to the string, we can ignore the internal structure of the string and use *low energy effective field theory*. String theory enters in determining the effective low energy action for the fields, as we will soon see. We have also implicitly used $\alpha'^{1/2}R_c^{-1} \ll 1$ when restricting our attention to *massless* backgrounds: when wavelengths are long compared to the string scale, massive string states are not created.

The nonlinear sigma model (3.7.2) is a renormalizable theory: the fields Y^μ have dimension zero and so the interactions all have dimension two. Nevertheless the couplings, which are the coefficients in the expansion of the metric, are infinite in number (unless some symmetry restricts the form of the metric). In fact it is more useful to think of the metric function itself, rather than its expansion coefficients, as defining the couplings of the theory.

Weyl Invariance of S_σ

We have emphasized that Weyl invariance is essential to the consistency of string theory. The action (3.7.6) will define a consistent string theory only if the two-dimensional quantum field theory is Weyl-invariant. This action happens to be the most general action that is classically invariant under a *rigid* Weyl transformation, with $\delta\omega$ independent of σ. This is easy to see: under a rigid Weyl transformation, the scaling of a term with n derivatives will be proportional to $2-n$.

Explanation ↘ ↪ Polchinski

The point is that adding additional derivatives can only come in pairs $\partial_a X^\mu \partial_b X^\nu$ combined with a worldsheet metric g^{ab}, otherwise we can't have a worldsheet diffeomorphism invariant action. If we have $n = 2k$ such derivatives then the action is of the form $\int d^2\sigma \sqrt{g} g^{a_1 b_1} \cdots g^{a_k b_k} \cdots$, where we have only written the terms that change under a Weyl transformation. But under $g_{ab} \to e^{2\omega}$, with ω constant, then this changes into $e^{(2-2k)\omega} \int d^2\sigma \sqrt{g} g^{a_1 b_1} \cdots g^{a_k b_k} \cdots$. We can thus only have invariance under rigid Weyl transformations if we have $n = 2k = 2$ derivatives.

Of course, it is necessary also to consider local Weyl transformations, under which the $G_{\mu\nu}$ and $B_{\mu\nu}$ terms in the action (3.7.6) are classically invariant while the Φ term is not, and to include the quantum contributions to the Weyl transformation properties. In the limit that $B_{\mu\nu}$ and Φ are small, and $G_{\mu\nu}$ is close to $\eta_{\mu\nu}$, we can use the results of the previous section to find the Weyl transformation. Write $S_\sigma = S_P - V_1 + \ldots$, with S_P the flat-space action and V_1 the vertex operator (3.6.14). Then

$$G_{\mu\nu}(X) = \eta_{\mu\nu} - 4\pi g_c s_{\mu\nu} e^{ik\cdot X} \ , \tag{3.7.11a}$$

$$B_{\mu\nu}(X) = -4\pi g_c a_{\mu\nu} e^{ik\cdot X} \ , \tag{3.7.11b}$$

$$\Phi(X) = -4\pi g_c \phi e^{ik\cdot X} \ . \tag{3.7.11c}$$

↳ Polchinski

Explanation ↘

Using the definition of the Polyakov action and the Vertex operator V_1, one find to linear order

$$\begin{aligned} S_\sigma = S_p - V_1 = & \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, g^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \\ & - \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{g} \left[(g^{ab} s_{\mu\nu} + i\epsilon^{ab} a_{\mu\nu}) \partial_a X^\mu \partial_b X^\nu e^{ik\cdot X} + \alpha' \phi R e^{ik\cdot X} \right] \\ = & \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \Big\{ \left[g^{ab} \left(\eta_{\mu\nu} - 4\pi g_c s_{\mu\nu} e^{ik\cdot X} \right) + i\epsilon^{ab} \left(-4\pi g_c a_{\mu\nu} e^{ik\cdot X} \right) \right] \\ & \times \partial_a X^\mu \partial_b X^\nu + \alpha' R \left(-4\pi g_c \phi e^{ik\cdot X} \right) \Big] \end{aligned} \tag{3.345}$$

Comparing this with the no nonlinear sigma model, we indeed find to linear order,

$$\begin{aligned} G_{\mu\nu}(X) &= \eta_{\mu\nu} - 4\pi g_c s_{\mu\nu} e^{ik\cdot X} \\ B_{\mu\nu}(X) &= -4\pi g_c a_{\mu\nu} e^{ik\cdot X} \\ \Phi(X) &= -4\pi g_c \phi e^{ik\cdot X} \end{aligned} \tag{3.346}$$

RG Beta Functions

We could of course take linear combinations of vertex operators with different momenta. To first order the Weyl variation is then given in eq. (3.6.16); we take for convenience a renormalization scheme in which $\gamma = 0$. Relating this Weyl transformation to $T^a{}_a$ as in eq. (3.4.6) gives

$$T^a{}_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu} g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R\,, \qquad (3.7.12)$$

where, to linear order in $\chi_{\mu\nu}$, $B_{\mu\nu}$ and Φ,

$$\beta^G_{\mu\nu} \approx -\frac{\alpha'}{2}\left(\partial^2\chi_{\mu\nu} - \partial_\nu\partial^\omega\chi_{\mu\omega} - \partial_\mu\partial^\omega\chi_{\omega\nu} + \partial_\mu\partial_\nu\chi^\omega{}_\omega\right) + 2\alpha'\partial_\mu\partial_\nu\Phi\,, \qquad (3.7.13a)$$

$$\beta^B_{\mu\nu} \approx -\frac{\alpha'}{2}\partial^\omega H_{\omega\mu\nu}\,, \qquad (3.7.13b)$$

$$\beta^\Phi \approx \frac{D-26}{6} - \frac{\alpha'}{2}\partial^2\Phi\,. \qquad (3.7.13c)$$

We have included in β^Φ the flat spacetime anomaly found in section 3.4, including the contribution proportional to 26 from the ghost fields. The symbol β is used for the coefficients in $T^a{}_a$ because these are essentially the renormalization group beta functions, governing the dependence of the physics on world-sheet scale. We will discuss this connection further, though it does not come up in detail until chapter 15.

The Weyl anomalies (3.7.13) have further contributions from higher orders in the fields. For example, expanding the path integral to second order in the cubic interaction from the expansion (3.7.10) will give rise to divergences when the two interaction vertices approach one another. Their OPE includes a singularity of the form $|z|^{-2}\partial X^\mu \bar\partial X^\nu$ times the square of the derivative of $G_{\alpha\beta}$. Integrating d^2z produces a logarithmic divergence. Diff invariance requires that the integral be cut off in terms of the invariant distance, $|z|\exp(\omega) > a_0$. This introduces a dependence on the scale of the metric, $\ln z_{\min} = -\omega + \ln a_0$. The Weyl variation $\beta^G_{\mu\nu}$ thus gets a contribution proportional to $O(G_{\alpha\beta,\gamma})^2$ which combines with the linear second derivative term to form the spacetime Ricci tensor.

Derivations from the notes of Stacy M Schrans

Recall first that all these equations are valid as operator equations, so we can use (3.4.6)

$$\delta_{\mathrm{W}}\langle\cdots\rangle = -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)\langle T_a^a(\sigma)\cdots\rangle \tag{3.347}$$

We split the S_σ again in the Polyakov and vertex part. The Weyl transformation of the Polyakov part is simply the Weyl anomaly (3.4.15)

$$(T_p)_a^a = -\frac{c}{12}R = -\frac{D-26}{12}R \tag{3.348}$$

The Weyl variation of the vertex part is given by (3.6.16)

$$\begin{aligned}\delta_{\mathrm{W}}V_1 &= \frac{g_c}{2}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)\Big\{(g^{ab}S_{\mu\nu}+i\epsilon^{ab}A_{\mu\nu})\left[\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+\alpha' FR\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\}\\ &= -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)(T_\sigma)_a^a(\sigma)\end{aligned} \tag{3.349}$$

with $S_{\mu\nu}, A_{\mu\nu}$ and F given by (3.6.17). We have written it is the form of (3.4.6) in the understanding that this is valid as an operator equation. We can thus write

$$(T_\sigma)_a^a = (-2\pi)\frac{g_c}{2}\Big\{(g^{ab}S_{\mu\nu}+i\epsilon^{ab}A_{\mu\nu})\left[\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}+\alpha' FR\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\Big\} \tag{3.350}$$

We thus have, combining both parts

$$T_a^a = -g_c\pi(g^{ab}S_{\mu\nu}+i\epsilon^{ab}A_{\mu\nu})\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} - g_c\pi\alpha' FRe^{ik\cdot X} - \frac{D-26}{12}R \tag{3.351}$$

We have dropped the renormalisation symbols for convenience. Let us write this in the form (3.7.12)

$$T_a^a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{1}{2}\beta^\Phi R \tag{3.352}$$

Let us break this down in parts top avoid long formula. We start with the last term

$$-\frac{1}{2}\beta^\Phi R = -g_c\pi\alpha' FRe^{ik\cdot X} - \frac{D-26}{12}R \tag{3.353}$$

Using (3.6.17c) for F and taking from here on $\gamma = 0$ we find

$$\beta^\Phi = 2g_c\pi\alpha'(-k^2\phi)e^{ik\cdot X} + \frac{D-26}{6} = 2g_c\pi\alpha'\phi\,\partial^2 e^{ik\cdot X} + \frac{D-26}{6} \tag{3.354}$$

Note that here ∂^2 and later ∂_μ denote derivatives w.r.t. the spacetime field X^μ and not the worldsheet coordinate σ^a. We now use the definition of the dilaton field $\Phi(X)$ in (3.7.11c)

$$\begin{aligned}\beta^\Phi &= 2g_c\pi\alpha'\left(-\frac{1}{4\pi g_c}\partial^2\Phi\right)+\frac{D-26}{6}\\ &= \frac{D-26}{6}-\frac{\alpha'}{2}\partial^2\Phi\end{aligned} \qquad [3.355]$$

Consider now the second term

$$-\frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu = -g_c\pi i\epsilon^{ab}A_{\mu\nu}\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} \qquad [3.356]$$

Using $A_{\mu\nu}$ from (3.6.16b) we find

$$\begin{aligned}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu &= 2\alpha' g_c\pi\epsilon^{ab}(-k^2 a_{\mu\nu}+k_\nu k^\omega a_{\mu\omega}-k_\mu k^\omega a_{\nu\omega})\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X}\\ &= 2\alpha' g_c\pi\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu(a_{\mu\nu}\partial^2-a_{\mu\omega}\partial_\nu\partial^\omega+a_{\nu\omega}\partial_\mu\partial^\omega)e^{ik\cdot X}\end{aligned} \qquad [3.357]$$

We use the expression for $B_{\mu\nu}(X)$ in (3.7.11b)

$$\begin{aligned}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu &= 2\alpha' g_c\pi\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\left(-\frac{1}{4\pi g_c}\right)\left(\partial^2 B_{\mu\nu}-\partial_\nu\partial^\omega B_{\mu\omega}+\partial_\mu\partial^\omega B_{\nu\omega}\right)\\ &= -\frac{\alpha'}{2}\epsilon^{ab}\partial_a X^\mu\partial_b X^\nu\left(\partial^2 B_{\mu\nu}-\partial_\nu\partial^\omega B_{\mu\omega}+\partial_\mu\partial^\omega B_{\nu\omega}\right)\end{aligned} \qquad [3.358]$$

from which we get

$$\begin{aligned}\beta^B_{\mu\nu} &= -\frac{\alpha'}{2}\left(\partial^2 B_{\mu\nu}-\partial_\nu\partial^\omega B_{\mu\omega}+\partial_\mu\partial^\omega B_{\nu\omega}\right)\\ &= -\frac{\alpha'}{2}\partial^\omega H_{\omega\mu\nu}\end{aligned} \qquad [3.359]$$

where we have used the definition of the field strength (3.7.8). Finally we take the first term

$$-\frac{1}{2\alpha'}\beta^G_{\mu\nu}g^{ab}\partial_a X^\mu\partial_b X^\nu = -g_c\pi g^{ab}S_{\mu\nu}\partial_a X^\mu\partial_b X^\nu e^{ik\cdot X} \qquad [3.360]$$

Using the definition of $S_{\mu\nu}$ in (3.6.17b), setting $\gamma=0$ and ignoring the $g^{ab}\partial_a X^\mu\partial_b X^\nu$ we have

$$\begin{aligned}\beta^G_{\mu\nu} &= 2\alpha' g_c\pi(-k^2 s_{\mu\nu}+k_\nu k^\omega s_{\mu\omega}+k_\mu k^\omega s_{\nu\omega}-k_\mu k_\nu s^\omega_\omega+4k_\mu k_\nu\phi)e^{ik\cdot X}\\ &= 2\alpha' g_c\pi(s_{\mu\nu}\partial^2-s_{\mu\omega}\partial_\nu\partial^\omega-s_{\nu\omega}\partial_\mu\partial^\omega+s^\omega_\omega\partial_\mu\partial_\nu-4\phi\partial_\mu\partial_\nu)e^{ik\cdot X}\\ &= -\frac{2\alpha' g_c\pi}{4\pi g_c}(\partial^2\chi_{\mu\nu}-\partial_\nu\partial^\omega\chi_{\mu\omega}-\partial_\mu\partial^\omega\chi_{\nu\omega}+\partial_\mu\partial_\nu\chi^\omega_\omega-4\partial_\mu\partial_\nu\Phi)\\ &= -\frac{\alpha'}{2}(\partial^2\chi_{\mu\nu}-\partial_\nu\partial^\omega\chi_{\mu\omega}-\partial_\mu\partial^\omega\chi_{\nu\omega}+\partial_\mu\partial_\nu\chi^\omega_\omega)+2\alpha'\partial_\mu\partial_\nu\Phi)\end{aligned} \qquad [3.361]$$

where we have used (3.7.11) and the fact that $G_{\mu\nu}=\eta_{\mu\nu}+\chi_{\mu\nu}$ to that order.

Deriving the Space-time field Equations

We will derive the following.

$$\beta^G_{\mu\nu} = \alpha' \boldsymbol{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} + O(\alpha'^2) \,, \tag{3.7.14a}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \Phi H_{\omega\mu\nu} + O(\alpha'^2) \,, \tag{3.7.14b}$$

$$\beta^\Phi = \frac{D - 26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\omega \Phi \nabla^\omega \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + O(\alpha'^2) \,. \tag{3.7.14c}$$

Several terms in (3.7.14) can be recognized from the linear approximation (3.7.13), but now made covariant under a change of spacetime coordinates. The spacetime Ricci tensor is $\boldsymbol{R}_{\mu\nu}$, distinguished from the world-sheet Ricci tensor R_{ab}. Terms with more derivatives are higher order in the world-sheet $\alpha'^{1/2} R_c^{-1}$ expansion. Incidentally, the most efficient method to carry the calculation of the Weyl anomaly to higher order is not the one we have described, but rather dimensional regularization.

The condition that the world-sheet theory be Weyl-invariant is thus

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0 \,. \tag{3.7.15}$$

These are sensible-looking equations of motion. The equation $\beta^G_{\mu\nu} = 0$ resembles Einstein's equation with source terms from the antisymmetric tensor field and the dilaton. The equation $\beta^B_{\mu\nu} = 0$ is the antisymmetric tensor generalization of Maxwell's equation, determining the divergence of the field strength.

↗ Polchinski

Derivation by Stony M Schorns below

3.51 p 111: Eq (3.7.14) The β Functions with two Spacetime Derivatives

We will not derive all these equations in detail. But let us remember that it was already argued before that the string action leads to spacetime general coordinate symmetry. We should thus not be surprised that if we add higher order terms, (part of) these will organise themselves in spacetime covariant derivatives. What we will do is two things and both are related to $\beta^G_{\mu\nu}$.

The first thing is just point out that if we would work out the spacetime Ricci tensor $\boldsymbol{R}_{\mu\nu}$ to first order in $G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}$ one indeed recovers the relevant part of (3.7.13a). This is a straightforward calculation which we now sketch. We expand the spacetime field $G_{\mu\nu} = \eta_{\mu\nu} + \chi_{\mu\nu}$, use it to raise and lower indices and work to linear order. We then have $G^{\mu\nu} = \eta_{\mu\nu} - \chi_{\mu\nu}$. The spacetime connection is then given by

$$\boldsymbol{\Gamma}^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) = \frac{1}{2}(\partial_\mu \chi^\sigma_\nu + \partial_\nu \chi^\sigma_\mu - \partial^\sigma \chi_{\mu\nu}) \tag{3.362}$$

The spacetime curvature tensor is

$$\begin{aligned} \boldsymbol{R}^\mu_{\nu\sigma\lambda} &= \partial_\sigma \Gamma^\mu_{\lambda\nu} - \partial_\lambda \Gamma^\mu_{\sigma\nu} + \Gamma^\mu_{\sigma\rho}\Gamma^\rho_{\lambda_\nu} - \Gamma^\mu_{\lambda\rho}\Gamma^\rho_{\nu\sigma} \\ &= \frac{1}{2}(\partial_\sigma\partial_\nu \chi^\mu_\lambda - \partial_\lambda\partial_\nu \chi^\mu_\sigma - \partial_\sigma\partial^\mu \chi_{\lambda\nu} + \partial_\lambda\partial^\mu \chi_{\sigma\nu}) \end{aligned} \tag{3.363}$$

and the spacetime Ricci tensor is

$$\boldsymbol{R}_{\nu\lambda} = R^\mu_{\nu\mu\lambda} = \frac{1}{2}(\partial_\mu\partial_\nu \chi^\mu_\lambda - \partial_\lambda\partial_\nu \chi^\mu_\mu - \partial_\mu\partial^\mu \chi_{\lambda\nu} + \partial_\lambda\partial^\mu \chi_{\mu\nu}) \tag{3.364}$$

We see form this that $\alpha' \boldsymbol{R}_{\mu\nu}$ gives exactly the χ dependent part in (3.7.13a)

The second part is more involved. We will show how the H^2 term appears in $\beta^G_{\mu\nu}$. This is actually the subject of exercise 3.11 and is quite instructive to show here. This solution is taken from Matthew Headrick who has published the solutions to about half of the exercises of Polchinki's book in arXiv:0812.4408 [hep-th].

We start by a simplification. Since our interest is in the H^2 term we might as well take $G_{\mu\nu}$ to be constant, Φ to vanish and $B_{\mu\nu}$ to be linear in X. This means that we can write $B_{\mu\nu} = b_{\omega\mu\nu} X^\omega$ for some constant $b_{\omega\mu\nu}$. The antisymmetric part of the non-linear sigma model action then becomes

$$S^{\rm A}_\sigma = \int d^2\sigma \, \sqrt{g}\, i\epsilon^{ab} b_{\omega\mu\nu} X^\omega \partial_a X^\mu \partial_b X^\nu \tag{3.365}$$

At the end of this section, we will argue that it is only this linear contribution that we need to be concerned about in checking the Weyl invariance. Let us now consider this, partially integrate and use the fact that $\partial_a(\sqrt{g}\epsilon^{bc}) = 0$,

$$\sqrt{g}\,\epsilon^{ab} X^\mu \partial_a X^\nu \partial_b X^\omega = \partial^a(\epsilon^{ab}\sqrt{g}\, X^\mu \partial_a X^\nu \partial_b X^\omega) - X^\nu \partial_a(\epsilon^{ab}\sqrt{g}\, X^\mu \partial_b X^\omega) \tag{3.366}$$

The first term vanishes as it is a boundary term and in the second term we use $\partial_a(\epsilon^{ab}\sqrt{g}) = 0$ and the fact that the $\partial_a\partial_b X^\omega$ vanishes when contracted with ϵ^{ab}. We do the same trick again

$$\epsilon^{ab}\sqrt{g}\, X^\mu \partial_a X^\nu \partial_b X^\omega = -\epsilon^{ab}\sqrt{g}\, X^\nu \partial_a X^\mu \partial_b X^\omega = +\epsilon^{ab}\sqrt{g}\, \partial_b X^\nu \partial_a X^\mu X^\omega \qquad [3.367]$$

We can therefore write

$$\begin{aligned} S_\sigma^{\rm A} &= \int d^2\sigma\, \sqrt{g}\, i\epsilon^{ab} b_{\omega\mu\nu}\frac{1}{3}(X^\omega \partial_a X^\mu \partial_b X^\nu + X^\mu \partial_a X^\nu \partial_b X^\omega + X^\nu \partial_a X^\omega \partial_b X^\mu) \\ &= \int d^2\sigma\, \sqrt{g}\, i\epsilon^{ab}\frac{1}{3}(b_{\omega\mu\nu} + b_{\mu\nu\omega} + b_{\nu\omega\mu}) X^\omega \partial_a X^\mu \partial_b X^\nu \end{aligned} \qquad [3.368]$$

In the linear approximation, the field strength is then given by

$$\begin{aligned} H_{\omega\mu\nu} &= \partial_\omega(b_{\sigma\mu\nu}X^\sigma) + \partial_\mu(b_{\sigma\nu\omega}X^\sigma) + \partial_\nu(b_{\sigma\omega\mu}X^\sigma) \\ &= b_{\sigma\mu\nu}\delta^\sigma_\omega + b_{\sigma\nu\omega}\delta^\sigma_\mu + b_{\sigma\omega\mu}\delta^\sigma_\nu = b_{\omega\mu\nu} + b_{\mu\nu\omega} + b_{\nu\omega\mu} \end{aligned} \qquad [3.369]$$

Therefore

$$S_\sigma^{\rm A} = \int d^2\sigma\, \sqrt{g}\, i\epsilon^{ab}\frac{1}{3} H_{\omega\mu\nu} X^\omega \partial_a X^\mu \partial_b X^\nu \qquad [3.370]$$

In this approximation the non-linear sigma model becomes

$$S_\sigma = \int d^2\sigma\, \sqrt{g}\, \left(G_{\mu\nu} g^{ab} \partial_a X^\mu \partial_b X^\nu + \frac{i}{3}\epsilon^{ab} H_{\omega\mu\nu} X^\omega \partial_a X^\mu \partial_b X^\nu \right) \qquad [3.371]$$

Now go to the conformal gauge and use complex coordinates. Remember that

$$\begin{aligned} \sqrt{g}\, g^{ab}\partial_a X^\mu \partial_b X^\nu &= 4(\partial X^\mu \bar\partial X^\nu + \bar\partial X^\mu \partial X^\nu) \\ \sqrt{g}\, \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu &= -4i(\partial X^\mu \bar\partial X^\nu - \bar\partial X^\mu \partial X^\nu) \end{aligned} \qquad [3.372]$$

plus $d^2z = 2d^2\sigma$ and we can write the action as $S_\sigma = S_0 + {}_i$ with

$$\begin{aligned} S_0 &= \frac{1}{2\pi\alpha'} G_{\mu\nu} \int d^2z\, \partial X^\mu \bar\partial X^\nu \\ S_i &= \frac{1}{6\pi\alpha'} H_{\omega\mu\nu} \int d^2z\, X^\omega \partial X^\mu \bar\partial X^\nu \end{aligned} \qquad [3.373]$$

S_0 is the the action for the free theory and S_i is the action for the interaction, which we will treat as a perturbation. We can expand the expectation value of some operator $\mathcal{O}(X)$ in the path integral formalism as

$$\begin{aligned} \langle \mathcal{O}(X)\rangle &= \int [dX]\, \mathcal{O}(X) e^{-S_0 - S_i} = \int [dX]\, \mathcal{O}(X) e^{-S_0}\left(1 - S_i + \frac{1}{2}S_i^2 + \cdots\right) \\ &= \langle \mathcal{O}(X)\rangle_0 - \langle S_i \mathcal{O}(X)\rangle_0 + \frac{1}{2}\langle S_i^2 \mathcal{O}(X)\rangle_0 \end{aligned} \qquad [3.374]$$

where $\langle\cdots\rangle_0$ denotes the expectation value taken in the free theory. We have already calculated the Weyl variation of the first two terms. The Weyl variation of the first term leads to the Weyl anomaly term proportional to $D-26$, whilst the Weyl variation of the second term gives a term linear in $H_{\omega\mu\nu}$, precisely the term that we find in $\beta^B_{\mu\nu}$ in (3.7.13b). It is the Weyl variation of the third term, that is quadratic in H, that interests us and in particular the part proportional to $\int d^2z\,\langle\partial X^\mu\bar\partial X^\nu\cdots\rangle_0$ as that is the part that will contribute to $\beta^G_{\mu\nu}$. Focussing on this term

$$\frac{1}{2}\langle S_i^2\cdots\rangle_0 = \frac{1}{2}\left(\frac{1}{6\pi\alpha'}\right)^2 H_{\omega\mu\nu}H_{\omega'\mu'\nu'}\int d^2z\,d^2z' \times\left\langle :X^\omega(z,\bar z)\partial X^\mu(z)\bar\partial X^\nu(\bar z):: X^{\omega'}(z',\bar z')\partial' X^{\mu'}(z')\bar\partial' X^{\nu'}(\bar z')\cdots\right\rangle_0 \quad [3.375]$$

We have written out the normal product signs explicitly and also notice that X is neither holomorphic nor antiholomorphic, but depends on z and $\bar z$. To calculate the Weyl variation of this, we will only need the singular parts of the OPE. Indeed the nonsingular parts becomes zero after the integration. In working out the OPE we need to identify the singular term with in the numerator a $\partial X\bar\partial X$ as that is the structure we are looking for. Hence we need to perform two cross-contractions. To calculate how many such terms there are, let us first identify how many possibilities there are to keep one uncontracted factor in each normal order product. This is clearly $3\times 3=9$. We then have two contractions to make between twice two factors and this can be done in two ways. In total there are thus $9\times 2=18$ possible terms.

Let us work out one such term. The contraction we need to use is from the free theory, i.e.

$$\overbrace{X^\mu(z,\bar z)X^\nu}(z',\bar z') = -\frac{\alpha'}{2}G^{\mu\nu}\ln|z-z'|^2 \quad [3.376]$$

consider the case where we contract $X^\omega(z,\bar z)$ with $\partial' X^{\mu'}(z')$ and $\bar\partial X^\nu(\bar z)$ with $X^{\omega'}(z',\bar z')$:

This term gives

$$\begin{aligned}
\frac{1}{2}\langle S_i^2\cdots\rangle_0^{(1)} =& \frac{1}{2}\left(\frac{1}{6\pi\alpha'}\right)^2 H_{\omega\mu\nu}H_{\omega'\mu'\nu'}\int d^2z\,d^2z' \\
&\times \overbrace{X^\omega(z,\bar z)\partial' X^{\mu'}(z')}\;\bar\partial\overbrace{X^\nu(\bar z)X^{\omega'}}(z',\bar z')\left\langle :\partial X^\mu(z)\bar\partial' X^{\nu'}(\bar z'):\cdots\right\rangle_0 \\
=& \frac{1}{2}\left(\frac{1}{6\pi\alpha'}\right)^2 H_{\omega\mu\nu}H_{\omega'\mu'\nu'}\int d^2z\,d^2z'\,\left(-\frac{\alpha'}{2}G^{\omega\mu'}\partial'\ln|z-z'|^2\right) \\
&\times\left(-\frac{\alpha'}{2}G^{\nu\omega'}\bar\partial\ln|z-z'|^2\right)\left\langle :\partial X^\mu(z)\bar\partial' X^{\nu'}(\bar z'):\right\rangle_0 \\
=& -\frac{1}{298\pi^2}H_{\omega\mu\nu}H_{\omega'\mu'\nu'}G^{\omega\mu'}G^{\nu\omega'}\int d^2z\,d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^\mu(z)\bar\partial' X^{\nu'}(\bar z'):\cdots\right\rangle_0 \\
=& -\frac{1}{298\pi^2}H^{\mu'}{}_{\mu\nu}H^{\nu}{}_{\mu'\nu'}\int d^2z\,d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^\mu(z)\bar\partial' X^{\nu'}(\bar z'):\cdots\right\rangle_0 \\
=& -\frac{1}{298\pi^2}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^\mu(z)\bar\partial' X^{\nu}(\bar z'):\cdots\right\rangle_0 \qquad [3.377]
\end{aligned}$$

In the last line we have renamed indices and used the symmetry of $H_{\omega\mu\nu}$. It so happens that the 17 other terms give the same contribution. Therefore

$$\frac{1}{2}\langle S_i^2\cdots\rangle_0 = -\frac{1}{16\pi^2}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,d^2z'\,\frac{1}{|z-z'|^2}\left\langle :\partial X^\mu(z)\bar\partial' X^{\nu}(\bar z'):\cdots\right\rangle_0 \qquad [3.378]$$

This integral is logarithmically divergent at $z=z'$. The expectation value has no other singularities as it is normal ordered. We can thus also expand z' around z in the expectation value and find

$$\frac{1}{2}\langle S_i^2\cdots\rangle_0 = -\frac{1}{16\pi^2}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,\left\langle :\partial X^\mu(z)\bar\partial X^{\nu}(\bar z):\cdots\right\rangle_0\int d^2z'\,\frac{1}{|z-z'|^2} \qquad [3.379]$$

We need to regularise the last integral in such a way that it remains diffeomorphism invariant (as we don't want to introduce an anomaly in that symmetry!). The diffeomorphism invariant distance at short distance is given by (3.6.9), i.e $d(z,z')=|z-z'|e^{\omega(z)}$. We can thus introduce a diffeomorphism invariant cut-off in the integral at $|z-z'|=\varepsilon e^{-\omega}$. In polar coordinates the last integral then becomes

$$\begin{aligned}
\int d^2z'\,\frac{1}{|z-z'|^2} &= \int_{\varepsilon e^{-\omega}}^\infty r dr\int_0^{2\pi}d\theta\,\frac{1}{r^2} = 2\pi\int_{\varepsilon e^{-\omega}}^\infty\frac{dr}{r} \\
&= -2\pi\ln\varepsilon e^{-\omega} + \text{terms independent of }\omega \\
&= -2\pi\ln\varepsilon + 2\pi\omega + \text{terms independent of }\omega \qquad [3.380]
\end{aligned}$$

We need the Weyl variation of the expectation value, i.e. the transformation under a change $\omega \to \omega + \delta\omega$. Thus

$$\begin{aligned}\delta_{\rm W}\frac{1}{2}\langle S_i^2\cdots\rangle_0 &= -\frac{1}{16\pi^2}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,\left\langle :\partial X^\mu(z)\bar\partial X^\nu(\bar z):\cdots\right\rangle_0\times 2\pi\delta\omega(z)\\ &= -\frac{1}{8\pi}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2z\,\delta\omega(z)\left\langle :\partial X^\mu(z)\bar\partial X^\nu(\bar z):\cdots\right\rangle_0\end{aligned}\qquad [3.381]$$

Rewriting this in a worldsheet covariant way and taking into account that an expectation value in the free theory will only differ by higher order terms to an expectation in the full theory, we can write

$$\begin{aligned}\delta_{\rm W}\frac{1}{2}\langle S_i^2\cdots\rangle_0 &= -\frac{1}{16\pi}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)g^{ab}\left\langle :\partial_a X^\mu\partial_b X^\nu:\cdots\right\rangle\\ &= -\frac{1}{2\pi}\int d^2\sigma\,\sqrt{g}\,\delta\omega(\sigma)\langle T^{[H^2]}{}_a^a\cdots\rangle\end{aligned}\qquad [3.382]$$

with

$$T^{[H^2]}{}_a^a = \frac{1}{8}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\qquad [3.383]$$

According to the definition (3.7.12) this gives a contribution to $\beta^G_{\mu\nu}$ of the form

$$\beta^G_{\mu\nu} = \cdots - \frac{\alpha'}{4}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega}\qquad [3.384]$$

Adding to this result the fact that the we already showed how the spacetime Ricci scalar appears and that we can replace the partial derivatives by covariant derivatives, we recover (3.7.14a).

In the above derivation we assumed that $B_{\mu\nu}$ was linear in X. Let us discuss what happens if we drop this requirement. Clearly a constant term will not impact the discussion as this reverts us back to the original antisymmetric part of the vertex operator. Recall what happened with our linear term. We basically had to work out an OPE of the symbolic form $(X\partial X\bar\partial X)(X\partial X\bar\partial X)$. In order to be left with a product $\partial X\bar\partial X$ we need the contractions $\overbrace{X\bar\partial X}\ \overbrace{\partial X X}$. This gives us a $z^{-1}\bar z^{-1} = |z|^{-2} = r^{-2}$ which led to the integral $\int dr/r$ giving us the logarithmic divergence we regularised.

Let us now assume that $B_{\mu\nu}$ depends on ∂X. Note that this is now a worldsheet derivative. We start with just one ∂X. We then have to take two contractions between $(\partial X\partial X\bar\partial X)(\partial X\partial X\bar\partial X)$. But if we are to be left with a product $\partial X\bar\partial X$, this means that we need to take two contractions between $\partial X\bar\partial X$ and $\partial X\partial X$. Clearly one of them needs to be $\partial\overbrace{X\bar\partial X}$ which vanishes. If $B_{\mu\nu}$ contains a mixture of $\partial^m X\,\bar\partial^n X$ we have the same

issue. Leaving out a $\partial X\bar{\partial}X$ from the product $(\partial^m X\,\bar{\partial}^n X\partial X\bar{\partial}X)(\partial^m X\,\bar{\partial}^n X\partial X\bar{\partial}X)$ leaves a different number of ∂X and $\bar{\partial}X$ on both sides to be contracted, and so we are necessarily left with a contraction that vanishes. Thus, terms of the form $\partial^m X\,\bar{\partial}^n X$ do not contribute and cannot cause an anomaly.

Let us now assume that $B_{\mu\nu}$ is of the $(n+1)$-th order in X. We then need to have contractions between $(X^{n+1}\partial X\bar{\partial}X)(X^{n+1}\partial X\bar{\partial}X)$. Leaving out a product $\partial X\bar{\partial}X$ we are left with one contraction $\overbrace{X\bar{\partial}X}$, one contraction $\partial\overbrace{X\,X}$ and n contractions $\overbrace{X\,X}$, thus an integral of the form

$$\begin{aligned}\int d^2z\,\frac{1}{|z|^2}\left(\ln|z|^2\right)^n &= 2\pi\int_{\varepsilon e^{-\omega}} dr\,\frac{\ln^n r^2}{r} = 2\pi\times\frac{\ln^{n+1} r^2}{2(n+1)}\Big|_{\varepsilon e^{-\omega}}\\ &= -\frac{\pi}{n+1}\left[\ln\left(\varepsilon e^{-\omega}\right)^2\right]^{n+1} + \text{terms independent of } \omega\\ &= -\frac{2^{n+1}\pi}{n+1}(\ln\varepsilon-\omega)^{n+1} + \text{terms independent of } \omega \qquad [3.385]\end{aligned}$$

This will result in a contribution to the Weyl variation of the form

$$2^{n+1}\pi(\ln\varepsilon-\omega)^2\delta\omega \sim 2^{n+1}\pi\delta\omega\ln^n\varepsilon \qquad [3.386]$$

In the last equation we have pushed the curvature to infinity, by setting $\omega=0$. We see that we recover the previous, linear, result if we set $n=0$, as we should. The upshot of all this is that that if $n\geq 1$, i.e. if we have more than one factor of X, then the possible Weyl anomaly terms comes coupled with the divergence and can be removed a the counterterm. We will leave it as an exercise to the reader to show that a general term of the form $X^k(\partial^m X)^{k_m}(\bar{\partial}^n X)^{k_n}$ can similarly be removed by a counterterm, unless it is only linear in X.

3.52 p 113: Eq (3.7.19) The β Function for the Linear Dilaton Model

Setting $G_{\mu\nu}=\eta_{\mu\nu}$, $B_{\mu\nu}=0$ and $\Phi=V_\mu X^\mu$, (3.7.14c) becomes

$$\beta^\Phi = \frac{D-26}{6} - 0 + \alpha'\boldsymbol{\nabla}_\mu\Phi\boldsymbol{\nabla}^\mu\Phi + 0 \qquad [3.387]$$

As Φ is a scalar field w.r.t. spacetime we have $\boldsymbol{\nabla}_\mu\Phi=\partial_\mu\Phi=V_\mu$ and thus we have

$$\beta^\Phi = \frac{D-26}{6} - 0 + \alpha' V_\mu V^\mu \qquad [3.388]$$

Requiring the β function to vanish give (3.7.19).

3.53 p 114: Eq (3.7.20) The Effective Spacetime Action

We do this in separate steps, starting with the variation of the dilaton field

$$\begin{aligned}
\delta_\Phi \boldsymbol{S} &= \frac{1}{2\kappa_0^2}\int d^Dx\,\delta_\Phi\Big\{\sqrt{-G}\,e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\Big\}\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,\Big\{e^{-2\Phi}(-2\delta\Phi)\sqrt{-G}\,\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\\
&\qquad\qquad +\sqrt{-G}\,e^{-2\Phi}8\partial_\mu\Phi\partial^\mu\delta\Phi\Big\}\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,\Big\{-2e^{-2\Phi}\sqrt{-G}\,\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\\
&\qquad\qquad -8\partial^\mu\Big[\sqrt{-G}\,e^{-2\Phi}\partial_\mu\Phi\Big]\Big\}\delta\Phi
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\kappa_0^2}\int d^Dx\,e^{-2\Phi}\sqrt{-G}\,\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\\
&\qquad\qquad +4(-G)^{-1/2}\big(\partial^\mu\sqrt{-G}\big)\partial_\mu\Phi+4\big(-2\partial^\mu\Phi\big)\partial_\mu\Phi+4\partial^\mu\partial_\mu\Phi\Big]\delta\Phi\\
&= -\frac{1}{\kappa_0^2}\int d^Dx\,e^{-2\Phi}\sqrt{-G}\,\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}-4\partial_\mu\Phi\partial^\mu\Phi\\
&\qquad\qquad +2G^{\nu\sigma}\partial^\mu G_{\nu\sigma}\partial_\mu\Phi+4\partial_\mu\partial^\mu\Phi\Big]\delta\Phi\\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,e^{-2\Phi}\sqrt{-G}\,2\delta\Phi\Big\{-4\Big[\frac{D-26}{6}+\alpha'\partial_\mu\Phi\partial^\mu\Phi-\frac{\alpha'}{2}\partial_\mu\partial^\mu\Phi-\frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\Big]\\
&\qquad\qquad +\alpha'\boldsymbol{R}+2\alpha'\partial^\mu\partial_\mu\Phi-\frac{\alpha'}{4}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\Big\}\\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,e^{-2\Phi}\sqrt{-G}\,2\delta\Phi\Big(-4\beta^\Phi+\beta^{G\,\mu}_{\ \ \mu}\Big) \qquad [3.389]
\end{aligned}$$

In the last line we have replaced the ordinary derivative by the covariant derivative, used the fact that the spacetime metric is covariantly constant, $\nabla_\mu G^{\mu\nu}=0$ and used the definition in (3.7.14).

Let us now consider

$$\begin{aligned}
\delta_B \boldsymbol{S} &= \frac{1}{2\kappa_0^2}\int d^D x\, \delta_B\Big\{\sqrt{-G}\, e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\Big\}\\
&= \frac{1}{2\kappa_0^2}\int d^D x\,\sqrt{-G}\, e^{-2\Phi}\left(-\frac{1}{6}H^{\mu\nu\lambda}\delta_B H_{\mu\nu\lambda}\right) = -\frac{1}{4\kappa_0^2}\int d^D x\,\sqrt{-G}\, e^{-2\Phi}H^{\mu\nu\lambda}\delta_B\partial_\mu B_{\nu\lambda}\\
&= \frac{1}{4\kappa_0^2}\int d^D x\,\partial_\mu\left(\sqrt{-G}\, e^{-2\Phi}H^{\mu\nu\lambda}\right)\delta B_{\nu\lambda}\\
&= \frac{1}{4\kappa_0^2}\int d^D x\,\sqrt{-G}\, e^{-2\Phi}\Big(+\frac{1}{2}G^{\nu\sigma}\partial_\mu G_{\nu\sigma}H^{\mu\nu\lambda}-2\partial_\mu\Phi H^{\mu\nu\lambda}+\partial_\mu H^{\mu\nu\lambda}\Big)\delta B_{\nu\lambda}\\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^D x\,\sqrt{-G}\, e^{-2\Phi}\Big(-\frac{\alpha'}{4}G^{\nu\sigma}\partial_\mu G_{\nu\sigma}H^{\mu\nu\lambda}+\alpha'\partial_\mu\Phi H^{\mu\nu\lambda}-\frac{\alpha'}{2}\partial_\mu H^{\mu\nu\lambda}\Big)\delta B_{\nu\lambda}\\
&= -\frac{1}{2\kappa_0^2\alpha'}\int d^D x\,\sqrt{-G}\, e^{-2\Phi}\beta^B_{\mu\nu}\delta B_{\nu\lambda}
\end{aligned} \tag{3.390}$$

Here also, we have replaced the ordinary derivative by the covariant derivative, used the fact that the spacetime metric is covariantly constant, $\nabla_\mu G^{\mu\nu}=0$ and used the definition in (3.7.14).

Finally we consider

$$\delta_G \boldsymbol{S} = \frac{1}{2\kappa_0^2}\int d^D x\,\delta_G\Big\{\sqrt{-G}\, e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'}+\boldsymbol{R}-\frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}+4\partial_\mu\Phi\partial^\mu\Phi\Big]\Big\} \tag{3.391}$$

Recall that we already worked out the variation of the Einstein-Hilbert action, see [1.35]

$$\delta_G\int d^D x\,\sqrt{-G}\,\boldsymbol{R} = \int d^D x\,\sqrt{-G}\left(\boldsymbol{R}_{\mu\nu}-\frac{1}{2}G_{\mu\nu}\boldsymbol{R}\right)\delta G^{\mu\nu} \tag{3.392}$$

But we need to be careful as we have an extra factor $e^{-2\Phi}$. Just as for [1.35] we split the calculation

$$\delta_G\int d^D x\,\sqrt{-G}\, e^{-2\Phi}G^{\mu\nu}\boldsymbol{R}_{\mu\nu} = \sum_{a=1}^{4}\delta_G I_a \tag{3.393}$$

with

$$\begin{aligned}
\delta_G I_1 &= \int d^D x\,(\delta_G\sqrt{-G})\, e^{-2\Phi}G^{\mu\nu}\boldsymbol{R}_{\mu\nu} = \int d^D x\,\frac{1}{2}\sqrt{-G}G^{\mu\nu}\delta G_{\mu\nu}e^{-2\Phi}G^{\rho\sigma}\boldsymbol{R}_{\rho\sigma}\\
&= \int d^D x\sqrt{-G}e^{-2\Phi}\Big[\frac{1}{2}G^{\mu\nu}\boldsymbol{R}\Big]\delta G_{\mu\nu}
\end{aligned} \tag{3.394}$$

Next,

$$\delta_G I_2 = \int d^D x \, \sqrt{-G} \, (\delta_G e^{-2\Phi}) G^{\mu\nu} \boldsymbol{R}_{\mu\nu} = 0 \qquad [3.395]$$

Third,

$$\begin{aligned} \delta_G I_3 &= \int d^D x \, \sqrt{-G} \, e^{-2\Phi} (\delta_G G^{\mu\nu}) \boldsymbol{R}_{\mu\nu} = \int d^D x \, \sqrt{-G} \, e^{-2\Phi} (-G^{\mu\rho} G^{\nu\sigma} \delta G_{\rho\sigma}) \boldsymbol{R}_{\mu\nu} \\ &= \int d^D x \sqrt{-G} e^{-2\Phi} \Big[- \boldsymbol{R}^{\mu\nu} \Big] \delta G_{\mu\nu} \end{aligned} \qquad [3.396]$$

Finally, and this is where the change occurs compared to the Einstein-Hilbert action

$$\delta_G I_4 = \int d^D x \, \sqrt{-G} \, e^{-2\Phi} G^{\mu\nu} \delta_G \boldsymbol{R}_{\mu\nu} = \qquad [3.397]$$

We already calculated the variation of the Ricci tensor in [1.31] $\delta \boldsymbol{R}_{\mu\nu} = \delta \boldsymbol{R}^{\rho}_{\mu\rho\nu} = \nabla_\rho \delta \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta \Gamma^\rho_{\mu\rho}$. Thus

$$\begin{aligned} \delta_G I_4 &= \int d^D x \, \sqrt{-G} \, e^{-2\Phi} G^{\mu\nu} (\nabla_\rho \delta_G \Gamma^\rho_{\mu\nu} - \nabla_\nu \delta_G \Gamma^\rho_{\mu\rho}) \\ &= \int d^D x \, e^{-2\Phi} \big[\sqrt{-G} \, \nabla_\rho (G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\nu}) - \sqrt{-G} \, \nabla_\nu (G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\rho}) \big] \\ &= \int d^D x \, e^{-2\Phi} \big[\partial_\rho (\sqrt{-G} \, G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\nu}) - \partial_\nu (\sqrt{-G} \, G^{\mu\nu} \delta_G \Gamma^\rho_{\mu\rho}) \big] \end{aligned} \qquad [3.398]$$

In the case of the Einstein-Hilbert action, i.e. $\Phi = 0$, this is a total derivative and vanishes. This times this is not the case as we get a contribution from the dilaton field upon partial integration

$$\delta_G I_4 = 2 \int d^D x \, e^{-2\Phi} \sqrt{-G} \, G^{\mu\nu} \big[\partial_\rho \Phi \delta_G \Gamma^\rho_{\mu\nu} - \partial_\nu \Phi \delta_G \Gamma^\rho_{\mu\rho} \big] \qquad [3.399]$$

Unfortunately, this time we need to work out the variations of the connections

$$\begin{aligned} \delta_G \Gamma^\rho_{\mu\nu} &= \delta_G \frac{1}{2} G^{\rho\sigma} (\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\sigma\mu} - \partial_\sigma G_{\mu\nu}) \\ &= \frac{1}{2} \Big[- G^{\rho\kappa} G^{\sigma\tau} \delta G_{\kappa\tau} (\partial_\mu G_{\sigma\nu} + \partial_\nu G_{\sigma\mu} - \partial_\sigma G_{\mu\nu}) \\ &\qquad + G^{\rho\sigma} (\partial_\mu \delta G_{\sigma\nu} + \partial_\nu \delta G_{\sigma\mu} - \partial_\sigma \delta G_{\mu\nu}) \Big] \end{aligned} \qquad [3.400]$$

Recall that by covariance we can, to that order, replace all partial derivatives by covariant derivatives. This means that we can ignore the first line. We then split the calculation in

two

$$\begin{aligned}\delta_G I_{4a} &= 2\int d^D x\, e^{-2\Phi}\sqrt{-G}\, G^{\mu\nu}\partial_\rho\Phi\frac{1}{2}G^{\rho\sigma}(\partial_\mu\delta G_{\sigma\nu}+\partial_\nu\delta G_{\sigma\mu}-\partial_\sigma\delta G_{\mu\nu})\\ &= -\int d^D x\,\sqrt{-G}\, G^{\mu\nu}G^{\rho\sigma}\Big[\partial_\mu\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\sigma\nu}+\partial_\nu\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\sigma\mu}\\ &\qquad -\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\mu\nu}\Big]\\ &= -\int d^D x\,\sqrt{-G}\, G^{\mu\nu}G^{\rho\sigma}\Big[2\partial_\mu\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\sigma\nu}-\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)\delta G_{\mu\nu}\Big]\\ &= -\int d^D x\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)(2G^{\sigma\nu}G^{\rho\mu}-G^{\mu\nu}G^{\rho\sigma})\delta G_{\mu\nu}\end{aligned}\tag{3.401}$$

The second part is

$$\begin{aligned}\delta_G I_{4b} &= -2\int d^D x\, e^{-2\Phi}\sqrt{-G}\, G^{\mu\nu}\partial_\nu\Phi\frac{1}{2}G^{\rho\sigma}(\partial_\mu\delta G_{\sigma\rho}+\partial_\rho\delta G_{\sigma\mu}-\partial_\sigma\delta G_{\rho\mu})\\ &= \int d^D x\,\sqrt{-G}\, G^{\mu\nu}G^{\rho\sigma}\Big[\partial_\mu\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\rho\sigma}+\partial_\rho\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\sigma\mu}\\ &\qquad -\partial_\sigma\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\rho\mu}\Big]\\ &= \int d^D x\,\sqrt{-G}\, G^{\mu\nu}G^{\rho\sigma}\partial_\mu\left(e^{-2\Phi}\partial_\nu\Phi\right)\delta G_{\rho\sigma}\\ &= \int d^D x\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)G^{\mu\nu}G^{\rho\sigma}\delta G_{\mu\nu}\end{aligned}\tag{3.402}$$

Therefore

$$\begin{aligned}\delta_G I_4 &= \int d^D x\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)(-2G^{\sigma\nu}G^{\rho\mu}+G^{\mu\nu}G^{\rho\sigma}+G^{\mu\nu}G^{\rho\sigma})\delta G_{\mu\nu}\\ &= \int d^D x\,\sqrt{-G}\,\partial_\sigma\left(e^{-2\Phi}\partial_\rho\Phi\right)2(G^{\mu\nu}G^{\rho\sigma}-G^{\sigma\nu}G^{\rho\mu})\delta G_{\mu\nu}\\ &= \int d^D x\,\sqrt{-G}\, e^{-2\Phi}(-2\partial_\sigma\Phi\partial_\rho\Phi+\partial_\rho\partial_\sigma\Phi)\,2(G^{\mu\nu}G^{\rho\sigma}-G^{\sigma\nu}G^{\rho\mu})\delta G_{\mu\nu}\\ &= \int d^D x\,\sqrt{-G}\, e^{-2\Phi}(-4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu}+4\partial^\mu\Phi\partial^\nu\Phi+2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu}-2\partial^\mu\partial^\nu\Phi)\delta G_{\mu\nu}\end{aligned}\tag{3.403}$$

Adding the four pieces together we find

$$\begin{aligned}\delta_G\int d^D x\,\sqrt{-G}\, e^{-2\Phi}G^{\mu\nu}\boldsymbol{R}_{\mu\nu} = \int d^D x\sqrt{-G}e^{-2\Phi}\Big(&\frac{1}{2}G^{\mu\nu}\boldsymbol{R}-\boldsymbol{R}^{\mu\nu}\\ &-4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu}+4\partial^\mu\Phi\partial^\nu\Phi+2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu}-2\partial^\mu\partial^\nu\Phi\Big)\delta G_{\mu\nu}\end{aligned}\tag{3.404}$$

As expected this gives back the Einstein equations when we have a constant Φ, but we see that the dilaton field gives a correction that includes its derivative only.

There is another thing that we need to be careful about as well. Any indices upstairs have been raised via the spacetime "metric" $G^{\mu\nu}$ so they also carry a metric dependence. For example

$$\begin{aligned}\delta_G H_{\mu\nu\lambda}H^{\mu\nu\lambda} &= H_{\mu\nu\lambda}\delta_G(G^{\mu\sigma}G^{\nu\rho}G^{\lambda\kappa}H_{\sigma\rho\kappa}) = 3H_{\mu\nu\lambda}G^{\mu\sigma}G^{\nu\rho}H_{\sigma\rho\kappa}\delta G^{\lambda\kappa}\\ &= -3H_{\mu\nu\lambda}G^{\mu\sigma}G^{\nu\rho}H_{\sigma\rho\kappa}G^{\lambda\tau}G^{\kappa\eta}\delta G_{\tau\eta}\\ &= -3H^{\sigma\rho\tau}H^{\eta}{}_{\rho\sigma}\delta G_{\tau\eta} = -3H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}\delta G_{\mu\nu}\end{aligned} \tag{3.405}$$

Similarly

$$\delta_G\partial_\mu\Phi\partial^\mu\Phi = \delta_G G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = -G^{\mu\rho}G^{\nu\sigma}\partial_\mu\Phi\partial_\nu\Phi\delta G_{\rho\sigma} = -\partial^\mu\Phi\partial^\nu\Phi\delta G_{\mu\nu} \tag{3.406}$$

So we have

$$\begin{aligned}\delta_G\boldsymbol{S} = \frac{1}{2\kappa_0^2}\int d^Dx\, e^{-2\Phi}\Big\{&\delta_G\big(\sqrt{-G}\big)\Big[-\frac{2(D-26)}{3\alpha'}-\frac{1}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}+4\partial_\sigma\Phi\partial^\sigma\Phi\Big]\\ &+\sqrt{-G}\Big(-\boldsymbol{R}^{\mu\nu}+\frac{1}{2}G^{\mu\nu}\boldsymbol{R}-4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu}+4\partial^\mu\Phi\partial^\nu\Phi+2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu}-2\partial^\mu\partial^\nu\Phi\\ &\qquad+\frac{1}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}-4\partial^\mu\Phi\partial^\nu\Phi\Big)\delta G_{\mu\nu}\Big\}\\ =\frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\Big\{&\frac{1}{2}G^{\mu\nu}\delta G_{\mu\nu}\Big[-\frac{2(D-26)}{3\alpha'}-\frac{1}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}+4\partial_\sigma\Phi\partial^\sigma\Phi\Big]\\ &+\Big(-\boldsymbol{R}^{\mu\nu}+\frac{1}{2}G^{\mu\nu}\boldsymbol{R}-4\partial_\sigma\Phi\partial^\sigma\Phi G^{\mu\nu}+2\partial_\sigma\partial^\sigma\Phi G^{\mu\nu}-2\partial^\mu\partial^\nu\Phi+\frac{1}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}\Big)\delta G_{\mu\nu}\Big\}\\ =-\frac{1}{2\kappa_0^2\alpha'}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\Big[&\alpha'\boldsymbol{R}^{\mu\nu}+2\alpha'\partial^\mu\partial^\nu\Phi-\frac{\alpha'}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma}\\ &-\frac{1}{2}G^{\mu\nu}\Big(\alpha'G^{\mu\nu}\boldsymbol{R}+4\alpha'\partial_\sigma\partial^\sigma\Phi-\frac{\alpha'}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}-\frac{2(D-26)}{3}-4\alpha'\partial_\sigma\Phi\partial^\sigma\Phi\Big)\Big]\end{aligned} \tag{3.407}$$

We can rewrite the term between brackets as

$$\mathfrak{B}^{\mu\nu} = \beta^{G\,\mu\nu}-\frac{1}{2}G^{\mu\nu}(\beta^{G}{}^{\sigma}_{\sigma}-4\beta^{\Phi}) \tag{3.408}$$

Indeed

$$\begin{aligned}\mathfrak{B}^{\mu\nu} &= \alpha' \boldsymbol{R}^{\mu\nu} + 2\alpha'\nabla^\mu\nabla^\nu\Phi - \frac{\alpha'}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma} - \frac{1}{2}G^{\mu\nu}\Big(\alpha'\boldsymbol{R} + 2\alpha'\nabla^2\Phi - \frac{\alpha'}{4}H^{\lambda\rho\sigma}H_{\lambda\rho\sigma} \\ &\quad - \frac{2(D-26)}{3} + 2\alpha'\nabla^2\Phi - 4\alpha'\nabla_\sigma\Phi\nabla^\sigma\Phi + \frac{\alpha'}{6}H^{\lambda\rho\sigma}H_{\lambda\rho\sigma}\Big) \\ &= \alpha' \boldsymbol{R}^{\mu\nu} + 2\alpha'\nabla^\mu\nabla^\nu\Phi - \frac{\alpha'}{4}H^{\mu\rho\sigma}H^{\nu}{}_{\rho\sigma} - \frac{1}{2}G^{\mu\nu}\Big(\alpha'\boldsymbol{R} + 4\alpha'\nabla^2\Phi - \frac{\alpha'}{12}H^{\lambda\rho\sigma}H_{\lambda\rho\sigma} \\ &\quad - \frac{2(D-26)}{3}i - 4\alpha'\nabla_\sigma\Phi\nabla^\sigma\Phi\Big) \end{aligned} \tag{3.409}$$

and we see that if we replace the covariant derivatives by partial derivatives we do indeed find the expression between brackets in [3.407].

3.54 p 114: Eq (3.7.23) The Ricci Scalar after a Weyl Transformation

I expect most of you are expecting yet another long and tedious calculation. But if that is the case then you have a short memory. Indeed we have used this formula many times for the specific case of two dimensions, but when we originally derived it in chapter one we did so for general dimensions D, see [1.21]. Happy and light-footed we move on to the next challenge.

3.55 p 114: Eq (3.7.25) The Space Time Action with Einstein Metric

Let us first work out the pre-factor in (3.7.20). We have

$$G_{\mu\nu} = e^{-2\omega}\tilde{G}_{\mu\nu} = e^{-4(\Phi_0-\Phi)/(D-2)}\tilde{G}_{\mu\nu} = e^{4\tilde{\Phi}/(D-2)}\tilde{G}_{\mu\nu} \tag{3.410}$$

implying that

$$\sqrt{-G} = e^{2D\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}} \tag{3.411}$$

so that

$$\sqrt{-G}e^{-2\Phi} = e^{2D\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\,e^{-2(\Phi_0+\tilde{\Phi})} = e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}} \tag{3.412}$$

We will absorb the $e^{-2\Phi_0}$ into the κ_0. Let us now do term by term in (3.7.20)

$$\begin{aligned}\mathfrak{T}_1 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}e^{-2\Phi}\left[-\frac{2(D-26)}{3\alpha'}\right] \\ &= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left[-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)}\right]\end{aligned} \tag{3.413}$$

Next, consider the term with the curvature. Using (3.7.23)

$$\begin{aligned}
\mathfrak{T}_2 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\boldsymbol{R}\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\left[e^{2\omega}\tilde{\boldsymbol{R}}+2(D-1)\nabla^2\omega+(D-2)(D-1)\partial\omega\cdot\partial\omega\right]\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\left[e^{-4\tilde{\Phi}/(D-2)}\tilde{\boldsymbol{R}}-\frac{4(D-1)}{D-2}\nabla^2\tilde{\Phi}+\frac{4(D-1)}{D-2}\partial\tilde{\Phi}\cdot\partial\tilde{\Phi}\right]\\
&= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left\{\tilde{\boldsymbol{R}}+4e^{4\tilde{\Phi}/(D-2)}\left[-\frac{D-1}{D-2}\nabla^2\tilde{\Phi}+\frac{D-1}{D-2}\partial\tilde{\Phi}\cdot\partial\tilde{\Phi}\right]\right\}
\end{aligned}\tag{3.414}$$

We need one more step. We write

$$\partial\tilde{\Phi}\cdot\partial\tilde{\Phi}=G^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}=e^{-4\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}=e^{-4\tilde{\Phi}/(D-2)}\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\tag{3.415}$$

Thus

$$\mathfrak{T}_2=\frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left[\tilde{\boldsymbol{R}}+4\frac{(D-1)}{D-2}\left(-e^{4\tilde{\Phi}/(D-2)}\nabla^2\tilde{\Phi}+\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\right)\right]\tag{3.416}$$

Next, the term with the antisymmetric field strength

$$\begin{aligned}
\mathfrak{T}_3 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}\left(-\frac{1}{12}H_{\mu\nu\lambda}G^{\mu\rho}G^{\nu\sigma}G^{\lambda\kappa}H_{\rho\sigma\kappa}\right)\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\left(-\frac{1}{12}H_{\mu\nu\lambda}e^{-12\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\rho}\tilde{G}^{\nu\sigma}\tilde{G}^{\lambda\kappa}\tilde{H}_{\rho\sigma\kappa}\right)\\
&= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left[-\frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}_{\rho\sigma\kappa}\right]
\end{aligned}\tag{3.417}$$

Finally, the term with the dilaton field

$$\begin{aligned}
\mathfrak{T}_4 &= \frac{1}{2\kappa_0^2}\int d^Dx\,\sqrt{-G}\,e^{-2\Phi}4G^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\\
&= \frac{1}{2\kappa_0^2}\int d^Dx\,e^{-2\Phi_0}e^{4\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\,4e^{-4\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\nu}\partial_\mu\tilde{\Phi}\partial_\nu\tilde{\Phi}\\
&= \frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\left[4\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\right]
\end{aligned}\tag{3.418}$$

Adding the four contributions we find

$$\begin{aligned}
\sum_{a=1}^{4}\mathfrak{T}_a=\frac{1}{2\kappa^2}\int d^Dx\,\sqrt{-\tilde{G}}\Big[&-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)}+\tilde{\boldsymbol{R}}-\frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}_{\rho\sigma\kappa}\\
&+4\frac{(D-1)}{D-2}\left(-e^{4\tilde{\Phi}/(D-2)}\nabla^2\tilde{\Phi}+\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\right)+4\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\Big]
\end{aligned}\tag{3.419}$$

We seem to be close but not exactly equal to (3.725). What saves us, as usual, is a partial integration. Indeed

$$\begin{aligned}
\mathfrak{X} &= \int d^D x\,\sqrt{-\tilde{G}}\,e^{4\tilde{\Phi}/(D-2)}\nabla^2\tilde{\Phi} = \int d^D x\,e^{-2D\tilde{\Phi}/(D-2)}\sqrt{-G}\,e^{4\tilde{\Phi}/(D-2)}\nabla_\mu\partial^\mu\tilde{\Phi}\\
&= \int d^D x\,e^{-2\tilde{\Phi}}\partial_\mu\left(\sqrt{-G}\,\partial^\mu\tilde{\Phi}\right) = -\int d^D x\,\partial_\mu\left(e^{-2\tilde{\Phi}}\right)\sqrt{-G}\,\partial^\mu\tilde{\Phi}\\
&= \int d^D x\,e^{-2\tilde{\Phi}}\left(-2\partial_\mu\tilde{\Phi}\right)\sqrt{-G}\,\partial^\mu\tilde{\Phi} = -2\int d^D x\,e^{-2\tilde{\Phi}}e^{2D\tilde{\Phi}/(D-2)}\sqrt{-\tilde{G}}\,\partial_\mu\tilde{\Phi}\partial^\mu\tilde{\Phi}\\
&= 2\int d^D x\,\sqrt{-\tilde{G}}\,e^{4\tilde{\Phi}/(D-2)}\partial_\mu\tilde{\Phi}G^{\mu\nu}\partial^\nu\tilde{\Phi} = 2\int d^D x\,\sqrt{-\tilde{G}}\,e^{4\tilde{\Phi}/(D-2)}\partial_\mu\tilde{\Phi}e^{-4\tilde{\Phi}/(D-2)}\tilde{G}^{\mu\nu}\partial_\nu\tilde{\Phi}\\
&= 2\int d^D x\,\sqrt{-\tilde{G}}\,\partial_\mu\tilde{\Phi}\tilde{\partial}^\mu\tilde{\Phi}
\end{aligned} \tag{3.420}$$

Note that we had to go back and forth between G and $\tilde{G}$ more than once in this last derivation. This means that we can write the last few terms of [3.419] as

$$\begin{aligned}
&\frac{1}{2\kappa^2}\int d^D x\,\sqrt{-\tilde{G}}\,4\left[\frac{D-1}{D-2}(-2+1)+1\right]\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\\
&= \frac{1}{2\kappa^2}\int d^D x\,\sqrt{-\tilde{G}}\left[-\frac{4}{D-2}\right]\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}
\end{aligned} \tag{3.421}$$

Therefore [3.419] becomes, finally,

$$\begin{aligned}
\sum_{a=1}^{4}\mathfrak{T}_a = \frac{1}{2\kappa^2}\int d^D x\,\sqrt{-\tilde{G}}\Big[&-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + \tilde{\boldsymbol{R}}\\
&-\frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}_{\rho\sigma\kappa} - \frac{4}{D-2}\partial\tilde{\Phi}\cdot\tilde{\partial}\tilde{\Phi}\Big]
\end{aligned} \tag{3.422}$$

which is exactly (3.7.25).

Various Backgrounds

One other qualitative feature of the field equations is readily understood: the field Φ always occurs differentiated, so there is an invariance under X^μ-independent shifts of Φ. This is because such a shift only changes the world-sheet action (3.7.6) by a term proportional to the Euler number, and so does not affect local properties like Weyl invariance. In particular, the background

$$G_{\mu\nu}(X) = \eta_{\mu\nu} \ , \quad B_{\mu\nu}(X) = 0 \ , \quad \Phi(X) = \Phi_0 \tag{3.7.16}$$

is exactly Weyl-invariant for any constant Φ_0. This is just the flat spacetime action (3.2.2), with

$$\lambda = \Phi_0 \ . \tag{3.7.17}$$

The value of λ determines the coupling strength between strings, but we now see that this does not mean that there are different string theories with different values of this parameter. Different values of λ correspond not to different *theories*, but to different *backgrounds* in a single theory. The other parameter appearing in the Nambu–Goto and Polyakov actions, the Regge slope α', is also not really a free parameter because it is dimensionful. It simply defines the unit of length, and can be absorbed in the definition of X^μ.

Changing the fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ in the world-sheet action would seem, from the two-dimensional point of view, to give a new theory. From the point of view of the full string theory, one is merely looking at a different background — a different state — in the same theory. This is one of the features that makes string theory attractive. In the Standard Model, and in most attempts to unify the Standard Model within quantum field theory, there are many constants that are not determined by the theory; any values for these constants give a consistent theory. In string theory there are no such free parameters — the coupling constants depend on the state and are determined in principle by the dynamics. Of course, for now this only moves the difficulty elsewhere, because we do not understand the dynamics well enough to know the values of the background fields.

It is striking that Einstein's equation turns up in what seems to be a rather out-of-the-way place, as the condition for Weyl invariance of a two-dimensional field theory. String theory provides the physical connection between these two ideas. It is also notable that, because of the need for Weyl invariance, strings can propagate consistently only in a background that satisfies appropriate field equations. This parallels our earlier discovery that only on-shell vertex operators make sense.

Another observation: the condition $D = 26$ came from the R term in $T^a{}_a$. In the nonlinear sigma model this is generalized to $\beta^\Phi = 0$. We see in eq. (3.7.14c) that the leading term in β^Φ is proportional to $D - 26$ but that there are corrections involving the gradients of the fields. Evidently we can have other values of D if the fields are not constant. This is true, although we cannot really conclude it from eq. (3.7.14c) because this has been derived in the approximation $\alpha'^{1/2} R_c^{-1} \ll 1$ where the correction terms in β^Φ are small. In fact, exact solutions with $D \neq 26$ are known. For now we give one simple example, saving others for later. This is

$$G_{\mu\nu}(X) = \eta_{\mu\nu} \ , \quad B_{\mu\nu}(X) = 0 \ , \quad \Phi(X) = V_\mu X^\mu \ . \tag{3.7.18}$$

The beta functions (3.7.14) vanish if

$$V_\mu V^\mu = \frac{26 - D}{6\alpha'} \ . \tag{3.7.19}$$

This result is actually *exact*, because the fields (3.7.18) are constant or linear in X^μ, so the world-sheet path integral remains Gaussian. Varying g_{ab} to determine T_{ab} for this theory, one finds that it is none other than the linear dilaton CFT (hence the name), whose central charge $c = D + 6\alpha' V_\mu V^\mu$ is indeed 26 under the condition (3.7.19). Because Φ must have a large gradient in order to change D, this CFT does not describe our roughly static and homogeneous four-dimensional spacetime, but it may have cosmological applications. This CFT is also important because the cases $D = 1$ and $D = 2$ are simple enough for the string theory to be exactly solvable, as we will discuss in section 9.9.

Space-time Action

The field equations (3.7.15) can be derived from the spacetime action

$$\boldsymbol{S} = \frac{1}{2\kappa_0^2}\int d^D x\,(-G)^{1/2} e^{-2\Phi}\Big[-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi + O(\alpha')\Big] . \qquad (3.7.20)$$

The normalization constant κ_0 is not determined by the field equations and has no physical significance since it can be changed by a redefinition of Φ. One can verify that

$$\delta\boldsymbol{S} = -\frac{1}{2\kappa_0^2\alpha'}\int d^D x\,(-G)^{1/2} e^{-2\Phi}\Big[\delta G_{\mu\nu}\beta^{G\mu\nu} + \delta B_{\mu\nu}\beta^{B\mu\nu} + \Big(2\delta\Phi - \tfrac{1}{2}G^{\mu\nu}\delta G_{\mu\nu}\Big)(\beta^{G\omega}{}_{\omega} - 4\beta^{\Phi})\Big] . \qquad (3.7.21)$$

The action is written in bold as a reminder that this is the effective action governing the low energy spacetime fields, unlike the world-sheet actions that have appeared everywhere previously in this book.

It is often useful to make a field redefinition of the form

$$\tilde{G}_{\mu\nu}(x) = \exp(2\omega(x))G_{\mu\nu}(x) , \qquad (3.7.22)$$

which is a spacetime Weyl transformation. The Ricci scalar constructed from $\tilde{G}_{\mu\nu}$ is

$$\tilde{\boldsymbol{R}} = \exp(-2\omega)\Big[\boldsymbol{R} - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega\Big] . \qquad (3.7.23)$$

For the special case $D = 2$, this is the Weyl transformation (3.3.5). Let $\omega = 2(\Phi_0 - \Phi)/(D-2)$ and define

$$\tilde{\Phi} = \Phi - \Phi_0 , \qquad (3.7.24)$$

which has vanishing expectation value. The action becomes

$$\boldsymbol{S} = \frac{1}{2\kappa^2}\int d^D X\,(-\tilde{G})^{1/2}\Big[-\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + \tilde{\boldsymbol{R}} - \frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} - \frac{4}{D-2}\partial_\mu\tilde{\Phi}\tilde{\partial}^\mu\tilde{\Phi} + O(\alpha')\Big] , \qquad (3.7.25)$$

where tildes have been inserted as a reminder that indices here are raised with $\tilde{G}^{\mu\nu}$. In terms of $\tilde{G}_{\mu\nu}$, the gravitational Lagrangian density takes the standard Hilbert form $(-\tilde{G})^{1/2}\tilde{\boldsymbol{R}}/2\kappa^2$. The constant $\kappa = \kappa_0 e^{\Phi_0}$ is the gravitational coupling, which in four-dimensional gravity has the value

$$\kappa = (8\pi G_{\rm N})^{1/2} = \frac{(8\pi)^{1/2}}{M_{\rm P}} = (2.43\times 10^{18}\ {\rm GeV})^{-1} . \qquad (3.7.26)$$

Compactification & CFT

The four string theories studied thus far (with or without boundaries, oriented or unoriented) have a number of features in common. One good feature is the automatic inclusion of general relativity. There are also several bad features: the need for 26 dimensions, the existence of the tachyon, and the absence of fermions in the spectrum. The elimination of the tachyon and the introduction of fermions will have to wait until volume two. However, the presence of general relativity provides a natural way to account for the extra dimensions. In general relativity the geometry of spacetime is dynamical, not fixed. Flat spacetime is only one of many solutions to the field equations. There may well be solutions in which some dimensions are large and flat and others are small and highly curved. The metric would be

$$g_{MN} = \begin{bmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{mn}(x^p) \end{bmatrix} . \tag{3.7.29}$$

We have divided the 26 coordinates $M, N = 0, \ldots, 25$ into four 'spacetime' coordinates $\mu, \nu = 0, \ldots, 3$ and 22 'internal' coordinates $m, n, p = 4, \ldots, 25$. The metric (3.7.29) is flat in the four spacetime dimensions and curved in the 22 internal dimensions. The internal dimensions are assumed to be compact. As one example, the background field equations (3.7.15) are satisfied to order α' if the dilaton field is constant, the antisymmetric tensor is zero, and the metric for the internal space is Ricci-flat, $\boldsymbol{R}_{mn} = 0$.

In other words, spacetime is of the form $M^d \times K$, where M^d is d-dimensional Minkowski space and K is some $(26-d)$-dimensional compact Riemannian space.[4] In such a spacetime, the physics on length scales much longer than the size of K is the same as in a d-dimensional Minkowski space. One says that $26 - d$ dimensions have been *compactified*, though cosmologically it seems more appropriate to say that the large dimensions have decompactified. As we discussed briefly in the introduction, there are good reasons, independent of string theory, to consider the possibility that our spacetime might be of this form. The compact dimensions must be small enough not to have been observed. In fact, we will see in chapter 8, and in more detail in chapter 18, that the size of any extra dimensions must be within a few orders of magnitude of the Planck length, 1.6×10^{-33} cm.

We can also think about generalizing this idea further. The basic physical consistency conditions are few in number. We require Lorentz invariance, and we require that quantum mechanical probabilities be positive and conserved — to date, nothing in string theory seems to require that the standard quantum mechanical framework be modified, though it would be interesting to learn otherwise. [illegible] ple

Although these conditions are simple there is a tension between them, in gauge theory and gravitation as well as in string theory. In a manifestly covariant gauge there are negative norm timelike excitations that must decouple from physical amplitudes. In the light-cone gauge the inner product is positive but covariance is no longer manifest.

One necessary condition is therefore two-dimensional diff invariance, so that the unphysical oscillations are merely oscillations of the coordinate system. Weyl invariance is a more technical requirement: the extra degree of freedom that appears when Weyl invariance is lost is less intuitive than an oscillation of the coordinate system, but we have argued in section 3.4 that it is incompatible with Lorentz invariance.

We will also make an additional technical assumption: that the embedding time X^0 appears in the world-sheet action only in the term

$$-\frac{1}{4\pi\alpha'}\int_M d^2\sigma\, g^{1/2} g^{ab}\partial_a X^0 \partial_b X^0 \ . \tag{3.7.30}$$

For example, in the sigma model this means that the fields are static with $G_{0\mu} = \eta_{0\mu}$ and $B_{0\mu} = 0$. This assumption is not much of a restriction. The form (3.7.30) is sufficient if one is interested in static states. Time-dependent backgrounds are certainly of interest for cosmological and other reasons, but they can be analyzed using low energy effective field theory except in the extreme case that the time scale is of the order of the string scale. The reason for making the assumption (3.7.30) is to put the main problem, the wrong-sign nature of X^0, in a rather explicit form.

The requirement that the local invariances be maintained leads to the following proposal for a more general string theory: *Replace the 25 spatial X^μ fields with any unitary CFT having $c = \tilde{c} = 25$.* This ensures that the full two-dimensional theory, with X^0 and the ghosts, can be coupled to a curved metric in a (diff×Weyl)-invariant way. The unitary (positive inner product) condition is necessary because there is only enough local symmetry to remove the unphysical X^0 excitation; the world-sheet inner product is relevant here because it becomes the spacetime inner product in the one-particle sector. We will see in volume two that further generalization is possible, enlarging simultaneously the local symmetry group and the set of unphysical excitations.

If we are interested in a bosonic string theory whose low energy physics looks four-dimensional, we keep the free fields X^μ for $\mu = 0, 1, 2, 3$, and replace the remaining X^μ with a compact unitary CFT having $c = \tilde{c} = 22$. By compact, we mean that the spectrum is discrete, just as for a quantum field theory in a finite volume. There have been some efforts to construct all CFTs, or all CFTs having various additional properties. We will describe some of these in chapter 15. Incidentally, all of these theories have tachyons, the product of the unit state $|1\rangle$ from the internal CFT with the ground state $|0; k\rangle$ from the four-dimensional CFT.

How large a generalization have we just made? If the local symmetries are the only essential restriction, we are free to introduce fields on the world-sheet with various world-sheet and spacetime quantum numbers. This seems to take us far beyond the original theory of 26 X^μ fields. However, in two dimensions there are many equivalences between seemingly different quantum field theories, so that the situation is not clear.

It is generally assumed that all string theories based on different CFTs but the same world-sheet gauge symmetries and topology are different ground states of the same theory. This causes a semantic problem: we often refer to strings with different world-sheet Lagrangians as different *theories*, though they are most likely just different vacua in one theory. We will take this even further in chapter 14, seeing that *all* string theories appear to be vacua of a single theory.

As we develop the techniques of string theory in volume one, we will usually take the simplest case of flat 26-dimensional spacetime in explicit examples. However, many results will be developed in such a way as to apply to the very general string theory we have described above.

For more details read David Tong's lecture notes on String Theory Ch-7.