Full Quantum Action for String

The String polyakov action is given by,

$$S_x = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{ab}(\partial_a X^\mu)(\partial_b X_\mu)$$

The gauge fields here are the world sheet metric tensor components $g_{ab}(\sigma^a)$. And the combined gauge transformations on $g_{ab}(\sigma^a)$ are given by the diff × Weyl transform,

$$g'^{\xi}_{ab}(\sigma') = e^{2\omega(\sigma)} \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b} g_{cd}(\sigma)$$

Infinitesimally, $\sigma^a \xrightarrow[\text{diff}]{} \sigma'^a = \sigma^a + \delta\sigma^a$

& $g_{ab}(\sigma) \xrightarrow{\text{Weyl}} e^{2\delta\omega} g_{ab}(\sigma) = (1 + 2\delta\omega) g_{ab}(\sigma)$

Then, $\delta g_{ab}(\sigma) = 2\delta\omega g_{ab} - (\nabla_a \delta\sigma_b + \nabla_b \delta\sigma_a)$

The Gauge fixing condition is

$$F_{ab} = \underbrace{\hat{g}_{ab}(\sigma)}_{\text{Gauge choice}} - g_{ab}(\sigma)$$

We fix the gauge by enforcing $F_{ab} = 0$
Thus, the Gauge fixing action is,

$$S_{g.f} = \frac{i}{4\pi}\int d^2\sigma \sqrt{g}\, B^{ab} F_{ab}$$

Since F_{ab} is symmetric, even B^{ab} is.
If we do the Path integral of 'B^{ab}', it results in gauge fixation.

Then, we should account for the ghost action. But in chapter 3, section 3, we got the Faddeev - Popov determinant in the form

$$\left(\Delta_{FP}[g_{ab}]\right)^{-1} = \int [\mathcal{D}2\omega][\mathcal{D}2\sigma^q][\mathcal{D}B^{ab}]\, e^{-i\left(S_{weyl}^{(boson)} + S_{diff}^{(boson)}\right)}$$

$$S_{weyl}^{(boson)} = \int d^2\sigma \sqrt{g}\, B^{ab} g_{ab} 2\omega$$

$$S_{diff}^{(boson)} = \int d^2\sigma \sqrt{g}\, B^{ab}\left[\nabla_a 2\sigma_b + \nabla_b 2\sigma_a\right]$$

$B^{ab} \longrightarrow$ Bosonic Symmetric 2-tensor field

But if we take inverse of the above, then we get the same but with fermionic fields.

$$\Delta_{FP}[g_{ab}] = \int [\mathcal{D}\Omega][\mathcal{D}b_{ab}][\mathcal{D}c^a]\, e^{-\left[S^{(fermion)}_{weyl} + S^{(fermion)}_{diff}\right]}$$

$$S^{(fermion)}_{weyl} = \frac{1}{2\pi}\int d^2\sigma \sqrt{g}\, b_{ab} g^{ab} \Omega \longrightarrow \begin{matrix}\text{Weyl}\\ \text{Ghost}\end{matrix}$$

$$S^{(fermion)}_{diff} = \frac{1}{2\pi}\int d^2\sigma \sqrt{g}\, b_{ab} \nabla^a c^b \longrightarrow \begin{matrix}\text{b-c (diff)}\\ \text{Ghost}\end{matrix}$$

Thus Weyl invariance results in the Weyl ghost action where Ω is the fermionic Weyl ghost field which just acts as a lagrange multiplier to make the b-ghost 2-tensor field a traceless - Symmetric field. But untill then b_{ab} is just Symmetric field.

So in total there are Six ghost fields. Three degrees of freedom in b_{ab} tensor, Two components of c^a & the Ω ghost.

Then we shall integrate the Weyl ghost & then fix b_{ab} to be also traceless.

(Note: Normalizing constants are ignored)

Then the ghost action is,

$$S_{ghost} = \frac{1}{2\pi}\int d^2\sigma \sqrt{g}\, b_{ab} \nabla^a c^b \longrightarrow \begin{matrix} b\text{-}c \text{ (diff)} \\ \text{Ghost} \end{matrix}$$

(b_{ab}: traceless-Symmetric)

Then, the full quantum action is,

$$S = S_X + S_{g.f} + S_{ghost}$$

We can also choose a Specific gauge to work on by doing the B^{ab} integral & replace B^{ab} in the BRST transformation using the g_{ab}'s eq^n of motion.

Gauge Fixing procedure

Here, $F_{ab} = \hat{g}_{ab} - g_{ab}$

So on doing the B^{ab} integral we should replace all g_{ab} with $\hat{g}_{ab}$. But before that we shall vary the full action with respect to g_{ab} & then find the equation for B^{ab} & finally replace all g_{ab} with $\hat{g}_{ab}$.

$$S = S_x + S_{ghost} + S_{g.f}$$

Then, $$\frac{\delta S}{\delta g_{ab}} = \frac{\delta S_x}{\delta g_{ab}} + \frac{\delta S_{ghost}}{\delta g_{ab}} + \frac{\delta S_{g.f}}{\delta g_{ab}}$$

But the eqn of motion because of varying the metric is $\frac{\delta S}{\delta g^{ab}} = 0 \rightarrow$ Variational Principle

Also, $$\frac{\delta S_x}{\delta g_{ab}} = -\frac{\sqrt{g}}{4\pi} T_x^{ab} \longrightarrow$$ Matter Part of Energy momentum tensor

& $\frac{\delta S_{ghost}}{\delta g_{ab}} = -\frac{\sqrt{g}}{4\pi} T_g^{ab} \rightarrow$ Ghost Part of Energy momentum tensor

Then, $\frac{\delta S_{g.f}}{\delta g_{ab}} = \frac{i}{4\pi}\int d^2\sigma \frac{\delta}{\delta g_{ab}}\left(\sqrt{g}\, B^{cd} F_{cd}\right)$

$$\frac{\delta}{\delta g_{ab}}\left(\sqrt{g}\, B^{ab} F_{ab}\right) = \left(\frac{\delta\sqrt{g}}{\delta g_{ab}}\right)\left(B^{cd} F_{cd}\right) + \sqrt{g}\, B^{cd}\frac{\delta F_{cd}}{\delta g_{ab}}$$

$$= \frac{1}{2}\sqrt{g}\, g^{ab}\left(B^{cd} F_{cd}\right) - \sqrt{g}\, B^{cd}\partial_c^a \partial_d^b$$

$$= \sqrt{g}\left(\frac{1}{2} g^{ab} B^{cd} F_{cd} - B^{ab}\right)$$

$$\longrightarrow \frac{\delta S_{g.f}}{\delta g_{ab}} = \frac{i}{4\pi}\sqrt{g}\left(\frac{1}{2} g^{ab} B^{cd} F_{cd} - B^{ab}\right)$$

Then, $\frac{\delta S_X}{\delta g_{ab}} + \frac{\delta S_{ghost}}{\delta g_{ab}} + \frac{\delta S_{g.f}}{\delta g_{ab}} = 0$ becomes,

$$\frac{\sqrt{g}}{4\pi}\left[T_X^{ab} + T_g^{ab} - i\left(\frac{1}{2} g^{ab} B^{cd} F_{cd} - B^{ab}\right)\right] = 0$$

$$\longrightarrow \frac{i}{2} g^{ab} B^{cd} F_{cd} - i B^{ab} = T_X^{ab} + T_g^{ab}$$

But, if we do gauge fixing, then we get $F_{cd} = 0$.

Thus after fixing to any gauge,

$$\boxed{B^{ab} = i\,T_x^{ab} + i\,T_g^{ab}}$$

Bosonic String in Unit gauge

Here, we choose $\hat{g}_{ab} = \eta_{ab}$

So, $$S_x = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \eta^{ab}(\partial_a X^\mu)(\partial_b X_\mu)$$

& $$S_{ghost} = \frac{1}{2\pi}\int d^2\sigma\, b_{ab}\eta^{ad}\partial_d c^b$$

Then the full Path integral after fixing to unit gauge is,

$$\boxed{Z = \int [\mathcal{D}X^\mu][\mathcal{D}b_{ab}][\mathcal{D}c^a]\, e^{-(S_x + S_{ghost})}}$$

BRST Invariance of String action

The transformation of X^μ field is just the diffeomorphism with parameter '$-i\epsilon c^a$'

Also exactly in parallel to what we got for point particle, we get,

- $\delta_B X^\mu = i\epsilon c^a \partial_a X^\mu$
- $\delta_B b_{ab} = i\epsilon \left(T^X_{ab} + T^{(g)}_{ab} \right)$
- $\delta_B c^a = i\epsilon c^b \partial_b c^a$

This set of transformation keeps the action invariant & the double BRST transformation is zero upto terms that are proportional to eq^n of motion.

To check this let us first go to the complex coordinates $z = \sigma^1 + i\sigma^2$

& $\bar{z} = \sigma^1 - i\sigma^2$

Then the metric becomes $g_{zz} = g_{\bar{z}\bar{z}} = 0$

$$g_{\bar{z}z} = g_{z\bar{z}} = \frac{1}{2}$$

Then $$S_x = \frac{1}{2\pi\alpha'}\int d^2z (\partial_z X^\mu)(\partial_{\bar{z}} X_\mu)$$

& $$S_{ghost} = \frac{1}{2\pi}\int d^2z \left(b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right)$$

Since b_{ab} is symmetric & traceless,

$$\underline{b_{z\bar{z}} = b_{\bar{z}z} = 0}$$

This also results in $T^X_{z\bar{z}} = T^X_{\bar{z}z} = 0$

& $$T^{(g)}_{z\bar{z}} = T^{(g)}_{\bar{z}z} = 0$$

The eqn. of motions set $\partial_z X^\mu \longrightarrow$ Holomorphic

& $\partial_{\bar{z}} X^\mu \longrightarrow$ anti-Holomorphic

For the ghosts b_{zz}, $c^z \longrightarrow$ Holomorphic

and $b_{\bar{z}\bar{z}}$, $c^{\bar{z}} \longrightarrow$ anti-Holomorphic

Thus $b_{zz} \longrightarrow b(z)$ & $b_{\bar{z}\bar{z}} \longrightarrow \tilde{b}(\bar{z})$

Similarly, $c^z \longrightarrow c(z)$ & $c^{\bar{z}} \longrightarrow \tilde{c}(\bar{z})$

Also the conservation of Energy-momentum tensors result in $\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0$

For both, matter & ghost:

$T_{zz} \equiv T(z) \longrightarrow$ Holomorphic

$T_{\bar{z}\bar{z}} \equiv \tilde{T}(\bar{z}) \longrightarrow$ Anti-Holomorphic

In the complex coordinates the on-shell BRST transform becomes,

- $\delta_B X^\mu = i\epsilon \left(c(z) \partial_z X^\mu + \tilde{c}(\bar{z}) \partial_{\bar{z}} X^\mu \right)$
- $\delta_B b(z) = i\epsilon \left(T^X(z) + T^{(g)}(z) \right)$
- $\delta_B \tilde{b}(\bar{z}) = i\epsilon \left(\tilde{T}^X(\bar{z}) + \tilde{T}^{(g)}(\bar{z}) \right)$
- $\delta_B c(z) = i\epsilon \, c(z) \partial_z c(z)$
- $\delta_B \tilde{c}(\bar{z}) = i\epsilon \, \tilde{c}(\bar{z}) \partial_{\bar{z}} \tilde{c}(\bar{z})$

Now, we shall check the BRST transform of S_X & S_{ghost}.

$$\delta_B S_X = \frac{1}{2\pi\alpha'}\int d^2z \left[\partial_z(\delta_B X^\mu)\right](\partial_{\bar z} X_\mu) + \frac{1}{4\pi\alpha'}\int d^2z (\partial_z X^\mu)\left[\partial_{\bar z}(\delta_B X_\mu)\right]$$

$$= \frac{i\epsilon}{2\pi\alpha'}\int d^2z \left[\partial_z\left(c(z)\partial_z X^\mu + \tilde c(\bar z)\partial_{\bar z} X^\mu\right)\right](\partial_{\bar z} X_\mu) + \frac{i\epsilon}{2\pi\alpha'}\int d^2z (\partial_z X^\mu)\left[\partial_{\bar z}\left(c(z)\partial_z X^\mu + \tilde c(\bar z)\partial_{\bar z} X^\mu\right)\right]$$

$$= \frac{i\epsilon}{2\pi\alpha'}\int d^2z \left[(\partial_z c(z))\partial_z X^\mu + c(z)(\partial_z^2 X^\mu)\right](\partial_{\bar z} X_\mu) + \frac{i\epsilon}{2\pi\alpha'}\int d^2z (\partial_z X_\mu)\left[(\partial_{\bar z}\tilde c(\bar z))\partial_{\bar z} X^\mu + \tilde c(\bar z)(\partial_{\bar z}^2 X^\mu)\right]$$

$$= \frac{i\epsilon}{2\pi\alpha'}\int d^2z \left[\begin{array}{l}(\partial_z c(z))(\partial_z X^\mu)(\partial_{\bar z} X_\mu) \\ \qquad + c(z)(\partial_z^2 X^\mu)(\partial_{\bar z} X_\mu) \\ + (\partial_z X_\mu)(\partial_{\bar z} X^\mu)(\partial_{\bar z}\tilde c(\bar z)) \\ \qquad + \tilde c(\bar z)(\partial_{\bar z}^2 X^\mu)(\partial_z X_\mu)\end{array}\right]$$

$$= \frac{i\epsilon}{2\pi\alpha'}\int d^2z \Big[\big[\partial_z c(z) + \partial_{\bar z}\tilde c(\bar z)\big](\partial_z X^\mu)(\partial_{\bar z} X_\mu) + \tilde c(\bar z)(\partial_{\bar z}^2 X^\mu)(\partial_z X_\mu) + c(z)(\partial_z^2 X^\mu)(\partial_{\bar z} X_\mu)\Big]$$

Now we shall integrate by parts

$$= \frac{(-i\epsilon)}{2\pi\alpha'}\int d^2z \Big[(\partial_z X^\mu) X_\mu\, \partial_{\bar z}\big[\partial_z c(z) + \partial_{\bar z}\tilde c(\bar z)\big] + \partial_z\underbrace{\big[\tilde c(\bar z)(\partial_{\bar z}^2 X^\mu)\big]}_{0} X_\mu + \partial_{\bar z}\underbrace{\big[c(z)(\partial_z^2 X^\mu)\big]}_{0} X_\mu\Big]$$

$$= \frac{(-i\epsilon)}{2\pi\alpha'}\int d^2z\, (\partial_z X^\mu) X_\mu \underbrace{\partial_{\bar z}^2 \tilde c(\bar z)}_{\text{Again Integrate by parts}}$$

$$= \frac{i\epsilon}{2\pi\alpha'}\int d^2z\, \partial_{\bar z}\big[(\partial_z X^\mu) X_\mu\big]\, \partial_{\bar z}\tilde c(\bar z)$$

$$= \frac{i\epsilon}{2\pi\alpha'}\int d^2z\, \underbrace{(\partial_z X^\mu)(\partial_{\bar z} X_\mu)}_{\text{Again Integrate by parts}}(\partial_{\bar z}\tilde c(\bar z))$$

$$= \frac{(-i\epsilon)}{2\pi\alpha'}\int d^2z\, X^\mu \partial_z\Big[\underbrace{(\partial_{\bar z} X_\mu)}_{0}(\partial_{\bar z}\tilde c(\bar z))\Big] = 0$$

$$\Longrightarrow \underline{\underline{\delta_B S_X = 0}}$$

Consider the Holomorphic ghost action

$$S^{(z)}_{ghost} = \frac{1}{2\pi}\int d^2z \left(b(z)\,\partial_{\bar z} c(z)\right)$$

$$\delta_B S^{(z)}_{ghost} = \frac{1}{2\pi}\int d^2z \left(\delta_B b(z)\right)\partial_{\bar z} c(z) + \frac{1}{2\pi}\int d^2z\, b(z)\,\partial_{\bar z}\left(\delta_B c(z)\right)$$

$$= \frac{1}{2\pi}\int d^2z \left(i\epsilon\left(T^X(z) + T^{(g)}(z)\right)\right)\partial_{\bar z} c(z)$$

Int. by Parts $\swarrow$ $+ \frac{1}{2\pi}\int d^2z\, b(z)\,\partial_{\bar z}\underbrace{\left(i\epsilon\, c(z)\,\partial_z c(z)\right)}_{0}$

$$= \frac{i\epsilon}{2\pi}\int d^2z\, \partial_{\bar z}\underbrace{\left(T^X(z) + T^{(g)}(z)\right)}_{0}\, c(z)$$

$$\longrightarrow \underline{\delta_B S^{(z)}_{ghost} = 0}$$

Consider the anti-Holomorphic ghost action

$$S^{(\bar{z})}_{ghost} = \frac{1}{2\pi}\int d^2z \left(\tilde{b}(\bar{z})\,\partial_z \tilde{C}(\bar{z})\right)$$

$$\delta_B S^{(\bar{z})}_{ghost} = \frac{1}{2\pi}\int d^2z \left(\delta_B \tilde{b}(\bar{z})\right)\partial_z \tilde{C}(\bar{z}) + \frac{1}{2\pi}\int d^2z\, \tilde{b}(\bar{z})\,\partial_z\left(\delta_B \tilde{C}(\bar{z})\right)$$

$$= \frac{1}{2\pi}\int d^2z \left(i\epsilon\left(\tilde{T}^X(\bar{z}) + \tilde{T}^{(g)}(\bar{z})\right)\right)\partial_z \tilde{C}(\bar{z})$$

Int. by Parts ↙

$$+ \frac{1}{2\pi}\int d^2z\, \tilde{b}(\bar{z})\,\partial_z\underbrace{\left(i\epsilon\,\tilde{C}(\bar{z})\,\partial_z \tilde{C}(\bar{z})\right)}_{0}$$

$$= \frac{i\epsilon}{2\pi}\int d^2z\, \partial_z\underbrace{\left(\tilde{T}^X(\bar{z}) + \tilde{T}^{(g)}(\bar{z})\right)}_{0}\tilde{C}(\bar{z})$$

$$\longrightarrow \underline{\delta_B S^{(\bar{z})}_{ghost} = 0}$$

Thus the full String action in unit gauge is on-shell BRST invariant.

BRST current for Bosonic String

By the usual Noether Procedure the variation of the Lagrangian utmost has to be proportional to a total derivative under a Symmetry transformation.

Let us first find the BRST variation of the full Lagrangian. In unit gauge, in complex coordinates it is given by,

$$\mathcal{L} = \frac{1}{2\pi\alpha'}(\partial_z X^\mu)(\partial_{\bar{z}} X_\mu) + \frac{1}{2\pi}\left(b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right)$$

$$\delta_B \mathcal{L} = \frac{1}{2\pi\alpha'}\delta_B\left[(\partial_z X^\mu)(\partial_{\bar{z}} X_\mu)\right]$$

$$+\frac{1}{2\pi}\delta_B\left(b_{zz}\partial_{\bar{z}} c^z\right) + \frac{1}{2\pi}\delta_B\left(b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right)$$

$$= \frac{1}{2\pi\alpha'}\left[(\partial_z X^\mu)\partial_{\bar{z}}(\delta_B X_\mu) + (\partial_{\bar{z}} X_\mu)\partial_z(\delta_B X_\mu)\right]$$

$$+\frac{1}{2\pi}\left[(\delta_B b_{zz})\partial_{\bar{z}} c^z + b_{zz}\partial_{\bar{z}}(\delta_B c^z)\right]$$

$$+\frac{1}{2\pi}\left[(\delta_B b_{\bar{z}\bar{z}})\partial_z c^{\bar{z}} + b_{\bar{z}\bar{z}}\partial_z(\delta_B c^{\bar{z}})\right]$$

(Use transformation formula & eq^n of motions)

$$= \frac{1}{2\pi\alpha'}(\partial_z X^\mu)\partial_{\bar z}\left[i\epsilon\left(c(z)\partial_z X^\mu + \tilde c(\bar z)\partial_{\bar z} X^\mu\right)\right]$$

$$+ \frac{1}{2\pi\alpha}(\partial_{\bar z} X_\mu)\partial_z\left[i\epsilon\left(c(z)\partial_z X^\mu + \tilde c(\bar z)\partial_{\bar z} X^\mu\right)\right]$$

$$+\frac{1}{2\pi}\left[(\delta_B b(z))\partial_{\bar z} c(z) + b(z)\partial_{\bar z}\left(i\epsilon c(z)\partial_z c(z)\right)\right]$$

$$+\frac{1}{2\pi}\left[(\delta_B \tilde b(\bar z))\partial_z \tilde c(\bar z) + \tilde b(\bar z)\partial_z\left(i\epsilon \tilde c(\bar z)\partial_{\bar z}\tilde c(\bar z)\right)\right]$$

$$= \frac{i\epsilon}{2\pi\alpha'}\left[(\partial_z X^\mu)\partial_{\bar z}\left(\tilde c(\bar z)\partial_{\bar z} X^\mu\right) + (\partial_{\bar z} X_\mu)\partial_z\left(c(z)\partial_z X^\mu\right)\right]$$

$$= \frac{i\epsilon}{2\pi\alpha'}\left(\partial_{\bar z}\left[\tilde c(\bar z)(\partial_z X^\mu)(\partial_{\bar z} X^\mu)\right] + \partial_z\left[c(z)(\partial_{\bar z} X_\mu)(\partial_z X^\mu)\right]\right)$$

$$= i\epsilon\left(\partial_{\bar z}\left[\tilde c(\bar z)\mathcal{L}_X\right] + \partial_z\left[c(z)\mathcal{L}_X\right]\right)$$

$$\longrightarrow \underline{\delta_B \mathcal{L} = \partial_a\left(i\epsilon c^a \mathcal{L}_X\right)} \longrightarrow \text{Total derivative}$$

where

$$\mathcal{L}_X = \frac{1}{4\pi\alpha'}\eta^{ab}(\partial_a X^\mu)(\partial_b X_\mu) = \frac{1}{2\pi\alpha'}(\partial_z X^\mu)(\partial_{\bar z} X^\mu)$$

But as $\mathcal{L}$ depends on X^μ, $b(z)$, $\tilde{b}(\bar{z})$ $c(z)$, $\tilde{c}(\bar{z})$ & their derivatives

$$\delta_B \mathcal{L} = \delta_B X^\mu \frac{\partial \mathcal{L}}{\partial X^\mu} + \delta_B(\partial_a X^\mu) \frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}$$

$$+ \delta_B c^a \frac{\partial \mathcal{L}}{\partial c^a} + \delta_B(\partial_b c^a) \frac{\partial \mathcal{L}}{\partial(\partial_b c^a)}$$

$$+ \delta_B b_{ab} \frac{\partial \mathcal{L}}{\partial b_{ab}} + \delta_B(\partial_d b_{ab}) \frac{\partial \mathcal{L}}{\partial(\partial_d b_{ab})}$$

$$= (\delta_B X^\mu)\left(\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_a\left[\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}\right]\right) + \partial_a\left[(\delta_B X^\mu)\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)}\right]$$

$$+ (\delta_B c^a)\left(\frac{\partial \mathcal{L}}{\partial c^a} - \partial_b\left[\frac{\partial \mathcal{L}}{\partial(\partial_b c^a)}\right]\right) + \partial_a\left[(\delta_B c^b)\frac{\partial \mathcal{L}}{\partial(\partial_a c^b)}\right]$$

$$+ (\delta_B b_{ab})\underbrace{\left(\frac{\partial \mathcal{L}}{\partial b_{ab}} - \partial_d\left[\frac{\partial \mathcal{L}}{\partial(\partial_d b_{ab})}\right]\right)}_{\text{Vanish by Eq}^{n}\text{. of motion}} + \partial_a\left[(\delta_B b_{cd})\frac{\partial \mathcal{L}}{\partial(\partial_a b_{cd})}\right]$$

$$= \partial_a\left[(\partial_B \chi^\mu)\frac{\partial \mathcal{L}}{\partial(\partial_a \chi^\mu)}\right] + \partial_a\left[(\partial_B c^b)\frac{\partial \mathcal{L}}{\partial(\partial_a c^b)}\right]$$

$$+ \partial_a\left[(\partial_B b_{cd})\frac{\partial \mathcal{L}}{\partial(\partial_a b_{cd})}\right]$$

$$\longrightarrow \partial_B \mathcal{L} = \partial_a\left[\begin{array}{r}(\partial_B \chi^\mu)\frac{\partial \mathcal{L}}{\partial(\partial_a \chi^\mu)} + (\partial_B c^b)\frac{\partial \mathcal{L}}{\partial(\partial_a c^b)} \\ + (\partial_B b_{cd})\frac{\partial \mathcal{L}}{\partial(\partial_a b_{cd})}\end{array}\right]$$

$$\longrightarrow \partial_a(i\epsilon c^a \mathcal{L}_\chi) =$$

$$= \partial_a\left[(\partial_B \chi^\mu)\frac{\partial \mathcal{L}}{\partial(\partial_a \chi^\mu)} + (\partial_B c^b)\frac{\partial \mathcal{L}}{\partial(\partial_a c^b)} + (\partial_B b_{cd})\frac{\partial \mathcal{L}}{\partial(\partial_a b_{cd})}\right]$$

$$\longrightarrow$$

$$\partial_a\left[\begin{array}{r}(\partial_B \chi^\mu)\frac{\partial \mathcal{L}}{\partial(\partial_a \chi^\mu)} + (\partial_B c^b)\frac{\partial \mathcal{L}}{\partial(\partial_a c^b)} + (\partial_B b_{cd})\frac{\partial \mathcal{L}}{\partial(\partial_a b_{cd})} \\ - i\epsilon c^a \mathcal{L}_\chi\end{array}\right] = 0$$

Thus the Noether Current J_B^a is given by

$$\frac{\epsilon}{2\pi i} J_B^a = (\delta_B X^\mu)\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} + (\delta_B c^b)\frac{\partial \mathcal{L}}{\partial(\partial_a c^b)} + (\delta_B b_{cd})\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_a b_{cd})}}_{0}$$

$$-i\epsilon c^a \mathcal{L}_X$$

where,

$$\mathcal{L} = \frac{1}{2\pi\alpha'}(\partial_z X^\mu)(\partial_{\bar{z}} X_\mu) + \frac{1}{2\pi}\left(b_{zz}\partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_z X^\mu)} = \frac{1}{2\pi\alpha'}(\partial_{\bar{z}} X_\mu) \qquad \frac{\partial \mathcal{L}}{\partial(\partial_{\bar{z}} X^\mu)} = \frac{1}{2\pi\alpha'}(\partial_z X_\mu)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_z c^{\bar{z}})} = -\frac{1}{2\pi} b_{\bar{z}\bar{z}} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_{\bar{z}} c^z)} = \frac{-1}{2\pi} b_{zz}$$

↳ -ve sign because of grassmann oddness ↲

$$\frac{\partial \mathcal{L}}{\partial(\partial_z c^z)} = 0 \qquad \frac{\partial \mathcal{L}}{\partial(\partial_{\bar{z}} c^{\bar{z}})} = 0$$

To get one component

$$\rightarrow \frac{\epsilon}{2\pi i} J_B^z = (\delta_B X^\mu)\frac{1}{2\pi\alpha'}(\partial_{\bar{z}} X_\mu) + (\delta_B c^{\bar{z}})\frac{\partial \mathcal{L}}{\partial(\partial_z c^{\bar{z}})}$$

$$-i\epsilon c^z \frac{1}{2\pi\alpha'}(\partial_z X^\mu)(\partial_{\bar{z}} X^\mu)$$

$$\rightarrow \frac{\epsilon}{2\pi i} J_B^z = i\epsilon \left(c(z)\partial_z X^\mu + \tilde{c}(\bar{z})\partial_{\bar{z}} X^\mu \right) \frac{1}{2\pi\alpha'} \left(\partial_{\bar{z}} X_\mu \right)$$

$$- i\epsilon \tilde{c}(\bar{z})\partial_{\bar{z}}\tilde{c}(\bar{z}) \frac{1}{2\pi} \tilde{b}(\bar{z})$$

$$- i\epsilon c^z \frac{1}{2\pi\alpha'} \left(\partial_z X^\mu \right)\left(\partial_{\bar{z}} X^\mu \right)$$

$$\rightarrow J_B^z = -\left(c(z)\partial_z X^\mu + \tilde{c}(\bar{z})\partial_{\bar{z}} X^\mu \right) \frac{1}{\alpha'} \left(\partial_{\bar{z}} X_\mu \right)$$

$$+ \tilde{c}(\bar{z})\partial_{\bar{z}}\tilde{c}(\bar{z})\tilde{b}(\bar{z}) + c(z)\frac{1}{\alpha'}\left(\partial_z X^\mu \right)\left(\partial_{\bar{z}} X^\mu \right)$$

$$\rightarrow J_B^z = -\tilde{c}(\bar{z})\frac{1}{\alpha'}\left(\partial_{\bar{z}} X^\mu \right)\left(\partial_{\bar{z}} X_\mu \right) + \tilde{c}(\bar{z})\partial_{\bar{z}}\tilde{c}(\bar{z})\tilde{b}(\bar{z})$$

Thus $\partial_z J_B^z = 0$ (on-shell)

$\Longrightarrow J_B^z \equiv \tilde{J}_B(\bar{z})$ is the anti-Holomorphic component given by,

$$\tilde{J}_B(\bar{z}) = -\tilde{c}(\bar{z})\frac{1}{\alpha'}\left(\partial_{\bar{z}} X^\mu \right)\left(\partial_{\bar{z}} X_\mu \right) + \tilde{c}(\bar{z})\left(\partial_{\bar{z}}\tilde{c}(\bar{z}) \right)\tilde{b}(\bar{z})$$

For another component we get,

$$\rightarrow \frac{\epsilon}{2\pi i} J_B^{\bar{z}} = (\delta_B X^\mu)\frac{1}{2\pi\alpha'}(\partial_z X_\mu) - (\delta_B c^z)\frac{1}{2\pi} b_{zz}$$

$$- i\epsilon c^{\bar{z}} \frac{1}{2\pi\alpha'}(\partial_z X^\mu)(\partial_{\bar{z}} X^\mu)$$

$$= i\epsilon\left(c(z)\partial_z X^\mu + \tilde{c}(\bar{z})\partial_{\bar{z}} X^\mu\right)\frac{1}{2\pi\alpha'}(\partial_z X_\mu)$$

$$- i\epsilon\, c(z)\left(\partial_z c(z)\right)\frac{1}{2\pi} b(z) - \frac{i\epsilon\,\tilde{c}(\bar{z})}{2\pi\alpha'}(\partial_z X^\mu)(\partial_{\bar{z}} X^\mu)$$

$$\rightarrow J_B^{\bar{z}} = -\frac{1}{\alpha'} c(z)(\partial_z X^\mu)(\partial_z X_\mu) + c(z)(\partial_z c(z)) b(z)$$

Thus $\partial_{\bar{z}} J_B^{\bar{z}} = 0$ (on-shell)

$\Longrightarrow J_B^{\bar{z}} \equiv J_B(z)$ is the Holomorphic component given by,

$$J_B(z) = -c(z)\frac{1}{\alpha'}(\partial_z X^\mu)(\partial_z X_\mu) + c(z)(\partial_z c(z)) b(z)$$

In terms of Energy momentum tensors we have,

$$T^{x}(z) = -\frac{1}{\alpha'}(\partial_z X^{\mu})(\partial_z X_{\mu})$$

$$\tilde{T}^{x}(\bar{z}) = -\frac{1}{\alpha'}(\partial_{\bar{z}} X^{\mu})(\partial_{\bar{z}} X_{\mu})$$

$$T^{g}(z) = 2[\partial_z c(z)]b(z) - [\partial_z b(z)]c(z)$$

$$\tilde{T}^{g}(\bar{z}) = 2[\partial_{\bar{z}} \tilde{c}(\bar{z})]\tilde{b}(\bar{z}) - [\partial_{\bar{z}} \tilde{b}(\bar{z})]\tilde{c}(\bar{z})$$

$$\Longrightarrow c(z)T^{g}(z) = 2c(z)(\partial_z c(z))b(z)$$

$$\& \quad \tilde{c}(\bar{z})\tilde{T}^{g}(\bar{z}) = 2\tilde{c}(\bar{z})(\partial_{\bar{z}} \tilde{c}(\bar{z}))\tilde{b}(\bar{z})$$

$$\Longrightarrow \underline{J_B(z) = c(z)T^{x}(z) + \frac{1}{2}c(z)T^{g}(z)}$$

$$\& \quad \underline{\tilde{J}_B(\bar{z}) = \tilde{c}(\bar{z})\tilde{T}^{x}(\bar{z}) + \frac{1}{2}\tilde{c}(\bar{z})\tilde{T}^{g}(\bar{z})}$$

We can add a total derivative term to the above current without affecting the BRST charge operator, but it can make the BRST current a Tensor operator.

The extra term we add is $\frac{3}{2}\partial_a(\partial_b c^b)'$.

So, $J_B(z) \longrightarrow J_B(z) + \frac{3}{2}\partial_z(\partial_z c^z + \underbrace{\partial_{\bar{z}} c^{\bar{z}}}_{0})$

& $\tilde{J}_B(\bar{z}) \longrightarrow \tilde{J}_B(\bar{z}) + \frac{3}{2}\partial_{\bar{z}}(\underbrace{\partial_z c^z}_{0} + \partial_{\bar{z}} c^{\bar{z}})$

For now let us write the BRST current as follows

$$J_B(z) = c(z)T^X(z) + \frac{1}{2}c(z)T^g(z) + \gamma\partial_z^2 c(z)$$

$$\tilde{J}_B(\bar{z}) = \tilde{c}(\bar{z})\tilde{T}^X(\bar{z}) + \frac{1}{2}\tilde{c}(\bar{z})\tilde{T}^g(\bar{z}) + \tilde{\gamma}\partial_{\bar{z}}^2\tilde{c}(\bar{z})$$

where we shall fix 'γ' & '$\tilde{\gamma}$' later.

OPE of various field operators with $:J_B(z):$

The OPE of various field operators with $:J_B(z):$ is given by,

- $$:J_B(z)::c(w): = T^x(z)\,\overbrace{c(z)c(w)} + \frac{1}{2}:c(z)T^g(z)::c(w): + \gamma\,\partial_z^2\big(\overbrace{c(z)c(w)}\big)$$

$$\sim \frac{1}{2}c(z)\,\overbrace{:T^g(z)::c(w):}$$

$$\sim \frac{1}{2}c(z)\left(\frac{-c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{(z-w)}\right)$$

$$\sim \frac{1}{2}\Big(c(w) + (z-w)\partial_w c(w)\Big)\left(\frac{-c(w)}{(z-w)^2} + \frac{\partial_w c(w)}{(z-w)}\right)$$

$$\sim \frac{1}{2}\left(\overbrace{\frac{-c(w)c(w)}{(z-w)^2}} + \frac{c(w)\,\partial_w c(w)}{(z-w)} + \frac{c(w)\partial_w c(w)}{(z-w)}\right)$$

$$\Rightarrow \quad :J_B(z)::c(w): \sim \frac{c(w)\,\partial_w c(w)}{(z-w)}$$

$:\widetilde{J}_B(z)::c(w):$ has no singular points

Then the BRST transformation of the c-ghost field is given by the coefficient of the simple pole.

$$\delta_B c(w) = \epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi} :J_B(z)::c(w):$$

$$\delta_B c(w) = i\epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi i} \frac{c(w)\,\partial_w c(w)}{(z-w)}$$

$$\Longrightarrow \underline{\delta_B c(w) = i\epsilon c(w)\,\partial_w c(w)}$$

Similarly, • $\underline{\delta_B \tilde{c}(\bar{z}) = i\epsilon \tilde{c}(\bar{z})\,\partial_{\bar{z}} \tilde{c}(\bar{z})}$

• $:J_B(z)::b(w): = :T^X(z): \overbrace{c(z)b(w)}$

$$+ \frac{1}{2}:c(z)T^g(z)::b(w): + \partial_z^2\left(\overbrace{c(z)b(w)}\right)$$

$$\sim \frac{:T^X(z):}{(z-w)} + \frac{1}{2}\frac{:T^g(z):}{(z-w)} + \frac{1}{2}:c(z)\left(\overbrace{:T^g(z)::b(w):}\right):$$

$$+ \gamma\,\partial_z^2\left(\frac{1}{z-w}\right)$$

$$\sim \frac{:T^x(z):}{(z-w)} + \frac{1}{2}\frac{:T^g(z):}{(z-w)} + \frac{2\gamma}{(z-w)^3}$$

$$+\frac{1}{2}:\left(c(w)+(z-w)\partial_w c(w)\right)\left(\frac{2b(w)}{(z-w)^2}+\frac{\partial_w b(w)}{(z-w)}\right):$$

$$\sim \frac{:T^x(z):}{(z-w)} + \frac{1}{2}\frac{:T^g(z):}{(z-w)} + \frac{2\gamma}{(z-w)^3}$$

$$+\frac{:c(w)b(w):}{(z-w)^2}+\frac{1}{2}\frac{:c(w)(\partial_w b(w)):}{(z-w)}+\frac{:(\partial_w c(w))b(w):}{(z-w)}$$

$$\sim \frac{:T^x(w):}{(z-w)} + \frac{1}{2}\frac{:T^g(w):}{(z-w)} + \frac{2\gamma}{(z-w)^3} + \frac{:c(w)b(w):}{(z-w)^2}$$

$$+\frac{1}{2}\left(\frac{2:(\partial_w c(w))b(w): - :(\partial_w b(w))c(w):}{(z-w)}\right)$$

Since $2:(\partial_w c(w))b(w): - :(\partial_w b(w))c(w): = :T^g(z):$

$$\sim \frac{2\gamma}{(z-w)^3} + \frac{:c(w)b(w):}{(z-w)^2} + \frac{:T^x(w):+:T^g(w):}{(z-w)}$$

$$\Rightarrow :J_B(z)::b(w): \sim \frac{2\gamma}{(z-w)^3} + \frac{:c(w)b(w):}{(z-w)^2}$$

$$+ \frac{:T^x(w): + :T^g(w):}{(z-w)}$$

$:\tilde{J}_B(\bar{z})::b(w):$ has no singular points

Then the BRST transformation of the b-ghost field is given by the coefficient of the simple Pole.

$$\delta_B b(w) = \epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi} :J_B(z)::b(w):$$

$$\delta_B b(w) = i\epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi i} \frac{:T^x(w): + :T^g(w):}{(z-w)}$$

$$\underline{\delta_B b(w) = i\epsilon \left(:T^x(w): + :T^g(w):\right)}$$

Similarly, $\bullet\ \underline{\delta_B \tilde{b}(\bar{z}) = i\epsilon \left(\tilde{T}^x(\bar{z}) + \tilde{T}^{(g)}(\bar{z})\right)}$

- $:J_B(z)::X^\mu(\omega,\bar{\omega}):\; = :c(z)T^X(z)::X^\mu(\omega,\bar{\omega}):$

$= :c(z)\overbrace{(:T^X(z)::\underbrace{X^\mu(\omega,\bar{\omega})}:)}:$

Primary of weight '0'

$\Rightarrow :J_B(z)::X^\mu(\omega,\bar{\omega}):\sim \dfrac{c(\omega)\,\partial X^\mu(\omega)}{(z-\omega)}$

Similarly $:\tilde{J}_B(\bar{z})::X^\mu(\omega,\bar{\omega}):\sim \dfrac{\tilde{c}(\bar{\omega})\,\bar{\partial} X^\mu(\bar{\omega})}{(\bar{z}-\bar{\omega})}$

Since the OPE of X^μ is non zero with both J_B & $\tilde{J}_B$, the complete BRST transformation of X^μ is,

$$\delta_B X^\mu(\omega,\bar{\omega}) = i\epsilon \oint_{C:|z-\omega|=\text{const.}} \frac{dz}{2\pi i} :J_B(z)::X^\mu(\omega,\bar{\omega}):$$

$$+ i\epsilon \oint_{\bar{C}:|\bar{z}-\bar{\omega}|=\text{const.}} \frac{d\bar{z}}{2\pi i} :\tilde{J}_B(\bar{z})::X^\mu(\omega,\bar{\omega}):$$

$$\Rightarrow \delta_B X^\mu(w,\bar{w}) = i\epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi i} \frac{c(w)\partial X^\mu(w)}{(z-w)}$$

$$+ i\epsilon \oint_{\bar{C}:|\bar{z}-\bar{w}|=\text{const.}} \frac{d\bar{z}}{2\pi i} \frac{\tilde{c}(\bar{w})\bar{\partial} X^\mu(\bar{w})}{(\bar{z}-\bar{w})}$$

$$\Rightarrow \underline{\delta_B X^\mu(w,\bar{w}) = i\epsilon\left(c(w)\partial X^\mu(w) + \tilde{c}(\bar{w})\bar{\partial} X^\mu(\bar{w})\right)}$$

- For a Primary operator $\theta^X(w,\bar{w})$ with weights $(h,\tilde{h})$ we have,

$$:J_B(z)::\theta^X(w,\bar{w}): \sim :c(z)\overbrace{(:T^X(z)::\theta^X(w,\bar{w}):)}:$$

$$\sim :\left(c(w)+(z-w)\partial_w c(w)\right)\left(\frac{h\theta^X(w,\bar{w})}{(z-w)^2} + \frac{\partial\theta^X(w,\bar{w})}{(z-w)}\right):$$

$$\sim \frac{h c(w)\theta^X(w,\bar{w})}{(z-w)^2} + \frac{c(w)(\partial\theta^X(w,\bar{w}))}{(z-w)}$$

$$+ \frac{h(\partial c(w))\theta^X(w,\bar{w})}{(z-w)}$$

Similarly,

$$:\tilde{J}_B(\bar{z}):\theta^x(w,\bar{w}): \sim \frac{\tilde{h}\tilde{c}(\bar{w})\theta^x(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\tilde{c}(\bar{w})(\bar{\partial}\theta^x(w,\bar{w}))}{(\bar{z}-\bar{w})} + \frac{\tilde{h}(\bar{\partial}\tilde{c}(\bar{w}))\theta^x(w,\bar{w})}{(\bar{z}-\bar{w})}$$

Since the OPE of θ^x is non zero with both J_B & $\tilde{J}_B$, the complete BRST transformation of θ^x is,

$$\delta_B\theta^x(w,\bar{w}) = i\epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi i} :J_B(z):\theta^x(w,\bar{w}): + i\epsilon \oint_{\bar{C}:|\bar{z}-\bar{w}|=\text{const.}} \frac{d\bar{z}}{2\pi i} :\tilde{J}_B(\bar{z}):\theta^x(w,\bar{w}):$$

Thus,

$$\delta_B\theta^x(w,\bar{w}) = i\epsilon\Big(c(w)(\partial\theta^x(w,\bar{w})) + h(\partial c(w))\theta^x(w,\bar{w}) + \tilde{c}(\bar{w})(\bar{\partial}\theta^x(w,\bar{w})) + \tilde{h}(\bar{\partial}\tilde{c}(\bar{w}))\theta^x(w,\bar{w})\Big)$$

For any operator $\mathcal{A}$

$$\delta_B \mathcal{A} = i\epsilon \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi i} :J_B(z):: \underbrace{\mathcal{A}(w,\bar{w})}_{\text{OPE}}:$$

$$+ i\epsilon \oint_{\bar{C}:|\bar{z}-\bar{w}|=\text{const.}} \frac{d\bar{z}}{2\pi i} :\tilde{J}_B(\bar{z}):: \underbrace{\mathcal{A}(w,\bar{w})}_{\text{OPE}}:$$

So, we get,

$$\boxed{\delta_B \mathcal{A} = i\epsilon \operatorname*{Res}_{z\to w} \left[:J_B(z):: \mathcal{A}(w,\bar{w}):\right] + i\epsilon \operatorname*{\overline{Res}}_{\bar{z}\to\bar{w}} \left[:\tilde{J}_B(\bar{z}):: \mathcal{A}(w,\bar{w}):\right]}$$

Also, we can write,

$$\delta_B \mathcal{A} = \frac{i\epsilon}{2\pi i} \oint_{C:|z-w|=\text{const.}} \Big(dz :J_B(z): - d\bar{z} :\tilde{J}_B(\bar{z}):\Big) \underbrace{:\mathcal{A}(w,\bar{w}):}_{\text{OPE}}$$

We shall define the BRST charge operator as

$$\hat{Q}_B = \frac{1}{2\pi i} \oint_{C:|z|=\text{const.}} \Big(dz :J_B(z): - d\bar{z} :\tilde{J}_B(\bar{z}):\Big)$$

where the above contour encloses the origin.

Then, $$\boxed{\delta_B \mathcal{A} = i\epsilon\, \hat{Q}_B \bullet \mathcal{A}(\omega,\bar{\omega})}$$

Here '$\bullet$' tells us define $\hat{Q}_B \bullet$ using a contour integral which encloses the location where the local operator is present

$\delta_B \mathcal{A}$ as a Graded Commutator

- If '$\mathcal{A}$' is Grassmann odd operator, then by the contour argument what we did in section 2.6, we can relate $\delta_B \mathcal{A}(\omega,\bar{\omega})$ with anti-commutator of $\hat{Q}_B$ with '$\mathcal{A}(\omega,\bar{\omega})$'. Let,

$$\hat{Q}_B(C_1) = \frac{1}{2\pi i}\oint_{C_1}\left(dz_1 :J_B(z_1): - d\bar{z}_1 :\tilde{J}_B(\bar{z}_1):\right)$$

$$\hat{Q}_B(C_2) = \frac{1}{2\pi i}\oint_{C_2}\left(dz_2 :J_B(z_2): - d\bar{z}_2 :\tilde{J}_B(\bar{z}_2):\right)$$

where $|z_1| > |\omega| > |z_2|$.

Since the BRST current is conserved, the charge operator is independent of contour.

$$\rightarrow \{\hat{Q}_B, \mathcal{A}\} = \hat{Q}_B(C_1)\mathcal{A}(\omega,\bar{\omega}) + \mathcal{A}(\omega,\bar{\omega})\hat{Q}_B(C_2)$$

$$= \frac{1}{2\pi i}\oint_{C_1}\left(dz_1 :J_B(z_1): - d\bar{z}_1 :\tilde{J}_B(\bar{z}_1):\right):\mathcal{A}(\omega,\bar{\omega}):$$

$$+ \frac{1}{2\pi i}\oint_{C_2} :\mathcal{A}(\omega,\bar{\omega}):\left(dz_2 :J_B(z_2): - d\bar{z}_2 :\tilde{J}_B(\bar{z}_2):\right)$$

$$= \frac{1}{2\pi i}\left(\oint_{C_1} dz_1 \overbrace{:J_B(z_1)::\mathcal{A}(\omega,\bar{\omega}):}^{\text{OPE}} + \oint_{C_2} dz_2 :\mathcal{A}(\omega,\bar{\omega})\underbrace{::J_B(z_2):}_{\text{OPE}}\right)$$

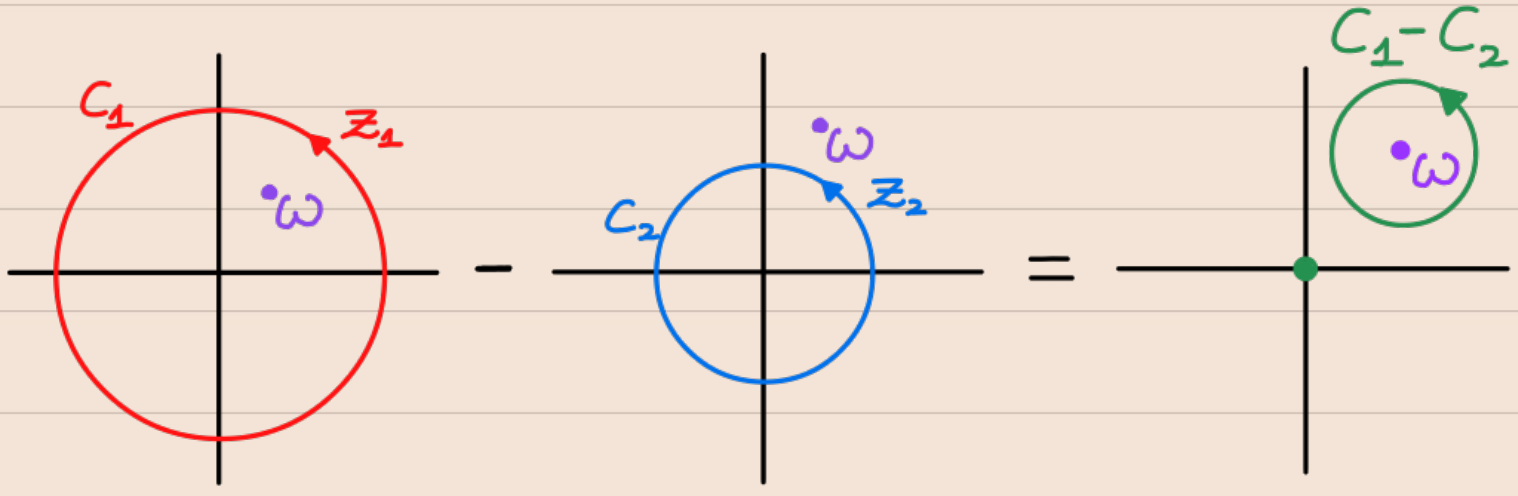
$$- \frac{1}{2\pi i}\left(\oint_{C_1} d\bar{z}_1 \overbrace{:\tilde{J}_B(\bar{z}_1)::\mathcal{A}(\omega,\bar{\omega}):}^{\text{OPE}} + \oint_{C_2} d\bar{z}_2 :\mathcal{A}(\omega,\bar{\omega})\underbrace{::\tilde{J}_B(\bar{z}_2):}_{\text{OPE}}\right)$$

$$= \frac{1}{2\pi i}\left(\oint_{C_1} dz_1 \, :J_B(z_1)\overbrace{::}^{\text{OPE}}\mathcal{A}(\omega,\bar{\omega}): \right.$$

$$\left. - \oint_{C_2} dz_2 \, :J_B(z_2)\underbrace{::\mathcal{A}(\omega,\bar{\omega}):}_{\text{OPE}} \right)$$

$$- \frac{1}{2\pi i}\left(\oint_{C_1} d\bar{z}_1 \, :\tilde{J}_B(\bar{z}_1)\overbrace{::}^{\text{OPE}}\mathcal{A}(\omega,\bar{\omega}): \right.$$

$$\left. - \oint_{C_2} d\bar{z}_2 \, :\tilde{J}_B(\bar{z}_2)\underbrace{::\mathcal{A}(\omega,\bar{\omega}):}_{\text{OPE}} \right)$$



$$\rightarrow \{\hat{Q}_B, \mathcal{A}\} = \frac{1}{2\pi i} \oint dz \, :J_B(z):: \underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$C_1 - C_2 \rightarrow |z-\omega| = \text{const.}$

$$- \frac{1}{2\pi i} \oint d\bar{z} \, :\tilde{J}_B(\bar{z}):: \underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$C_1 - C_2 \rightarrow |z-\omega| = \text{const.}$

$$= \frac{1}{2\pi i} \oint \Big(dz \, :J_B(z): - d\bar{z} \, :\tilde{J}_B(\bar{z}): \Big) : \underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$C: |z-\omega| = \text{const.}$

$$= \oint \frac{dz}{2\pi i} \, :J_B(z):: \underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$C: |z-\omega| = \text{const.}$

$$+ \oint \frac{d\bar{z}}{2\pi i} \, :\tilde{J}_B(\bar{z}):: \underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$\bar{C}: |\bar{z}-\bar{\omega}| = \text{const.}$

$$= \operatorname*{Res}_{z \rightarrow \omega} \left[:J_B(z):: \mathcal{A}(\omega,\bar{\omega}): \right]$$

$$+ \overline{\operatorname*{Res}_{\bar{z} \rightarrow \bar{\omega}}} \left[:\tilde{J}_B(\bar{z}):: \mathcal{A}(\omega,\bar{\omega}): \right]$$

$$\rightarrow i\epsilon\{\hat{Q}_B, \mathcal{A}\} = i\epsilon \underset{z\to\omega}{Res}\left[:J_B(z)::\mathcal{A}(\omega,\bar{\omega}):\right]$$

$$+ i\epsilon \overline{\underset{\bar{z}\to\bar{\omega}}{Res}}\left[:\tilde{J}_B(\bar{z})::\mathcal{A}(\omega,\bar{\omega}):\right]$$

$$\rightarrow \underline{\underline{i\epsilon\{\hat{Q}_B, \mathcal{A}\} = \delta_B \mathcal{A}}} \quad \text{(If } \mathcal{A} \text{ is grassmann odd)}$$

- If '$\mathcal{A}$' is bosonic operator which is Grassmann-even, then

$$\rightarrow \left[\hat{Q}_B, \mathcal{A}\right] = \hat{Q}_B(C_1)\mathcal{A}(\omega,\bar{\omega}) - \mathcal{A}(\omega,\bar{\omega})\hat{Q}_B(C_2)$$

$$= \frac{1}{2\pi i}\oint_{C_1}\left(dz_1 :J_B(z_1): - d\bar{z}_1 :\tilde{J}_B(\bar{z}_1):\right):\mathcal{A}(\omega,\bar{\omega}):$$

$$- \frac{1}{2\pi i}\oint_{C_2} :\mathcal{A}(\omega,\bar{\omega}):\left(dz_2 :J_B(z_2): - d\bar{z}_2 :\tilde{J}_B(\bar{z}_2):\right)$$

$$= \frac{1}{2\pi i}\oint dz\ :J_B(z)::\underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$C_1 - C_2 \rightarrow |z-\omega| = \text{const.}$

$$- \frac{1}{2\pi i}\oint d\bar{z}\ :\tilde{J}_B(\bar{z})::\underbrace{\mathcal{A}(\omega,\bar{\omega})}_{\text{OPE}}:$$

$C_1 - C_2 \rightarrow |z-\omega| = \text{const.}$

$$= \frac{1}{2\pi i} \oint_{C:|z-w|=\text{const.}} \Big(dz :J_B(z): - d\bar{z} :\tilde{J}_B(\bar{z}):\Big) :\mathcal{A}(w,\bar{w}): \quad \underbrace{\qquad}_{\text{OPE}}$$

$$= \oint_{C:|z-w|=\text{const.}} \frac{dz}{2\pi i} :J_B(z)::\mathcal{A}(w,\bar{w}): \quad \underbrace{\qquad}_{\text{OPE}}$$

$$+ \oint_{\bar{C}:|\bar{z}-\bar{w}|=\text{const.}} \frac{d\bar{z}}{2\pi i} :\tilde{J}_B(\bar{z})::\mathcal{A}(w,\bar{w}): \quad \underbrace{\qquad}_{\text{OPE}}$$

$$= \operatorname*{Res}_{z\to w} \left[:J_B(z)::\mathcal{A}(w,\bar{w}): \right] + \overline{\operatorname*{Res}_{\bar{z}\to\bar{w}}} \left[:\tilde{J}_B(\bar{z})::\mathcal{A}(w,\bar{w}): \right]$$

$$\rightarrow i\epsilon \left[\hat{Q}_B, \mathcal{A}\right] = i\epsilon \operatorname*{Res}_{z\to w} \left[:J_B(z)::\mathcal{A}(w,\bar{w}): \right] + i\epsilon \overline{\operatorname*{Res}_{\bar{z}\to\bar{w}}} \left[:\tilde{J}_B(\bar{z})::\mathcal{A}(w,\bar{w}): \right]$$

$$\rightarrow \underline{i\epsilon \left[\hat{Q}_B, \mathcal{A}\right] = \delta_B \mathcal{A}} \quad \text{(If } \mathcal{A} \text{ is grassmann even)}$$

Thus,

$$\delta_B \mathcal{A}(\omega,\bar{\omega}) = i\epsilon\left(\hat{Q}_B \mathcal{A}(\omega,\bar{\omega}) - (-1)^{\#\mathcal{A}} \mathcal{A}(\omega,\bar{\omega})\hat{Q}_B\right)$$

Where, $\#\mathcal{A} \longrightarrow$ Grassmann Parity of $\mathcal{A}$

$$\left(\hat{Q}_B \mathcal{A}(\omega,\bar{\omega}) - (-1)^{\#\mathcal{A}} \mathcal{A}(\omega,\bar{\omega})\hat{Q}_B\right) = \left[\hat{Q}_B, \mathcal{A}\right]_{\text{graded}}$$

Graded commutator

$$\Longrightarrow \underline{\delta_B \mathcal{A}(\omega,\bar{\omega}) = i\epsilon\left[\hat{Q}_B, \mathcal{A}\right]_{\text{graded}}}$$

Eg:

$$\left[\hat{Q}_B, \theta^x\right] = c(\partial\theta^x) + h(\partial c)\theta^x + \tilde{c}(\bar{\partial}\theta^x) + \tilde{h}(\bar{\partial}\tilde{c})\theta^x$$

$$\{\hat{Q}_B, b(z)\} = T^x(z) + T^{(g)}(z)$$

$$\{\hat{Q}_B, c(z)\} = c(z)\partial c(z)$$

Tensor Property of BRST current

Here we shall find those conditions to be satisfied to make BRST currents a Primary fields.

For now let us write the BRST current as follows

$$J_B(z) = c(z)T^x(z) + \frac{1}{2}c(z)T^g(z) + \gamma\partial_z^2 c(z)$$

where we shall fix 'γ' later.

Consider the OPE $(:T^x(z):+:T^g(z):):J_B(w):$

$$= :T^x(z)::J_B(w): + :T^g(z)::J_B(w):$$

$$\sim :c(w)\overbrace{(:T^x(z)::T^x(w):)}: + :T^x(w)\overbrace{(:T^g(z):c(w))}:$$

$$+ :T^g(z)::c(w)(\partial_w c(w))b(w): + \gamma\overbrace{:T^g(z):\partial_w^2 c(w)}$$

1st term: $:c(w)\overbrace{:T^x(z)::T^x(w):}:$

$$\sim :c(w)\left(\frac{c^x}{2(z-w)^4}+\frac{2T^x(w)}{(z-w)^2}+\frac{\partial T^x(w)}{(z-w)}\right):$$

$$\sim \frac{:c(w):\,c^x}{2(z-w)^4}+2\frac{:c(w)T^x(w):}{(z-w)^2}+\frac{:c(w)\partial T^x(w):}{(z-w)}$$

2nd term:

$$:T^x(w)\overbrace{:T^g(z):c(w)}:\sim :T^x(w)\left(\frac{-c(w)}{(z-w)^2}+\frac{\partial c(w)}{(z-w)}\right):$$

$$\sim -\frac{:T^x(w)c(w):}{(z-w)^2}+\frac{:T^x(w)\partial c(w):}{(z-w)}$$

3rd term:

$$:T^g(z)::c(w)(\partial_w c(w))b(w):$$

$$=2:[\partial_z c(z)]b(z)::c(w)(\partial_w c(w))b(w):$$

$$-:[\partial_z b(z)]c(z)::c(w)(\partial_w c(w))b(w):$$

double contractions

$$\sim -2:(\partial_w c(w)):\overbrace{b(z)c(w)}\overbrace{[\partial_z c(z)]b(w)}$$

$$+2:c(w):\overbrace{[\partial_z c(z)]b(w)}\overbrace{b(z)(\partial_w c(w))}$$

$$-:(\partial_w c(w)):\overbrace{[\partial_z b(z)]c(w)}\overbrace{c(z)b(w)}$$

$$+:c(w):\overbrace{[\partial_z b(z)](\partial_w c(w))}\overbrace{c(z)b(w)}$$

$$\sim 2:\frac{(\partial_w c(w))}{(z-w)^3}:-\frac{2:c(w):}{(z-w)^4}+\frac{:(\partial_w c(w)):}{(z-w)^3}-\frac{2:c(w):}{(z-w)^4}$$

$$\text{double contractions} \sim -\frac{4:c(w):}{(z-w)^4}+3:\frac{(\partial_w c(w))}{(z-w)^3}:$$

Single contractions

$$\sim 2:\overbrace{[\partial_z c(z)]b(z)\ c(w)(\partial_w c(w))b(w)}:$$

$$+2:[\partial_z c(z)]\overbrace{b(z)\ c(w)(\partial_w c(w))}b(w):$$

$$-:[\partial_z b(z)]\overbrace{c(z)\ c(w)(\partial_w c(w))}b(w):$$

$$-:\overbrace{[\partial_z b(z)]c(z)}\ c(w)(\partial_w c(w))b(w):$$

$$-:\overbrace{[\partial_z b(z)]c(z)\ c(w)}(\partial_w c(w))b(w):$$

$$\sim -2:c(w)(\partial_w c(w))b(z): \partial_z\big(\overbrace{c(z)b(w)}\big)$$

$$+2:c(w)[\partial_z c(z)]b(w): \partial_w\big(\overbrace{b(z)c(w)}\big)$$

$$-:c(w)(\partial_w c(w))[\partial_z b(z)]: \overbrace{c(z)\,b(w)}$$

$$-:c(z)c(w)b(w): \partial_z\partial_w\big(\overbrace{b(z)c(w)}\big)$$

$$+:c(z)(\partial_w c(w))b(w): \partial_z\big(\overbrace{b(z)c(w)}\big)$$

$$\sim \frac{2:c(w)(\partial_w c(w))b(z):}{(z-w)^2} + 2:c(w)[\partial_z c(z)]b(w): \frac{}{(z-w)^2}$$

$$-\frac{:c(w)(\partial_w c(w))[\partial_w b(w)]:}{(z-w)}$$

$$+2\frac{:c(z)c(w)b(w):}{(z-w)^3} - \frac{:c(z)(\partial_w c(w))b(w):}{(z-w)^2}$$

$$\sim \frac{2:c(w)(\partial_w c(w))b(w):}{(z-w)^2} + \frac{2:c(w)(\partial_w c(w))(\partial_w b(w)):}{(z-w)}$$

$$+ \frac{2:c(w)(\partial_w c(w))b(w):}{(z-w)^2} + \frac{2:c(w)(\partial_w^2 c(w))b(w):}{(z-w)}$$

$$- \frac{:c(w)(\partial_w c(w))[\partial_w b(w)]:}{(z-w)} + \frac{2:c(w)c(w)b(w):}{(z-w)^3}$$

$$+ 2\frac{:(\partial_w c(w))c(w)b(w):}{(z-w)^2} + \frac{:(\partial_w^2 c(w))c(w)b(w):}{(z-w)}$$

$$- \frac{:c(w)(\partial_w c(w))b(w):}{(z-w)^2}$$

$$\sim \frac{:c(w)(\partial_w c(w))b(w):}{(z-w)^2}$$

$$+ \frac{:c(w)(\partial_w^2 c(w))b(w): + :c(w)(\partial_w c(w))(\partial_w b(w)):}{(z-w)}$$

Single contractions

$$\sim \frac{:c(w)(\partial_w c(w))b(w):}{(z-w)^2} + \frac{:c(w)\partial_w[(\partial_w c(w))b(w)]:}{(z-w)}$$

Thus,

$$:T^g(z)::c(w)(\partial_w c(w))b(w): \sim -\frac{4:c(w):}{(z-w)^4}$$

$$+ 3\frac{:(\partial c(w)):}{(z-w)^3} + \frac{:c(w)(\partial c(w))b(w):}{(z-w)^2}$$

$$+ \frac{:c(w)\partial[(\partial c(w))b(w)]:}{(z-w)}$$

4th term $\quad \gamma\overline{:T^g(z):\partial_w^2 c(w)}$

$$\sim \gamma:\partial_w^2\left(\frac{-c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)}\right):$$

$$\sim -\frac{6\gamma:c(w):}{(z-w)^4} - 2\gamma\frac{:\partial c(w):}{(z-w)^3} + \frac{\gamma:\partial^2 c(w):}{(z-w)^2}$$

$$+ \gamma\frac{:\partial^3 c(w):}{(z-w)}$$

Thus,

$$(:T^x(z):+:T^g(z):):J_B(w):$$

$$\sim \frac{:c(w):\,c^x}{2(z-w)^4} + 2\frac{:c(w)T^x(w):}{(z-w)^2} + \frac{:c(w)\partial T^x(w):}{(z-w)}$$

$$- \frac{:T^x(w)c(w):}{(z-w)^2} + \frac{:T^x(w)\partial c(w):}{(z-w)}$$

$$- \frac{4:c(w):}{(z-w)^4} + 3\frac{:(\partial c(w)):}{(z-w)^3} + \frac{:c(w)(\partial c(w))b(w):}{(z-w)^2}$$

$$+ \frac{:c(w)\partial[(\partial c(w))b(w)]:}{(z-w)} - \frac{6\gamma:c(w):}{(z-w)^4}$$

$$- 2\gamma\frac{:\partial c(w):}{(z-w)^3} + \frac{\gamma:\partial^2 c(w):}{(z-w)^2} + \gamma\frac{:\partial^3 c(w):}{(z-w)}$$

Now, we shall group the terms by order

Using, $c(w)(\partial c(w))b(w) = \frac{1}{2}c(w)T^g(w)$

& $:c(w)\partial[(\partial c(w))b(w)]: = \partial:[c(w)(\partial c(w))b(w)]:$

$$= \frac{1}{2}\partial:[c(w)T^g(w)]:$$

$$\longrightarrow \quad (:T^x(z):+:T^g(z):):J_B(w):$$

$$\sim \left(\frac{c^x}{2}-4-6\gamma\right)\frac{:c(w):}{(z-w)^4}+(3-2\gamma)\frac{:\partial c(w):}{(z-w)^3}$$

$$+\frac{:c(w)T^x(w):+:c(w)(\partial c(w))b(w):+\gamma:\partial^2 c(w):}{(z-w)^2}$$

$$+\frac{\left[\partial:(c(w)T^x(w)):+:c(w)\partial[(\partial c(w))b(w)]:+\gamma:\partial^3 c(w):\right]}{(z-w)}$$

$$\longrightarrow (:T^x(z):+:T^g(z):):J_B(w):$$

$$\sim \left(\frac{c^x}{2}-4-6\gamma\right)\frac{:c(w):}{(z-w)^4}+(3-2\gamma)\frac{:\partial c(w):}{(z-w)^3}$$

$$+\frac{:J_B(w):}{(z-w)^2}+\frac{\partial:J_B(w):}{(z-w)}$$

Thus to make $:J_B(w):$ a tensor, we need

$$\frac{c^x}{2}-4-6\gamma=0=3-2\gamma$$

$$\longrightarrow \gamma=\frac{3}{2} \text{ \& } c^x=26$$

Then J_B becomes a conformal tensor with weight $h_{J_B} = 1$

In the exact same way for the $\tilde{J}_B$ anti-holomorphic component,

$$\tilde{J}_B(\bar{z}) = \tilde{c}(\bar{z})\tilde{T}^x(\bar{z}) + \frac{1}{2}\tilde{c}(\bar{z})\tilde{T}^g(\bar{z}) + \tilde{\gamma}\partial_{\bar{z}}^2\tilde{c}(\bar{z})$$

$$\longrightarrow (:\tilde{T}^x(\bar{z}):+:\tilde{T}^g(\bar{z}):):\tilde{J}_B(\bar{w}):$$

$$\sim \left(\frac{\tilde{c}^x}{2} - 4 - 6\tilde{\gamma}\right)\frac{:\tilde{c}(\bar{w}):}{(\bar{z}-\bar{w})^4} + (3-2\tilde{\gamma})\frac{:\bar{\partial}\tilde{c}(\bar{w}):}{(\bar{z}-\bar{w})^3}$$

$$+ \frac{:\tilde{J}_B(\bar{w}):}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}:\tilde{J}_B(\bar{w}):}{(\bar{z}-\bar{w})}$$

Thus to make $:\tilde{J}_B(\bar{w}):$ a tensor, we need

$$\tilde{\gamma} = \frac{3}{2} \;\&\; \tilde{c}^x = 26 \longrightarrow \tilde{h}_{\tilde{J}_B} = 1$$

Nilpotency of BRST charge

$$\hat{Q}_B^2 = \hat{Q}_B \hat{Q}_B = \frac{1}{2}\{\hat{Q}_B, \hat{Q}_B\}$$

Since $\hat{Q}_B$ is Grassman odd,

$$\delta_B \hat{Q}_B = i\epsilon\{\hat{Q}_B, \hat{Q}_B\}$$

Also, $\delta_B \hat{Q}_B = \frac{1}{2\pi i}\oint \left(dw\, \delta_B :J_B(w): - d\bar{w}\, \delta_B :\tilde{J}_B(\bar{w}): \right)$

$$\delta_B :J_B(w): = i\epsilon \operatorname*{Res}_{z\to w} [:J_B(z)::J_B(w):]$$

$$\delta_B :\tilde{J}_B(\bar{w}): = i\epsilon \overline{\operatorname*{Res}_{\bar{z}\to \bar{w}}} [:\tilde{J}_B(\bar{z})::\tilde{J}_B(\bar{w}):]$$

$$\Longrightarrow \{\hat{Q}_B, \hat{Q}_B\} = \frac{1}{2\pi i}\oint dw \operatorname*{Res}_{z\to w} [:J_B(z)::J_B(w):]$$

$$+ \frac{1}{2\pi i}\oint d\bar{w} \overline{\operatorname*{Res}_{\bar{z}\to \bar{w}}} [:\tilde{J}_B(\bar{z})::\tilde{J}_B(\bar{w}):]$$

Thus to have $\hat{Q}_B^2 = 0$, we require

$$\underset{z\to w}{\text{Res}}\,[:J_B(z)::J_B(w):] = 0 = \overline{\underset{\bar{z}\to\bar{w}}{\text{Res}}}\,[:\tilde{J}_B(\bar{z})::\tilde{J}_B(\bar{w}):]$$

For now let us write the BRST current as follows

$$J_B(z) = c(z)T^X(z) + \frac{1}{2}c(z)T^g(z) + \gamma\partial_z^2 c(z)$$

where we shall fix 'γ' later.

$$\begin{aligned}
:J_B(z)::J_B(w): = {} & :c(z)T^X(z)::c(w)T^X(w): \\
& + \frac{1}{2}:c(z)T^X(z)::c(w)T^g(w): \\
& + \frac{1}{2}:c(z)T^g(z)::c(w)T^X(w): \\
& + \frac{1}{2^2}:c(z)T^g(z)::c(w)T^g(w): \\
& + \frac{1}{2}\gamma:c(z)T^g(z)::\partial_w^2 c(w): \\
& + \frac{1}{2}\gamma:\partial_z^2 c(z)::c(w)T^g(w):
\end{aligned}$$

1st term:

$$:c(z)T^x(z)::c(w)T^x(w):$$

$$\sim :c(z)c(w)\overbrace{(:T^x(z)::T^x(w):)}:$$

$$\sim :c(z)c(w)\left(\frac{c^x}{2(z-w)^4}+\frac{2T^x(w)}{(z-w)^2}+\frac{\partial T^x(w)}{(z-w)}\right):$$

(expand $c(z)$ about 'w')

$$\sim :\left(c(w)+(z-w)\partial c(w)+\frac{(z-w)^2\partial^2 c(w)}{2}+\frac{(z-w)^3\partial^3 c(w)}{6}\right)$$

$$\times c(w)\left(\frac{c^x}{2(z-w)^4}+\frac{2T^x(w)}{(z-w)^2}+\frac{\partial T^x(w)}{(z-w)}\right):$$

$$\sim \frac{:\partial c(w)\,c(w):c^x}{2(z-w)^3}+\frac{:\partial^2 c(w)\,c(w):c^x}{4(z-w)^2}$$

$$+\frac{2:\partial c(w)\,c(w)T^x(w):}{(z-w)}+\frac{\partial^3 c(w)\,c(w)c^x}{12(z-w)}$$

2nd term:

$$\frac{1}{2}:c(z)T^x(z)::c(w)T^g(w):$$

$$=:c(z)T^x(z)::c(w)(\partial_w c(w))b(w):$$

$$\sim:\overbrace{c(z)b(w)}\,c(w)(\partial_w c(w))T^x(z):$$

$$\sim:\frac{c(w)(\partial_w c(w))T^x(z):}{(z-w)}$$

$$\frac{1}{2}:c(z)T^x(z)::c(w)T^g(w):\sim:\frac{c(w)(\partial_w c(w))T^x(w):}{(z-w)}$$

3rd term:

$$\frac{1}{2}:c(z)T^g(z)::c(w)T^x(w):$$

$$\sim:c(z)(\partial_z c(z))T^x(w)\overbrace{b(z)c(w)}:$$

$$\sim:\frac{c(z)(\partial_z c(z))T^x(w):}{(z-w)}\sim:\frac{c(w)(\partial_w c(w))T^x(w):}{(z-w)}$$

4th term: $\frac{1}{2^2} :c(z)T^g(z)::c(w)T^g(w):$

$= :c(z)(\partial_z c(z))b(z)::c(w)(\partial_w c(w))b(w):$

$\sim :\overbrace{c(z)(\partial_z c(z))b(z)\ c(w)(\partial_w c(w))b(w)}:$

$+ :c(z)\overbrace{(\partial_z c(z))b(z)\ c(w)(\partial_w c(w))b(w)}:$

$+ :c(z)(\partial_z c(z))\overbrace{b(z)\ c(w)}(\partial_w c(w))b(w):$

$+ :c(z)(\partial_z c(z))\overbrace{b(z)\ c(w)}(\partial_w c(w))b(w):$

$+ :\overbrace{c(z)(\partial_z c(z))b(z)\ c(w)}(\partial_w c(w))b(w):$

$+ :c(z)\overbrace{(\partial_z c(z))\ b(z)\ c(w)}(\partial_w c(w))b(w):$

$+ :\overbrace{c(z)(\partial_z c(z))b(z)\ c(w)(\partial_w c(w))}b(w):$

$+ :c(z)\overbrace{(\partial_z c(z))b(z)\ c(w)(\partial_w c(w))}b(w):$

$$\sim \overbrace{b(z)c(w)}\,\overbrace{c(z)b(w)} :(\partial_z c(z))(\partial_w c(w)):$$

$$- :c(z)(\partial_w c(w)):\overbrace{b(z)c(w)}\,\overbrace{(\partial_z c(z))b(w)}$$

$$- :(\partial_z c(z))c(w):\overbrace{c(z)b(w)}\,\overbrace{b(z)(\partial_w c(w))}$$

$$+ :c(z)c(w):\overbrace{(\partial_z c(z))b(w)}\,\overbrace{b(z)(\partial_w c(w))}$$

$$\sim \frac{:(\partial_z c(z))(\partial_w c(w)):}{(z-w)^2} + \frac{:c(z)(\partial_w c(w)):}{(z-w)^3}$$

$$- \frac{:(\partial_z c(z))c(w):}{(z-w)^3} - \frac{:c(z)c(w):}{(z-w)^4}$$

$$\sim \frac{:(\partial_w^2 c(w))(\partial_w c(w)):}{(z-w)} + \frac{:c(w)(\partial_w c(w)):}{(z-w)^3}$$

$$+ \frac{:(\partial_w^2 c(w))(\partial_w c(w)):}{2(z-w)} - \frac{:(\partial_w c(w))c(w):}{(z-w)^3} - \frac{:(\partial_w^2 c(w))c(w):}{(z-w)^2}$$

$$- \frac{:(\partial_w^3 c(w))c(w):}{2(z-w)} - \frac{:c(w)c(w):}{(z-w)^4} - \frac{:(\partial_w c(w))c(w):}{(z-w)^3}$$

$$- \frac{:(\partial_w^2 c(w))c(w):}{2(z-w)^2} - \frac{:(\partial_w^3 c(w))c(w):}{6(z-w)}$$

$$\sim -3:\frac{(\partial_w c(w))c(w):}{(z-w)^3} - \frac{3:(\partial_w^2 c(w))c(w):}{2(z-w)^2}$$

$$+\frac{3:(\partial_w^2 c(w))(\partial_w c(w)):}{2(z-w)} - \frac{2:(\partial_w^3 c(w))c(w):}{3(z-w)}$$

Thus,

$$\frac{1}{2^2}:c(z)T^g(z)::c(w)T^g(w):$$

$$\sim -3:\frac{(\partial_w c(w))c(w):}{(z-w)^3} - \frac{3:(\partial_w^2 c(w))c(w):}{2(z-w)^2}$$

$$+\frac{\frac{3}{2}:(\partial_w^2 c(w))(\partial_w c(w)): - \frac{2}{3}:(\partial_w^3 c(w))c(w):}{(z-w)}$$

5th term:

$$\frac{1}{2}\gamma:c(z)T^g(z)::\partial_w^2 c(w):$$

$$=\gamma:c(z)(\partial_z c(z))b(z)::\partial_w^2 c(w):$$

$$\sim\gamma:c(z)(\partial_z c(z))\partial_w^2\left(\overbrace{b(z)c(w)}\right):$$

$$\sim \frac{2\gamma :c(z)(\partial_z c(z)):}{(z-w)^3}$$

Thus,

$$\frac{1}{2}\gamma :c(z)T^g(z)::\partial_w c(w):$$

$$\sim \frac{2\gamma :c(w)(\partial_w c(w)):}{(z-w)^3} + \frac{2\gamma :c(w)(\partial_w^2 c(w)):}{(z-w)^2}$$

$$+ \frac{\gamma :(\partial_w c(w))(\partial_w^2 c(w)): + \gamma :c(w)(\partial_w^3 c(w)):}{(z-w)}$$

6th term:

$$\frac{1}{2}\gamma :\partial_z^2 c(z)::c(w)T^g(w):$$

$$= \gamma :\partial_z^2 c(z)::c(w)(\partial_w c(w))b(w):$$

$$\sim \gamma :c(w)(\partial_w c(w)):\ \partial_z^2\overline{(c(z)b(w))}$$

$$\frac{1}{2}\gamma :\partial_z^2 c(z)::c(w)T^g(w): \sim 2\gamma :c(w)(\partial_w c(w)):\frac{}{(z-w)^3}$$

Thus, $:J_B(z)::J_B(w):=$

$$\sim \frac{:\partial c(w)\, c(w): c^x}{2(z-w)^3} + \frac{:\partial^2 c(w)\, c(w): c^x}{4(z-w)^2}$$

$$+ \frac{2:\partial c(w)\, c(w)\, T^x(w):}{(z-w)} + \frac{\partial^3 c(w)\, c(w) c^x}{12(z-w)}$$

$$+ \frac{:c(w)(\partial_w c(w))\, T^x(w):}{(z-w)} + \frac{:c(w)(\partial_w c(w))\, T^x(w):}{(z-w)}$$

$$- 3\frac{:(\partial_w c(w))\, c(w):}{(z-w)^3} - \frac{3:(\partial_w^2 c(w))\, c(w):}{2(z-w)^2}$$

$$+ \frac{\frac{3}{2}:(\partial_w^2 c(w))(\partial_w c(w)): - \frac{2}{3}:(\partial_w^3 c(w))\, c(w):}{(z-w)}$$

$$+ \frac{2\gamma:c(w)(\partial_w c(w)):}{(z-w)^3} + \frac{2\gamma:c(w)(\partial_w^2 c(w)):}{(z-w)^2}$$

$$+ \frac{\gamma:(\partial_w c(w))(\partial_w^2 c(w)): + \gamma:c(w)(\partial_w^3 c(w)):}{(z-w)}$$

$$+ 2\gamma\frac{:c(w)(\partial_w c(w)):}{(z-w)^3}$$

On Simplifying we get,

$$:J_B(z)::J_B(w):\sim\left(\frac{1}{2}c^x-3-4\gamma\right)\frac{:(\partial c(w))c(w):}{(z-w)^3}$$

$$+\left(\frac{c^x}{4}-\frac{3}{2}-2\gamma\right)\frac{:(\partial^2 c(w))c(w):}{(z-w)^2}$$

$$+\frac{\left[\left(\frac{c^x}{12}-\frac{2}{3}-\gamma\right):(\partial^3 c(w))c(w):+\left(\frac{3}{2}-\gamma\right):(\partial^2 c(w))(\partial c(w)):\right]}{(z-w)}$$

Similarly,

$$:\tilde{J}_B(\bar{z})::\tilde{J}_B(\bar{w}):\sim\left(\frac{1}{2}\tilde{c}^x-3-4\tilde{\gamma}\right)\frac{:(\bar{\partial}\tilde{c}(\bar{w}))\tilde{c}(\bar{w}):}{(\bar{z}-\bar{w})^3}$$

$$+\left(\frac{\tilde{c}^x}{4}-\frac{3}{2}-2\tilde{\gamma}\right)\frac{:(\bar{\partial}^2\tilde{c}(\bar{w}))\tilde{c}(\bar{w}):}{(\bar{z}-\bar{w})^2}$$

$$+\frac{\left[\left(\frac{\tilde{c}^x}{12}-\frac{2}{3}-\tilde{\gamma}\right):(\bar{\partial}^3\tilde{c}(\bar{w}))\tilde{c}(\bar{w}):+\left(\frac{3}{2}-\tilde{\gamma}\right):(\bar{\partial}^2\tilde{c}(\bar{w}))(\bar{\partial}\tilde{c}(\bar{w})):\right]}{(\bar{z}-\bar{w})}$$

Now, to have a Nilpotent BRST charge we require,

$$\frac{1}{2\pi i}\oint dw \operatorname*{Res}_{z\to w}\left[:J_B(z)::J_B(w):\right]=0$$

(replace $z\leftrightarrow w$)

$$\longrightarrow \left(\frac{c^x}{12}-\frac{2}{3}-\gamma\right)\frac{1}{2\pi i}\oint dz:(\partial^3 c(z))c(z):$$

$$+\left(\frac{3}{2}-\gamma\right)\frac{1}{2\pi i}\oint dz:(\partial^2 c(z))(\partial c(z)):=0$$

Now, $$c(z)=\sum_{n=-\infty}^{\infty}\frac{\hat{c}_n}{z^{n-1}}$$

$$\longrightarrow \partial^3 c(z)=-\sum_{n=-\infty}^{\infty}\frac{(n-1)n(n+1)\hat{c}_n}{z^{n+2}}$$

$$\longrightarrow \partial^3 c(z)=-\sum_{n=-\infty}^{\infty}\frac{(n^2-1)n\hat{c}_n}{z^{n+2}}$$

$$\longrightarrow C(z)\partial^3 C(z) = -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\hat{C}_m (n^2-1) n \hat{C}_n}{z^{m-1} z^{n+2}}$$

$$\longrightarrow (\partial^3 C(z)) C(z) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\hat{C}_m (n^2-1) n \hat{C}_n}{z^{m+n+1}}$$

$$\longrightarrow \frac{1}{2\pi i}\oint dz\, (\partial^3 C(z)) C(z) =$$

$$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\hat{C}_m (n^2-1) n \hat{C}_n \underbrace{\frac{1}{2\pi i}\oint \frac{dz}{z^{m+n+1}}}_{\delta_{m,-n}}$$

$$\longrightarrow \frac{1}{2\pi i}\oint dz\, :(\partial^3 C(z)) C(z):$$

$$= \sum_{n=-\infty}^{\infty}(n^2-1) n :\hat{C}_{-n}\hat{C}_n:$$

Similarly, $$C(z) = \sum_{n=-\infty}^{\infty}\frac{\hat{C}_n}{z^{n-1}}$$

$$\partial C(z) = -\sum_{n=-\infty}^{\infty} (n-1)\frac{\hat{C}_n}{z^n}$$

$$\partial^2 C(z) = \sum_{n=-\infty}^{\infty} \frac{(n-1)n\hat{C}_n}{z^{n+1}}$$

$$\longrightarrow (\partial C(z))(\partial^2 C(z))$$

$$= -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{(m-1)(n-1)n\hat{C}_m\hat{C}_n}{z^{m+n+1}}$$

$$\longrightarrow \frac{1}{2\pi i}\oint dz :(\partial C(z))(\partial^2 C(z)):$$

$$= -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(m-1)(n-1)n:\hat{C}_m\hat{C}_n:\delta_{m,-n}$$

Thus, $\frac{-1}{2\pi i}\oint dz(\partial^2 C(z))(\partial C(z))$

$$= -\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(m-1)(n-1)n:\hat{C}_m\hat{C}_n:\delta_{m,-n}$$

So,

$$\frac{1}{2\pi i}\oint dz(\partial^2 C(z))(\partial C(z)) = -\sum_{n=-\infty}^{\infty}(n^2-1)n:\hat{C}_{-n}\hat{C}_n:$$

So, for having $\hat{Q}_B^2 = 0$, we require

$$\rightarrow \left(\frac{c^x}{12} - \frac{2}{3} - \gamma\right) \sum_{n=-\infty}^{\infty} (n^2-1) n : \hat{c}_{-n} \hat{c}_n :$$

$$- \left(\frac{3}{2} - \gamma\right) \sum_{n=-\infty}^{\infty} (n^2-1) n : \hat{c}_{-n} \hat{c}_n : = 0$$

$$\longrightarrow \text{we require } \gamma = \frac{3}{2}$$

$$\rightarrow \left(\frac{c^x}{12} - \frac{2}{3} - \frac{3}{2}\right) \sum_{n=-\infty}^{\infty} (n^2-1) n : \hat{c}_{-n} \hat{c}_n : = 0$$

$$\rightarrow \left(\frac{c^x - 26}{12}\right) \sum_{n=-\infty}^{\infty} (n^2-1) n : \hat{c}_{-n} \hat{c}_n : = 0$$

Similarly for the anti holomorphic part we require $\tilde{\gamma} = \frac{3}{2}$

$$\rightarrow \left(\frac{\tilde{c}^x - 26}{12}\right) \sum_{n=-\infty}^{\infty} (n^2-1) n : \hat{\tilde{c}}_{-n} \hat{\tilde{c}}_n : = 0$$

Thus to make $\hat{Q}_B^2 = 0$, we finally need

$$\boxed{c^x = \tilde{c}^x = 26} \rightarrow \text{Critical dimension}$$

BRST charge in terms of modes

The BRST charge is given by,

$$\hat{Q}_B = \frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} \left(dz :J_B(z): - d\bar{z} :\tilde{J}_B(\bar{z}): \right)$$

For the holomorphic part,

$$J_B(z) = c(z)T^X(z) + \frac{1}{2}c(z)T^g(z) + \gamma\partial_z^2 c(z)$$

$$c(z) = \sum_{m=-\infty}^{\infty} \frac{\hat{c}_m}{z^{m-1}} \quad , \quad T^X(z) = \sum_{n=-\infty}^{\infty} \frac{\hat{L}^M_n}{z^{n+2}}$$

$$\frac{1}{2}c(z)T^g(z) = c(z)\left(\partial_z c(z)\right)b(z)$$

$$b(z) = \sum_{l=-\infty}^{\infty} \frac{\hat{b}_l}{z^{l+2}}$$

$$\partial^3 c(z) = -\sum_{n=-\infty}^{\infty} \frac{(n^2-1)n\hat{c}_n}{z^{n+2}}$$

$$:c(z)T^X(z): = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{:\hat{c}_m \hat{L}^M_n:}{z^{m+n+1}}$$

$$\frac{1}{2}:C(z)T^g(z):$$

$$= -\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(m-1)\frac{:\hat{C}_n\hat{C}_m\hat{b}_l:}{z^{l+m+n+1}}$$

Thus,

$$\frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} dz\,:J_B(z): =$$

$$= \frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} dz\left(C(z)T^x(z)+\frac{1}{2}:C(z)T^g(z):+\gamma\partial_z^2 C(z)\right)$$

$$= \frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} dz\sum_{m,n}\frac{:\hat{C}_m\hat{L}^{\mathbb{M}}_n:}{z^{m+n+1}}$$

$$-\frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} dz\sum_{l,m,n}(m-1)\frac{:\hat{C}_n\hat{C}_m\hat{b}_l:}{z^{l+m+n+1}}$$

$$-\frac{\gamma}{2\pi i}\oint_{C:|z|=\text{const.}} dz\sum_{n=-\infty}^{\infty}\frac{(n^2-1)n\hat{C}_n}{z^{n+2}}$$

$$\longrightarrow \frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} dz :J_{\mathcal{B}}(z): = \sum_{m,n} \mathcal{J}_{m+n,0} :\hat{c}_m \hat{L}^{\mathbb{M}}_n:$$

$$- \sum_{l,m,n} (n-1)\mathcal{J}_{l+m+n,0} :\hat{c}_n \hat{c}_m \hat{b}_l:$$

$$-\gamma \sum_{n=-\infty}^{\infty} \mathcal{J}_{n+1,0} \underbrace{(n^2-1)}_{0} n \hat{c}_n$$

Thus,

$$\frac{1}{2\pi i}\oint_{C:|z|=\text{const.}} dz :J_{\mathcal{B}}(z): = \sum_n :\hat{c}_{-n}\hat{L}^{\mathbb{M}}_n:$$

$$- \sum_{m,n} (m-1) :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}:$$

Now,

$$\sum_{m,n} :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}: = -\sum_{m,n} :\underbrace{\hat{c}_m \hat{c}_n \hat{b}_{-m-n}}_{\text{replace } m \leftrightarrow n}:$$

$$\longrightarrow \sum_{m,n} :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}: = -\sum_{m,n} :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}:$$

$$\longrightarrow \sum_{m,n} :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}: = 0$$

$$\sum_{m,n} m :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}: = -\sum_{m,n} m :\hat{c}_m \hat{c}_n \hat{b}_{-m-n}:$$

replace $m \leftrightarrow n$

$$= -\sum_{m,n} n :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}:$$

$$\longrightarrow \sum_{m,n} m :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}: =$$

$$= \sum_{m,n} \left(\frac{m-n}{2}\right) :\hat{c}_n \hat{c}_m \hat{b}_{-m-n}:$$

$$= -\sum_{m,n} \left(\frac{m-n}{2}\right) :\hat{c}_m \hat{c}_n \hat{b}_{-m-n}:$$

Thus,

$$\frac{1}{2\pi i} \oint_{C:|z|=\text{const.}} dz :J_B(z): = \sum_n :\hat{c}_{-n} \hat{L}^M_n:$$

$$+ \sum_{m,n} \left(\frac{m-n}{2}\right) :\hat{c}_m \hat{c}_n \hat{b}_{-m-n}:$$

In the 2nd term when $m=0$ or $n=0$, there is a normal ordering ambiguity.

If $m=0$, $\sum_{n=-\infty}^{\infty}\left(\frac{-n}{2}\right):\hat{c}_0\hat{c}_n\hat{b}_{-n}:$

$\hat{c}_n\hat{b}_{-n}+\hat{b}_{-n}\hat{c}_n=1$

$$=\sum_{n=-\infty}^{\infty}\left(\frac{n}{2}\right):\hat{c}_0\hat{b}_{-n}\hat{c}_n: - \hat{c}_0\underbrace{\sum_{n=-\infty}^{\infty}\left(\frac{n}{2}\right)}_{\text{normal ordering constant}}$$

Thus, we should actually write

$$\sum_{m,n}\left(\frac{m-n}{2}\right):\hat{c}_m\hat{c}_n\hat{b}_{-m-n}: \sim$$

$$\sim\sum_{m,n}\left(\frac{m-n}{2}\right):\hat{c}_m\hat{c}_n\hat{b}_{-m-n}: + \hat{c}_0 a^B$$

a^B = normal ordering constant

But no normal ordering ambiguity from $:\hat{c}_{-n}\hat{L}^M_n:$ term as matter virasoro modes & ghost modes commute.

We get very similar expansion for the anti-holomorphic part. Thus, combining we get,

$$\hat{Q}_B = \sum_{n\in\mathbb{Z}} \left(:\hat{c}_{-n}\hat{L}^M_n: + :\hat{\tilde{c}}_{-n}\hat{\tilde{L}}^M_n: \right)$$

$$+ \sum_{m,n\in\mathbb{Z}} \frac{(m-n)}{2} :\left(\hat{c}_m \hat{c}_n \hat{b}_{-m-n} + \hat{\tilde{c}}_m \hat{\tilde{c}}_n \hat{\tilde{b}}_{-m-n} \right):$$

$$+ a^B \left(\hat{c}_0 + \hat{\tilde{c}}_0 \right)$$

We can also write it in terms of the ghost virasoro modes. Using,

$$J_B(z) = c(z) T^X(z) + \frac{1}{2} c(z) T^g(z) + \gamma \partial_z^2 c(z)$$

we get,

$$\hat{Q}_B = \sum_{n\in\mathbb{Z}} \left(:\hat{c}_{-n}\hat{L}^M_n: + :\hat{\tilde{c}}_{-n}\hat{\tilde{L}}^M_n: \right)$$

$$+ \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(:\hat{c}_{-n}\hat{L}^{(g)}_n: + :\hat{\tilde{c}}_{-n}\hat{\tilde{L}}^{(g)}_n: \right)$$

But the 2nd term has normal ordering ambiguity when $n=0$. Since,

$$\left[\hat{C}_{-n}, \hat{L}^{(g)}_{n}\right] = n\hat{C}_0$$

$$\Longrightarrow \frac{1}{2}\sum_{n\in\mathbb{Z}} :\hat{C}_{-n}\hat{L}^{(g)}_{n}: \text{ should actually be written as,}$$

$$\frac{1}{2}\sum_{n\in\mathbb{Z}} :\hat{C}_{-n}\hat{L}^{(g)}_{n}: + \hat{C}_0 \underbrace{\sum_{n=-\infty}^{\infty}\left(\frac{n}{2}\right)}_{a^B}$$

Thus,

$$\hat{Q}_B = \sum_{n\in\mathbb{Z}}\left(:\hat{C}_{-n}\hat{L}^{M}_{n}: + :\hat{\tilde{C}}_{-n}\hat{\tilde{L}}^{M}_{n}:\right) + \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(:\hat{C}_{-n}\hat{L}^{(g)}_{n}: + :\hat{\tilde{C}}_{-n}\hat{\tilde{L}}^{(g)}_{n}:\right) + a^B\left(\hat{C}_0 + \hat{\tilde{C}}_0\right)$$

To find the normal ordering constant a^B, consider $\delta_B b(z) = i\epsilon\left(T^X(z) + T^{(g)}(z)\right)$

But, $i\epsilon\{\hat{Q}_B, b(z)\} = \partial_B b(z)$

$$\longrightarrow \{\hat{Q}_B, b(z)\} = T^X(z) + T^{(g)}(z)$$

For the zero modes, $\{\hat{Q}_B, \hat{b}_0\} = \hat{L}_0^M + \hat{L}_0^{(g)}$

So, to find $\{\hat{Q}_B, \hat{b}_0\}$,

$$\{\hat{Q}_B, \hat{b}_0\} = \sum_{n\in\mathbb{Z}} \hat{L}_n^M \underbrace{\{\hat{c}_{-n}, \hat{b}_0\}}_{\partial_{n,0}}$$

$$+ \frac{1}{2}\sum_{n\in\mathbb{Z}} :\{\hat{c}_{-n}\hat{L}_n^{(g)}, \hat{b}_0\}:$$

$$+ a^B \underbrace{\{\hat{c}_0, \hat{b}_0\}}_{1}$$

To find,

$$\{\hat{c}_{-n}\hat{L}_n^{(g)}, \hat{b}_0\} = \hat{c}_{-n}\hat{L}_n^{(g)}\hat{b}_0 + \hat{b}_0\hat{c}_{-n}\hat{L}_n^{(g)}$$

$$= \hat{c}_{-n}\hat{L}_n^{(g)}\hat{b}_0 - \hat{c}_{-n}\hat{b}_0\hat{L}_n^{(g)} + \hat{c}_{-n}\hat{b}_0\hat{L}_n^{(g)}$$

$$+ \hat{b}_0\hat{c}_{-n}\hat{L}_n^{(g)}$$

$$\{\hat{c}_{-n}\hat{L}^{(g)}_n, \hat{b}_0\} = \hat{c}_{-n}\underbrace{[\hat{L}^{(g)}_n, \hat{b}_0]}_{n\hat{b}_n} + \underbrace{\{\hat{c}_{-n}, \hat{b}_0\}}_{\delta_{n,0}}\hat{L}^{(g)}_n$$

$$\longrightarrow \{\hat{c}_{-n}\hat{L}^{(g)}_n, \hat{b}_0\} = n\hat{c}_{-n}\hat{b}_n + \delta_{n,0}\hat{L}^{(g)}_n$$

$$\{\hat{Q}_B, \hat{b}_0\} = \sum_{n\in\mathbb{Z}} \hat{L}^{M}_n \delta_{n,0}$$

$$+ \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(n{:}\hat{c}_{-n}\hat{b}_n{:} + \delta_{n,0}\hat{L}^{(g)}_n\right) + a^B$$

$$= \hat{L}^{M}_0 + \frac{1}{2}\underbrace{\sum_{n\in\mathbb{Z}} n{:}\hat{c}_{-n}\hat{b}_n{:}}_{\hat{L}^{(g)}_0 - A^g} + \frac{1}{2}\hat{L}^{(g)}_0 + a^B$$

$\hat{L}^{(g)}_0 - A^g \rightarrow$ ghost normal ordering

Thus,

$$\{\hat{Q}_B, \hat{b}_0\} = \hat{L}^{M}_0 + \hat{L}^{(g)}_0 - A^g + a^B$$

So, to satisfy $\{\hat{Q}_B, \hat{b}_0\} = \hat{L}^{M}_0 + \hat{L}^{(g)}_0$

we need, $\boxed{a^B = A^g = -1}$

Thus, the final expression is,

$$\hat{Q}_{\mathcal{B}} = \sum_{n\in\mathbb{Z}} \left(\hat{c}_{-n} \hat{L}^{\mathbb{M}}_{n} + \hat{\tilde{c}}_{-n} \hat{\tilde{L}}^{\mathbb{M}}_{n} \right) + \sum_{m,n\in\mathbb{Z}} \frac{(m-n)}{2} :\left(\hat{c}_m \hat{c}_n \hat{b}_{-m-n} + \hat{\tilde{c}}_m \hat{\tilde{c}}_n \hat{\tilde{b}}_{-m-n} \right): - \left(\hat{c}_0 + \hat{\tilde{c}}_0 \right)$$

In terms of virasoro operators,

$$\hat{Q}_{\mathcal{B}} = \sum_{n\in\mathbb{Z}} \left(\hat{c}_{-n} \hat{L}^{\mathbb{M}}_{n} + \hat{\tilde{c}}_{-n} \hat{\tilde{L}}^{\mathbb{M}}_{n} \right) + \frac{1}{2} \sum_{n\in\mathbb{Z}} :\left(\hat{c}_{-n} \hat{L}^{(g)}_{n} + \hat{\tilde{c}}_{-n} \hat{\tilde{L}}^{(g)}_{n} \right): - \left(\hat{c}_0 + \hat{\tilde{c}}_0 \right)$$

BRST cohomology of the String

We have $(\hat{L}^{M}_{n})^{\dagger} = \hat{L}^{M}_{-n}$ & $(\alpha^{\mu}_{n})^{\dagger} = \alpha^{\mu}_{-n}$
as X^{μ}-CFT is unitary. ↳ X^{μ} modes

The hermiticity of the BRST charge operator implies that that

$$(\hat{c}_n)^{\dagger} = \hat{c}_{-n} \quad \& \text{ necessitates } \quad (\hat{b}_n)^{\dagger} = \hat{b}_{-n}$$

$$\Longrightarrow (\hat{L}^{(g)}_{n})^{\dagger} = \hat{L}^{(g)}_{-n}$$

Thus, $(\hat{\alpha}^{\mu}_{n})^{\dagger} = \hat{\alpha}^{\mu}_{-n}$ & $(\hat{\tilde{\alpha}}^{\mu}_{n})^{\dagger} = \hat{\tilde{\alpha}}^{\mu}_{-n}$

$$(\hat{c}_n)^{\dagger} = \hat{c}_{-n} \quad \& \quad (\hat{\tilde{c}}_n)^{\dagger} = \hat{\tilde{c}}_{-n}$$

$$(\hat{b}_n)^{\dagger} = \hat{b}_{-n} \quad \& \quad (\hat{\tilde{b}}_n)^{\dagger} = \hat{\tilde{b}}_{-n}$$

The ground state of X^{μ}-CFT is $|0, k^{\mu}\rangle$, where

$$\hat{\alpha}^{\mu}_{n}|0,k^{\mu}\rangle = 0 \quad \forall \quad n>0$$

$$\& \quad \hat{L}^{M}_{n}|0,k^{\mu}\rangle = 0 \quad \forall \quad n>0$$

Similarly for the anti-holomorphic modes.

And for the bc-CFT, we have $|\uparrow\rangle$ & $|\downarrow\rangle$, where,

$$\hat{c}_0|\uparrow\rangle = 0 \quad \Big| \quad \hat{b}_0|\downarrow\rangle = 0$$

$$\hat{c}_0|\downarrow\rangle = |\uparrow\rangle \quad \Big| \quad \hat{b}_0|\uparrow\rangle = |\downarrow\rangle$$

$$\hat{b}_n|\downarrow\rangle = 0 = \hat{c}_n|\downarrow\rangle \quad , n>0$$

$$\hat{b}_n|\uparrow\rangle = 0 = \hat{c}_n|\uparrow\rangle \quad , n>0$$

$$\rightarrow \hat{L}_n^{(g)}|\uparrow\rangle = 0 = \hat{L}_n^{(g)}|\downarrow\rangle, \; n>0$$

Now,

$$|\uparrow\rangle = \hat{c}_0|\downarrow\rangle \rightarrow \langle\uparrow| = \langle\downarrow|\hat{c}_0^{+} = \langle\downarrow|\hat{c}_0$$

$$\rightarrow \langle\uparrow|\uparrow\rangle = \langle\downarrow|\underbrace{\hat{c}_0\hat{c}_0}_{0}|\downarrow\rangle \rightarrow \langle\uparrow|\uparrow\rangle = 0$$

Similarly, $\langle\downarrow|\downarrow\rangle = 0$

But, $\langle\downarrow|\uparrow\rangle = \langle\downarrow|\hat{c}_0|\downarrow\rangle = \langle\uparrow|\hat{b}_0\hat{c}_0|\downarrow\rangle$

Then, $\langle\downarrow|\uparrow\rangle^{*} = \langle\uparrow|\downarrow\rangle = \langle\downarrow|\hat{c}_0|\downarrow\rangle = \langle\downarrow|\uparrow\rangle$

In the case of open string, where $\hat{\tilde{c}}_0$ does not exist, we choose,

$$\langle\downarrow|\hat{c}_0|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 1$$

$$\longrightarrow \langle\downarrow|\hat{c}_0|\downarrow\rangle^* = \langle\downarrow|\uparrow\rangle^* = 1$$

So, for open string, we define an inner product of some state $|\psi\rangle$ as,

$$\| |\psi\rangle \|^2_{BPZ} = \langle\psi|\hat{c}_0|\psi\rangle$$

(open-string)

$\longrightarrow$ is real

$\longrightarrow$ BPZ inner product for open-string

In the case of closed string, which has, $(bc \otimes \tilde{b}\tilde{c})$-CFT, consider,

$$\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle = \langle\downarrow,\tilde{\downarrow}|\uparrow,\tilde{\uparrow}\rangle$$

So that, $\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle^* = \langle\downarrow,\tilde{\downarrow}|\hat{c}_0\hat{\tilde{c}}_0|\downarrow,\tilde{\downarrow}\rangle$

$$= -\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle = -\langle\downarrow,\tilde{\downarrow}|\uparrow,\tilde{\uparrow}\rangle$$

$$\Longrightarrow \langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle^* = -\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle$$

$$\longrightarrow \langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle = i$$

$$\Longrightarrow \left(-i\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle\right)^* = -i\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle$$

Thus, if we define

$$\underset{\text{(closed-string)}}{\||\psi\rangle\|^2_{BPZ}} = \underbrace{-i\langle\psi|\hat{\tilde{c}}_0\hat{c}_0|\psi\rangle}_{\text{real under hermitian conjugation}}$$

$$\Longrightarrow \underline{\||\downarrow,\tilde{\downarrow}\rangle\|^2_{BPZ} = -i\langle\downarrow,\tilde{\downarrow}|\hat{\tilde{c}}_0\hat{c}_0|\downarrow,\tilde{\downarrow}\rangle = 1}$$

For the $(X^\mu \otimes bc \otimes \tilde{b}\tilde{c})$-CFT, there are four ground states

$$|0,k^\mu,\uparrow,\tilde{\uparrow}\rangle,\ |0,k^\mu,\downarrow,\tilde{\uparrow}\rangle,$$

$$|0,k^\mu,\uparrow,\tilde{\downarrow}\rangle,\ |0,k^\mu,\downarrow,\tilde{\downarrow}\rangle$$

Same as in the point particle case, we require physical states $|\psi\rangle$ to satisfy

$$\hat{b}_0|\psi\rangle = \hat{\tilde{b}}_0|\psi\rangle = 0$$

Thus, the physical ground state is,

$$\underline{|0,k^\mu,\downarrow,\tilde{\downarrow}\rangle}$$

Consider the BPZ inner product of the above closed string state $|0, k^\mu, \downarrow, \tilde{\downarrow}\rangle$

$$\left\| |0, k^\mu, \downarrow, \tilde{\downarrow}\rangle \right\|^2_{BPZ} =$$

$$= -i \langle 0, k^\mu, \downarrow, \tilde{\downarrow} | \hat{\tilde{c}}_0 \hat{c}_0 | 0, k^\mu, \downarrow, \tilde{\downarrow} \rangle$$

$$= -i \langle 0, k^\mu | 0, k^\mu \rangle \cdot \underbrace{\langle \downarrow, \tilde{\downarrow} | \hat{\tilde{c}}_0 \hat{c}_0 | \downarrow, \tilde{\downarrow} \rangle}_{i}$$

$$= \langle 0, k^\mu | 0, k^\mu \rangle$$

Then,

$$\langle 0, q^\mu, \downarrow, \tilde{\downarrow} | 0, k^\mu, \downarrow, \tilde{\downarrow} \rangle_{BPZ}$$

$$= -i \langle 0, q^\mu, \downarrow, \tilde{\downarrow} | \hat{\tilde{c}}_0 \hat{c}_0 | 0, k^\mu, \downarrow, \tilde{\downarrow} \rangle$$

$$= -i \langle 0, q^\mu | 0, k^\mu \rangle \cdot \underbrace{\langle \downarrow, \tilde{\downarrow} | \hat{\tilde{c}}_0 \hat{c}_0 | \downarrow, \tilde{\downarrow} \rangle}_{i}$$

$$= \langle 0, q^\mu | 0, k^\mu \rangle = (2\pi)^D \delta^D (q^\mu - k^\mu)$$

Thus, for the closed string, we get,

$$\underline{\langle 0, q^\mu, \downarrow, \tilde{\downarrow} | \hat{\tilde{c}}_0 \hat{c}_0 | 0, k^\mu, \downarrow, \tilde{\downarrow} \rangle = i (2\pi)^D \delta^D (q^\mu - k^\mu)}$$

Consider the BPZ inner product of the open string state $|0, k^\mu, \downarrow\rangle$

$$\left\| |0, k^\mu, \downarrow\rangle \right\|^2_{BPZ} = \langle 0, k^\mu, \downarrow | \hat{c}_0 | 0, k^\mu, \downarrow \rangle$$

$$= \langle 0, k^\mu | 0, k^\mu \rangle \cdot \underbrace{\langle \downarrow | \hat{c}_0 | \downarrow \rangle}_{1}$$

$$= \langle 0, k^\mu | 0, k^\mu \rangle$$

Then, $\langle 0, q^\mu, \downarrow | \hat{c}_0 | 0, k^\mu, \downarrow \rangle_{BPZ}$

$$= \langle 0, q^\mu, \downarrow | \hat{c}_0 | 0, k^\mu, \downarrow \rangle$$

$$= \langle 0, q^\mu | 0, k^\mu \rangle \cdot \underbrace{\langle \downarrow | \hat{c}_0 | \downarrow \rangle}_{1}$$

$$= \langle 0, q^\mu | 0, k^\mu \rangle = (2\pi)^D \delta^D (q^\mu - k^\mu)$$

Thus, for the open string, we get,

$$\underline{\langle 0, q^\mu, \downarrow | \hat{c}_0 | 0, k^\mu, \downarrow \rangle = (2\pi)^D \delta^D (q^\mu - k^\mu)}$$

Same as in the point particle case, we require physical states $|\psi\rangle$ to satisfy

$$\hat{b}_0|\psi\rangle = \hat{\tilde{b}}_0|\psi\rangle = 0$$

Also the BRST quantization requires the physical states to be closed under $\hat{Q}_B$'s action. i.e., $\hat{Q}_B|\psi\rangle = 0$ is needed. Also, as seen before,

$$\{\hat{Q}_B, \hat{b}_0\} = \hat{L}_0^M + \hat{L}_0^{(g)} = \hat{L}_0 \rightarrow \text{Total virasoro}$$

$$\{\hat{Q}_B, \hat{\tilde{b}}_0\} = \hat{\tilde{L}}_0^M + \hat{\tilde{L}}_0^{(g)} = \hat{\tilde{L}}_0$$

Now, $\hat{L}_0|\psi\rangle = \{\hat{Q}_B, \hat{b}_0\}|\psi\rangle$

$$= \hat{Q}_B \underbrace{\hat{b}_0|\psi\rangle}_{0} + \hat{b}_0 \underbrace{\hat{Q}_B|\psi\rangle}_{0}$$

Thus, the pysical states should satisfy

$$\underline{\hat{L}_0|\psi\rangle = \left(\hat{L}_0^M + \hat{L}_0^{(g)}\right)|\psi\rangle = 0}$$

&

$$\underline{\hat{\tilde{L}}_0|\psi\rangle = \left(\hat{\tilde{L}}_0^M + \hat{\tilde{L}}_0^{(g)}\right)|\psi\rangle = 0}$$

BRST cohomology of Open String

For the critical bosonic theory, $D = 26$.
And for the open string, we have,

$$\hat{L}_0^M = \alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \eta_{\mu\nu} \hat{\alpha}_{-n}^\mu \hat{\alpha}_n^\nu$$

For the minkowski background,

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots 1)$$

$$\Longrightarrow \hat{L}_0^M = \alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} \left(-\hat{\alpha}_{-n}^0 \hat{\alpha}_n^0 + \sum_{i=1}^{25} \hat{\alpha}_{-n}^i \hat{\alpha}_n^i \right)$$

The commutation relation is,

$$\left[\hat{\alpha}_n^\mu, \hat{\alpha}_k^\nu \right] = n \eta^{\mu\nu} \delta_{k,-n}$$

$$\longrightarrow \left[\frac{\hat{\alpha}_n^\mu}{\sqrt{n}}, \frac{\hat{\alpha}_k^\nu}{\sqrt{n}} \right] = \eta^{\mu\nu} \delta_{k,-n}$$

Thus, if $a_n^\mu = \dfrac{\hat{\alpha}_n^\mu}{\sqrt{n}}$, then,

$$\left[a_n^\mu, a_k^\nu \right] = \eta^{\mu\nu} \delta_{k,-n}$$

But, $[a^0_n, a^0_k] = -\delta_{k,-n}$

or $[a^0_{-n}, a^0_k] = -\delta_{k,n}$

$\longrightarrow [(a^0_n)^\dagger, a^0_k] = -\delta_{k,n}$

Thus zero-th mode oscillator has a wrong sign of the commutator of the usual oscillator algebra.

So, if define the number operator along the i-th direction in n^{th}-mode as,

$$N^i_n = \hat{a}^i_{-n}\hat{a}^i_n \quad \text{for} \quad i = 1, 2, 3, \cdots 25$$

Then,

$$[N^i_n, (a^j_n)^\dagger] = \delta^{ij}(a^j_n)^\dagger \,, \quad n > 0$$

$$[N^i_n, a^j_n] = -\delta^{ij} a^j_n \,, \quad n > 0$$

Thus, $(a^j_n)^\dagger \rightarrow$ creation operator for $n > 0$

$a^j_n \rightarrow$ annihilation operator for $n > 0$

But, '$\hat{a}^0_{-n}\hat{a}^0_n$' due to a wrong sign in the commutator

$$\left[(a^0_n)^\dagger, a^0_k\right] = -\delta_{k,n}$$

makes $\left[(a^0_n)^\dagger a^0_n, (a^0_n)^\dagger\right] = -(a^0_n)^\dagger, \quad n>0$

and $\left[(a^0_n)^\dagger a^0_n, a^0_n\right] = a^0_n \quad , n>0$

Thus, to make, $(a^0_n)^\dagger \longrightarrow$ creation operator
& $a^0_n \longrightarrow$ annihilation operator
we should define,

$N^0_n = -\hat{a}^0_{-n}\hat{a}^0_n$ as the number operator along the zeroth direction.

$$\Longrightarrow N^i_n = \hat{a}^i_{-n}\hat{a}^i_n = \frac{\hat{\alpha}^i_{-n}\hat{\alpha}^i_n}{n}$$

$$\& \quad N^0_n = -\hat{a}^0_{-n}\hat{a}^0_n = \frac{-\hat{\alpha}^0_{-n}\hat{\alpha}^0_n}{n}$$

$$\Longrightarrow \hat{L}_0^{M} = \alpha' \hat{p}^{\mu}\hat{p}_{\mu} + \sum_{n=1}^{\infty}\left(n N_n^0 + \sum_{i=1}^{25} n N_n^i\right)$$

$$\Longrightarrow \boxed{\hat{L}_0^{M} = \alpha' \hat{p}^{\mu}\hat{p}_{\mu} + \sum_{n=1}^{\infty} n \sum_{\mu=0}^{25} N_n^{\mu}}$$

For the ghost virasoro zero-th mode,

$$\hat{L}_0^{(g)} = \sum_{n=-\infty}^{\infty} n :\hat{c}_{-n}\hat{b}_n: - 1$$

$$\hat{L}_0^{(g)} = \sum_{n=1}^{\infty} n\left(:\hat{c}_{-n}\hat{b}_n: - :\hat{c}_n\hat{b}_{-n}:\right) - 1$$

$$\hat{L}_0^{(g)} = \sum_{n=1}^{\infty} n\left(\hat{c}_{-n}\hat{b}_n + \hat{b}_{-n}\hat{c}_n\right) - 1$$

Let, $N_n^c = \hat{c}_{-n}\hat{b}_n$,

$$\left[N_n^c, \hat{c}_k\right] = \left[\hat{c}_{-n}\hat{b}_n, \hat{c}_k\right] = \hat{c}_{-n}\hat{b}_n\hat{c}_k - \hat{c}_k\hat{c}_{-n}\hat{b}_n$$

$$= \hat{c}_{-n}\hat{b}_n\hat{c}_k + \hat{c}_{-n}\hat{c}_k\hat{b}_n - \left(\hat{c}_{-n}\hat{c}_k\hat{b}_n + \hat{c}_k\hat{c}_{-n}\hat{b}_n\right)$$

$$= \hat{c}_{-n}\underbrace{\left\{\hat{b}_n, \hat{c}_k\right\}}_{\delta_{n,-k}} - \underbrace{\left\{\hat{c}_{-n}, \hat{c}_k\right\}}_{0}\hat{b}_n$$

$$\rightarrow \left[\hat{c}_{-n}\hat{b}_n, \hat{c}_k\right] = \hat{c}_{-n}\delta_{k,-n}$$

$$\rightarrow \underline{\left[N_n^c, \hat{c}_k\right] = 0 \text{ , for } n,k>0}$$

But, $\left[\hat{c}_{-n}\hat{b}_n, \hat{c}_{-k}\right] = \hat{c}_{-n}\delta_{n,k}$

$$\Longrightarrow \underline{\left[N_n^c, \hat{c}_{-k}\right] = \hat{c}_{-n}\delta_{n,k} \text{ , for } n,k>0}$$

Then,

$$\left[N_n^c, \hat{b}_k\right] = \left[\hat{c}_{-n}\hat{b}_n, \hat{b}_k\right] = \hat{c}_{-n}\hat{b}_n\hat{b}_k - \hat{b}_k\hat{c}_{-n}\hat{b}_n$$

$$= \hat{c}_{-n}\hat{b}_n\hat{b}_k + \hat{c}_{-n}\hat{b}_k\hat{b}_n - \left(\hat{c}_{-n}\hat{b}_k\hat{b}_n + \hat{b}_k\hat{c}_{-n}\hat{b}_n\right)$$

$$= \hat{c}_{-n}\underbrace{\left\{\hat{b}_n, \hat{b}_k\right\}}_{0} - \underbrace{\left\{\hat{c}_{-n}, \hat{b}_k\right\}}_{\delta_{n,k}}\hat{b}_n$$

$$\rightarrow \underline{\left[N_n^c, \hat{b}_k\right] = -\hat{b}_n\delta_{n,k} \text{ , for } n,k>0}$$

&

$$\underline{\left[N_n^c, \hat{b}_{-k}\right] = 0 \text{ , for } n,k>0}$$

Now, consider the operator, $N_n^b = \hat{b}_{-n}\hat{c}_n$

$$[N_n^b, \hat{c}_k] = [\hat{b}_{-n}\hat{c}_n, \hat{c}_k] = \hat{b}_{-n}\hat{c}_n\hat{c}_k - \hat{c}_k\hat{b}_{-n}\hat{c}_n$$

$$= \hat{b}_{-n}\hat{c}_n\hat{c}_k + \hat{b}_{-n}\hat{c}_k\hat{c}_n - \left(\hat{b}_{-n}\hat{c}_k\hat{c}_n + \hat{c}_k\hat{b}_{-n}\hat{c}_n\right)$$

$$= \hat{b}_{-n}\underbrace{\{\hat{c}_n, \hat{c}_k\}}_{0} - \underbrace{\{\hat{b}_{-n}, \hat{c}_k\}}_{\delta_{n,k}}\hat{c}_n$$

$$\rightarrow \underline{[N_n^b, \hat{c}_k] = -\hat{c}_n\delta_{n,k}}$$

$$\& \quad [\hat{b}_{-n}\hat{c}_n, \hat{c}_{-k}] = -\hat{c}_n\delta_{n,-k}$$

$$\rightarrow \underline{[N_n^b, \hat{c}_{-k}] = 0, \text{ for } n,k>0}$$

Then, $[N_n^b, \hat{b}_k] = [\hat{b}_{-n}\hat{c}_n, \hat{b}_k]$

$$= \hat{b}_{-n}\underbrace{\{\hat{c}_n, \hat{b}_k\}}_{\delta_{n,-k}} - \underbrace{\{\hat{b}_{-n}, \hat{b}_k\}}_{0}\hat{c}_n$$

$$\rightarrow \underline{[N_n^b, \hat{b}_k] = \hat{b}_{-n}\delta_{n,-k} = 0, \text{ for } n,k>0}$$

$$\& \quad \underline{[N_n^b, \hat{b}_{-k}] = \hat{b}_{-n}\delta_{n,k}, \text{ for } n,k>0}$$

Thus, for $n, k > 0$

$$[N_n^c, \hat{c}_k] = [N_n^c, \hat{b}_{-k}] = 0$$

$$[N_n^c, \hat{c}_{-k}] = \hat{c}_{-n} \delta_{n,k}$$

$$[N_n^c, \hat{b}_k] = -\hat{b}_n \delta_{n,k}$$

& $$[N_n^b, \hat{c}_{-k}] = [N_n^b, \hat{b}_k] = 0$$

$$[N_n^b, \hat{c}_k] = -\hat{c}_n \delta_{n,k}$$

$$[N_n^b, \hat{b}_{-k}] = \hat{b}_{-n} \delta_{n,k}$$

N_n^c counts the number of $\hat{c}_{-n}$ excitation & N_n^c ignores $\hat{b}_{-n}$ completely.

N_n^b counts the number of $\hat{b}_{-n}$ excitation & N_n^b ignores $\hat{c}_{-n}$ completely.

The ghost virasoro zeroth mode can be written as,

$$\hat{L}_0^{(g)} = \sum_{n=1}^{\infty} n\left(N_n^c + N_n^b\right) - 1$$

where

$$\left[\hat{L}_0^{(g)}, \hat{c}_{-n}\right] = n\hat{c}_{-n}\,, \text{ for } n \in \mathbb{Z}$$

$$\left[\hat{L}_0^{(g)}, \hat{b}_{-n}\right] = n\hat{b}_{-n}\,, \text{ for } n \in \mathbb{Z}$$

The total virasoro zeroth mode is,

$$\hat{L}_0 = \hat{L}_0^{M} + \hat{L}_0^{(g)} = \alpha' \hat{p}^{\mu}\hat{p}_{\mu} + \sum_{n=1}^{\infty} n \sum_{\mu=0}^{25} N_n^{\mu} + \sum_{n=1}^{\infty} n\left(N_n^c + N_n^b\right) - 1$$

Thus,

$$\boxed{\hat{L}_0 = \alpha' \hat{p}^{\mu}\hat{p}_{\mu} + \sum_{n=1}^{\infty} n\left(N_n^c + N_n^b + \sum_{\mu=0}^{25} N_n^{\mu}\right) - 1}$$

But the space-time mass of the string is given by, $\underbrace{\hat{p}^\mu \hat{p}_\mu = -m^2}_{\text{mass-shell}}$

But physical states should have,

$$\hat{L}_0 |\psi\rangle = 0$$

Thus,

$$\alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} n\left(N_n^c + N_n^b + \sum_{\mu=0}^{25} N_n^\mu\right) - 1 = 0$$

$$\longrightarrow \boxed{\alpha' m^2 = \sum_{n=1}^{\infty} n\left(N_n^c + N_n^b + \sum_{\mu=0}^{25} N_n^\mu\right) - 1}$$

This is the mass-shell condition for the open string.

The space of states that satisfy the condition $\hat{b}_0 |\psi\rangle = 0$ and the mass-shell condition $\hat{L}_0 |\psi\rangle = 0$ are denoted by $\hat{\mathbb{H}}$.

Commutator of $\hat{Q}_B$ & $\hat{L}_0$

$$\left[\hat{Q}_B, \hat{L}_0^{M}\right] = \sum_{n\in\mathbb{Z}} \hat{c}_{-n} \underbrace{\left[\hat{L}_n^{M}, \hat{L}_0^{M}\right]}_{n\hat{L}_n^{M}} = \sum_{n\in\mathbb{Z}} n\,\hat{c}_{-n}\hat{L}_n^{M}$$

$$\left[\hat{Q}_B, \hat{L}_0^{(g)}\right] = \sum_{n\in\mathbb{Z}} \hat{L}_n^{M} \underbrace{\left[\hat{c}_{-n}, \hat{L}_0^{(g)}\right]}_{-n\hat{c}_{-n}}$$

$$+\frac{1}{2}\sum_{n\in\mathbb{Z}} :\left[\hat{c}_{-n}\hat{L}_n^{(g)}, \hat{L}_0^{(g)}\right]: - \underbrace{\left[\hat{c}_0, \hat{L}_0^{(g)}\right]}_{0}$$

$$= -\sum_{n\in\mathbb{Z}} n\,\hat{c}_{-n}\hat{L}_n^{M}$$

$$+\frac{1}{2}\sum_{n\in\mathbb{Z}} :\left(\hat{c}_{-n}\underbrace{\left[\hat{L}_n^{(g)}, \hat{L}_0^{(g)}\right]}_{n\hat{L}_n^{(g)}} + \underbrace{\left[\hat{c}_{-n}, \hat{L}_0^{(g)}\right]}_{-n\hat{c}_{-n}}\hat{L}_n^{(g)}\right):$$

$$\longrightarrow \left[\hat{Q}_B, \hat{L}_0^{(g)}\right] = -\sum_{n\in\mathbb{Z}} n\,\hat{c}_{-n}\hat{L}_n^{M} = -\left[\hat{Q}_B, \hat{L}_n^{M}\right]$$

$$\longrightarrow \underline{\left[\hat{Q}_B, \left(\hat{L}_0^{M} + \hat{L}_0^{(g)}\right)\right] = \left[\hat{Q}_B, \hat{L}_0\right] = 0}$$

Then, consider, '$\hat{Q}_B|\psi\rangle$' where $|\psi\rangle \in \hat{\mathbb{H}}$

i.e.,

$$\hat{b}_0|\psi\rangle = 0 \quad \& \quad \hat{L}_0|\psi\rangle = 0$$

Now, $\hat{b}_0(\hat{Q}_B|\psi\rangle) = \underbrace{\{\hat{Q}_B, \hat{b}_0\}}_{\hat{L}_0}|\psi\rangle - \hat{Q}_B\hat{b}_0|\psi\rangle$

$$= \underbrace{\hat{L}_0|\psi\rangle}_{0} - \hat{Q}_B\underbrace{\hat{b}_0|\psi\rangle}_{0}$$

$$\Longrightarrow \underline{\hat{b}_0(\hat{Q}_B|\psi\rangle) = 0}$$

$$\& \quad \hat{L}_0(\hat{Q}_B|\psi\rangle) = \hat{Q}_B\underbrace{\hat{L}_0|\psi\rangle}_{0}$$

$$\Longrightarrow \underline{\hat{L}_0(\hat{Q}_B|\psi\rangle) = 0}$$

Thus, if $|\psi\rangle \in \hat{\mathbb{H}} \Longrightarrow \underline{\hat{Q}_B|\psi\rangle \in \hat{\mathbb{H}}}$

So, $\hat{Q}_B$ maps $\hat{\mathbb{H}}$ to $\hat{\mathbb{H}}$ itself.

Reduced inner product for the $\hat{\mathbb{H}}$ space

The mass-shell condition $\alpha'\left(\hat{p}^\mu \hat{p}_\mu + m^2\right) = 0$ fixes one momentum component out of all 'k^μ'. So, if we take inner product b/w the states of '$\hat{\mathbb{H}}$', then, the usual inner product is proportional to $\delta^{26}(q^\mu - k^\mu)$. But one component is fixed in both k^μ & q^μ Since,

$$\alpha'\left(k^\mu k_\mu + m^2\right) = 0 \quad \& \quad \alpha'\left(q^\mu q_\mu + m^2\right) = 0$$

So, the usual inner product will be proportional to '$\delta(0)\delta^{25}(q^i - k^i)$'. The presence of $\delta(0)$ is not good.

Moreover, the states will be of the type $|\chi^\mu$ CFT states$\rangle \otimes$ descendants of $|\downarrow\rangle$. Then, usual inner products become zero due to anti-commutation relations of ghost modes.

So, to avoid the above problems, we can define a new inner product for states in the $\hat{\mathbb{H}}$ space denoted by $\langle\Psi\|\Psi\rangle$. In this we simply ignore the X^0 + ghost modes. It is this inner product which is used for probabilistic interpretation. '$\hat{Q}_B$' is still Hermitian in this new inner product.

This inner product is given as follows.

$$\langle 0, q^\mu, \downarrow \| 0, k^\mu, \downarrow \rangle = 2k^0 (2\pi)^{25} \delta^{25}(q^i - k^i)$$

where we have chosen positive energy states whose $k^0 > 0$.

This normalization is lorentz '$SO(1,25)$' invariant with the completeness relation of

$$\int \underbrace{\frac{d^{25}k}{(2\pi)^{25} 2k^0}} |0, k^i, \downarrow\rangle\langle 0, k^i, \downarrow| = 1$$

Lorentz invariant measure.

Tachyon State

At the lowest level in $D = 26$ minkowski space-time, the physical state is $|0, k^\mu, \downarrow\rangle$ which satisfies the mass shell condition of

$$k^\mu k_\mu + m^2 = 0$$

where,

$$\alpha' m^2 = \sum_{n=1}^{\infty} n\left(N_n^c + N_n^b + \sum_{\mu=0}^{25} N_n^\mu\right) - 1$$

But $N_n^c = N_n^b = N_n^\mu = 0$. Thus,

$$k^\mu k_\mu = -m^2 = \frac{1}{\alpha'}$$

This state is invariant,

$$Q_B|0;\mathbf{k}\rangle = 0\,, \tag{4.3.24}$$

because every term in Q_B contains either lowering operators or L_0. This also shows that there are no exact states at this level, so each invariant state corresponds to a cohomology class. These are just the states of the tachyon. The mass-shell condition is the same as found in the light-cone quantization of section 1.3 and from the open string vertex operators in section 3.6. The tachyon mass-squared, determined in a heuristic way in chapter 1, is here fixed by the normal ordering constant in Q_B, which is determined by the requirement of nilpotence.

Using the expansion of the BRST charge in modes (4.3.7) we have

$$Q_{\mathrm{B}} \left|0;\boldsymbol{k}\right\rangle = \left[\sum_{n=-\infty}^{\infty} c_n L_{-n}^m + \frac{1}{2}\sum_{m,n=-\infty}^{\infty} (m-n)\,{:}c_m c_n b_{-m-n}{:} - c_0\right]\left|0;\boldsymbol{k}\right\rangle \qquad [4.182]$$

The first term, for $n \neq 0$, always has either c_k or L_k with $k \geq 1$ which annihilates $|0;k\rangle$. The $n = 0$ term gives $c_0 L_0^m \left|0;\boldsymbol{k}\right\rangle$ with L_0^m given by (2.7.7), or at least the equivalent relation for the open string,

$$L_0^m = \alpha' p^2 + \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{\mu\, n} \qquad [4.183]$$

Thus $c_0 L_0^m \left|0;\boldsymbol{k}\right\rangle = \alpha' k^2 c_0 \left|0;\boldsymbol{k}\right\rangle = c_0 \left|0;\boldsymbol{k}\right\rangle$. This cancels with the third term. In the second term all the terms that have $m \neq 0$ and $m \neq 0$ clearly annihilate $|0;k\rangle$ due to the normal ordering. The $m = n = 0$ term has a c_0^2 that is zero by itself. So what remains is terms with either m or n equal to zero, but not both. These are of the form $\sum_{n=-\infty, n\neq 0}^{\infty} {:}c_0 c_n b_{-n}{:}$ but by normal ordering there is always an annihilation operator to the right so these terms also annihilate $|0;k\rangle$. We thus find that indeed $Q_{\mathrm{B}} \left|0;\boldsymbol{k}\right\rangle = 0$.

The Level 1 string states are very nicely explained by the "Companion reader to Polchinski's string theory" of Schranz.