

Physical & Unphysical States

The world sheet fields in play after gauge fixing are D number of X^μ fields and the Faddeev-Popov ghost fields b_{ab} & c^a .

But due to Covariant quantization we have oscillator modes $\hat{\alpha}^\mu$ & $\hat{\tilde{\alpha}}^\mu$ corresponding to each X^μ field. The presence of time-like oscillators leads to -ve normed states due to -ve sign in the commutation of ' α^0 ' oscillators.

$$\text{i.e. } [\hat{\alpha}_n^{\mu\dagger}, \hat{\alpha}_m^\nu] = n \delta_{nm} \eta^{\mu\nu}, \quad (n > 0)$$

$$\text{and } \langle 0 | \alpha_n^\circ \alpha_n^{\circ\dagger} | 0 \rangle = n \eta^{\circ\circ} + \underbrace{\langle 0 | \alpha_n^{\circ\dagger} \alpha_n^\circ | 0 \rangle}_0$$

$$\rightarrow \|\alpha_n^{\circ\dagger} | 0 \rangle\|^2 = -n \quad (-\text{ve norm state})$$

Also Ghost fields b & c have -ve normed states because the weight of c is -1 . So state corresponding to ' c ' has $\|\hat{L}_{-1} | c \rangle\|^2 = 2h_c = -2$ (-ve norm).

So the current Hilbert Space has a large number of unphysical / ghost states. To get the correct spectrum of string, we only need the Hilbert Space of physical states.

To identify the smaller physical Hilbert Space, consider a long cylindrical world sheet where we have initially used the local symmetries to fix the metric to $\hat{g}_{ab}(\sigma)$ by Faddeev-Popov gauge fixing.

Now if a state $|\psi_i\rangle$ propagates to a different state $|\psi_f\rangle$ on the cylinder, then there exists some amplitude $\langle\psi_f|\psi_i\rangle$.

If we then change the gauge fixed metric by $\delta\hat{g}_{ab}(\sigma)$ then, as seen in section 3.4, the amplitude changes as follows

$$\delta\langle\psi_f|\psi_i\rangle = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \delta\hat{g}^{ab} \langle\psi_f|\hat{T}_{ab}|\psi_i\rangle$$

Now a physical amplitude should never depend on the choice of gauge fixed metric.

$\rightarrow \delta \langle \psi_f | \psi_i \rangle = 0$ for any arbitrary $\delta \hat{g}^{ab}$ changes in the metric.

$$\rightarrow \langle \psi_f | \hat{T}_{ab} | \psi_i \rangle = 0$$

So those states that satisfy the above condition are called as physical states.

Note that, if we would have not done any gauge fixing, then we would get the eqⁿ of motion for gauge fields $g_{ab}(\sigma)$ to be $T_{ab} = 0$ which appears as an operator eqⁿ. (inside the path integral). But, after gauge fixing, we don't have freedom to vary the gauge fixed metric $\hat{g}_{ab}(\sigma)$. Thus, $\hat{T}_{ab} = 0$ doesn't hold as an operator eqⁿ anymore. So, not all states in the gauge fixed theory can satisfy $\langle \hat{T}_{ab} \rangle = 0$.

But only those states that have $\langle \hat{T}_{ab} \rangle = 0$ are called Physical states.

After gauge fixing, we will have even the Ghost action in play which will contribute to physical action.

$$\text{i.e. } S = S_x + S_g$$

$$\rightarrow T_{ab} = \underbrace{T_{ab}^x}_{\text{for } X^\mu \text{ field}} + \underbrace{T_{ab}^g}_{\text{for Ghost field}}$$

The X^μ field is the matter field. So the below OCQ analysis & formulation is valid for any 'D' matter fields. Then,

$$T_{ab} = T_{ab}^M + T_{ab}^g$$

'M' represents matter field. & 'g' for the Ghost fields.

Ad-hoc Approach of OCQ

In this approach, we will simply ignore Ghost energy-momentum tensor & restrict the matter energy-momentum tensor matrix elements go to zero for physical states.

In terms of Laurent coefficients $\hat{L}_n^M$ i.e. the Virasoro generators for the matter sector, we need its matrix elements to be zero.

It is too hard to restrict $\hat{L}_n^M |\psi\rangle = 0$ for all 'n'. And it won't go well with the Virasoro algebra because of the central charge term which need not be zero.

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{m+n} + \frac{C}{12} m(m^2-1)\delta_{m,-n}$$

So we can atleast restrict $\hat{L}_n^M |\psi\rangle = 0$ for 'n > 0'. And $(\hat{L}_0^M + A^M) |\psi\rangle = 0$

Here A^M is the normal ordering constant that we came across earlier which was present only for 0^{th} mode.

We require that any physical state to obey

$$\underline{(\hat{L}_n^M + A^M \delta_{n,0})|\psi\rangle = 0 \text{ for } n \geq 0}$$

This is enough to make those matrix elements of $\hat{L}_n^M$ w.r.t. the physical states to go to zero for all $n \neq 0$.

We know that X^μ CFT is unitary. So, we have

$$\hat{L}_{-n}^M = \hat{L}_n^{M\dagger} \quad \forall 'n'$$

Consider $\langle\psi_f|\hat{L}_n^M|\psi_i\rangle$ where $|\psi_i\rangle \neq |\psi_f\rangle$ are physical states. i.e. for $n > 0$, $\hat{L}_n^M|\psi_i\rangle = 0$

$\Rightarrow \langle\psi_f|\hat{L}_n^M|\psi_i\rangle = 0$ for $n > 0$ by definition.

Then, for $n < 0$,

$$\langle\psi_f|\hat{L}_n^M = (\hat{L}_n^{M\dagger}|\psi_f\rangle)^\dagger = (\hat{L}_{-n}^M|\psi_f\rangle)^\dagger = 0$$

$$\rightarrow \langle \psi_f | \hat{L}_n^M | \psi_i \rangle = 0 \text{ even for } n < 0$$

$$\text{Thus, } \langle \psi_f | \hat{L}_n^M | \psi_i \rangle = 0 \text{ for all } n \neq 0$$

$$\text{For } n=0, \langle \psi_f | (\hat{L}_0^M + A^M) | \psi_i \rangle = 0$$

$$\rightarrow \langle \psi_f | \hat{L}_0^M | \psi_i \rangle = -A^M \langle \psi_f | \psi_i \rangle$$

$$\rightarrow \langle \psi | \hat{L}_0^M | \psi \rangle = -A^M \text{ as } \langle \psi | \psi \rangle = 1$$

So, if $|\psi_i\rangle$ & $|\psi_f\rangle$ are orthonormal, then

$$\langle \psi_f | \hat{L}_0^M | \psi_i \rangle = -A^M \delta_{ij}$$

Thus the matrix elements of all non zero mode ' $\hat{L}_n^M$ ' is zero.

But in chapter-2 we saw that those states which are eigen states of $\hat{L}_0^M$ are the highest weight primary states with weight $-A^M$.

Spurious States

Consider all possible states as an infinite set $\{|\chi_n\rangle\}$ $n=1,2,3,\dots$

Then, the state $|\chi\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n}^M |\chi_n\rangle$ is orthogonal to all the physical states.

Let $|\psi\rangle$ be some physical state.

$$\Rightarrow \hat{L}_n^M |\psi\rangle = 0 \text{ for all } n > 0.$$

$$\begin{aligned} \text{Then } \langle\psi|\chi\rangle &= \langle\psi|\chi\rangle = \sum_{n=1}^{\infty} \langle\psi|\hat{L}_{-n}^M|\chi_n\rangle \\ &= \sum_{n=1}^{\infty} (\hat{L}_{-n}^{M\dagger}|\psi\rangle)^\dagger |\chi_n\rangle = \sum_{n=1}^{\infty} (\underbrace{\hat{L}_n^M|\psi\rangle}_0)^\dagger |\chi_n\rangle \end{aligned}$$

$$\Rightarrow \langle\psi|\chi\rangle = 0 \text{ for any Physical } |\psi\rangle.$$

Such a state ' $|\chi\rangle$ ' is called Spurious.

$$\text{And, } \hat{L}_k^M |\chi\rangle = \sum_{n=1}^{\infty} \hat{L}_k^M \hat{L}_{-n}^M |\chi_n\rangle$$

$$[\hat{L}_k^M, \hat{L}_{-n}^M] = (k+n) \hat{L}_{k-n}^M + \frac{C}{12} k(k^2-1) \delta_{k,n}$$

$$\hat{L}_k^M \hat{L}_{-n}^M = (k+n) \hat{L}_{k-n}^M + \frac{C}{12} k(k^2-1) \delta_{k,n} + \hat{L}_{-n}^M \hat{L}_k^M$$

$$= \sum_{n=1}^{\infty} \left[(k+n) \hat{L}_{k-n}^M |\chi_n\rangle + \frac{C}{12} k(k^2-1) \delta_{k,n} |\chi_n\rangle + \hat{L}_{-n}^M \hat{L}_k^M |\chi_n\rangle \right]$$

$$= \sum_{n=1}^{\infty} (k+n) \hat{L}_{k-n}^M |\chi_n\rangle + \frac{C}{12} k(k^2-1) |\chi_k\rangle + \sum_{n=1}^{\infty} \hat{L}_{-n}^M \hat{L}_k^M |\chi_n\rangle$$

Put $k-n \rightarrow -n$

$$= \sum_{n=0}^{\infty} (2k+n) \hat{L}_{-n}^M |\chi_{k+n}\rangle + \frac{C}{12} k(k^2-1) |\chi_k\rangle + \sum_{n=1}^{\infty} \hat{L}_{-n}^M \hat{L}_k^M |\chi_n\rangle$$

$$\rightarrow \hat{L}_k^M |\chi\rangle = \left[2k \hat{L}_0^M + \frac{C}{12} k(k^2-1) \right] |\chi_k\rangle + \sum_{n=1}^{\infty} \left[(2k+n) \hat{L}_{-n}^M |\chi_{k+n}\rangle + \hat{L}_{-n}^M \hat{L}_k^M |\chi_n\rangle \right]$$

Null State

A Physical state which is also Spurious is called a Null state.

i.e. $\langle \psi | \chi \rangle = 0$ for any Physical $|\psi\rangle$.

$$\& \quad \underline{(\hat{L}_n^M + A^M \alpha_{n,0})|\chi\rangle = 0 \text{ for } n \geq 0}$$

Then, consider ' $|\psi\rangle + |\chi\rangle$ ' state.
It is also physical &

$$\langle \psi' | (|\psi\rangle + |\chi\rangle) = \langle \psi' | \psi \rangle + \underbrace{\langle \psi' | \chi \rangle}_0$$

$$\Rightarrow \underline{\langle \psi' | (|\psi\rangle + |\chi\rangle) = \langle \psi' | \psi \rangle}$$

$$\begin{aligned} & (\langle \psi | + \langle \chi |)(|\psi\rangle + |\chi\rangle) \\ &= \langle \psi | \psi \rangle + \underbrace{\langle \chi | \psi \rangle}_0 + \underbrace{\langle \psi | \chi \rangle}_0 + \langle \chi | \chi \rangle \end{aligned}$$

$$(\langle \psi | + \langle \chi |)(|\psi\rangle + |\chi\rangle) = \langle \psi | \psi \rangle + \langle \chi | \chi \rangle$$

If $\langle \chi | \chi \rangle \sim 0$ (null-ness)

Then $(\langle \psi | + \langle \chi |)(|\psi\rangle + |\chi\rangle) = \langle \psi | \psi \rangle$

Thus $|\psi\rangle + |\chi\rangle$ behaves exactly same as ' $|\psi\rangle$ '. So, effectively

$$|\psi\rangle + |\chi\rangle \simeq |\psi\rangle$$

These two states are indistinguishable

Then the real physical Hilbert space is the set of equivalence classes or the quotient group $\mathbb{H}_{OCQ} = \frac{\mathbb{H}_{\text{Physical}}}{\mathbb{H}_{\text{null}}}$

$\mathbb{H}_{\text{Physical}} \rightarrow$ Set of all physical states

$\mathbb{H}_{\text{null}} \rightarrow$ Set of all null states

$\mathbb{H}_{OCQ} \rightarrow$ Set of all physical states
which are not null states

OCC for 1st two mass levels

We saw in chapter 2, that

$$\hat{L}_m^M = \frac{1}{2} \left[:\hat{\alpha}_m^\mu \hat{\alpha}_0^\nu: + \sum_{n=1}^{\infty} (\hat{\alpha}_{m-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{m+n}^\mu) \right] \eta_{\mu\nu}$$

$$\rightarrow \hat{L}_0^M = \frac{1}{2} \hat{\alpha}_0^\mu \hat{\alpha}_0^\nu \eta_{\mu\nu} + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}$$

But $\hat{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}} \hat{p}^\mu = \hat{\tilde{\alpha}}_0^\mu$ for closed string

& $\hat{\alpha}_0^\mu = \sqrt{2\alpha'} \hat{p}^\mu$ for open string

$$\rightarrow \hat{L}_0^M = \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu} \text{ (closed string)}$$

There is also the anti-holomorphic part for the closed string case.

$$\hat{\tilde{L}}_0^M = \frac{\alpha'}{4} \hat{\tilde{p}}^\mu \hat{\tilde{p}}_\mu + \sum_{n=1}^{\infty} \hat{\tilde{\alpha}}_{-n}^\mu \hat{\tilde{\alpha}}_n^\nu \eta_{\mu\nu} \text{ (closed string)}$$

Then, for open string, we get,

$$\hat{L}_0^M = \alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu} \text{ (open string)}$$

And,

$$\begin{aligned}\hat{L}_{\pm 1}^M &= \frac{1}{2} \left[\hat{\alpha}_{\pm 1}^\mu \hat{\alpha}_0^\nu + \sum_{n=1}^{\infty} (\hat{\alpha}_{\pm 1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\pm 1+n}^\mu) \right] \eta_{\mu\nu} \\ &= \frac{1}{2} \left[\hat{\alpha}_{\pm 1}^\mu \hat{\alpha}_0^\nu + (\hat{\alpha}_{\pm 1-1}^\mu \hat{\alpha}_1^\nu + \hat{\alpha}_{-1}^\nu \hat{\alpha}_{\pm 1+1}^\mu) \right. \\ &\quad \left. + \sum_{n=2}^{\infty} (\hat{\alpha}_{\pm 1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\pm 1+n}^\mu) \right] \eta_{\mu\nu}\end{aligned}$$

$$\begin{aligned}\Rightarrow \hat{L}_{+1}^M &= \frac{1}{2} \left[\hat{\alpha}_{+1}^\mu \hat{\alpha}_0^\nu + \hat{\alpha}_0^\mu \hat{\alpha}_1^\nu + \hat{\alpha}_{-1}^\nu \hat{\alpha}_2^\mu \right. \text{combine} \\ &\quad \left. + \sum_{n=2}^{\infty} \underbrace{\hat{\alpha}_{+1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{+1+n}^\mu}_{1-n \rightarrow -n} \right] \eta_{\mu\nu}\end{aligned}$$

$$\hat{L}_{+1}^M = \hat{\alpha}_0^\mu \hat{\alpha}_1^\nu \eta_{\mu\nu} + \frac{\eta_{\mu\nu}}{2} \left[\sum_{n=1}^{\infty} (\hat{\alpha}_{1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{+1+n}^\mu) \right]$$

Similarly,

$$\hat{L}_{-1}^M = \hat{\alpha}_{-1}^\nu \hat{\alpha}_0^\mu \eta_{\mu\nu} + \frac{\eta_{\mu\nu}}{2} \left[\sum_{n=1}^{\infty} (\hat{\alpha}_{-1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{-1+n}^\mu) \right]$$

Thus,

$$\hat{L}_{\pm 1}^M = :\hat{\alpha}_{\pm 1}^\mu \hat{\alpha}_0^\nu: \eta_{\mu\nu} + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{\pm 1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\pm 1+n}^\mu)$$

For closed string,

$$\hat{L}_{\pm 1}^M = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{\pm 1}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} (\hat{\alpha}_{\pm 1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\pm 1+n}^\mu) \frac{\eta_{\mu\nu}}{2}$$

$$\hat{L}_{\pm 1}^M = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{\pm 1}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} (\hat{\alpha}_{\pm 1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\pm 1+n}^\mu) \frac{\eta_{\mu\nu}}{2}$$

For open string,

$$\hat{L}_{\pm 1}^M = \sqrt{2\alpha'} \hat{\alpha}_{\pm 1}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} (\hat{\alpha}_{\pm 1-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\pm 1+n}^\mu) \frac{\eta_{\mu\nu}}{2}$$

Now we shall see how the Old Covariant quantization works for the 1st two levels.

Tachyon State

Open String

The lowest mass level state is $|0; k^\mu\rangle$

$$\hat{L}_0^M |0; k^\mu\rangle = \left[\alpha' \hat{p}^\mu \hat{p}_\mu + \underbrace{\sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}}_0 \right] |0; k^\mu\rangle$$

$$= \alpha' \hat{p}^\mu \hat{p}_\mu |0; k^\mu\rangle = \alpha' k^\mu k_\mu |0; k^\mu\rangle$$
$$(k^\mu k_\mu = -M^2)$$

$$\rightarrow \hat{L}_0^M |0; k^\mu\rangle = -\alpha' M^2 |0; k^\mu\rangle$$

Then, consider for $l > 0$,

$$\hat{L}_l^M = \frac{1}{2} \left[:\hat{\alpha}_l^\mu \hat{\alpha}_0^\nu: + \sum_{n=1}^{\infty} (\hat{\alpha}_{l-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{l+n}^\mu) \right] \eta_{\mu\nu}$$

$$= \frac{1}{2} \left[\hat{\alpha}_0^\mu \hat{\alpha}_l^\nu + \sum_{n=1}^{\infty} (\hat{\alpha}_{l-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{l+n}^\mu) \right] \eta_{\mu\nu}$$

$$\rightarrow \hat{L}_l^M |0; k^\mu\rangle = 0 \quad \text{for all } l > 0.$$

Thus we got that $|0; k^\mu\rangle$ is a physical state that has the form

$$[\hat{L}_\ell^M + \delta_{\ell,0} \alpha' M^2] |0; k^\mu\rangle = 0$$

$$\Rightarrow A^M = \alpha' M^2 \Rightarrow M^2 = \frac{A^M}{\alpha'}$$

We cannot find a spurious state at this level because such a state is defined as,

$$|\chi\rangle = \sum_{\ell=1}^{\infty} \hat{L}_{-\ell}^M |\chi_\ell\rangle$$

where

$$\hat{L}_{-\ell}^M = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-\ell}^\mu \hat{p}_\mu + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{-\ell-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{-\ell+n}^\mu)$$

There are creation operators in $\hat{L}_{-\ell}^M$, so they increase the mass level. So we cannot find a state that is spurious & physical at this mass level other than $|0; k^\mu\rangle$. Thus there is only one element in $\mathbb{H}_{OCQ}$ at level '0' corresponding to $|0; k^\mu\rangle$.

For closed String

The lowest mass level state is $|0;0;k^\mu\rangle$

$$\begin{aligned}\hat{L}_0^M |0;0;k^\mu\rangle &= \left[\frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu + \underbrace{\sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu}}_0 \right] |0;0;k^\mu\rangle \\ &= \frac{\alpha'}{4} \hat{p}^\mu \hat{p}_\mu |0;0;k^\mu\rangle = \frac{\alpha'}{4} k^\mu k_\mu |0;0;k^\mu\rangle \\ &\quad (k^\mu k_\mu = -M^2)\end{aligned}$$

$$\rightarrow \hat{L}_0^M |0;0;k^\mu\rangle = -\frac{\alpha' M^2}{4} |0;0;k^\mu\rangle$$

Similarly,

$$\hat{L}_0^M |0;0;k^\mu\rangle = -\frac{\alpha' M^2}{4} |0;0;k^\mu\rangle$$

$$\& \hat{L}_\ell^M |0;0;k^\mu\rangle = 0 = \hat{L}_\ell^M |0;0;k^\mu\rangle \quad (\ell > 0).$$

$$\text{Thus } \left[\hat{L}_\ell^M + \delta_{\ell,0} \frac{\alpha' M^2}{4} \right] |0;0;k^\mu\rangle = 0$$

$$\& \left[\hat{L}_\ell^M + \delta_{\ell,0} \frac{\alpha' M^2}{4} \right] |0;0;k^\mu\rangle = 0$$

Thus $|0;0;k^\mu\rangle$ is a physical state & the mass is connected to A^M by,

$$\rightarrow A^M = \frac{\alpha' M^2}{4} \rightarrow M^2 = 4 \frac{A^M}{\alpha'}$$

Even here we cannot find a spurious state at this level because

$$|\chi\rangle = \sum_{\ell=1}^{\infty} \hat{L}_{-\ell}^M |\chi_\ell\rangle$$

where $\hat{L}_{-\ell}^M$ contains creation operators. So they increase the mass level. So we cannot find a state that is spurious & physical at this mass level, other than $|0;0;k^\mu\rangle$. Thus there is only one element in $\mathbb{H}_{OCQ}$ at level '0' corresponding to $|0;0;k^\mu\rangle$.

Photon State of Open String

The 1st excited state is got by acting a creation operator $\hat{\alpha}_{-1}^{\mu}$ on the level zero state $|0; k^{\nu}\rangle$ with a polarization vector ε_{μ} (constant)

$$\rightarrow \varepsilon_{\mu} \hat{\alpha}_{-1}^{\mu} |0; k^{\nu}\rangle \equiv |\varepsilon; k^{\nu}\rangle$$

$$\text{Similarly } f_{\rho} \hat{\alpha}_{-1}^{\rho} |0; q^{\sigma}\rangle \equiv |f; q^{\sigma}\rangle$$

Then the amplitude $\langle f; q^{\sigma} | \varepsilon; k^{\nu} \rangle$

$$\langle f; q^{\sigma} | \varepsilon; k^{\nu} \rangle = \langle 0; q^{\sigma} | f_{\rho}^{*} \hat{\alpha}_{-1}^{\rho} \varepsilon_{\mu} \hat{\alpha}_{-1}^{\mu} |0; k^{\nu}\rangle$$

$$\text{use } [\hat{\alpha}_n^{\mu}, \hat{\alpha}_{-m}^{\nu}] = m \delta_{n,m} \eta^{\mu\nu}$$

$$\begin{aligned} \Rightarrow \langle \varepsilon; q^{\sigma} | \varepsilon; k^{\nu} \rangle &= \\ &= f_{\rho}^{*} \varepsilon_{\mu} \langle 0; q^{\sigma} | (\hat{\alpha}_{-1}^{\mu} \hat{\alpha}_{-1}^{\rho} + \eta^{\rho\mu}) |0; k^{\nu}\rangle \\ &= f_{\rho}^{*} \varepsilon_{\mu} \underbrace{\langle 0; q^{\sigma} | \hat{\alpha}_{-1}^{\mu} \hat{\alpha}_{-1}^{\rho} |0; k^{\nu}\rangle}_0 + f_{\rho}^{*} \varepsilon_{\mu} \eta^{\mu\rho} \langle 0; q^{\sigma} | 0; k^{\nu} \rangle \end{aligned}$$

$$\Rightarrow \langle f; q^\sigma | \varepsilon; k^\nu \rangle = f_\mu^* \varepsilon^\mu \langle 0; q^\sigma | 0; k^\nu \rangle$$

$$\text{But } \langle 0; q^\sigma | 0; k^\nu \rangle = (2\pi)^D \delta^{(D)}(k^\nu - q^\nu)$$

$$\Rightarrow \langle f; q^\sigma | \varepsilon; k^\nu \rangle = f_\mu^* \varepsilon^\mu (2\pi)^D \delta^{(D)}(k^\nu - q^\nu)$$

Then the norm of the state $|\varepsilon; k^\nu\rangle$ is given by

$$\begin{aligned} \langle \varepsilon; k^\sigma | \varepsilon; k^\nu \rangle &= \varepsilon_\mu^* \varepsilon^\mu (2\pi)^D \delta^{(D)}(0) \\ &= (-|\varepsilon^0|^2 + |\vec{\varepsilon}|^2) (2\pi)^D \delta^{(D)}(0) \end{aligned}$$

But if ε^μ was purely time-like, then $\vec{\varepsilon} = 0$, & $\varepsilon^0 = 1$.

$$\rightarrow \langle \varepsilon; k^\sigma | \varepsilon; k^\nu \rangle = \underset{\substack{\text{Pure} \\ \text{time-like}}}{-(2\pi)^D \delta^{(D)}(0)}$$

Thus the pure-time like excitation has -ve norm.

$\hat{L}_0^M$ condition

$$\hat{L}_0^M |\varepsilon; k^\mu\rangle = \left[\alpha' \hat{p}^\nu \hat{p}_\nu + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\rho \hat{\alpha}_n^\nu \eta_{\rho\nu} \right] |\varepsilon; k^\mu\rangle$$

$$= \alpha' \hat{p}^\nu \hat{p}_\nu |\varepsilon; k^\mu\rangle + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\rho \hat{\alpha}_n^\nu \eta_{\rho\nu} \varepsilon_\sigma \hat{\alpha}_{-1}^\sigma |0; k^\mu\rangle$$

$$\begin{aligned} \rightarrow \hat{L}_0^M |\varepsilon; k^\mu\rangle &= -\alpha' k^\nu k_\nu |\varepsilon; k^\mu\rangle \\ &\quad + \varepsilon_\sigma \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\nu n} \hat{\alpha}_{-1}^\sigma |0; k^\mu\rangle \end{aligned}$$

use $[\hat{\alpha}_{-n}^\nu \hat{\alpha}_{\nu n}, \hat{\alpha}_{-1}^\sigma] = n \delta_{n1} \hat{\alpha}_{-1}^\sigma$

$$\begin{aligned} \rightarrow \hat{L}_0^M |\varepsilon; k^\mu\rangle &= \alpha' k^\nu k_\nu |\varepsilon; k^\mu\rangle \\ &\quad + \varepsilon_\sigma \sum_{n=1}^{\infty} \left(n \delta_{n,-1} \hat{\alpha}_{-n}^\sigma + \underbrace{\hat{\alpha}_{-n}^\nu \hat{\alpha}_{\nu n}}_0 \right) |0; k^\mu\rangle \end{aligned}$$

$$= \alpha' k^\nu k_\nu |\varepsilon; k^\mu\rangle + \varepsilon_\nu \hat{\alpha}_{-1}^\nu |0; k^\mu\rangle$$

$$\rightarrow \hat{L}_0^M |\varepsilon; k^\mu\rangle = (\alpha' k^\nu k_\nu + 1) |\varepsilon; k^\mu\rangle$$

$$\rightarrow (\hat{L}_0^M + A^M) |\varepsilon; k^\mu\rangle = 0$$

where, $A^M = -\alpha' k^\nu k_\nu - 1$

But $k^\nu k_\nu = -M^2$

$$\Rightarrow A^M = M^2 \alpha' - 1$$

$$\Rightarrow \boxed{M^2 = \frac{A^M + 1}{\alpha'}} \Rightarrow \text{Mass level for Photon state}$$

Thus for the above mass level, the $\hat{L}_0^M$ condition is satisfied.

$\hat{L}_l^M$ ($l > 0$) condition

Now,

$$\hat{L}_l^M = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_l^\mu \hat{p}_\mu + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{l-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{l+n}^\mu)$$

Then,

$$\hat{L}_l^M |\varepsilon; k^\nu\rangle \text{ for } l > 0 \text{ is}$$

$$\hat{L}_l^M |\varepsilon; k^\nu\rangle = \hat{L}_l^M \varepsilon_\mu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle$$

$$\hat{L}_\ell^M |\varepsilon; k^\nu\rangle = \varepsilon_\mu \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_\ell^\nu \hat{p}_\nu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle$$

$$+ \frac{\eta_{\nu\rho} \varepsilon_\mu}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{\ell-n}^\nu \hat{\alpha}_n^\rho \hat{\alpha}_{-1}^\mu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{\ell+n}^\rho \hat{\alpha}_{-1}^\mu) |0; k^\nu\rangle$$

$$\left(\hat{\alpha}_\ell^\nu \hat{p}_\nu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle = \hat{\alpha}_\ell^\nu \hat{\alpha}_{-1}^\mu k_\nu |0; k^\nu\rangle \right.$$

$$\left. = \eta^{\mu\nu} \delta_{\ell,1} k_\nu |0; k^\nu\rangle + \underbrace{\hat{\alpha}_{-1}^\mu \hat{\alpha}_\ell^\nu}_{0} |0; k^\nu\rangle \right)$$

use $[\hat{\alpha}_n^\mu, \hat{\alpha}_{-m}^\nu] = m \delta_{n,m} \eta^{\mu\nu}$

$$\hat{\alpha}_n^\rho \hat{\alpha}_{-1}^\mu = \eta^{\rho\mu} \delta_{n,1} + \hat{\alpha}_{-1}^\mu \hat{\alpha}_n^\rho$$

$$\hat{\alpha}_{\ell+n}^\rho \hat{\alpha}_{-1}^\mu = \eta^{\rho\mu} \delta_{\ell+n,1} + \hat{\alpha}_{-1}^\mu \hat{\alpha}_{\ell+n}^\rho$$

$$\Rightarrow \hat{L}_\ell^M |\varepsilon; k^\nu\rangle = \sqrt{\frac{\alpha'}{2}} \delta_{\ell,1} \varepsilon_\mu \eta^{\mu\nu} k_\nu |0; k^\nu\rangle$$

$$+ \frac{\eta_{\nu\rho} \varepsilon_\mu}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{\ell-n}^\nu \eta^{\rho\mu} \delta_{n,1} |0; k^\nu\rangle$$

$$+ \frac{\eta_{\nu\rho} \varepsilon_\mu}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\nu \eta^{\rho\mu} \underbrace{\delta_{\ell+n,1}}_{n=1-\ell, \ell>0} |0; k^\nu\rangle$$

is never satisfied
as $n \geq 1$

(other terms don't contribute)

$$\Rightarrow \hat{L}_\ell^M |\varepsilon; k^\nu\rangle = \sqrt{\frac{\alpha'}{2}} \varepsilon_\mu \delta_{\ell,1} \eta^{\mu\nu} k_\nu |0; k^\nu\rangle + \frac{\varepsilon_\nu}{2} \hat{\alpha}_{\ell-1}^\nu |0; k^\nu\rangle$$

$$\Rightarrow \hat{L}_\ell^M |\varepsilon; k^\nu\rangle = \left(\sqrt{\frac{\alpha'}{2}} \delta_{\ell,1} \varepsilon_\mu k^\mu + \frac{\varepsilon_\mu}{2} \hat{\alpha}_{\ell-1}^\mu \right) |0; k^\nu\rangle$$

For $\ell=1$,

$$\begin{aligned} \hat{L}_1^M |\varepsilon; k^\nu\rangle &= \left(\sqrt{\frac{\alpha'}{2}} \varepsilon_\mu k^\mu + \frac{\varepsilon_\mu}{2} \hat{\alpha}_0^\mu \right) |0; k^\nu\rangle \\ &= \left(\sqrt{\frac{\alpha'}{2}} \varepsilon_\mu k^\mu + \frac{\varepsilon_\mu \sqrt{2\alpha'}}{2} \hat{p}^\mu \right) |0; k^\nu\rangle \\ &= \sqrt{2\alpha'} \varepsilon_\mu k^\mu |0; k^\nu\rangle \end{aligned}$$

And for $\ell > 1$,

$$\hat{L}_\ell^M |\varepsilon; k^\nu\rangle = \frac{\varepsilon_\mu \hat{\alpha}_{\ell-1}^\mu}{2} |0; k^\nu\rangle$$



 doesn't contribute

$$\Rightarrow \hat{L}_\ell^M |\varepsilon; k^\nu\rangle = 0 \quad (\ell > 1)$$

Thus for $\ell > 0$

$$\hat{L}_\ell^M |\varepsilon; k^\nu\rangle = \delta_{\ell,1} \sqrt{2\alpha'} \varepsilon_\mu k^\mu |0; k^\nu\rangle$$

Hence, for $\hat{L}_l^M |\varepsilon; k^\nu\rangle = 0$ for all $l > 0$
 we require $\boxed{\varepsilon_\mu k^\mu = 0}$

Thus for $|\varepsilon; k^\nu\rangle$ to be a physical state
 we require, $-k^\mu k_\mu = M^2 = \frac{A^M + 1}{\alpha'}$
 $\& \varepsilon_\mu k^\mu = 0$

Only then, $(\hat{L}_l^M + A^M \delta_{l,0}) |\varepsilon; k^\mu\rangle = 0$

Spurious state at 1st level

$$\hat{L}_{-l}^M = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^\mu \hat{p}_\mu + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{-l-n}^\mu \hat{\alpha}_n^\nu + \hat{\alpha}_{-n}^\nu \hat{\alpha}_{n-l}^\mu)$$

Consider $\hat{L}_{-l}^M |0; k^\nu\rangle$, it is given by,

$$= \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^\mu k_\mu |0; k^\nu\rangle + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^{\infty} \underbrace{\hat{\alpha}_{-n}^\nu \hat{\alpha}_{n-l}^\mu}_{\text{goes to zero when } n > l} |0; k^\nu\rangle$$

goes to zero when $n > l$

$$= \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^\mu k_\mu |0; k^\nu\rangle + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^l \hat{\alpha}_{-n}^\nu \hat{\alpha}_{n-l}^\mu |0; k^\nu\rangle$$

$$\hat{L}_{-\ell}^M |0; k^\nu\rangle = \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-\ell}^\mu |0; k^\nu\rangle + \frac{\eta_{\mu\nu}}{2} \sum_{n=1}^{\ell-1} \hat{\alpha}_{-n}^\nu \hat{\alpha}_{n-\ell}^\mu |0; k^\nu\rangle$$

To get spurious state of level '1', we consider only $\hat{L}_{-1}^M |0; k^\nu\rangle$

$$\hat{L}_{-1}^M |0; k^\nu\rangle = \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle$$

Now, for any physical state $|\psi\rangle$, we have

$$\langle \psi | \hat{L}_{-1}^M |0; k^\nu\rangle = 0$$

Thus, $\hat{L}_{-1}^M |0; k^\nu\rangle = \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle$ is a spurious level 1 state.

So level-1 states of the form, $E_\mu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle$ whose $E_\mu = \sqrt{2\alpha'} k_\mu$ is spurious

For it to be physical ' $k_\mu k^\mu = 0$ ' (light-like D-momenta) is needed.

Then, it implies we need mass to be $k_\mu k^\mu = -M^2 = 0 \rightarrow A^M = -1$. Only then,

$$(\hat{L}_\ell^M + A^M \partial_{\ell,0}) \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle = 0$$

& $\sqrt{2\alpha'} k_\mu \hat{\alpha}_{-1}^\mu |0; k^\nu\rangle$ is a Null state.

Hence Such a State whose polarization is proportional to light-like D-momenta is Null. This won't appear as a state in H_{OCQ} at level-1. As it is equivalent to any other physical state.

Thus if $A^M = -1$, there are two restrictions on ε^μ (polarization) for $|\varepsilon; k^\nu\rangle$ to appear in H_{OCQ} . It should satisfy $\varepsilon_\mu k^\mu = 0$ and $k_\mu k^\mu = 0$. So, it only has D-2 degrees of freedom with all +ve norms as Pure time-like excitation is forbidden.

Also $\varepsilon^\mu \simeq \varepsilon^\mu + k^\mu$, this reflects the Space-time Gauge Symmetry.

There are no other spurious states at level one. And other spurious states of the form

$$\hat{L}_{-2}^M |0; k^\nu\rangle = \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-2}^\mu |0; k^\nu\rangle + \frac{\eta_{\mu\nu} \hat{\alpha}_{-1}^\nu \hat{\alpha}_{-1}^\mu}{2} |0; k^\nu\rangle$$

$$\hat{L}_{-3}^M |0; k^\nu\rangle = \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-3}^\mu |0; k^\nu\rangle + \eta_{\mu\nu} \hat{\alpha}_{-1}^\nu \hat{\alpha}_{-2}^\mu |0; k^\nu\rangle$$

$$\hat{L}_{-4}^M |0; k^\nu\rangle = \sqrt{2\alpha'} k_\mu \hat{\alpha}_{-4}^\mu |0; k^\nu\rangle + \eta_{\mu\nu} \hat{\alpha}_{-1}^\nu \hat{\alpha}_{-3}^\mu |0; k^\nu\rangle + \frac{\eta_{\mu\nu} \hat{\alpha}_{-2}^\nu \hat{\alpha}_{-2}^\mu}{2} |0; k^\nu\rangle$$

and so on are not level 1 & also need not be physical unless some other extra conditions are imposed.

Consequences of other values of ' A^M '

(i) If $A > -1$, the mass-squared is positive. Going to the rest frame, the physical state condition removes the negative norm timelike polarization. The spurious state (4.1.18) is not physical, $k \cdot k \neq 0$, so there are no null states and the spectrum consists of the $D - 1$ positive-norm states of a massive vector particle.

(ii) If $A = -1$, the mass-squared is zero. Now $k \cdot k = 0$ so the spurious state is physical and null. Thus,

$$k \cdot e = 0, \quad e_\mu \cong e_\mu + \gamma k_\mu. \quad (4.1.19)$$

This describes the $D - 2$ positive-norm states of a massless vector particle.

(iii) If $A < -1$, the mass-squared is negative. The momentum is spacelike, so the physical state condition removes a positive-norm *spacelike* polarization. The spurious state is not physical, so we are left with a tachyonic vector particle with $D - 2$ positive-norm states and one negative-norm state.

Case (iii) is unacceptable. Case (ii) is the same as the light-cone quantization. Case (i) is not the same as the light-cone spectrum, having different masses and an extra state at the first level, but does not have any obvious inconsistency. The difficulty is that there is no known way to give such a theory consistent interactions. Case (ii) is the one of interest in string theory, and the one we will recover from the more general approach of the next section.

Only for $A^M = -1$ & $D = 26$ we get
$$\mathbb{H}_{\text{OCQ}} = \mathbb{H}_{\text{light cone}}$$

For level-2 states in open string, refer to "A Companion reader to Polchinski's string Theory" by Stony M Schrans.

Level-1 Excitations for Closed String

This level is got by acting both $\hat{\alpha}_{-1}^{\mu}$ and $\hat{\alpha}_{-1}^{\nu}$ on the closed string ground state $|0;0;k^{\nu}\rangle$.

Let $f_{\mu\nu} \hat{\alpha}_{-1}^{\mu} \hat{\alpha}_{-1}^{\nu} |0;0;k^{\nu}\rangle \equiv |f_{\mu\nu};k\rangle$ be the level 1 state for closed string.

$$\hat{L}_0^M |f_{\mu\nu};k\rangle = \left[\frac{\alpha'}{4} \hat{p}^{\rho} \hat{p}_{\rho} + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^{\mu} \hat{\alpha}_n^{\nu} \eta_{\mu\nu} \right] |f_{\mu\nu};k\rangle$$

$$= \frac{\alpha'}{4} \hat{p}^{\rho} \hat{p}_{\rho} f_{\mu\nu} \hat{\alpha}_{-1}^{\mu} \hat{\alpha}_{-1}^{\nu} |0;0;k\rangle + \eta_{\rho\sigma} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^{\rho} \hat{\alpha}_n^{\sigma} \hat{\alpha}_{-1}^{\mu} \hat{\alpha}_{-1}^{\nu} |0;0;k\rangle$$

$$\hat{\alpha}_n^{\sigma} \hat{\alpha}_{-1}^{\mu} = \delta_{n,1} \eta^{\sigma\mu} + \hat{\alpha}_{-1}^{\mu} \hat{\alpha}_n^{\sigma}$$

$$= \frac{\alpha'}{4} k^{\rho} k_{\rho} |f_{\mu\nu};k\rangle$$

$$+ \eta_{\rho\sigma} f_{\mu\nu} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^{\rho} \delta_{n,1} \eta^{\sigma\mu} \hat{\alpha}_{-1}^{\nu} |0;0;k\rangle$$

$$\begin{aligned} \rightarrow \hat{L}_0^M |f_{\mu\nu}; k\rangle &= \frac{\alpha' k^\rho k_\rho}{4} |f_{\mu\nu}; k\rangle \\ &\quad + f_{\mu\nu} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0; 0; k\rangle \end{aligned}$$

$$\rightarrow \hat{L}_0^M |f_{\mu\nu}; k\rangle = \left(\frac{\alpha' k^\rho k_\rho}{4} + 1 \right) |f_{\mu\nu}; k\rangle$$

$\hat{L}_0^M$ condition :- $(\hat{L}_0^M + A^M) |f_{\mu\nu}; k\rangle = 0$

where $A^M = -\frac{\alpha' k^\rho k_\rho}{4} - 1 = \frac{\alpha' M^2}{4} - 1$

$$\rightarrow \boxed{M^2 = \frac{4}{\alpha'} (A^M + 1)}$$

In the exact same way we can find

$$\hat{L}_0^M |f_{\mu\nu}; k\rangle = \left[\frac{\alpha' \hat{p}^\rho \hat{p}_\rho}{4} + \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu \eta_{\mu\nu} \right] |f_{\mu\nu}; k\rangle$$

$$= \frac{\alpha' \hat{p}^\rho \hat{p}_\rho}{4} f_{\mu\nu} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0; 0; k\rangle$$

$$+ \eta_{\rho\sigma} f_{\mu\nu} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\rho \hat{\alpha}_{-1}^\mu \hat{\alpha}_n^\sigma \hat{\alpha}_{-1}^\nu |0; 0; k\rangle$$

$$\hat{\alpha}_n^\sigma \hat{\alpha}_{-1}^\nu = \delta_{n,1} \eta^{\sigma\nu} + \hat{\alpha}_{-1}^\nu \hat{\alpha}_n^\sigma$$

$$\begin{aligned} \Rightarrow \hat{L}_0^M |f_{\mu\nu}; k\rangle &= \frac{\alpha' k^\rho k_\rho}{4} |f_{\mu\nu}; k\rangle \\ &+ \eta_{\rho\sigma} f_{\mu\nu} \eta^{\sigma\gamma} \sum_{n=1}^{\infty} \delta_{n,1} \hat{\alpha}_{-n}^\rho \hat{\alpha}_{-1}^\mu |0;0;k\rangle \end{aligned}$$

$$\Rightarrow \hat{L}_0^M |f_{\mu\nu}; k\rangle = \frac{\alpha' k^\rho k_\rho}{4} |f_{\mu\nu}; k\rangle + |f_{\mu\nu}; k\rangle$$

$$\Rightarrow (\hat{L}_0^M + \tilde{A}^M) |f_{\mu\nu}; k\rangle = 0$$

$$\text{where } \tilde{A}^M = -\frac{\alpha' k^\rho k_\rho}{4} - 1 = \frac{\alpha' M^2}{4} - 1$$

$$\Rightarrow \boxed{M^2 = \frac{4}{\alpha'} (\tilde{A}^M + 1)}$$

$$\underline{\hat{L}_l^M, l > 0 \text{ condition}}$$

$$\hat{L}_l^M = \frac{1}{2} \left[\sqrt{\frac{\alpha'}{2}} \hat{\alpha}_l^\rho \hat{p}_\rho + \sum_{n=1}^{\infty} \eta_{\rho\sigma} (\hat{\alpha}_{l-n}^\rho \hat{\alpha}_n^\sigma + \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{l+n}^\rho) \right]$$

$$\hat{L}_l^M |f_{\mu\nu}; k\rangle = \frac{f_{\mu\nu}}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_l^\rho \hat{p}_\rho \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0;0;k\rangle$$

$$+ \frac{\eta_{\rho\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{l-n}^\rho \hat{\alpha}_n^\sigma + \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{l+n}^\rho) \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0;0;k\rangle$$

$$\hat{L}_l^M |f_{\mu\nu}; k\rangle = \frac{f_{\mu\nu} k_\rho}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\nu \hat{\alpha}_l^\rho \hat{\alpha}_{-1}^\mu |0; 0; k\rangle$$

$$+ \frac{\eta_{\rho\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{l-n}^\rho \hat{\alpha}_n^\sigma + \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{l+n}^\rho) \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0; 0; k\rangle$$

$$= \frac{f_{\mu\nu} k_\rho}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\nu \eta^{\rho\mu} \delta_{l,1} |0; 0; k\rangle$$

$$+ \frac{\eta_{\rho\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{l-n}^\rho \hat{\alpha}_{-1}^\nu \hat{\alpha}_n^\sigma \hat{\alpha}_{-1}^\mu |0; 0; k\rangle$$

$$+ \frac{\eta_{\rho\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{-1}^\nu \hat{\alpha}_{l+n}^\rho \hat{\alpha}_{-1}^\mu |0; 0; k\rangle$$

$$= \frac{f_{\mu\nu} k^\mu}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\nu \delta_{l,1} |0; 0; k\rangle$$

$$+ \frac{\eta_{\rho\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{l-n}^\rho \hat{\alpha}_{-1}^\nu \delta_{n,1} \eta^{\sigma\mu} |0; 0; k\rangle$$

$$+ \frac{\eta_{\rho\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{-1}^\nu \underbrace{\delta_{l+n,1}}_{\text{for } l>0 \neq n>0, l+n \neq 1} \eta^{\rho\mu} |0; 0; k\rangle$$

for $l > 0 \neq n > 0, l+n \neq 1$

$$\Rightarrow \hat{L}_l^M |f_{\mu\nu}; k\rangle = \frac{f_{\mu\nu} k^\mu}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\nu \delta_{l,1} |0;0;k\rangle + \frac{f_{\mu\nu}}{2} \hat{\alpha}_{-1}^\nu \hat{\alpha}_{l-1}^\mu |0;0;k\rangle$$

For $l > 1$, $\hat{L}_l^M |f_{\mu\nu}; k\rangle = 0$ trivially

But if $l = 1$,

$$\hat{L}_1^M |f_{\mu\nu}; k\rangle = \frac{f_{\mu\nu} k^\mu}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\nu |0;0;k\rangle + \frac{f_{\mu\nu}}{2} \hat{\alpha}_{-1}^\nu \sqrt{\frac{\alpha'}{2}} k^\mu |0;0;k\rangle$$

$$\Rightarrow \hat{L}_1^M |f_{\mu\nu}; k\rangle = f_{\mu\nu} k^\mu \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\nu |0;0;k\rangle$$

Thus if $\hat{L}_l^M |f_{\mu\nu}; k\rangle = 0$ is needed for $l > 0$ then we require,

$$\boxed{f_{\mu\nu} k^\mu = 0}$$

$$\underline{\hat{L}_l^M, l > 0 \text{ condition}}$$

$$\hat{L}_l^M = \frac{1}{2} \left[\sqrt{\frac{\alpha'}{2}} \hat{\alpha}_l^p \hat{p}_p + \sum_{n=1}^{\infty} \eta_{p\sigma} (\hat{\alpha}_{l-n}^p \hat{\alpha}_n^\sigma + \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{l+n}^p) \right]$$

$$\begin{aligned} \hat{L}_l^M |f_{\mu\nu}; k\rangle &= \frac{f_{\mu\nu}}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu \hat{\alpha}_l^p \hat{\alpha}_{-1}^\nu \hat{p}_p |0; 0; k\rangle \\ &+ \frac{\eta_{p\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} (\hat{\alpha}_{l-n}^p \hat{\alpha}_n^\sigma + \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{l+n}^p) \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0; 0; k\rangle \end{aligned}$$

$$= \frac{f_{\mu\nu} k_p}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu \eta^{p\nu} \delta_{l,1} |0; 0; k\rangle$$

$$+ \frac{\eta_{p\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{l-n}^p \hat{\alpha}_n^\sigma \hat{\alpha}_{-1}^\nu |0; 0; k\rangle$$

$$+ \frac{\eta_{p\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{l+n}^p \hat{\alpha}_{-1}^\nu |0; 0; k\rangle$$

$$= \frac{f_{\mu\nu} k^\nu}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu \delta_{l,1} |0; 0; k\rangle$$

$$+ \frac{\eta_{p\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{l-n}^p \delta_{n,1} \eta^{\sigma\nu} |0; 0; k\rangle$$

$$+ \frac{\eta_{p\sigma} f_{\mu\nu}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-n}^\sigma \underbrace{\delta_{l+n,1}}_{\text{for } l > 0 \neq n > 0, l+n \neq 1} \eta^{p\nu} |0; 0; k\rangle$$

$$\begin{aligned} \Rightarrow \hat{L}_l^M |f_{\mu\nu}; k\rangle &= \frac{f_{\mu\nu} k^\nu}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu \delta_{l,1} |0;0;k\rangle \\ &\quad + \frac{f_{\mu\nu}}{2} \hat{\alpha}_{-1}^\mu \hat{\alpha}_{l-1}^\nu |0;0;k\rangle \end{aligned}$$

For $l > 1$, $\hat{L}_l^M |f_{\mu\nu}; k\rangle = 0$ trivially

But if $l = 1$,

$$\begin{aligned} \hat{L}_1^M |f_{\mu\nu}; k\rangle &= \frac{f_{\mu\nu} k^\nu}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu |0;0;k\rangle \\ &\quad + \frac{f_{\mu\nu}}{2} \hat{\alpha}_{-1}^\mu \sqrt{\frac{\alpha'}{2}} k^\nu |0;0;k\rangle \end{aligned}$$

$$\Rightarrow \hat{L}_1^M |f_{\mu\nu}; k\rangle = f_{\mu\nu} k^\nu \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu |0;0;k\rangle$$

Thus if $\hat{L}_l^M |f_{\mu\nu}; k\rangle = 0$ is needed for $l > 0$ then we require,

$$\boxed{f_{\mu\nu} k^\nu = 0}$$

Closed String Spurious State at level 1

We have,

$$\hat{L}_{-l}^M = \frac{1}{2} \left[\sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^u \hat{p}_u + \sum_{n=1}^{\infty} \eta_{uv} (\hat{\alpha}_{-l-n}^u \hat{\alpha}_n^v + \hat{\alpha}_{-n}^v \hat{\alpha}_{n-l}^u) \right]$$

$$\hat{L}_{-l}^M = \frac{1}{2} \left[\sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^p \hat{p}_p + \sum_{n=1}^{\infty} \eta_{p\sigma} (\hat{\alpha}_{-l-n}^p \hat{\alpha}_n^\sigma + \hat{\alpha}_{-n}^\sigma \hat{\alpha}_{n-l}^p) \right]$$

$$\hat{L}_{-l}^M |0;0;k\rangle = \frac{1}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^u \hat{p}_u |0;0;k\rangle$$

$$+ \frac{\eta_{uv}}{2} \sum_{n=1}^{\infty} \underbrace{(\hat{\alpha}_{-l-n}^u \hat{\alpha}_n^v + \hat{\alpha}_{-n}^v \hat{\alpha}_{n-l}^u)}_0 |0;0;k\rangle$$

$$= \frac{1}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^u k_u |0;0;k\rangle + \frac{\eta_{uv}}{2} \sum_{n=1}^{\infty} \hat{\alpha}_{-n}^v \hat{\alpha}_{n-l}^u |0;0;k\rangle$$

$$\rightarrow \hat{L}_{-1}^M |0;0;k\rangle = \frac{k_u}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^u |0;0;k\rangle + \frac{\eta_{uv}}{2} \hat{\alpha}_{-1}^v \hat{\alpha}_0^u |0;0;k\rangle$$

$$\begin{aligned} \rightarrow \hat{L}_{-1}^M |0;0;k\rangle &= \frac{k_u}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^u |0;0;k\rangle \\ &\quad + \frac{1}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^u k_u |0;0;k\rangle \end{aligned}$$

$$\rightarrow \hat{L}_{-1}^M |0;0;k\rangle = k_u \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^u |0;0;k\rangle$$

Then,

$$\hat{L}_{-1}^M \hat{L}_{-1}^M |0;0;k\rangle = k_u \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^u \hat{L}_{-1}^M |0;0;k\rangle$$

$$\begin{aligned} \text{Now, } \hat{L}_{-l}^M |0;0;k\rangle &= \frac{1}{2} \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-l}^y k_y |0;0;k\rangle \\ &\quad + \sum_{n=1}^{\infty} \eta_{p\sigma} \hat{\alpha}_{-n}^{\sigma} \hat{\alpha}_{n-l}^p |0;0;k\rangle \end{aligned}$$

$$\rightarrow \hat{L}_{-1}^M |0;0;k\rangle = \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^y k_y |0;0;k\rangle$$

$$\rightarrow \hat{L}_{-1}^M \hat{L}_{-1}^M |0;0;k\rangle = \frac{\alpha'}{2} k_u k_y \hat{\alpha}_{-1}^u \hat{\alpha}_{-1}^y |0;0;k\rangle$$

This is a level-1 state whose Polarization tensor is

$$f_{uv} = \frac{\alpha'}{2} k_u k_y$$

This is a Spurious state. Since, for any physical state $|\psi\rangle$, we have

$$\langle\psi|\hat{L}_{-1}^M\hat{L}_{-1}^M|0;0;k\rangle=0$$

For $\hat{L}_{-1}^M\hat{L}_{-1}^M|0;0;k\rangle$ to be physical we need,

$$\frac{\alpha'}{2}k_\mu k_\nu k^\mu = \frac{\alpha'}{2}k_\mu k_\nu k^\nu = 0$$

$$\Rightarrow k_\mu k^\mu = 0$$

$$\Rightarrow M^2 = -k^\mu k_\mu = 0 = \frac{4}{\alpha'}(A^M + 1) = \frac{4}{\alpha'}(\tilde{A}^M + 1)$$

$$\Rightarrow \underline{A^M = \tilde{A}^M = -1}$$

Thus when the above condition is satisfied we have Null states. So then, these states whose $f_{\mu\nu} \propto k_\mu k_\nu$ & $k_\mu k^\mu = 0$ are not seen in the HOCQ space.

There are even more spurious states in this level. We previously got

$$\hat{L}_{-1}^M |0;0;k\rangle = k_\mu \sqrt{\frac{\alpha'}{2}} \hat{\alpha}_{-1}^\mu |0;0;k\rangle$$

Consider $\mathcal{E}_\nu \hat{\alpha}_{-1}^\nu \hat{L}_{-1}^M + \mathcal{E}_\mu \hat{\alpha}_{-1}^\mu \hat{L}_{-1}^M |0;0;k\rangle$

$$= \sqrt{\frac{\alpha'}{2}} k_\mu \mathcal{E}_\nu \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0;0;k\rangle + \sqrt{\frac{\alpha'}{2}} \mathcal{E}_\mu k_\nu \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0;0;k\rangle$$

$$= \sqrt{\frac{\alpha'}{2}} (k_\mu \mathcal{E}_\nu + k_\nu \mathcal{E}_\mu) \hat{\alpha}_{-1}^\mu \hat{\alpha}_{-1}^\nu |0;0;k\rangle$$

This is a level-1 spurious state whose

$$f_{\mu\nu} = \sqrt{\frac{\alpha'}{2}} (k_\mu \mathcal{E}_\nu + k_\nu \mathcal{E}_\mu)$$

If $A^M = \tilde{A}^M = -1$, then for it to be physical $k_\mu k^\mu = 0$ is needed. And if $\mathcal{E}_\nu k^\nu = 0$ & also if $\mathcal{E}_\mu k^\mu = 0$, then

$$\underline{f_{\mu\nu} k^\nu = f_{\mu\nu} k^\mu = 0}$$

Thus in the case $A^M = \tilde{A}^M = -1$,
we get $f_{\mu\nu} = \sqrt{\frac{\alpha'}{2}} (k_\mu \xi_\nu + k_\nu \xi_\mu)$

whose $\xi_\nu k^\nu = \xi_\mu k^\mu = 0$ & $k_\mu k^\mu = 0$
to be physical & spurious. Thus
they are null states at level 1.

Hence, $f_{\mu\nu} \simeq f_{\mu\nu} + \underbrace{\sqrt{\frac{\alpha'}{2}} (k_\mu \xi_\nu + k_\nu \xi_\mu)}_{\text{equivalent to } f_{\mu\nu}}$

Given $A^M = \tilde{A}^M = -1$ & $k_\mu k^\mu = 0$ out of D^2
degrees of freedom for $f_{\mu\nu}$ we have
restrictions $f_{\mu\nu} k^\mu = 0, f_{\mu\nu} k^\nu = 0$. Note
that in k^μ there are $D-1$ independent
components. So by above two conditions
we lost $2(D-1)$ degrees of freedom.

As seen before $f_{\mu\nu} \propto k_\mu k_\nu$ gives a
null state. But this actually included
in the general case of null states when
 $f_{\mu\nu} \propto k_\mu \xi_\nu + k_\nu \xi_\mu$.

Lastly, $f_{\mu\nu} \propto k_\mu \xi_\nu + k_\nu \xi_\mu$ result in null states provided $\xi_\nu k^\nu = \xi_\mu k^\mu = 0$.

Thus there are $2D-2$ null states.

So a total of $2(D-1) + 2D-2$ degrees of freedom of $f_{\mu\nu}$ is gone

$\rightarrow D^2 - 4D + 4 = (D-2)^2$ degrees of freedom remains in 1st level in the form of $f_{\mu\nu}$.

$$\text{Now, } f_{\mu\nu} = \underbrace{g_{\mu\nu}}_{\substack{\downarrow \\ \text{Symmetric} \\ \text{graviton field}}} + \underbrace{a_{\mu\nu}}_{\substack{\downarrow \\ \text{Anti-Symmetric} \\ \text{field}}} + \underbrace{\frac{g_{\mu\nu} f}{D}}_{\substack{\nearrow \text{Dilaton} \\ \text{field}}}$$