

General Faddeev - Popov Formalism

Consider a path Integral with local gauge symmetries depending on the fields ϕ_j where 'j' represents different set of fields also the coordinates. In Polyakov path integral ϕ_j represents $\chi^\mu(\sigma^1, \sigma^2) \neq g_{ab}(\sigma^1, \sigma^2)$ for all possible 'u', 'a', 'b', $\sigma^1 \neq \sigma^2$.

The original path integral can be written as,

$$\int \frac{[D\phi_j] e^{-S_1[\phi_j]}}{\text{Vol}_{\text{gauge}}}$$

The gauge transformation is represented as

$$\phi_j \xrightarrow[\text{transform}]{\text{gauge}} \phi_j + \omega^\alpha \partial_\alpha \phi_j$$

where ∂_α is the gauge transforming lie algebra generator. α also includes $\sigma^1 \neq \sigma^2$ which implies the lie algebra generators are functions of σ^a coordinates. $\omega^\alpha \rightarrow$ infinitesimal gauge transform parameters

Now, $\partial_\alpha \partial_\beta \phi_j - \partial_\beta \partial_\alpha \phi_j = f^\gamma_{\alpha\beta} \partial_\gamma \phi_j$ is the closure property of gauge lie group.

$$\Rightarrow (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) = [\partial_\alpha, \partial_\beta] = f^\gamma_{\alpha\beta} \partial_\gamma$$

$f^\gamma_{\alpha\beta}$ are the structure constants of the gauge theory (lie algebra).

To perform gauge fixing we need to impose some conditions on the path integral fields. So, we can consider the gauge fixing equations as,

$$\underline{F^A(\hat{\phi}) = 0}$$

where 'A' are the different equations & also includes σ^1 & σ^2 . Here $\hat{\phi}_j$ is the special value such that it is the root of F^A . So then, we can consider a dirac delta functional which imposes the gauge fixing conditions as, $\delta[F^A(\phi)] = 0$

which forces $\phi_j \rightarrow \hat{\phi}_j$ (gauge fixing)

If we try to do the Faddeev - Popov gauge fixing procedure, then we will be introducing the Faddeev - Popov determinant by the equation,

$$1 = \int [D\zeta] \delta[F^A(\phi_j^{\zeta})] \Delta_{FP}(\phi)$$

This is analogous to

$$1 = \Delta_{FP}(\partial_{ab}) \int d\xi d\zeta \delta[\partial_{ab} - \hat{g}_{ab}^{\zeta}(\xi)]$$

Now, in the eqⁿ above, the delta functional forces $\phi_j^{\zeta} \rightarrow \hat{\phi}_j$. So, we shall expand the gauge fixing function $F^A(\phi_j^{\zeta})$ about $\hat{\phi}_j$. And replace the $\Delta_{FP}(\phi)$ with $\Delta_{FP}(\hat{\phi})$. Also, Δ_{FP} is gauge invariant.

To reach $\hat{\phi}_j$ from ϕ_j^{ζ} we should do some gauge transformation. But, very close to $\hat{\phi}_j$ infinitesimal gauge transforms ' $\partial_{\alpha}\phi_j$ ' are enough. So,

$$F^A(\phi^{\zeta}) = \underbrace{F^A(\hat{\phi})}_0 + \omega^{\alpha} \partial_{\alpha} F^A(\hat{\phi})$$

where $\delta_\alpha F^A = \frac{\partial F^A}{\partial \phi_j} \delta_\alpha \phi_j$

Thus,

$$[\Delta_{FP}(\hat{\phi})]^{-1} = \int [D\omega^\alpha] \delta[\omega^\alpha \delta_\alpha F^A(\hat{\phi})]$$

Using the path integral repⁿ of delta functional, we get,

$$\delta[\omega^\alpha \delta_\alpha F^A] = \int [DB_A] e^{[i B_A \omega^\alpha \delta_\alpha F^A]}$$

$$\Rightarrow [\Delta_{FP}(\phi)]^{-1} = \int [D\omega^\alpha DB_A] e^{[i B_A \omega^\alpha \delta_\alpha F^A(\phi)]}$$

In the above, ω^α & B_A are commuting objects, but the same $\Delta_{FP}(\phi)$ can be represented as a fermionic path integral

$$\Delta_{FP}(\phi) = \int [\mathcal{D}b_A \mathcal{D}c^\alpha] e^{[-b_A c^\alpha \partial_\alpha F^A(\phi)]}$$

Then, consider the eqⁿ

$$1 = \int [\mathcal{D}\xi] \delta[F^A(\phi^\xi)] \Delta_{FP}(\phi)$$

take inverse gauge throughout

$$1 = \int \underbrace{[\mathcal{D}\xi \xi^{-1}]} \delta[F^A(\phi^\xi \xi^{-1})] \Delta_{FP}(\phi^\xi \xi^{-1})$$

measure is gauge invariant

$\nexists \Delta_{FP}(\phi^\xi \xi^{-1}) = \Delta_{FP}(\phi)$ it is gauge invariant.

$$\Rightarrow 1 = \int [\mathcal{D}\xi] \underbrace{\delta[F^A(\phi)] \Delta_{FP}(\phi)}_{\text{Independent of } \xi}$$

Thus, $1 = \delta[F^A(\phi)] \Delta_{FP}(\phi) \underbrace{\int [\mathcal{D}\xi]}_{\text{Vol. gauge}}$

$$\Rightarrow 1 = \delta[F^A(\phi)] \Delta_{FP}(\phi) \times \text{Vol.}_{\text{gauge}}$$

Again, using the path integral representation of dirac-delta functional, we write,

$$\delta[F^A(\phi)] = \int [\mathcal{D}B_A] e^{[iB_A F^A(\phi)]}$$

$$\Rightarrow 1 = \delta[F^A(\phi)] \Delta_{FP}(\phi) \times \text{Vol.}_{\text{gauge}}$$

$$= \int [\mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha] e^{[iB_A F^A(\phi)]} e^{[-b_A c^\alpha \partial_\alpha F^A(\phi)]} \times \text{Vol.}_{\text{gauge}}$$

Then, we can replace the $\frac{1}{\text{Vol.}_{\text{gauge}}}$

in the original path integral $\int \frac{[\mathcal{D}\phi_j] e^{-S_1[\phi_j]}}{\text{Vol.}_{\text{gauge}}}$ with the above relation.

Thus, the Normalized path integral

$$Z = \frac{\int [\mathcal{D}\phi_j] e^{-S_1[\phi_j]}}{\text{Vol}_{\text{gauge}}}$$

becomes,

$$Z = \int [\mathcal{D}\phi_j \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha] e^{-(S_1[\phi_j] + S_2[B_A] + S_3[b_A, c^\alpha])}$$

where

$S_1[\phi_j] \longrightarrow$ Original gauge invariant action

$S_2[B_A] = -iB_A F^A(\phi) \longrightarrow$ gauge fixing action

$S_3[b_A, c^\alpha] = b_A c^\alpha \partial_\alpha F^A(\phi) \longrightarrow$ Faddeev Popov Ghost action

Then the path integral is done over commuting fields $\phi_j \nparallel B_A$. And over the anti-commuting fields $b_A \nparallel c^\alpha$.

BRST invariance of the full quantum action

The full quantum action is given by,

$$'S_1[\phi_i] + S_2[B_A] + S_3[b_A, c^\alpha]'$$

$S_1[\phi_i] \longrightarrow$ Original gauge invariant action

$S_2[B_A] = iB_A F^A(\phi) \longrightarrow$ gauge fixing action

$S_3[b_A, c^\alpha] = b_A c^\alpha \partial_\alpha F^A(\phi) \longrightarrow$ Faddeev Popov Ghost action

The BRST transformation is

$\delta_B \phi_j = -i\epsilon c^\alpha \partial_\alpha \phi_j \longrightarrow$ Gauge transform

$$\delta_B B_A = 0$$

$\delta_B b_A = \epsilon B_A \longrightarrow$ Makes ' ϵ ' anti-commuting

$\delta_B c^\alpha = \frac{i}{2} \epsilon f_{\beta\gamma}^\alpha c^\beta c^\gamma \longrightarrow$ By lie algebra.

Under these changes, we shall see how the full quantum action is preserved.

$S_1[\phi_j]$ is the original gauge invariant action. And $\delta_B \phi_j = -i\epsilon C^\alpha \partial_\alpha \phi_j$ is the usual gauge transformation where ' $-i\epsilon C^\alpha$ ' is equivalent to the gauge transform parameter w^α .

$S_1[\phi_j] \rightarrow$ doesn't depend on any other fields
 $\Rightarrow S_1[\phi_j]$ has to be invariant under BRST transformation. So, $\delta_B S_1[\phi_j] = 0$

$$\text{Then, } S_2[B_A] = -iB_A F^A(\phi)$$

$$\delta_B S_2[B_A] = -iB_A \delta_B F^A(\phi) - i \underbrace{(\delta_B B_A)}_0 F^A(\phi)$$

$$\begin{aligned} \delta_B F^A(\phi) &= \frac{\partial F^A(\phi)}{\partial \phi_j} \delta_B \phi_j = -i \frac{\partial F^A(\phi)}{\partial \phi_j} \epsilon C^\alpha \partial_\alpha \phi_j \\ &= -i\epsilon C^\alpha \frac{\partial F^A(\phi)}{\partial \phi_j} \partial_\alpha \phi_j = -i\epsilon C^\alpha \partial_\alpha F^A(\phi) \end{aligned}$$

$$\Rightarrow \delta_B F^A(\phi) = -i\epsilon C^\alpha \partial_\alpha F^A(\phi)$$

$$\Rightarrow \delta_B S_2[B_A] = -B_A \epsilon C^\alpha \partial_\alpha F^A(\phi)$$

$$\begin{aligned}\text{Finally, } \partial_B S_3[b_A, c^\alpha] &= \partial_B (b_A c^\alpha \partial_\alpha F^A(\phi)) \\ &= (\partial_B b_A) c^\alpha \partial_\alpha F^A(\phi) + b_A (\partial_B c^\alpha) \partial_\alpha F^A(\phi) \\ &\quad + b_A c^\alpha \partial_B (\partial_\alpha F^A(\phi))\end{aligned}$$

$$\begin{aligned}&= \epsilon B_A c^\alpha \partial_\alpha F^A(\phi) + \frac{i}{2} b_A f_{\beta\gamma}^\alpha c^\beta c^\gamma \partial_\alpha F^A(\phi) \\ &\quad + b_A c^\alpha \partial_B (\partial_\alpha F^A(\phi))\end{aligned}$$

$$\text{use } \partial_B (\partial_\alpha F^A(\phi)) = -i \epsilon c^\beta \partial_\beta (\partial_\alpha F^A(\phi))$$

$$\& f_{\beta\gamma}^\alpha \partial_\alpha = [\partial_\beta, \partial_\gamma] = \partial_\beta \partial_\gamma - \partial_\gamma \partial_\beta$$

$$\begin{aligned}\text{Then, } c^\beta c^\gamma f_{\beta\gamma}^\alpha \partial_\alpha F^A(\phi) &= c^\beta c^\gamma (\partial_\beta \partial_\gamma - \partial_\gamma \partial_\beta) F^A(\phi) \\ &= [c^\beta c^\gamma \partial_\beta \partial_\gamma F^A(\phi) - c^\beta c^\gamma \partial_\gamma \partial_\beta F^A(\phi)] \\ &= [c^\beta c^\gamma \partial_\beta \partial_\gamma F^A(\phi) + \underbrace{c^\gamma c^\beta \partial_\gamma \partial_\beta F^A(\phi)}_{\gamma \leftrightarrow \beta}]\end{aligned}$$

$$\underline{\underline{\longrightarrow c^\beta c^\gamma f_{\beta\gamma}^\alpha \partial_\alpha F^A(\phi) = 2 c^\beta c^\gamma \partial_\beta \partial_\gamma F^A(\phi)}}$$

$$\text{Thus, } \delta_B S_3[b_A, c^\alpha] = \epsilon B_A c^\alpha \delta_\alpha F^A(\phi)$$

$$+ i b_A \epsilon c^\beta c^\gamma \delta_\beta \delta_\gamma F^A(\phi) - i b_A \underbrace{c^\alpha \epsilon c^\beta}_{\text{(Interchange } c^\alpha \epsilon \text{ \& then } c^\alpha c^\beta)}} \delta_\beta (\delta_\alpha F^A(\phi))$$

$$= \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + i b_A \epsilon c^\beta c^\gamma \cancel{\delta_\beta \delta_\gamma} F^A(\phi)$$

$$- i b_A \epsilon c^\beta \cancel{c^\alpha} \delta_\beta \delta_\alpha F^A(\phi)$$

$$\text{Thus, } \delta_B S_3[b_A, c^\alpha] = \epsilon B_A c^\alpha \delta_\alpha F^A(\phi)$$

But,

$$\delta_B S_2[B_A] = -B_A \epsilon c^\alpha \delta_\alpha F^A(\phi) = -\epsilon B_A c^\alpha \delta_\alpha F^A(\phi)$$

(Note: Commuting fields also commutes with anti-commuting fields.)

$$\rightarrow \underline{\delta_B S_2[B_A] + \delta_B S_3[b_A, c^\alpha] = 0}$$

$$\text{Thus } \delta_B S_1[\phi_i] + \delta_B S_2[B_A] + \delta_B S_3[b_A, c^\alpha] = 0$$

Hence the full quantum action & the path-integral is invariant under BRST transform.

BRST Ghost number conservation

The fields b_A & c^α are called Ghost fields which are Grassmann fields, where as the fields ϕ_j & B_A & $F^A(\phi)$ are ordinary commuting objects (not ghosts)

Now, the BRST transformation of ordinary fields are,

$$\left. \begin{aligned} \delta_B \phi_j &= -i \epsilon c^\alpha \delta_\alpha \phi_j \\ \delta_B B_A &= 0 \\ \delta_B F^A(\phi) &= -i \epsilon c^\alpha \delta_\alpha F^A(\phi) \end{aligned} \right\} \begin{array}{l} \text{A ghost} \\ \text{field appears} \\ \text{under BRST} \\ \text{variation} \end{array}$$

Thus a BRST variation gives rise to a c^α Ghost field. So we can assign a Ghost number of $+1$ to c^α field. But the overall action should have Ghost number '0'.

Also, fields ' ϕ_j ' & B_A have Ghost number '0'.

Consider the BRST variations

- $\delta_B S_1[\phi_i] = 0 \rightarrow$ Maintains zero Ghosts

So, ghost number will not change under the BRST variations.

- $\underbrace{\delta_B \phi_j}_{\text{ghost number Zero}} = -i \epsilon c^\alpha \delta_\alpha \phi_j \quad \left. \vphantom{\delta_B \phi_j} \right\} \begin{array}{l} \text{'}\epsilon\text{' should have Ghost} \\ \text{number } -1 \text{ to cancel} \\ c^\alpha \text{'s } +1 \text{ number} \end{array}$

- $\delta_B b_A = \epsilon \widetilde{B}_A \rightarrow b_A \text{ \& } \epsilon \text{ have the same Ghost number of } -1.$

- $S_3[b_A, c^\alpha] = b_A c^\alpha \delta_\alpha F^A(\phi) \quad \left. \vphantom{S_3[b_A, c^\alpha]} \right\} S_3 \text{ has ghost number Zero}$

This ghost number assignment to the fields is consistent where all actions & the ordinary fields have Zero ghost number.

To show $\partial_{B_2} \partial_{B_1}(\phi_j, B_A, b_A, c^\alpha) = 0$

$$\partial_{B_1} \phi_j = -i\epsilon_1 c^\alpha \partial_\alpha \phi_j$$

$$\partial_{B_2} \partial_{B_1} \phi_j = -i\epsilon_1 \partial_{B_2} (c^\alpha \partial_\alpha \phi_j)$$

$$= -i\epsilon_1 (\partial_{B_2} c^\alpha) \partial_\alpha \phi_j - i\epsilon_1 c^\alpha (\partial_{B_2} \partial_\alpha \phi_j)$$

$$= \frac{\epsilon_1 \epsilon_2}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma \partial_\alpha \phi_j - \epsilon_1 c^\alpha \epsilon_2 c^\beta \partial_\beta \partial_\alpha \phi_j$$

$$= \frac{\epsilon_1 \epsilon_2}{2} c^\beta c^\gamma \underbrace{f_{\beta\gamma}^\alpha}_{\partial_\beta \partial_\gamma - \partial_\gamma \partial_\beta} \partial_\alpha \phi_j - \epsilon_1 c^\alpha \epsilon_2 c^\beta \partial_\beta \partial_\alpha \phi_j$$

$$= \frac{\epsilon_1 \epsilon_2}{2} c^\beta c^\gamma (\partial_\beta \partial_\gamma - \partial_\gamma \partial_\beta) \phi_j - \epsilon_1 c^\alpha \epsilon_2 c^\beta \partial_\beta \partial_\alpha \phi_j$$

$$= \epsilon_1 \epsilon_2 c^\gamma c^\beta \partial_\gamma \partial_\beta \phi_j + \underbrace{\epsilon_1 \epsilon_2 c^\alpha c^\beta \partial_\beta \partial_\alpha \phi_j}_{\alpha \rightarrow \beta, \beta \rightarrow \gamma}$$

$$= \epsilon_1 \epsilon_2 c^\gamma c^\beta \partial_\gamma \partial_\beta \phi_j + \epsilon_1 \epsilon_2 c^\beta c^\gamma \partial_\gamma \partial_\beta \phi_j$$

$$= \epsilon_1 \epsilon_2 c^\gamma c^\beta \partial_\gamma \partial_\beta \phi_j - \epsilon_1 \epsilon_2 c^\gamma c^\beta \partial_\gamma \partial_\beta \phi_j = 0$$

$$\longrightarrow \underline{\partial_{B_2} \partial_{B_1} \phi_j = 0}$$

Also $\delta_{B_2} \delta_{B_1} B_A = 0$ as $\delta_B B_A = 0$

Then, $\delta_{B_2} \delta_{B_1} b_A = \delta_{B_2} \epsilon_1 B_A = \epsilon_1 \underbrace{\delta_{B_2} B_A}_0 = 0$

$\Rightarrow$ $\delta_{B_2} \delta_{B_1} b_A = 0$

Finally, $\delta_{B_2} \delta_{B_1} c^\alpha = \delta_{B_2} \left(\frac{i\epsilon_1}{2} f_{\beta\gamma}^\alpha c^\beta c^\gamma \right)$

$= \frac{i\epsilon_1}{2} f_{\beta\gamma}^\alpha \delta_{B_2} (c^\beta c^\gamma)$

$= \frac{i\epsilon_1}{2} f_{\beta\gamma}^\alpha \left[(\delta_{B_2} c^\beta) c^\gamma + c^\beta (\delta_{B_2} c^\gamma) \right]$

$= \frac{i\epsilon_1}{2} f_{\beta\gamma}^\alpha \left[\frac{i\epsilon_2}{2} f_{\rho\sigma}^\beta c^\rho c^\sigma c^\gamma + c^\beta \frac{i\epsilon_2}{2} f_{\rho\sigma}^\gamma c^\rho c^\sigma \right]$

$= \frac{i\epsilon_1}{2} f_{\beta\gamma}^\alpha \left[\frac{i\epsilon_2}{2} f_{\rho\sigma}^\beta c^\rho c^\sigma c^\gamma - \frac{i\epsilon_2}{2} c^\beta f_{\rho\sigma}^\gamma c^\rho c^\sigma \right]$

$= \frac{-1}{4} \epsilon_1 \epsilon_2 f_{\beta\gamma}^\alpha \left[f_{\rho\sigma}^\beta c^\rho c^\sigma c^\gamma - f_{\rho\sigma}^\gamma c^\beta c^\rho c^\sigma \right]$

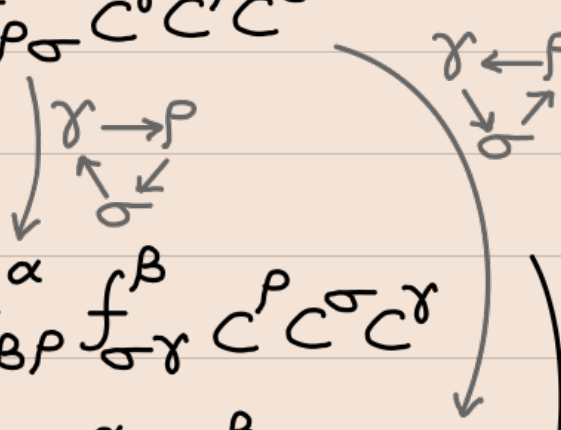
$= \frac{-1}{4} \epsilon_1 \epsilon_2 f_{\beta\gamma}^\alpha \left[f_{\rho\sigma}^\beta c^\gamma - f_{\rho\sigma}^\gamma c^\beta \right] c^\rho c^\sigma$

$$\text{Now, } f_{\beta\gamma}^{\alpha} \partial_{\alpha} = \partial_{\beta} \partial_{\gamma} - \partial_{\gamma} \partial_{\beta}$$

$$\text{So, } f_{\beta\gamma}^{\alpha} \partial_{\alpha} = -f_{\gamma\beta}^{\alpha} \partial_{\alpha} \Rightarrow f_{\beta\gamma}^{\alpha} = -f_{\gamma\beta}^{\alpha}$$

Thus $f_{\beta\gamma}^{\alpha}$ is anti-symmetric in its lower two indices.

$$\begin{aligned} \rightarrow f_{\beta\gamma}^{\alpha} [f_{\rho\sigma}^{\beta} C^{\gamma} - f_{\rho\sigma}^{\gamma} C^{\beta}] &= f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} C^{\gamma} - f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\gamma} C^{\beta} \\ &= f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} C^{\gamma} + f_{\gamma\beta}^{\alpha} f_{\rho\sigma}^{\gamma} C^{\beta} \\ &= 2 f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} C^{\gamma} \end{aligned}$$

$$\begin{aligned} \rightarrow \partial_{\beta_2} \partial_{\beta_1} C^{\alpha} &= \frac{-1}{2} \epsilon_1 \epsilon_2 f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} C^{\gamma} C^{\rho} C^{\sigma} \\ &= \frac{-1}{2} \frac{\epsilon_1 \epsilon_2}{3} \left(f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} C^{\gamma} C^{\rho} C^{\sigma} + f_{\beta\rho}^{\alpha} f_{\sigma\gamma}^{\beta} C^{\rho} C^{\sigma} C^{\gamma} + f_{\beta\sigma}^{\alpha} f_{\gamma\rho}^{\beta} C^{\sigma} C^{\gamma} C^{\rho} \right) \end{aligned}$$


$$= \frac{-\epsilon_1 \epsilon_2}{6} (f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} + f_{\beta\rho}^{\alpha} f_{\sigma\gamma}^{\beta} + f_{\beta\sigma}^{\alpha} f_{\gamma\rho}^{\beta}) C^{\gamma} C^{\rho} C^{\sigma}$$

$$\text{But, } f_{\beta\gamma}^{\alpha} f_{\rho\sigma}^{\beta} + f_{\beta\rho}^{\alpha} f_{\sigma\gamma}^{\beta} + f_{\beta\sigma}^{\alpha} f_{\gamma\rho}^{\beta} = 0$$

Jacobi identity

$$\rightarrow \underline{\delta_{B_2} \delta_{B_1} c^{\alpha} = 0}$$

Thus the BRST variation of already BRST varied fields is zero.

$$\underline{\delta_{B_2} \delta_{B_1} (\phi_j, B_A, b_A, c^{\alpha}) = 0}$$

Jacobi Identity (lie algebra of Gauge group)

$$[\delta_{\gamma}, [\delta_{\rho}, \delta_{\sigma}]] = [\delta_{\gamma}, \delta_{\beta}] f_{\rho\sigma}^{\beta} = f_{\gamma\beta}^{\alpha} f_{\rho\sigma}^{\beta} \delta_{\alpha}$$

$$[\delta_{\rho}, [\delta_{\sigma}, \delta_{\gamma}]] = [\delta_{\rho}, \delta_{\beta}] f_{\sigma\gamma}^{\beta} = f_{\rho\beta}^{\alpha} f_{\sigma\gamma}^{\beta} \delta_{\alpha}$$

$$[\delta_{\sigma}, [\delta_{\gamma}, \delta_{\rho}]] = [\delta_{\sigma}, \delta_{\beta}] f_{\gamma\rho}^{\beta} = f_{\sigma\beta}^{\alpha} f_{\gamma\rho}^{\beta} \delta_{\alpha}$$

Lie algebra generators follow the sum of the above three eqⁿ to be zero.

$$\rightarrow f_{\gamma\beta}^{\alpha} f_{\rho\sigma}^{\beta} + f_{\rho\beta}^{\alpha} f_{\sigma\gamma}^{\beta} + f_{\sigma\beta}^{\alpha} f_{\gamma\rho}^{\beta} = 0$$

To show $\delta_B(b_A F^A(\phi)) = i \in (S_2[B_A] + S_3[b_A, c^\alpha])$

$$\delta_B(b_A F^A(\phi)) = (\delta_B b_A) F^A(\phi) + b_A (\delta_B F^A(\phi))$$

$$= \epsilon B_A F^A(\phi) - i b_A \epsilon c^\alpha \delta_\alpha F^A(\phi)$$

$$= i \in \underbrace{(-i B_A F^A(\phi))}_{S_2[B_A]} + i \in \underbrace{b_A c^\alpha \delta_\alpha F^A(\phi)}_{S_3[b_A, c^\alpha]}$$

$$\text{Thus, } \delta_B(b_A F^A(\phi)) = i \in (S_2[B_A] + S_3[b_A, c^\alpha])$$

$$\text{Then, } S_1[\phi_j] + S_2[B_A] + S_3[b_A, c^\alpha]$$

$$= S_1[\phi_j] + \# \delta_B(b_A F^A(\phi))$$

Such that

$$i \in (S_1[\phi_j] + S_2[B_A] + S_3[b_A, c^\alpha]) = i \in S_1[\phi_j] + \delta_B(b_A F^A(\phi))$$

BRST Conserved charge ' $\hat{Q}_B$ '

We can consider an operator ' $\hat{Q}_B$ ' which generates the BRST transformation on any object made of the fields $\phi_j, B_A, b_A, c^\alpha$ as follows,

$$\underline{\delta_B \mathcal{A} = i \epsilon \{ \mathcal{A}, \hat{Q}_B \}}$$

$$\text{i.e. } \delta_B \mathcal{A} = i \epsilon (\mathcal{A} \hat{Q}_B + \hat{Q}_B \mathcal{A})$$

Then the below eqⁿ can be written in terms of $\hat{Q}_B$ as follows,

$$i \epsilon (S_1[\phi_j] + S_2[B_A] + S_3[b_A, c^\alpha]) = i \epsilon S_1[\phi_j] + \delta_B (b_A F^A(\phi))$$

$$= i \epsilon S_1[\phi_j] + i \epsilon \{ b_A F^A(\phi), \hat{Q}_B \}$$

$$\rightarrow S_1[\phi_j] + S_2[B_A] + S_3[b_A, c^\alpha]$$

$$= S_1[\phi_j] + \{ b_A F^A(\phi), \hat{Q}_B \}$$

Physical states are BRST Invariant

The Gauge fixing condition is enforced by the action

$$S_2 = iB_A F^A(\phi)$$

where it forces ϕ_j 's $\longrightarrow \underbrace{\hat{\phi}_j}_{\text{gauge fixed field}}$'s

But if we change the gauge fixing slightly by doing

$$F^A(\phi) \longrightarrow F^A(\phi) + \delta_g F^A(\phi)$$

then for physical states the amplitude should remain invariant.

Consider the transition amplitude $\langle \psi' | \psi \rangle$ expressed as a path integral with relevant boundary conditions.

$$\langle \psi' | \psi \rangle = \int [\mathcal{D}\phi_j \mathcal{D}B_A \mathcal{D}b_A \mathcal{D}c^\alpha] e^{-(S_1 + S_2 + S_3)}$$

$$\text{Now, } \delta_g \langle \psi' | \psi \rangle = \langle \psi' | \psi \rangle_{F^A + \delta_g F^A} - \langle \psi' | \psi \rangle_{F^A}$$

$$= \int [\delta \phi_j \delta B_A \delta b_A \delta c^\alpha] e^{\left(\begin{aligned} &-S_1 + i B_A F^A + i B_A \delta_g F^A \\ &-b_A c^\alpha \partial_\alpha F^A - b_A c^\alpha \partial_\alpha \delta_g F^A \end{aligned} \right)}$$

$$- \int [\delta \phi_j \delta B_A \delta b_A \delta c^\alpha] e^{\left(-S_1 + i B_A F^A - b_A c^\alpha \partial_\alpha F^A \right)}$$

(upto 1st order in $\delta_g F^A$)

$$= \int [\delta \phi_j \delta B_A \delta b_A \delta c^\alpha] e^{\left(-S_1 + i B_A F^A - b_A c^\alpha \partial_\alpha F^A \right)}$$

$$\times \left(i B_A \delta_g F^A - b_A c^\alpha \partial_\alpha \delta_g F^A \right)$$

Then,

$$\in \delta_g \langle \psi' | \psi \rangle =$$

$$= \int [\delta \phi_j \delta B_A \delta b_A \delta c^\alpha] e^{\left(-S_1 + i B_A F^A - b_A c^\alpha \partial_\alpha F^A \right)}$$

$$\times \left(\underbrace{i (\in B_A)}_{(\delta_B b_A)} \delta_g F^A - \in b_A c^\alpha \partial_\alpha \delta_g F^A \right)$$

$$\text{Then, } -\epsilon b_A c^\alpha \partial_\alpha \partial_g F^A = b_A \epsilon c^\alpha \partial_\alpha \partial_g F^A$$

$$\text{Since, } \partial_B F^A = -i \epsilon c^\alpha \partial_\alpha F^A$$

$$\Rightarrow i \partial_B \partial_g F^A = \epsilon c^\alpha \partial_\alpha \partial_g F^A$$

$$\Rightarrow -\epsilon b_A c^\alpha \partial_\alpha \partial_g F^A = i b_A (\partial_B \partial_g F^A)$$

$$\Rightarrow \epsilon \partial_g \langle \psi' | \psi \rangle =$$

$$= \int [\partial \phi_j \partial B_A \partial b_A \partial c^\alpha] e^{-(S_1 + S_2 + S_3)} \times (i(\partial_B b_A) \partial_g F^A + i b_A (\partial_B \partial_g F^A))$$

$$= \int [\partial \phi_j \partial B_A \partial b_A \partial c^\alpha] e^{-(S_1 + S_2 + S_3)} i \partial_B (b_A \partial_g F^A)$$

$$= -\epsilon \int [\partial \phi_j \partial B_A \partial b_A \partial c^\alpha] e^{-(S_1 + S_2 + S_3)} \{b_A \partial_g F^A, \hat{Q}_B\}$$

$$\Rightarrow \partial_g \langle \psi' | \psi \rangle =$$

$$= -\int [\partial \phi_j \partial B_A \partial b_A \partial c^\alpha] e^{-(S_1 + S_2 + S_3)} \{b_A \partial_g F^A, \hat{Q}_B\}$$

$$\Rightarrow \boxed{\partial_g \langle \psi' | \psi \rangle = -\langle \psi' | \{b_A \partial_g F^A, \hat{Q}_B\} | \psi \rangle}$$

Thus, the physical states should satisfy

$$\delta_g \langle \psi' | \psi \rangle = 0$$

Physical amplitudes should be gauge invariant

Hence, $\langle \psi' | \{ b_A \delta_g F^A, \hat{Q}_B \} | \psi \rangle = 0$

$$\Rightarrow \langle \psi' | b_A \delta_g F^A \hat{Q}_B | \psi \rangle + \langle \psi' | \hat{Q}_B b_A \delta_g F^A | \psi \rangle = 0$$

For this to hold for arbitrary ' $\delta_g F^A$ ' & ' b_A ' we need

$$\hat{Q}_B | \psi \rangle = 0 \quad \& \quad \langle \psi' | \hat{Q}_B = (\hat{Q}_B^+ | \psi' \rangle)^+ = 0$$

If $\hat{Q}_B = \hat{Q}_B^+$, then the above condition becomes

$$\underline{\hat{Q}_B | \psi \rangle = 0 = \hat{Q}_B | \psi \rangle}$$

Thus the physical states have to be BRST invariant with zero Eigen value.

Nilpotency of BRST Conserved charge ' $\hat{Q}_B$ '

In the previous argument we have seen that under a change of gauge, the path integral changes as the matrix element of the operator ' $\{b_A \delta_g F^A, \hat{Q}_B\}$ '

To move around different gauge choices, the BRST charge has to be conserved.

Now, for ' $\hat{Q}_B$ ' to remain invariant under the change of gauge choices, it has to commute with the generator of different gauge choices, i.e. $[\hat{Q}_B, \{b_A \delta_g F^A, \hat{Q}_B\}] = 0$

$$\rightarrow \hat{Q}_B \{b_A \delta_g F^A, \hat{Q}_B\} - \{b_A \delta_g F^A, \hat{Q}_B\} \hat{Q}_B = 0$$

$$\begin{aligned} \rightarrow \hat{Q}_B (b_A \delta_g F^A) \hat{Q}_B + \hat{Q}_B^2 (b_A \delta_g F^A) \\ - \hat{Q}_B (b_A \delta_g F^A) \hat{Q}_B - (b_A \delta_g F^A) \hat{Q}_B^2 = 0 \end{aligned}$$

$$\rightarrow [\hat{Q}_B^2, b_A \delta_g F^A] = 0$$

For the above to hold for arbitrary gauge change, we need $\hat{Q}_B^2$ to be constant or zero. Now, the charge operator is defined by, $\delta_B A = i \in \{A, \hat{Q}_B\}$

To conserve ghost numbers both side, the ghost number of $\hat{Q}_B$ should cancel that of 'E'. Since 'E' has ghost number -1, we get the ghost number of ' $\hat{Q}_B$ ' to be +1.

Thus, if $\hat{Q}_B^2$ is non-zero, then, it would have ghost number of '2'. But that is not consistent with

$$\underbrace{[\hat{Q}_B^2, b_A \delta_B F^A]}_{\text{ghost number 2 if } \hat{Q}_B^2 \neq 0} = 0 \rightarrow \text{ghost number zero}$$

ghost number -1

Thus to have consistent ghost numbers, we require $\hat{Q}_B$ to be Nilpotent. i.e. $\hat{Q}_B^2 = 0$

Cohomology of $\hat{Q}_B \neq H_{BRST}$

Consider any random state $|\chi\rangle$. Then $\hat{Q}_B|\chi\rangle$ is a physical state as

$$\hat{Q}_B(\hat{Q}_B|\chi\rangle) = \hat{Q}_B^2|\chi\rangle = 0$$

Thus, the state $\hat{Q}_B|\chi\rangle$ is BRST invariant. And also, it is orthogonal to any other physical state $|\psi\rangle$, as,

$$\langle\psi|\hat{Q}_B|\chi\rangle = (\hat{Q}_B^+|\psi\rangle)^+|\chi\rangle = (\underbrace{\hat{Q}_B|\psi\rangle}_0)^+|\chi\rangle$$

$$\rightarrow \langle\psi|(\hat{Q}_B|\chi\rangle) = 0 \text{ for any Physical } |\psi\rangle$$

Moreover $\hat{Q}_B|\chi\rangle$ is orthogonal to itself.

Thus the state $\hat{Q}_B|\chi\rangle$ is a Null state.

Then, any two physical states differing by the Null state $\hat{Q}_B|\chi\rangle$ are equivalent.

$$|\psi'\rangle \simeq |\psi\rangle + \hat{Q}_B|\chi\rangle$$


Equivalent

So, same like old covariant quantization, we identify the true BRST physical states as the set of equivalence classes with two states differing by the Null state formed by the BRST conserved charge operator $\hat{Q}_B$ acting on any general state $|\chi\rangle$ as equivalent.

This is a natural construction for a Nilpotent conserved charge operator ' $\hat{Q}_B$ ', thus the set of equivalence classes is called as 'Cohomology' of ' $\hat{Q}_B$ '.

The states of the form ' $\hat{Q}_B|\chi\rangle$ ' are called 'Exact' states. And states ' $|\psi\rangle$ ' that are annihilated by $\hat{Q}_B$ are called 'Closed' states.

Then the Hilbert space of BRST physical states is the Quotient group of $\mathbb{H}_{\text{closed}}$ with $\mathbb{H}_{\text{exact}}$. $\Rightarrow \underbrace{\mathbb{H}_{\text{BRST}}}_{\text{(Closed \& not exact)} \rightarrow \text{Cohomology of } \hat{Q}_B} = \frac{\mathbb{H}_{\text{closed}}}{\mathbb{H}_{\text{exact}}}$

Operator Cohomology of $\hat{Q}_B$

Consider Path integrals with arbitrary operator insertions given by,

$$\langle \Psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{\Theta}_n | \Psi \rangle$$

The physical observable operators have to be such that the above matrix element is gauge invariant.

Following the same procedure as before, under arbitrary infinitesimal change of the gauge fixing condition, $F^A \rightarrow F^A + \delta_g F^A$, we get

$$\begin{aligned} \delta_g \langle \Psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{\Theta}_n | \Psi \rangle &= \langle \Psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{\Theta}_n | \Psi \rangle_{F^A + \delta_g F^A} \\ &\quad - \langle \Psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{\Theta}_n | \Psi \rangle_{F^A} \\ &= - \langle \Psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{\Theta}_n \{ b_A \delta_g F^A, \hat{Q}_B \} | \Psi \rangle \end{aligned}$$

But for physical operators, their matrix elements have to be gauge invariant.

$$\Rightarrow \delta_g \langle \psi' | \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 \dots \hat{\mathcal{O}}_n | \psi \rangle = 0 \text{ is needed}$$

where $|\psi\rangle$ and $|\psi'\rangle$ are physical states.

$$\rightarrow \langle \psi' | \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 \dots \hat{\mathcal{O}}_n \{b_A \delta_g F^A, \hat{Q}_B\} | \psi \rangle = 0$$

$$\Rightarrow \langle \psi' | \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 \dots \hat{\mathcal{O}}_n \hat{Q}_B (b_A \delta_g F^A) | \psi \rangle = 0$$

Thus, for the above to be zero for arbitrary ' $\delta_g F^A$ ', we require

$$\hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 \dots \hat{\mathcal{O}}_n \hat{Q}_B = \hat{Q}_B \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \hat{\mathcal{O}}_3 \dots \hat{\mathcal{O}}_n$$

i.e. Each and every operator insertion has to commute with ' $\hat{Q}_B$ '. $\Rightarrow \underline{\underline{[\hat{\mathcal{O}}_m, \hat{Q}_B] = 0}}$

Thus only those operator insertions are allowed inside the path integral which will commute with the BRST charge operator.

So those operators that commute with the BRST charge operator ' $\hat{Q}_B$ ' are called as 'Closed Operators'.

Now, consider a general operator $\hat{\Omega}$ which need not be $\hat{Q}_B$ -closed. i.e. $[\hat{\Omega}, \hat{Q}_B] \neq 0$.

But the operator $\{\hat{\Omega}, \hat{Q}_B\}$ is $\hat{Q}_B$ -closed.

$$[\{\hat{\Omega}, \hat{Q}_B\}, \hat{Q}_B] = [\hat{\Omega} \hat{Q}_B + \hat{Q}_B \hat{\Omega}, \hat{Q}_B]$$

$$= \underbrace{\hat{\Omega} \hat{Q}_B^2}_0 + \cancel{\hat{Q}_B \hat{\Omega} \hat{Q}_B} - \cancel{\hat{Q}_B \hat{\Omega} \hat{Q}_B} - \underbrace{\hat{Q}_B^2 \hat{\Omega}}_0$$

$$\Rightarrow [\{\hat{\Omega}, \hat{Q}_B\}, \hat{Q}_B] = 0$$

$$\Rightarrow \{\hat{\Omega}, \hat{Q}_B\} \text{ is } \hat{Q}_B\text{-closed.}$$

Then, consider inserting this $\{\hat{\Omega}, \hat{Q}_B\}$ operator inside the path integral even in the presence of arbitrary insertions of other $\hat{Q}_B$ -closed operators.

Consider, $\langle \psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \{ \hat{\Omega}, \hat{Q}_B \} \dots \hat{\Theta}_n | \psi \rangle$

where $|\psi\rangle$ and $|\psi'\rangle$ are physical states.

$$\begin{aligned} & \langle \psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \{ \hat{\Omega}, \hat{Q}_B \} \dots \hat{\Theta}_n | \psi \rangle \\ &= \langle \psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{\Omega} \hat{Q}_B \dots \hat{\Theta}_n | \psi \rangle \quad \text{annihilates} \\ & \quad + \langle \psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \hat{Q}_B \hat{\Omega} \dots \hat{\Theta}_n | \psi \rangle \quad \text{annihilates} \end{aligned}$$

$$\Rightarrow \langle \psi' | \hat{\Theta}_1 \hat{\Theta}_2 \hat{\Theta}_3 \dots \{ \hat{\Omega}, \hat{Q}_B \} \dots \hat{\Theta}_n | \psi \rangle = 0$$

Thus, the operator $\{ \hat{\Omega}, \hat{Q}_B \}$ is always annihilated by physical states even in the presence of insertions.

We call operators of the form $\{ \hat{\Omega}, \hat{Q}_B \}$ which makes the path integral go to zero even in the presence of arbitrary insertions as $\hat{Q}_B$ -exact operators.

Now consider two operators $\hat{O}_m$ & $\hat{O}_m + \{\hat{\Omega}, \hat{Q}_B\}$ where both are $\hat{Q}_B$ -closed.

$$\begin{aligned} & \langle \psi' | \hat{O}_1 \hat{O}_2 \hat{O}_3 \dots (\hat{O}_m + \{\hat{\Omega}, \hat{Q}_B\}) \dots \hat{O}_n | \psi \rangle \\ &= \langle \psi' | \hat{O}_1 \hat{O}_2 \hat{O}_3 \dots \hat{O}_m \dots \hat{O}_n | \psi \rangle \\ & \quad + \underbrace{\langle \psi' | \hat{O}_1 \hat{O}_2 \hat{O}_3 \dots \{\hat{\Omega}, \hat{Q}_B\} \dots \hat{O}_n | \psi \rangle}_0 \end{aligned}$$

$$\begin{aligned} \Rightarrow & \langle \psi' | \hat{O}_1 \hat{O}_2 \hat{O}_3 \dots \hat{O}_m \dots \hat{O}_n | \psi \rangle \\ &= \langle \psi' | \hat{O}_1 \hat{O}_2 \hat{O}_3 \dots (\hat{O}_m + \{\hat{\Omega}, \hat{Q}_B\}) \dots \hat{O}_n | \psi \rangle \end{aligned}$$

So, the operators $\hat{O}_m$ & $\hat{O}_m + \{\hat{\Omega}, \hat{Q}_B\}$ are equivalent w.r.t physical/ $\hat{Q}_B$ -closed states. for any $\hat{\Omega}$ operator.

Hence two $\hat{Q}_B$ -closed operators that differ by a $\hat{Q}_B$ -exact operator are Equivalent with respect to $\hat{Q}_B$ -closed states.

$$\underline{\hat{O}_m \simeq \hat{O}_m + \{\hat{\Omega}, \hat{Q}_B\}}$$

BRST quantization for Point Particle

The point particle action is given by

$$S_{\text{Particle}}^{\text{Minkowski}} = \frac{1}{2} \int d\tau [\mathcal{E}(\tau)^{-1} \dot{X}^\mu \dot{X}_\mu - \mathcal{E}(\tau) m^2]$$

In euclidean signature $\tau \rightarrow -i\tau$

$$d\tau \rightarrow -i d\tau, \quad \dot{X}^\mu \rightarrow i \dot{X}^\mu \quad \& \quad \dot{X}^\mu \dot{X}_\mu \rightarrow -\dot{X}^\mu \dot{X}_\mu$$

$$S_{\text{Particle}}^{\text{Minkowski}} = \frac{-i}{2} \int d\tau [-\mathcal{E}(\tau)^{-1} \dot{X}^\mu \dot{X}_\mu - \mathcal{E}(\tau) m^2]$$

Then the Euclidean action is $-S^E = i S^{\text{Minkowski}}$

$$\Rightarrow S_{\text{Particle}}^E = \frac{1}{2} \int [\mathcal{E}(\tau)^{-1} \dot{X}^\mu \dot{X}_\mu + \mathcal{E}(\tau) m^2]$$

& the only gauge symmetry is the world line diffeomorphism or reparameterization which changes the coordinate $\tau \rightarrow \tau'(\tau)$ & the auxiliary gauge field $\mathcal{E}(\tau) \rightarrow \frac{\partial \tau'}{\partial \tau} \mathcal{E}'(\tau')$

Let us see how the fields transform under an infinitesimal reparameterization.

$$\text{Let } \tau \rightarrow \tau' = \tau'(\tau) = \tau + \underbrace{\omega(\tau)}_{\text{Infinitesimal}}$$

$$\Rightarrow \chi^\mu(\tau) \rightarrow \underbrace{\chi'^\mu(\tau')}_{\chi^\mu(\tau) \text{ as it is world line scalar}} = \chi'^\mu(\tau + \omega(\tau))$$

$$\Rightarrow \chi^\mu(\tau) = \chi'^\mu(\tau) + \omega(\tau) \dot{\chi}^\mu(\tau)$$

$$\chi'^\mu(\tau) = \chi^\mu(\tau) - \omega(\tau) \dot{\chi}^\mu$$

$$\omega(\tau) \dot{\chi}^\mu = \omega(\tau) \dot{\chi}^\mu - \omega(\tau) \underbrace{\frac{d}{d\tau} [\omega(\tau) \dot{\chi}^\mu]}_{2^{\text{nd}} \text{ order}}$$

$$\Rightarrow \omega(\tau) \dot{\chi}^\mu = \omega(\tau) \dot{\chi}^\mu$$

$$\text{Thus, } \delta \chi^\mu(\tau) = \chi'^\mu(\tau) - \chi^\mu(\tau) = -\omega(\tau) \dot{\chi}^\mu(\tau)$$

So the gauge transform of $\chi^\mu(\tau)$ is,

$$\underline{\underline{\delta \chi^\mu(\tau) = -\omega(\tau) \dot{\chi}^\mu(\tau)}}$$

Now, the gauge field $\mathcal{E}(\tau)$ changes in the following way,

$$\mathcal{E}(\tau) \rightarrow \mathcal{E}'(\tau') = \frac{\partial \tau}{\partial \tau'} \mathcal{E}(\tau) = \frac{\mathcal{E}(\tau)}{1 + \frac{\partial \omega(\tau)}{\partial \tau}}$$

$$\Rightarrow \mathcal{E}'(\tau') = \mathcal{E}(\tau) - \mathcal{E}(\tau) \frac{\partial \omega(\tau)}{\partial \tau}$$

But

$$\mathcal{E}'(\tau') = \mathcal{E}'(\tau + \omega(\tau)) = \mathcal{E}'(\tau) + \omega(\tau) \frac{\partial \mathcal{E}'(\tau)}{\partial \tau}$$

$$\Rightarrow \mathcal{E}'(\tau) + \omega(\tau) \frac{\partial \mathcal{E}'(\tau)}{\partial \tau} = \mathcal{E}(\tau) - \mathcal{E}(\tau) \frac{\partial \omega(\tau)}{\partial \tau}$$

To find $\omega(\tau) \frac{\partial \mathcal{E}'(\tau)}{\partial \tau}$, use the above eqⁿ,

$$\rightarrow \omega(\tau) \frac{\partial \mathcal{E}'(\tau)}{\partial \tau} + \cancel{\omega(\tau) \frac{\partial}{\partial \tau} \left(\omega(\tau) \frac{\partial \mathcal{E}'(\tau)}{\partial \tau} \right)}$$

$$= \omega(\tau) \frac{\partial \mathcal{E}(\tau)}{\partial \tau} - \cancel{\omega(\tau) \frac{\partial}{\partial \tau} \left(\mathcal{E}(\tau) \frac{\partial \omega(\tau)}{\partial \tau} \right)}$$

$$\rightarrow \omega(\tau) \frac{\partial \mathcal{E}'(\tau)}{\partial \tau} = \omega(\tau) \frac{\partial \mathcal{E}(\tau)}{\partial \tau}$$

$$\Rightarrow \mathcal{E}'(\tau) - \mathcal{E}(\tau) = -\omega(\tau) \frac{\partial \mathcal{E}(\tau)}{\partial \tau} - \mathcal{E}(\tau) \frac{\partial \omega(\tau)}{\partial \tau}$$

$$\Rightarrow \underline{\underline{\partial \mathcal{E}(\tau) = -\frac{\partial}{\partial \tau} (\omega(\tau) \mathcal{E}(\tau))}}$$

$$\phi_j \Rightarrow \{\chi^\mu(\tau), \mathcal{E}(\tau)\}$$

Now, we should find the α -index.

The reparameterization is given by,

$$\tau \rightarrow \tau'(\tau) = \tau + \omega(\tau)$$

For each possible ' $\omega(\tau)$ ' we have those many different reparameterizations. Let us try to build $\omega(\tau)$ using a dirac delta basis as follows,

$$\omega(\tau) = \int d\alpha \overset{\text{Expansion Coefficients}}{\omega(\alpha)} \underbrace{\delta(\tau - \alpha)}_{\text{Basis}}$$

So we can think of reparameterization parameters to be the set of dirac delta function centered at ' α '.
 ($\omega \sim \delta(\tau - \alpha)$)

Thus, $\partial_\alpha \tau = \partial(\tau - \alpha)$, $\alpha \rightarrow$ Parameterizes ' τ '

So, in terms of α -index, the gauge transform $\partial \chi^\mu(\tau) = -\omega(\tau) \dot{\chi}^\mu(\tau)$ can be written as,

$$\underline{\underline{\partial_\alpha \chi^\mu(\tau) = -\partial(\tau - \alpha) \dot{\chi}^\mu(\tau)}}$$

Similarly, for gauge field

$$\underline{\underline{\partial_\alpha \varepsilon(\tau) = -\frac{\partial}{\partial \tau} (\partial(\tau - \alpha) \varepsilon(\tau))}}$$

The lie algebra structure constants are found by evaluating

$$[\partial_\alpha, \partial_\beta] \chi^\mu(\tau) = \partial_\alpha \partial_\beta \chi^\mu(\tau) - \partial_\beta \partial_\alpha \chi^\mu(\tau)$$

$$\neq [\partial_\alpha, \partial_\beta] \chi^\mu(\tau) = \int d\gamma f_{\alpha\beta}^\gamma \partial_\gamma \chi^\mu(\tau)$$

$$\text{Consider } \partial_1 \chi^\mu(\tau) = -\omega_1(\tau) \dot{\chi}^\mu(\tau)$$

Then we shall again do a reparameterization $\tau \rightarrow \tau + \omega_2(\tau)$.

$$\partial_1 \chi^\mu(\tau) \xrightarrow{\tau \rightarrow \tau + \omega_2(\tau)} \partial_1 \chi'^\mu(\tau') = -\omega_1'(\tau') \frac{\partial \chi'^\mu(\tau')}{\partial \tau'}$$

$$\partial_1 \chi'^\mu(\tau') = -\omega_1'(\tau + \omega_2(\tau)) \frac{\partial \tau}{\partial \tau'} \frac{\partial \chi'^\mu(\tau + \omega_2(\tau))}{\partial \tau}$$

$$= -\left(\omega_1'(\tau) + \omega_2(\tau) \frac{\partial \omega_1(\tau)}{\partial \tau}\right) \left(1 - \frac{\partial \omega_2(\tau)}{\partial \tau}\right)$$

$$\times \frac{\partial}{\partial \tau} \left(\chi'^\mu(\tau) + \omega_2(\tau) \frac{\partial \chi^\mu(\tau)}{\partial \tau} \right)$$

$$= -\omega_1'(\tau) \frac{\partial \chi'^\mu(\tau)}{\partial \tau} - \omega_2(\tau) \frac{\partial \omega_1(\tau)}{\partial \tau} \frac{\partial \chi^\mu(\tau)}{\partial \tau}$$

$$+ \omega_1(\tau) \frac{\partial \omega_2(\tau)}{\partial \tau} \frac{\partial \chi^\mu(\tau)}{\partial \tau} - \omega_1(\tau) \frac{\partial}{\partial \tau} \left(\omega_2(\tau) \frac{\partial \chi^\mu(\tau)}{\partial \tau} \right)$$

$$\partial_1 \chi'^\mu(\tau') = \partial_1 \chi^\mu(\tau) \rightarrow \text{World line scalar}$$

$$\& -\omega_1'(\tau) \frac{\partial \chi'^\mu(\tau)}{\partial \tau} = \partial_1 \chi'^\mu(\tau). \text{ Thus we get,}$$

$$\partial_1 \chi^\mu(\tau) = \partial_1 \chi'^\mu(\tau) - \omega_2(\tau) \frac{\partial \omega_1(\tau)}{\partial \tau} \frac{\partial \chi^\mu(\tau)}{\partial \tau}$$

$$+ \omega_1(\tau) \frac{\partial \omega_2(\tau)}{\partial \tau} \frac{\partial \chi^\mu(\tau)}{\partial \tau} - \omega_1(\tau) \frac{\partial}{\partial \tau} \left(\omega_2(\tau) \frac{\partial \chi^\mu(\tau)}{\partial \tau} \right)$$

$$\begin{aligned}
 \text{Then, } \partial_2 \partial_1 \chi^\mu(\tau) &= \partial_1 \chi'^\mu(\tau) - \partial_1 \chi^\mu(\tau) \\
 &= \omega_2(\tau) \frac{\partial \omega_1(\tau)}{\partial \tau} \frac{\partial \chi^\mu(\tau)}{\partial \tau} - \omega_1(\tau) \frac{\partial \omega_2(\tau)}{\partial \tau} \frac{\partial \chi^\mu(\tau)}{\partial \tau} \\
 &\quad + \omega_1(\tau) \frac{\partial}{\partial \tau} \left(\omega_2(\tau) \frac{\partial \chi^\mu(\tau)}{\partial \tau} \right)
 \end{aligned}$$

$$\begin{aligned}
 \partial_2 \partial_1 \chi^\mu(\tau) &= \left(\omega_2(\tau) \frac{\partial \omega_1(\tau)}{\partial \tau} - \omega_1(\tau) \frac{\partial \omega_2(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) \\
 &\quad + \omega_1(\tau) \frac{\partial}{\partial \tau} \left(\omega_2(\tau) \dot{\chi}^\mu(\tau) \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\omega_2(\tau) \frac{\partial \omega_1(\tau)}{\partial \tau} - \omega_1(\tau) \frac{\partial \omega_2(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) \\
 &\quad + \omega_1(\tau) \left(\frac{\partial \omega_2(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) + \omega_1(\tau) \omega_2(\tau) \ddot{\chi}^\mu(\tau)
 \end{aligned}$$

Thus, finally we get,

$$\begin{aligned}
 \partial_2 \partial_1 \chi^\mu(\tau) &= \omega_2(\tau) \left(\frac{\partial \omega_1(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) \\
 &\quad + \omega_1(\tau) \omega_2(\tau) \ddot{\chi}^\mu(\tau)
 \end{aligned}$$

Note that the above result can be got in a clever way as follows

$$\text{Since } \partial_1 \chi^\mu(\tau) = -\omega_1(\tau) \dot{\chi}^\mu(\tau)$$

$$\begin{aligned} \partial_2(\partial_1 \chi^\mu(\tau)) &= \omega_2(\tau) \frac{\partial}{\partial \tau} (\omega_1(\tau) \dot{\chi}^\mu(\tau)) \\ &= \omega_2(\tau) \left(\frac{\partial \omega_1(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) + \omega_2(\tau) \omega_1(\tau) \ddot{\chi}^\mu(\tau) \end{aligned}$$

Similarly,

$$\begin{aligned} \partial_1 \partial_2 \chi^\mu(\tau) &= \omega_2(\tau) \omega_1(\tau) \ddot{\chi}^\mu(\tau) \\ &\quad + \omega_1(\tau) \left(\frac{\partial \omega_2(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) \end{aligned}$$

$$\text{Now, } \partial_1 \partial_2 \chi^\mu(\tau) - \partial_2 \partial_1 \chi^\mu(\tau) =$$

$$= \omega_1(\tau) \left(\frac{\partial \omega_2(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau) - \omega_2(\tau) \left(\frac{\partial \omega_1(\tau)}{\partial \tau} \right) \dot{\chi}^\mu(\tau)$$

In the dirac-delta reparameterization

$$\text{where } \partial_\alpha : \omega_1(\tau) \longrightarrow \delta(\tau - \alpha)$$

$$\partial_\beta : \omega_2(\tau) \longrightarrow \delta(\tau - \beta)$$

$$\Rightarrow \partial_\alpha \partial_\beta \chi^\mu(\tau) - \partial_\beta \partial_\alpha \chi^\mu(\tau) =$$

$$= \left(\partial(\tau - \alpha) \frac{\partial \partial(\tau - \beta)}{\partial \tau} - \partial(\tau - \beta) \frac{\partial \partial(\tau - \alpha)}{\partial \tau} \right) \dot{\chi}^\mu(\tau)$$

$$= \int d\gamma \left(\partial(\gamma - \alpha) \frac{\partial \partial(\gamma - \beta)}{\partial \gamma} - \partial(\gamma - \beta) \frac{\partial \partial(\gamma - \alpha)}{\partial \gamma} \right) \partial(\tau - \gamma) \dot{\chi}^\mu(\tau)$$

$$\text{Use } \partial_\gamma \chi^\mu(\tau) = -\partial(\tau - \gamma) \dot{\chi}^\mu(\tau)$$

$$\text{Thus, } (\partial_\alpha \partial_\beta - \partial_\beta \partial_\alpha) \chi^\mu(\tau) = \int d\gamma f_{\alpha\beta}^\gamma \partial_\gamma \chi^\mu(\tau)$$

where

$$f_{\alpha\beta}^\gamma = -\partial(\gamma - \alpha) \frac{\partial \partial(\gamma - \beta)}{\partial \gamma} + \partial(\gamma - \beta) \frac{\partial \partial(\gamma - \alpha)}{\partial \gamma}$$

BRST Transformation of Point particle

Since there is only one Gauge field ' $\mathcal{E}(\tau)$ ', we have only one gauge fixing condition.

If we fix the gauge to ' $\hat{\mathcal{E}}(\tau)$ ' then

$$'F^A \rightarrow F(\tau) = \hat{\mathcal{E}}(\tau) - \mathcal{E}(\tau)'$$

where the 'A' index just represents ' τ '.

So, the gauge fixing action is

$$\underline{S_2 = -i \int d\tau B(\tau) F(\tau)}$$

And the ghost action $S_3 = \int d\tau b(\tau) c^\alpha \partial_\alpha F(\tau)$
where,

$$\partial_\alpha F(\tau) = - \frac{\partial}{\partial \tau} (\delta(\tau - \alpha) F(\tau))$$

becomes

$$S_3 = - \int d\tau d\alpha b(\tau) c(\alpha) \frac{\partial}{\partial \tau} (\delta(\tau - \alpha) F(\tau))$$

The BRST transformation for $\chi^\mu(\tau)$ is

$$\begin{aligned}\delta_B \chi^\mu(\tau) &= -i\epsilon C^\alpha \delta_\alpha \chi^\mu(\tau) \\ &= i\epsilon \int d\alpha C(\alpha) \delta(\tau - \alpha) \dot{\chi}^\mu(\tau)\end{aligned}$$

$$\implies \underline{\delta_B \chi^\mu(\tau) = i\epsilon C(\tau) \dot{\chi}^\mu(\tau)}$$

(reparameterization with $-i\epsilon C(\tau)$ parameter)

Similarly for $\varepsilon(\tau)$, it becomes

$$\begin{aligned}\delta_B \varepsilon(\tau) &= -i\epsilon C^\alpha \delta_\alpha \varepsilon(\tau) \\ &= i\epsilon \int d\alpha C(\alpha) \frac{\partial}{\partial \tau} (\delta(\tau - \alpha) \varepsilon(\tau)) \\ &= i\epsilon \frac{\partial}{\partial \tau} \left[\int d\alpha C(\alpha) \delta(\tau - \alpha) \varepsilon(\tau) \right]\end{aligned}$$

$$\implies \underline{\delta_B \varepsilon(\tau) = i\epsilon \frac{\partial}{\partial \tau} (C(\tau) \varepsilon(\tau))}$$

(reparameterization with $-i\epsilon C(\tau)$ parameter)

For the Ghost field 'c(τ)'

$$\delta_B c(\tau) = \frac{i\epsilon}{2} f_{\alpha\beta}^\tau C(\alpha)C(\beta)$$

$$\begin{aligned} &= -\frac{i\epsilon}{2} \int d\alpha d\beta \delta(\tau - \alpha) \frac{\partial \delta(\tau - \beta)}{\partial \tau} C(\alpha)C(\beta) \\ &\quad + \frac{i\epsilon}{2} \int d\alpha d\beta \delta(\tau - \beta) \frac{\partial \delta(\tau - \alpha)}{\partial \tau} C(\alpha)C(\beta) \end{aligned}$$

$$\begin{aligned} \Rightarrow \delta_B c(\tau) &= -i\epsilon C(\tau) \underbrace{\int d\alpha \left[\frac{\partial \delta(\tau - \alpha)}{\partial \tau} \right]}_{\text{Int: by parts}} C(\alpha) \end{aligned}$$

$$= i\epsilon C(\tau) \int d\alpha \delta(\tau - \alpha) \frac{\partial C(\alpha)}{\partial \tau}$$

$$\Rightarrow \underline{\delta_B c(\tau) = i\epsilon C(\tau) \frac{\partial C(\tau)}{\partial \tau}}$$

Then for remaining fields it is,

$$\delta_B B(\tau) = 0 \quad \& \quad \delta_B b(\tau) = \epsilon B(\tau)$$

Thus the BRST transformation for a point particle is,

- $\delta_B \chi^\mu(\tau) = i\epsilon C(\tau) \dot{\chi}^\mu(\tau)$
- $\delta_B \mathcal{E}(\tau) = i\epsilon \frac{\partial}{\partial \tau} (C(\tau) \mathcal{E}(\tau))$
- $\delta_B B(\tau) = 0$
- $\delta_B b(\tau) = \epsilon B(\tau)$
- $\delta_B C(\tau) = i\epsilon C(\tau) \frac{\partial C(\tau)}{\partial \tau}$

This transformation keeps the full quantum action invariant.

Point particle in unit Gauge

This is equivalent of fixing the string world sheet to unit gauge.

$$\text{Here, } \hat{E}(\tau) = 1 \implies F(\tau) = 1 - \mathcal{E}(\tau)$$

$$\implies S_2 = -i \int d\tau B(\tau) (1 - \mathcal{E}(\tau))$$

$$S_3 = \int d\tau b(\tau) c^\alpha \partial_\alpha F(\tau) = - \int d\tau b(\tau) c^\alpha \partial_\alpha \mathcal{E}(\tau)$$

$$= \int d\tau d\alpha b(\tau) c(\alpha) \frac{\partial}{\partial \tau} (\underbrace{\delta(\tau - \alpha)}_{\text{Int: by parts}} \mathcal{E}(\tau))$$

$$= - \int d\tau d\alpha \delta(\tau - \alpha) \mathcal{E}(\tau) \left[\frac{\partial b(\tau)}{\partial \tau} \right] c(\alpha)$$

$$\implies \underline{S_3 = - \int d\tau \mathcal{E}(\tau) \dot{b}(\tau) c(\tau)}$$

Then the full quantum action which fixes the Point particle action in unit gauge is

$$S = \int d\tau \left(\frac{1}{2} [\mathcal{E}(\tau)^{-1} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + \mathcal{E}(\tau) m^2] - iB(\tau)(1 - \mathcal{E}(\tau)) - \mathcal{E}(\tau) \dot{b}(\tau) C(\tau) \right)$$

The eqⁿ of motion for $\mathcal{E}(\tau)$ is,

$$\frac{1}{2} [-\mathcal{E}(\tau)^{-2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2] + iB(\tau) - \dot{b}(\tau) C(\tau) = 0$$

$$\Rightarrow B(\tau) = \frac{-i}{2} [\mathcal{E}(\tau)^{-2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - m^2] - i\dot{b}(\tau) C(\tau)$$

In the path integral if we do the ' $B(\tau)$ ' integral, then we should replace ' $\mathcal{E}(\tau) = 1$ ' which then implies we should also replace

$$\underline{B(\tau) = \frac{-i}{2} [\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - m^2] - i\dot{b}(\tau) C(\tau)}$$

Now, the gauge fixing action will not appear in the path integral.

Then, the gauge fixed action is,

$$S = \int d\tau \left(\frac{1}{2} [\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2] - \dot{b}(\tau) C(\tau) \right)$$

Now, the BRST transformation after gauge fixing is,

- $\delta_B \chi^\mu(\tau) = i\epsilon C(\tau) \dot{\chi}^\mu(\tau)$
- $\delta_B b(\tau) = -i\epsilon \left[\frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - \frac{1}{2} m^2 + \dot{b}(\tau) C(\tau) \right]$
- $\delta_B C(\tau) = i\epsilon C(\tau) \dot{C}(\tau)$

We can show that this transformation is Nilpotent

$$\begin{aligned} \delta_{B_2} \delta_{B_1} \chi^\mu(\tau) &= i\epsilon_1 \delta_{B_2} [C(\tau) \dot{\chi}^\mu(\tau)] \\ &= i\epsilon_1 C(\tau) [\delta_{B_2} \dot{\chi}^\mu(\tau)] + i\epsilon_1 [\delta_{B_2} C(\tau)] \dot{\chi}^\mu(\tau) \end{aligned}$$

$$= i\epsilon_1 C(\tau) \frac{\partial}{\partial \tau} [\partial_{\beta_2} \chi^\mu(\tau)] - \epsilon_1 \epsilon_2 C(\tau) \dot{C}(\tau) \dot{\chi}^\mu(\tau)$$

$$= i\epsilon_1 C(\tau) \frac{\partial}{\partial \tau} [i\epsilon_2 C(\tau) \dot{\chi}^\mu(\tau)] - \epsilon_1 \epsilon_2 C(\tau) \dot{C}(\tau) \dot{\chi}^\mu(\tau)$$

$$= i\epsilon_1 C(\tau) i\epsilon_2 [\dot{C}(\tau) \dot{\chi}^\mu(\tau) + C(\tau) \ddot{\chi}^\mu(\tau)] - \epsilon_1 \epsilon_2 C(\tau) \dot{C}(\tau) \dot{\chi}^\mu(\tau)$$

$$= \epsilon_1 \epsilon_2 C(\tau) [\cancel{\dot{C}(\tau) \dot{\chi}^\mu(\tau)} + C(\tau) \ddot{\chi}^\mu(\tau)] - \epsilon_1 \epsilon_2 C(\tau) \cancel{\dot{C}(\tau) \dot{\chi}^\mu(\tau)}$$

Since, $C(\tau)C(\tau) = 0 \Rightarrow \partial_{\beta_2} \partial_{\beta_1} \chi^\mu(\tau) = 0$

$$\bullet \partial_{\beta_2} \partial_{\beta_1} C(\tau) = i\epsilon_1 \partial_{\beta_2} [C(\tau) \dot{C}(\tau)]$$

$$= i\epsilon_1 [\partial_{\beta_2} C(\tau)] \dot{C}(\tau) + i\epsilon_1 C(\tau) [\partial_{\beta_2} \dot{C}(\tau)]$$

$$= i\epsilon_1 i\epsilon_2 C(\tau) \dot{C}(\tau) \dot{C}(\tau)$$

$$+ i\epsilon_1 C(\tau) \frac{\partial}{\partial \tau} [i\epsilon_2 C(\tau) \dot{C}(\tau)]$$

$$\begin{aligned}
&= -\epsilon_1 \epsilon_2 C(\tau) \dot{C}(\tau) \dot{C}(\tau) \\
&\quad + i\epsilon_1 C(\tau) i\epsilon_2 \left[\dot{C}(\tau) \dot{C}(\tau) + C(\tau) \ddot{C}(\tau) \right]
\end{aligned}$$

$$\begin{aligned}
&= -\epsilon_1 \epsilon_2 C(\tau) \cancel{\dot{C}(\tau)} \dot{C}(\tau) + \epsilon_1 \epsilon_2 \cancel{C(\tau)} \dot{C}(\tau) \dot{C}(\tau) \\
&\quad + \epsilon_1 \epsilon_2 C(\tau) C(\tau) \ddot{C}(\tau) \\
&\quad \quad \quad (C(\tau)C(\tau) = 0)
\end{aligned}$$

$$\Rightarrow \underline{\delta_{B_2} \delta_{B_1} C(\tau) = 0}$$

Then, for the b -ghost,

$$\delta_{B_1} b(\tau) = -i\epsilon_1 \left[\frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - \frac{1}{2} m^2 + \dot{b}(\tau) C(\tau) \right]$$

$$\delta_{B_2} \delta_{B_1} b(\tau) = -i\epsilon_1 \left(\frac{1}{2} \delta_{B_2} \left[\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) \right] + \delta_{B_2} \left[\dot{b}(\tau) C(\tau) \right] \right)$$

$$\begin{aligned}
&= -i\epsilon_1 \left(\left[\frac{\partial}{\partial \tau} \delta_{B_2} \chi^\mu(\tau) \right] \dot{\chi}_\mu(\tau) + \left[\frac{\partial}{\partial \tau} \delta_{B_2} b(\tau) \right] C(\tau) \right. \\
&\quad \left. + \dot{b}(\tau) \left[\delta_{B_2} C(\tau) \right] \right)
\end{aligned}$$

$$= -i\epsilon_1 i\epsilon_2 \frac{\partial}{\partial \tau} \left[C(\tau) \dot{\chi}^\mu(\tau) \right] \dot{\chi}_\mu(\tau)$$

$$+ i\epsilon_1 i\epsilon_2 \frac{\partial}{\partial \tau} \left[\frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - \cancel{\frac{1}{2} m^2} + \dot{b}(\tau) C(\tau) \right] C(\tau) \\ + i\epsilon_1 i\epsilon_2 \dot{b}(\tau) C(\tau) \dot{C}(\tau)$$

$$= -i\epsilon_1 i\epsilon_2 \left[\dot{C}(\tau) \dot{\chi}^\mu(\tau) + C(\tau) \ddot{\chi}^\mu(\tau) \right] \dot{\chi}_\mu(\tau)$$

$$+ i\epsilon_1 i\epsilon_2 \left[\ddot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + \cancel{\ddot{b}(\tau) C(\tau)} + \dot{b}(\tau) \dot{C}(\tau) \right] C(\tau) \\ + i\epsilon_1 i\epsilon_2 \dot{b}(\tau) C(\tau) \dot{C}(\tau)$$

$$= \epsilon_1 \epsilon_2 \dot{C}(\tau) \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + \epsilon_1 \epsilon_2 \cancel{C(\tau)} \ddot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) \\ - \epsilon_1 \epsilon_2 \cancel{\ddot{\chi}^\mu(\tau)} \dot{\chi}_\mu(\tau) C(\tau) - \epsilon_1 \epsilon_2 \dot{b}(\tau) \cancel{\dot{C}(\tau)} C(\tau) \\ - \epsilon_1 \epsilon_2 \dot{b}(\tau) \cancel{C(\tau)} \dot{C}(\tau)$$

$$\Rightarrow \underline{\underline{\delta_{\mathcal{B}_2} \delta_{\mathcal{B}_1} b(\tau) = \epsilon_1 \epsilon_2 \dot{C}(\tau) \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau)}}$$

Then on-shell condition on 'c' is $\dot{C}(\tau) = 0$.

Thus, $\delta_{\mathcal{B}_2} \delta_{\mathcal{B}_1} b(\tau) = 0$ on shell.

BRST Symmetry for Particle in Unit gauge

$$S = \int d\tau \left(\frac{1}{2} [\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2] - \underbrace{\dot{b}(\tau) C(\tau)}_{\text{Int. by Parts}} \right)$$

$$S = \int d\tau \left(\frac{1}{2} [\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2] + b(\tau) \dot{C}(\tau) \right)$$

$$\delta_B S = \int d\tau \left[\frac{1}{2} \delta_B (\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau)) + \delta_B (b(\tau) \dot{C}(\tau)) \right]$$

$$= \int d\tau \left[\dot{\chi}_\mu(\tau) \frac{\partial}{\partial \tau} (\delta_B \chi^\mu(\tau)) + (\delta_B b(\tau)) \dot{C}(\tau) + b(\tau) \frac{\partial}{\partial \tau} (\delta_B C(\tau)) \right]$$

$$= \int d\tau \left[\dot{\chi}_\mu(\tau) \frac{\partial}{\partial \tau} (i \epsilon C(\tau) \dot{\chi}^\mu(\tau)) - i \epsilon \left[\frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - \frac{1}{2} m^2 + \dot{b}(\tau) C(\tau) \right] \dot{C}(\tau) + b(\tau) i \epsilon \frac{\partial}{\partial \tau} (C(\tau) \dot{C}(\tau)) \right]$$

$$= i \epsilon \int d\tau \left[\begin{aligned} & \dot{\chi}_\mu(\tau) \dot{C}(\tau) \ddot{\chi}^\mu(\tau) + \dot{\chi}_\mu(\tau) C(\tau) \ddot{\chi}^\mu(\tau) \\ & - \frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) \dot{C}(\tau) + \frac{1}{2} m^2 \dot{C}(\tau) \\ & - \dot{b}(\tau) C(\tau) \dot{C}(\tau) - b(\tau) \dot{C}(\tau) \dot{C}(\tau) \\ & - b(\tau) C(\tau) \ddot{C}(\tau) \end{aligned} \right]$$

$$= i \epsilon \int d\tau \left[\begin{aligned} & \frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) \dot{C}(\tau) + \dot{\chi}_\mu(\tau) C(\tau) \ddot{\chi}^\mu(\tau) \\ & + \frac{1}{2} m^2 \dot{C}(\tau) - \dot{b}(\tau) C(\tau) \dot{C}(\tau) \\ & - b(\tau) \dot{C}(\tau) \dot{C}(\tau) - b(\tau) C(\tau) \ddot{C}(\tau) \end{aligned} \right]$$

$$= i \epsilon \int d\tau \left[\begin{aligned} & \frac{1}{2} \dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) \dot{C}(\tau) + \dot{\chi}_\mu(\tau) C(\tau) \ddot{\chi}^\mu(\tau) \\ & + \frac{1}{2} m^2 \dot{C}(\tau) + b(\tau) \cancel{C(\tau)} \ddot{C}(\tau) \\ & + b(\tau) \cancel{\dot{C}(\tau)} \dot{C}(\tau) - b(\tau) \dot{C}(\tau) \cancel{\dot{C}(\tau)} \\ & - b(\tau) \cancel{C(\tau)} \ddot{C}(\tau) \end{aligned} \right]$$

$$= i \epsilon \int d\tau \left[\begin{aligned} & \frac{1}{2} (\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2) \dot{C}(\tau) \\ & + \dot{\chi}_\mu(\tau) \ddot{\chi}^\mu(\tau) C(\tau) \end{aligned} \right]$$

$$\Rightarrow \delta_B S = i\epsilon \int d\tau \frac{\partial}{\partial \tau} \left[\frac{1}{2} (\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2) C(\tau) \right]$$

Thus, the BRST transformation is a total derivative.

But on-shell we get $\dot{C}(\tau) = 0$

$$\& \quad \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) = 0$$

$$\Rightarrow \delta_B S = 0 \quad (\text{on-shell})$$

Then to find the corresponding charge, we shall do the usual Noether procedure.

For this we shall allow the BRST parameter to be a function of ' τ '. And then do the BRST transform. Then the coefficient of $\dot{C}(\tau)$ will give us the current.

BRST charge Operator

$$S = \int d\tau \left(\frac{1}{2} [\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2] + b(\tau) \dot{C}(\tau) \right)$$

$$\delta_B S = \int d\tau \left[\dot{\chi}_\mu(\tau) \frac{\partial}{\partial \tau} (\delta_B \chi^\mu(\tau)) + (\delta_B b(\tau)) \dot{C}(\tau) \right. \\ \left. + b(\tau) \frac{\partial}{\partial \tau} (\delta_B C(\tau)) \right]$$

$$= \int d\tau \left[\dot{\chi}_\mu(\tau) \frac{\partial}{\partial \tau} (i\epsilon(\tau) C(\tau) \dot{\chi}^\mu(\tau)) \right. \\ \left. - i\epsilon(\tau) \left[\frac{1}{2} (\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - m^2) + \dot{b}(\tau) C(\tau) \right] \dot{C}(\tau) \right. \\ \left. + b(\tau) \frac{\partial}{\partial \tau} (i\epsilon(\tau) C(\tau) \dot{C}(\tau)) \right]$$

$$= \int d\tau \left[\dot{\chi}_\mu(\tau) i\dot{\epsilon}(\tau) C(\tau) \dot{\chi}^\mu(\tau) + \dot{\chi}_\mu(\tau) i\epsilon(\tau) \dot{C}(\tau) \dot{\chi}^\mu(\tau) \right. \\ \left. + \dot{\chi}_\mu(\tau) i\epsilon(\tau) C(\tau) \ddot{\chi}^\mu(\tau) \right. \\ \left. - i\epsilon(\tau) \left[\frac{1}{2} (\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) - m^2) + \dot{b}(\tau) C(\tau) \right] \dot{C}(\tau) \right. \\ \left. + b(\tau) (i\dot{\epsilon}(\tau) C(\tau) \dot{C}(\tau) + i\epsilon(\tau) \dot{C}(\tau) \dot{C}(\tau)) \right. \\ \left. + i\epsilon(\tau) C(\tau) \ddot{C}(\tau) \right]$$

$$= \int d\tau \left[\begin{aligned} & i\dot{E}(\tau) \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) C(\tau) + iE(\tau) \frac{1}{2} m^2 \dot{C}(\tau) \\ & + iE(\tau) \left(\frac{1}{2} \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) \dot{C}(\tau) + \cancel{\dot{\chi}_\mu(\tau) C(\tau) \dot{\chi}^\mu(\tau)} \right) \\ & - iE(\tau) \cancel{\dot{b}(\tau) C(\tau) \dot{C}(\tau)} \\ & + b(\tau) \left(\cancel{i\dot{E}(\tau) C(\tau) \dot{C}(\tau)} + \cancel{iE(\tau) \dot{C}(\tau) \dot{C}(\tau)} \right) \\ & \quad + iE(\tau) C(\tau) \ddot{C}(\tau) \end{aligned} \right]$$

↓
vanish by
Eqⁿ of motion

$$= \int d\tau \left[\begin{aligned} & i\dot{E}(\tau) \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) C(\tau) + iE(\tau) \frac{1}{2} m^2 \dot{C}(\tau) \\ & + iE(\tau) \frac{1}{2} \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) \dot{C}(\tau) \end{aligned} \right]$$

do Integration by parts

$$= \int d\tau \left[\begin{aligned} & i\dot{E}(\tau) \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) C(\tau) - i\dot{E}(\tau) \frac{1}{2} m^2 C(\tau) \\ & - \frac{1}{2} i\dot{E}(\tau) \dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) C(\tau) \\ & + iE(\tau) \cancel{\dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) C(\tau)} \end{aligned} \right]$$

vanish by Eqⁿ of motion

$$\delta_B S = \int d\tau C(\tau) \left(\frac{1}{2} \left[-\dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) + m^2 \right] \right) i\dot{c}(\tau)$$

So, we shall define the coefficient of ' $i\dot{c}(\tau)$ ' as the BRST charge density

$$\text{Thus, } \hat{Q}_B = C(\tau) \left[\frac{-\dot{\chi}_\mu(\tau) \dot{\chi}^\mu(\tau) + m^2}{2} \right]$$

In the upcoming derivation it will be shown that the Hamiltonian for the Point particle in unit gauge is,

$$\mathcal{H} = \frac{1}{2} \left[-\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2 \right]$$

Thus the final form for BRST generator is

$$\hat{Q}_B = C \mathcal{H}$$

It is clearly Nilpotent as $c^2 = 0$.

Canonical Commutation Relations

$$S^E = \int d\tau \left(\frac{1}{2} [\dot{\chi}^\mu(\tau) \dot{\chi}_\mu(\tau) + m^2] + b(\tau) \dot{C}(\tau) \right)$$

The above is in Euclidean signature. To write it in Minkowski signature we should replace, $\tau \rightarrow it$, and get S^M such that

$$e^{-S^E} = e^{iS^M}$$

Under $\tau \rightarrow it$, we have,

$$S^E = i \int dt \left(\frac{1}{2} [-\dot{\chi}^\mu(t) \dot{\chi}_\mu(t) + m^2] - i b(t) \dot{C}(t) \right)$$

$$S^M = i S^E$$

$$\Rightarrow S^M = \int dt \left(\frac{1}{2} [\dot{\chi}^\mu(t) \dot{\chi}_\mu(t) - m^2] + i b(t) \dot{C}(t) \right)$$

$$\Rightarrow \boxed{\mathcal{L}^M = \frac{1}{2} [\dot{\chi}^\mu(t) \dot{\chi}_\mu(t) - m^2] + i b(t) \dot{C}(t)}$$

The conjugate momenta for χ^μ is,

$$\pi_x^\mu = \frac{\partial \mathcal{L}}{\partial \dot{\chi}^\mu} = \dot{\chi}^\mu(t) = \frac{\partial \chi^\mu}{\partial t} = i \frac{\partial \chi^\mu(\tau)}{\partial \tau} = i \dot{\chi}^\mu(\tau)$$

$$\Rightarrow \boxed{p^\mu = \dot{\chi}^\mu(t) = i \dot{\chi}^\mu(\tau)}$$

The conjugate momenta for c is,

$$\pi_c = \frac{\partial \mathcal{L}}{\partial \dot{c}(t)} = ib(t) = ib(\tau)$$

Then the usual canonical relations are

$$[\hat{\chi}^\mu, \hat{p}^\nu] = i \eta^{\mu\nu} \Rightarrow [\hat{\chi}^\mu, i \hat{\chi}^\nu(\tau)] = i \eta^{\mu\nu}$$

$$\Rightarrow \boxed{[\hat{\chi}^\mu, \hat{\chi}^\nu(\tau)] = \eta^{\mu\nu}}$$

Similarly,

$$\{\hat{c}(t), \hat{\pi}_c(t)\} = -i$$

$$\Rightarrow \{\hat{c}(t), i \hat{b}(t)\} = -i \Rightarrow \boxed{\{\hat{b}(\tau), \hat{c}(\tau)\} = 1}$$

$$\& \quad \underline{\{\hat{b}(\tau), \hat{b}(\tau)\} = \{\hat{c}(\tau), \hat{c}(\tau)\} = 0}$$

Then, the Hamiltonian is given by,

$$\hat{H} = \hat{p}^\mu \hat{\chi}_\mu(t) + \hat{\Pi}_c \hat{C}(t) - \hat{\mathcal{L}}$$

$$\begin{aligned} \hat{H} &= \hat{\chi}^\mu(t) \hat{\chi}_\mu(t) + i\cancel{\hat{b}(t)} \hat{C}(t) \\ &\quad - \frac{1}{2} (\hat{\chi}^\mu(t) \hat{\chi}_\mu(t) - m^2) - i\cancel{\hat{b}(t)} \hat{C}(t) \end{aligned}$$

$$\underline{\hat{H} = \frac{1}{2} (\hat{\chi}^\mu(t) \hat{\chi}_\mu(t) + m^2) = \frac{1}{2} (\hat{p}^\mu \hat{p}_\mu + m^2)}$$

In Euclidean Signature, $\hat{p}^\mu = i \hat{\chi}^\mu(\tau)$

$$\Rightarrow \boxed{\hat{H} = \frac{1}{2} (-\hat{\chi}^\mu(\tau) \hat{\chi}_\mu(\tau) + m^2)}$$

$$\underline{[\hat{p}^\mu, \hat{p}^\nu] = [\hat{p}^\mu, \hat{b}] = [\hat{p}^\mu, \hat{C}] = 0}$$

$$\underline{[\hat{H}, \hat{p}^\mu] = [\hat{H}, \hat{b}] = [\hat{H}, \hat{C}] = 0}$$

$$[\hat{H}, \hat{\chi}^\mu] = \frac{1}{2} [\hat{p}^\nu \hat{p}_\nu, \hat{\chi}^\mu] = \hat{p}_\nu [\hat{p}^\nu, \hat{\chi}^\mu] = -i \hat{p}_\nu \eta^{\mu\nu}$$

$$\Rightarrow \underline{[\hat{H}, \hat{\chi}^\mu] = -i \hat{p}^\mu}$$

Exact & Closed states of a Point Particle

The eigen states of ' $\hat{p}^\mu$ ' is given by ' $|k\rangle$ ' where

$$\underline{\hat{p}^\mu |k\rangle = k^\mu |k\rangle}$$

$$\text{Now, } \hat{H}|k\rangle = \frac{1}{2}(k^\mu k_\mu + m^2)|k\rangle$$

Then as seen in 2nd chapter we saw that the 0th mode b-c ghosts form a 2-state fermionic system denoted by $|\uparrow\rangle$ & $|\downarrow\rangle$

$$\text{They satisfy } \hat{c}|\uparrow\rangle = 0, \quad \hat{b}|\downarrow\rangle = 0$$

$$\& \quad \hat{c}|\downarrow\rangle = |\uparrow\rangle, \quad \hat{b}|\uparrow\rangle = |\downarrow\rangle$$

We can take the tensor product of momentum space states $|k\rangle$ & the fermionic states $\{|\uparrow\rangle, |\downarrow\rangle\}$ & get a larger space given by $\{|k; \uparrow\rangle, |k; \downarrow\rangle\}$.

$$\hat{p}^\mu |k; \uparrow\rangle = k^\mu |k; \uparrow\rangle \neq \hat{p}^\mu |k; \downarrow\rangle = k^\mu |k; \downarrow\rangle$$

$$\hat{c} |k; \uparrow\rangle = 0, \quad \hat{b} |k; \downarrow\rangle = 0$$

$$\hat{c} |k; \downarrow\rangle = |k; \uparrow\rangle, \quad \hat{b} |k; \uparrow\rangle = |k; \downarrow\rangle$$

$$\hat{H} |k; \uparrow\rangle = \frac{1}{2} (k^\mu k_\mu + m^2) |k; \uparrow\rangle$$

$$\hat{H} |k; \downarrow\rangle = \frac{1}{2} (k^\mu k_\mu + m^2) |k; \downarrow\rangle$$

Thus in the complete space, we have two copies of states with same energy.

All the above mentioned operators are all Hermitian.

$$\text{We know } \langle q | k \rangle = \delta^D(q^\mu - k^\mu)$$

But it is not straight forward for the fermionic states or for $\{|k; \uparrow\rangle, |k; \downarrow\rangle\}$.

Now consider the action of the BRST operator $\hat{Q}_B$ on these states.

$$\hat{Q}_B = \hat{C} \hat{H} = \hat{H} \hat{C}$$

Closed States

$\hat{Q}_B |k; \uparrow\rangle = 0$. So, $|k; \uparrow\rangle$ any ' k^μ ' is closed.

$$\hat{Q}_B |k; \downarrow\rangle = \frac{1}{2} (k^\mu k_\mu + m^2) |k; \uparrow\rangle$$

$\Rightarrow |k; \downarrow\rangle$ is closed state only if $k^\mu k_\mu = -m^2$

Exact States

$\hat{Q}_B |k; \downarrow\rangle$ is Exact state when $k^\mu k_\mu \neq -m^2$

$$\Rightarrow \frac{1}{2} (k^\mu k_\mu + m^2) |k; \uparrow\rangle = \hat{Q}_B |k; \downarrow\rangle \text{ is}$$

Exact state when $k^\mu k_\mu \neq -m^2$

$$\text{or } |k; \uparrow\rangle = \frac{\hat{Q}_B |k; \downarrow\rangle}{\frac{1}{2} (k^\mu k_\mu + m^2)} \text{ is Exact state } (k^\mu k_\mu \neq -m^2)$$

when ' $k^\mu k_\mu \neq -m^2$ ' & for any ' q '.

$$\langle q; \uparrow | k; \uparrow \rangle = \frac{\langle q; \uparrow | \hat{Q}_B | k; \downarrow \rangle}{\frac{1}{2}(k^\mu k_\mu + m^2)} = 0$$

So, for up-states,

$|k; \uparrow\rangle, (k^\mu k_\mu = -m^2)$ is closed & not exact.

$|k; \uparrow\rangle, (k^\mu k_\mu \neq -m^2)$ is closed & exact.

For down states

$|k; \downarrow\rangle, (k^\mu k_\mu = -m^2)$ is closed & not exact.

It can't be written as a state where $\hat{Q}_B$ acting on something. So it is not exact.

$|k; \downarrow\rangle, (k^\mu k_\mu \neq -m^2)$ is not closed.

Thus Cohomology of $\hat{Q}_B$ or H_{BRST} for Point particle is,

$$\underline{\{|k; \uparrow\rangle \text{ \& } |k; \downarrow\rangle\} \text{ whose } k^\mu k_\mu = -m^2}$$

So only those particles that are on-shell are present in the BRST Hilbert space.

That is, physical states must satisfy the mass-shell condition, but we have *two copies* of the expected spectrum. In fact, only states $|k, \downarrow\rangle$ satisfying the additional condition

$$b|\psi\rangle = 0 \tag{4.2.33}$$

appear in physical amplitudes. The origin of this additional condition is kinematic. For $k^2 + m^2 \neq 0$, the states $|k, \uparrow\rangle$ are exact — they are orthogonal to all physical states and their amplitudes must vanish identically. So the amplitudes can only be proportional to $\delta(k^2 + m^2)$. But amplitudes in field theory and in string theory, while they may have poles and cuts, never have delta functions (except in $D = 2$ where the kinematics is special). So they must vanish identically.