

Compactification & CFT

Although the Bosonic string is embedded in a 26 dimensional background, not all dimensions have to be large and observable.

Instead some dimensions can be curled up into a very small compact manifold.

So we consider the target space time to be of the form

$$M^{(26)} = M^{(d)} \times \mathcal{K}^{(26-d)}$$

$M^{(d)}$ has coordinates X^μ ($\mu = 0, 1, 2, \dots, d-1$) with one time-like coordinate X^0 & $d-1$ space-like coordinates $\vec{X} = \{X^1, X^2, X^3, \dots, X^{d-1}\}$. And $\mathcal{K}^{(26-d)}$ is a $26-d$ dimensional Euclidean compact manifold.

So, the Polyakov string action splits into two CFTs. An external CFT with a central charge 'd' & an internal compact CFT with a central charge '26-d'.

But the ghost CFT still exists with a central charge '-26'.

Thus, the total string action after gauge fixing is

$$S = S^M + S^K + S^g$$

$\downarrow \quad \quad \downarrow \quad \quad \rightarrow$
External Internal Ghost
CFT CFT CFT

So, the total Energy momentum tensor also splits into

$$T = T^M + T^K + T^g$$

Similarly, the ~~virasoro~~ generators also split as

$$\hat{L}_n = \underbrace{\hat{L}_n^M + \hat{L}_n^K}_{\hat{L}_n^M} + \hat{L}_n^{(g)}$$

Now, K -CFT is compact. So, $\hat{L}_0^K$ has a discrete spectrum whose eigenvalues are 'I'.

So a general state in the $\hat{H}$ space is of the form,

$$|N, I, k^\mu, \downarrow\rangle \Rightarrow \text{Open String}$$

$$|N, \tilde{N}, I, k^\mu, \downarrow\rangle \Rightarrow \text{Closed String}$$

where,

$$N = \{N_n^\mu, N_n^c, N_n^b\} \text{ \& } \tilde{N} = \{\tilde{N}_n^\mu, \tilde{N}_n^c, \tilde{N}_n^b\}$$

The total Virasoro 0^{th} mode is given by,

$$\hat{L}_0 = \alpha' \hat{p}^\mu \hat{p}_\mu + \sum_{n=1}^{\infty} n \left(N_n^c + N_n^b + \sum_{\mu=0}^{d-1} N_n^\mu \right) + \hat{L}_0^{\mathcal{H}} - 1$$

For the open-string, the space-time mass is,

$$\hat{p}^\mu \hat{p}_\mu = -m^2 \} \text{ mass-shell}$$

The physical states should satisfy,

$$\hat{L}_0 |\psi\rangle = 0$$

$$\rightarrow \boxed{\alpha' m^2 = \sum_{n=1}^{\infty} n \left(N_n^c + N_n^b + \sum_{\mu=0}^{d-1} N_n^\mu \right) + \hat{L}_0^{\mathcal{H}} - 1}$$

$\hookrightarrow$ open-string mass-shell condition.

For the closed-string, the space-time mass is,

$$\hat{p}^\mu \hat{p}_\mu = -m^2 = -\tilde{m}^2 \} \text{mass-shell}$$

The physical states should satisfy,

$$\hat{L}_0 |\psi\rangle = \hat{\tilde{L}}_0 |\psi\rangle = 0$$

$$\rightarrow \alpha' m^2 = \sum_{n=1}^{\infty} n \left(N_n^c + N_n^b + \sum_{\mu=0}^{d-1} N_n^\mu \right) + \hat{L}_0^{\mathcal{H}} - 1$$

$$\& \quad \alpha' \tilde{m}^2 = \sum_{n=1}^{\infty} n \left(\tilde{N}_n^c + \tilde{N}_n^b + \sum_{\mu=0}^{d-1} \tilde{N}_n^\mu \right) + \hat{\tilde{L}}_0^{\mathcal{H}} - 1$$

This is the mass-shell condition for the closed-string.

The only information about the compact internal $\mathcal{H}$ -CFT that we need is that it has appropriate central charge to make the BRST charge Nilpotent & $\mathcal{H}$ -CFT is Conformally invariant, unitary & $\hat{L}_0^{\mathcal{H}}$ has a discrete spectrum satisfying the level matching condition of $I = \tilde{I}$.

The reduced inner product in the $\hat{H}$ space is,
 $\langle 0, q^\mu, I, \downarrow \| 0, k^\mu, I, \downarrow \rangle = 2k^0 (2\pi)^{d-1} \delta^{d-1}(\vec{q} - \vec{k}) \delta_{I, I'}$
 with positive energy states $k^0 > 0$.

So, once the vacuum normalization is fixed, every excited-state inner product is obtained algebraically by using the commutation relations.

Statement of No-Ghost Theorem

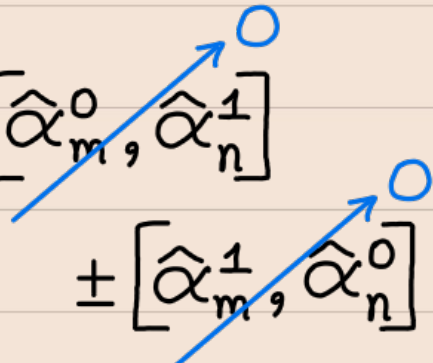
If we define a transverse hilbert space $H^\perp$ which has all states with no longitudinal & ghost excitations (X^0, X^1, b, c) . Then, the light-cone hilbert space & the BRST cohomology is isomorphic to $H^\perp$. And the inner product of states in $H^\perp$ is positive. Finally BRST-cohomology is isomorphic to the old Covariant quantized hilbert space whose $A^M = \tilde{A}^M = -1$.

$$H^\perp \simeq H_{\text{BRST}} \simeq H_{\text{OCQ}} \simeq H_{\text{l.c}}$$

Proof: Define the light-cone oscillators

$$\hat{\alpha}_n^{\pm} = \frac{\hat{\alpha}_n^0 \pm \hat{\alpha}_n^1}{\sqrt{2}}$$

$$\begin{aligned} [\hat{\alpha}_m^{\pm}, \hat{\alpha}_n^{\pm}] &= \frac{1}{2} [\hat{\alpha}_m^0 \pm \hat{\alpha}_m^1, \hat{\alpha}_n^0 \pm \hat{\alpha}_n^1] \\ &= \frac{1}{2} \left([\hat{\alpha}_m^0, \hat{\alpha}_n^0] \pm [\hat{\alpha}_m^0, \hat{\alpha}_n^1] \right. \\ &\quad \left. \pm [\hat{\alpha}_m^1, \hat{\alpha}_n^0] + [\hat{\alpha}_m^1, \hat{\alpha}_n^1] \right) \end{aligned}$$

$$= \frac{1}{2} \left([\hat{\alpha}_m^0, \hat{\alpha}_n^0] \pm [\hat{\alpha}_m^0, \hat{\alpha}_n^1] \pm [\hat{\alpha}_m^1, \hat{\alpha}_n^0] + [\hat{\alpha}_m^1, \hat{\alpha}_n^1] \right)$$


$$= \frac{1}{2} (m\eta^{00} \delta_{m+n,0} + m\eta^{11} \delta_{m+n,0})$$

$$= \frac{1}{2} m \delta_{m+n,0} \underbrace{(\eta^{00} + \eta^{11})}_0$$

$$\Rightarrow \underline{[\hat{\alpha}_m^{\pm}, \hat{\alpha}_n^{\pm}] = 0}$$

$$\begin{aligned}
[\hat{\alpha}_m^+, \hat{\alpha}_n^-] &= \frac{1}{2} [\hat{\alpha}_m^0 + \hat{\alpha}_m^1, \hat{\alpha}_n^0 - \hat{\alpha}_n^1] \\
&= \frac{1}{2} \left([\hat{\alpha}_m^0, \hat{\alpha}_n^0] - [\hat{\alpha}_m^0, \hat{\alpha}_n^1] \right. \\
&\quad \left. + [\hat{\alpha}_m^1, \hat{\alpha}_n^0] - [\hat{\alpha}_m^1, \hat{\alpha}_n^1] \right) \\
&= \frac{1}{2} \left(\underbrace{m\eta^{00}}_{-1} \delta_{m+n,0} - \underbrace{m\eta^{11}}_1 \delta_{m+n,0} \right)
\end{aligned}$$

$$\Rightarrow \underline{[\hat{\alpha}_m^+, \hat{\alpha}_n^-] = -m \delta_{m+n,0}}$$

$\hat{\alpha}_{m>0}^+$: Annihilates '-' light-cone excitation
 $\hat{\alpha}_{m<0}^+$: Creates '+' light-cone excitation
 $\hat{\alpha}_{m>0}^-$: Annihilates '+' light-cone excitation
 $\hat{\alpha}_{m<0}^-$: Creates '-' light-cone excitation

Consider the quantum number operator

$$N^{l.c} = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} : \hat{\alpha}_{-n}^+ \hat{\alpha}_n^- :$$

$$\rightarrow \underline{N^{l.c} = \sum_{n=1}^{\infty} \frac{1}{n} (\hat{\alpha}_{-n}^+ \hat{\alpha}_n^- - \hat{\alpha}_{-n}^- \hat{\alpha}_n^+)}$$

Now,

$$\begin{aligned}
 [N^{l.c}, \hat{\alpha}_m^\pm] &= \sum_{n=1}^{\infty} \frac{1}{n} [(\hat{\alpha}_{-n}^+ \hat{\alpha}_n^- - \hat{\alpha}_{-n}^- \hat{\alpha}_n^+), \hat{\alpha}_m^\pm] \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} ([\hat{\alpha}_{-n}^+ \hat{\alpha}_n^-, \hat{\alpha}_m^\pm] - [\hat{\alpha}_{-n}^- \hat{\alpha}_n^+, \hat{\alpha}_m^\pm]) \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \left([\hat{\alpha}_{-n}^+, \hat{\alpha}_m^\pm] \hat{\alpha}_n^- + \hat{\alpha}_{-n}^+ [\hat{\alpha}_n^-, \hat{\alpha}_m^\pm] \right. \\
 &\quad \left. - \hat{\alpha}_{-n}^- [\hat{\alpha}_n^+, \hat{\alpha}_m^\pm] - [\hat{\alpha}_{-n}^-, \hat{\alpha}_m^\pm] \hat{\alpha}_n^+ \right)
 \end{aligned}$$

$$[N^{l.c}, \hat{\alpha}_m^+] = \sum_{n=1}^{\infty} \frac{1}{n} \left(\underbrace{\hat{\alpha}_{-n}^+ [\hat{\alpha}_n^-, \hat{\alpha}_m^+]}_{m \delta_{m+n,0}} - \underbrace{[\hat{\alpha}_{-n}^-, \hat{\alpha}_m^+] \hat{\alpha}_n^+}_{m \delta_{mn}} \right)$$

$$[N^{l.c}, \hat{\alpha}_m^+] = \sum_{n=1}^{\infty} \frac{m}{n} (\hat{\alpha}_{-n}^+ \delta_{n,-m} - \delta_{mn} \hat{\alpha}_n^+)$$

$$\rightarrow [N^{l.c}, \hat{\alpha}_{m>0}^+] = -\hat{\alpha}_m^+, \quad [N^{l.c}, \hat{\alpha}_{m<0}^+] = -\hat{\alpha}_m^+$$

$$\& \quad [N^{l.c}, \hat{\alpha}_0^+] = 0$$

$$\rightarrow \underline{[N^{l.c}, \hat{\alpha}_m^+] = -\hat{\alpha}_m^+ (1 - \delta_{m0})}$$

$$\begin{aligned}
 [N^{l.c}, \hat{\alpha}_m^-] &= \sum_{n=1}^{\infty} \frac{1}{n} ([\hat{\alpha}_{-n}^+, \hat{\alpha}_m^-] \hat{\alpha}_n^- - \hat{\alpha}_{-n}^- [\hat{\alpha}_n^+, \hat{\alpha}_m^-]) \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\underbrace{[\hat{\alpha}_{-n}^+, \hat{\alpha}_m^-]}_{n\delta_{mn}} \hat{\alpha}_n^- - \hat{\alpha}_{-n}^- \underbrace{[\hat{\alpha}_n^+, \hat{\alpha}_m^-]}_{-n\delta_{m+n,0}} \right)
 \end{aligned}$$

$$\rightarrow [N^{l.c}, \hat{\alpha}_m^-] = \sum_{n=1}^{\infty} (\delta_{mn} \hat{\alpha}_n^- + \delta_{m,-n} \hat{\alpha}_{-n}^-)$$

$$\rightarrow [N^{l.c}, \hat{\alpha}_{m>0}^-] = \hat{\alpha}_m^- , \quad [N^{l.c}, \hat{\alpha}_{m<0}^+] = \hat{\alpha}_m^-$$

$$\& [N^{l.c}, \hat{\alpha}_0^-] = 0$$

$$\rightarrow \underline{[N^{l.c}, \hat{\alpha}_m^-] = \hat{\alpha}_m^- (1 - \delta_{m0})}$$

Thus, we can combine the two relations

$$\underline{[N^{l.c}, \hat{\alpha}_m^{\pm}] = \mp \hat{\alpha}_m^{\pm} (1 - \delta_{m0})}$$

So, every $\hat{\alpha}_m^-$ excitation raises $N^{l.c}$ by '1'
 & every $\hat{\alpha}_m^+$ excitation lowers $N^{l.c}$ by '1'.

$$\text{Hence, } N^{l.c} = N^- - N^+$$

We can decompose the BRST charge operator using the quantum number $N^{l.c.}$.

$$\hat{Q}_B = \hat{Q}_1 + \hat{Q}_0 + \hat{Q}_{-1}$$

where, $[N^{l.c.}, \hat{Q}_1] = \hat{Q}_1$

$$[N^{l.c.}, \hat{Q}_0] = 0$$

$$[N^{l.c.}, \hat{Q}_{-1}] = -\hat{Q}_{-1}$$

$$\Rightarrow [N^{l.c.}, \hat{Q}_j] = j \hat{Q}_j, \text{ where } j = \{1, 0, -1\}$$

Also, we require each $\hat{Q}_j$ increases the ghost number by '1'. i.e.

$$[N^g, \hat{Q}_j] = \hat{Q}_j$$

$$\text{where } N^g = \hat{c}_0 \hat{b}_0 - \frac{1}{2} + \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{b}_n - \hat{b}_{-n} \hat{c}_n)$$

is the ghost number operator.

The ghost number of ' $\hat{b}_n$ ' is -1 .

$$\rightarrow [N^g, \hat{b}_n] = -\hat{b}_n$$

And the ghost number of ' $\hat{c}_n$ ' is $+1$.

$$\rightarrow [N^g, \hat{c}_n] = \hat{c}_n$$

Thus, ' N^g ' counts the difference b/w the c -type excitations & b -type excitations.

Now, consider $\hat{Q}_B = \hat{Q}_1 + \hat{Q}_0 + \hat{Q}_{-1}$

$$\rightarrow \hat{Q}_B^2 = (\hat{Q}_1 + \hat{Q}_0 + \hat{Q}_{-1})(\hat{Q}_1 + \hat{Q}_0 + \hat{Q}_{-1})$$

$$\begin{aligned} &= \hat{Q}_1^2 + \hat{Q}_0 \hat{Q}_1 + \hat{Q}_{-1} \hat{Q}_1 + \hat{Q}_1 \hat{Q}_0 + \hat{Q}_0^2 \\ &\quad + \hat{Q}_{-1} \hat{Q}_0 + \hat{Q}_1 \hat{Q}_{-1} + \hat{Q}_0 \hat{Q}_{-1} + \hat{Q}_{-1}^2 \end{aligned}$$

$$= \hat{Q}_1^2 + \{\hat{Q}_1, \hat{Q}_0\} + \{\hat{Q}_1, \hat{Q}_{-1}\} + \hat{Q}_0^2$$

$$+ \{\hat{Q}_0, \hat{Q}_{-1}\} + \hat{Q}_{-1}^2$$

Since $\hat{Q}_2^2 = 0$,

$$\begin{aligned} \hat{Q}_1^2 + (\{\hat{Q}_1, \hat{Q}_0\}) + (\{\hat{Q}_1, \hat{Q}_{-1}\} + \hat{Q}_0^2) \\ + (\{\hat{Q}_0, \hat{Q}_{-1}\}) + \hat{Q}_{-1}^2 = 0 \end{aligned}$$

$$\begin{aligned} [N^{l.c}, \hat{Q}_1^2] &= \hat{Q}_1 [N^{l.c}, \hat{Q}_1] + [N^{l.c}, \hat{Q}_1] \hat{Q}_1 \\ &= \hat{Q}_1^2 + \hat{Q}_1^2 \end{aligned}$$

$$\rightarrow [N^{l.c}, \hat{Q}_1^2] = 2\hat{Q}_1^2$$

- So, $\hat{Q}_1^2$ has light-cone number of 2.

$$\text{Similarly, } [N^{l.c}, \hat{Q}_0^2] = 0$$

- So, $\hat{Q}_0^2$ has light-cone number of 0.

$$\text{And, } [N^{l.c}, \hat{Q}_{-1}^2] = -2\hat{Q}_{-1}^2$$

- So, $\hat{Q}_{-1}^2$ has light-cone number of -2.

$$\begin{aligned}
[N^{l.c}, \{\hat{Q}_1, \hat{Q}_0\}] &= [N^{l.c}, \hat{Q}_1 \hat{Q}_0] + [N^{l.c}, \hat{Q}_0 \hat{Q}_1] \\
&= [N^{l.c}, \hat{Q}_1] \hat{Q}_0 + \hat{Q}_0 [N^{l.c}, \hat{Q}_1] \\
&= \hat{Q}_1 \hat{Q}_0 + \hat{Q}_0 \hat{Q}_1
\end{aligned}$$

$$\rightarrow [N^{l.c}, \{\hat{Q}_1, \hat{Q}_0\}] = \{\hat{Q}_1, \hat{Q}_0\}$$

- So, $\{\hat{Q}_1, \hat{Q}_0\}$ has light-cone number '1'.

$$\text{Similarly, } [N^{l.c}, \{\hat{Q}_0, \hat{Q}_{-1}\}] = -\{\hat{Q}_0, \hat{Q}_{-1}\}$$

- So, $\{\hat{Q}_0, \hat{Q}_{-1}\}$ has light-cone number '-1'.

Finally,

$$\begin{aligned}
[N^{l.c}, \{\hat{Q}_1, \hat{Q}_{-1}\}] &= [N^{l.c}, \hat{Q}_1 \hat{Q}_{-1}] + [N^{l.c}, \hat{Q}_{-1} \hat{Q}_1] \\
&= [N^{l.c}, \hat{Q}_1] \hat{Q}_{-1} + \hat{Q}_1 [N^{l.c}, \hat{Q}_{-1}] \\
&\quad + \hat{Q}_{-1} [N^{l.c}, \hat{Q}_1] + [N^{l.c}, \hat{Q}_{-1}] \hat{Q}_1
\end{aligned}$$

$$[N^{l.c}, \{\hat{Q}_1, \hat{Q}_{-1}\}] = \cancel{\hat{Q}_1 \hat{Q}_{-1}} - \cancel{\hat{Q}_1 \hat{Q}_{-1}} + \cancel{\hat{Q}_{-1} \hat{Q}_1} - \cancel{\hat{Q}_{-1} \hat{Q}_1}$$

$$\rightarrow [N^{l.c}, \{\hat{Q}_1, \hat{Q}_{-1}\}] = 0$$

- So, $\{\hat{Q}_1, \hat{Q}_{-1}\}$ has light-cone number '0'.

Thus,

$$\begin{aligned} \hat{Q}_1^2 + (\{\hat{Q}_1, \hat{Q}_0\}) + (\{\hat{Q}_1, \hat{Q}_{-1}\} + \hat{Q}_0^2) \\ + (\{\hat{Q}_0, \hat{Q}_{-1}\}) + \hat{Q}_{-1}^2 = 0 \end{aligned}$$

Each term has different light-cone number. They cannot cancel each other. Thus they should vanish independently.

$$\rightarrow \hat{Q}_1^2 = 0, \{\hat{Q}_1, \hat{Q}_0\} = 0, \{\hat{Q}_0, \hat{Q}_{-1}\} = 0,$$

$$\{\hat{Q}_1, \hat{Q}_{-1}\} + \hat{Q}_0^2 = 0 \quad \& \quad \hat{Q}_{-1}^2 = 0$$

- Thus, $\hat{Q}_1$ is Nilpotent & forms a Cohomology of itself.
- Similarly, $\hat{Q}_{-1}$ is Nilpotent & forms a Cohomology of itself.
- $\hat{Q}_1$ & $\hat{Q}_0$ anti-commute with each other.
- $\hat{Q}_0$ & $\hat{Q}_{-1}$ anti-commute with each other.

Now, we should find the explicit expressions for $\hat{Q}_1$, $\hat{Q}_0$, $\hat{Q}_{-1}$ such that all commutation and anti-commutation relations that we discussed above holds.

For the open string,

$$\hat{Q}_B = \sum_{n \in \mathbb{Z}} \hat{C}_{-n} \hat{L}_n^M + \sum_{m, n \in \mathbb{Z}} \frac{(m-n)}{2} : \hat{C}_m \hat{C}_n \hat{b}_{-m-n} : - \hat{C}_0$$

$$\text{where, } \hat{L}_n^M = \hat{L}_n^{\mathcal{M}} + \hat{L}_n^{\mathcal{K}}$$

$$\begin{aligned}
\text{with } \hat{L}_n^M &= \frac{1}{2} \sum_{m \in \mathbb{Z}} \eta_{\mu\nu} : \hat{\alpha}_{n-m}^\mu \hat{\alpha}_m^\nu : \\
&= \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(- : \hat{\alpha}_{n-m}^0 \hat{\alpha}_m^0 : + : \hat{\alpha}_{n-m}^1 \hat{\alpha}_m^1 : \right) \\
&\quad + \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{i=2}^{d-1} : \hat{\alpha}_{n-m}^i \hat{\alpha}_m^i :
\end{aligned}$$

$$\text{use, } \hat{\alpha}_n^0 = \frac{\hat{\alpha}_n^+ + \hat{\alpha}_n^-}{\sqrt{2}} \quad \hat{\alpha}_n^1 = \frac{\hat{\alpha}_n^+ - \hat{\alpha}_n^-}{\sqrt{2}}$$

$$\hat{L}_n^M = \frac{1}{2} \sum_{m \in \mathbb{Z}} \left(- : \hat{\alpha}_{n-m}^+ \hat{\alpha}_m^- : - : \hat{\alpha}_{n-m}^- \hat{\alpha}_m^+ : + \sum_{i=2}^{d-1} : \hat{\alpha}_{n-m}^i \hat{\alpha}_m^i : \right)$$

Thus,

$$\begin{aligned}
\hat{Q}_B &= -\frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{C}_{-n} \left(: \hat{\alpha}_{n-m}^+ \hat{\alpha}_m^- : + : \hat{\alpha}_{n-m}^- \hat{\alpha}_m^+ : \right) \\
&\quad + \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sum_{i=2}^{d-1} \hat{C}_{-n} : \hat{\alpha}_{n-m}^i \hat{\alpha}_m^i : \\
&\quad + \sum_{n \in \mathbb{Z}} \hat{C}_{-n} \hat{L}_n^{\mathcal{H}} + \sum_{m, n \in \mathbb{Z}} \frac{(m-n)}{2} : \hat{C}_m \hat{C}_n \hat{b}_{-m-n} : - \hat{C}_0
\end{aligned}$$

All those terms without any light-cone oscillators ' $\hat{\alpha}^+$ or $\hat{\alpha}^-$ ' commute with ' $N^{l.c.}$ ' and will be a part of $\hat{Q}_0$.

Consider the terms ' $\hat{\alpha}_m^+ \hat{\alpha}_l^-$ ' whose $m, l \neq 0$,

$$\begin{aligned}
 [N^{l.c.}, \hat{\alpha}_m^+ \hat{\alpha}_l^-] &= \sum_{n=1}^{\infty} \frac{1}{n} [(\hat{\alpha}_{-n}^+ \hat{\alpha}_n^- - \hat{\alpha}_{-n}^- \hat{\alpha}_n^+), \hat{\alpha}_m^+ \hat{\alpha}_l^-] \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} ([\hat{\alpha}_{-n}^+ \hat{\alpha}_n^-, \hat{\alpha}_m^+ \hat{\alpha}_l^-] - [\hat{\alpha}_{-n}^- \hat{\alpha}_n^+, \hat{\alpha}_m^+ \hat{\alpha}_l^-]) \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\hat{\alpha}_{-n}^+ [\hat{\alpha}_n^-, \hat{\alpha}_m^+] \hat{\alpha}_l^- + \hat{\alpha}_m^+ [\hat{\alpha}_{-n}^+, \hat{\alpha}_l^-] \hat{\alpha}_n^- \right. \\
 &\quad \left. - \hat{\alpha}_{-n}^- \hat{\alpha}_m^+ [\hat{\alpha}_n^+, \hat{\alpha}_l^-] - [\hat{\alpha}_{-n}^-, \hat{\alpha}_m^+] \hat{\alpha}_n^+ \hat{\alpha}_l^- \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{n} \left(\cancel{m \delta_{m+n,0} \hat{\alpha}_{-n}^+ \hat{\alpha}_l^-} + \cancel{n \delta_{nl} \hat{\alpha}_m^+ \hat{\alpha}_n^-} \right. \\
 &\quad \left. + \cancel{n \delta_{n+l,0} \hat{\alpha}_{-n}^- \hat{\alpha}_m^+} - \cancel{m \delta_{nm} \hat{\alpha}_n^+ \hat{\alpha}_l^-} \right) \\
 &= \cancel{-\hat{\alpha}_m^+ \hat{\alpha}_l^-} + \cancel{\hat{\alpha}_m^+ \hat{\alpha}_l^-} - \cancel{\hat{\alpha}_l^- \hat{\alpha}_m^+} - \cancel{\hat{\alpha}_m^+ \hat{\alpha}_l^-}
 \end{aligned}$$

Thus, $[N^{l.c.}, \hat{\alpha}_m^+ \hat{\alpha}_l^-] = 0$ when both $m, l \neq 0$.

So, the ' $\hat{Q}_1$ ' & ' $\hat{Q}_{-1}$ ' parts occur when $m=0$ in the below term.

$$e^{-\frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \hat{C}_{-n} \left(\underbrace{\hat{\alpha}_{n-m}^+ \hat{\alpha}_m^-}_{\uparrow} + \underbrace{\hat{\alpha}_{n-m}^- \hat{\alpha}_m^+}_{\uparrow} \right)}$$

• $m=0 \neq n \neq 0$

gives $\hat{Q}_{-1}$ part

■ $m=0 \neq n \neq 0$

gives $\hat{Q}_1$ part

• $m \neq 0 \neq n=m$

gives $\hat{Q}_1$ part

■ $m \neq 0 \neq n=m$

gives $\hat{Q}_{-1}$ part

$$\hat{Q}_1 = -\frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{C}_{-n} (\hat{\alpha}_0^+ \hat{\alpha}_n^- + \hat{\alpha}_n^- \hat{\alpha}_0^+)$$

For open strings, $\frac{\hat{\alpha}_0^\mu}{\sqrt{2\alpha'}} = \hat{p}^\mu \sim k^\mu$ ↘ Eigen value

$$\rightarrow \boxed{\hat{Q}_1 = -\sqrt{2\alpha'} k^+ \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{C}_{-n} \hat{\alpha}_n^-}$$

Similarly,

$$\boxed{\hat{Q}_{-1} = -\sqrt{2\alpha'} k^- \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{C}_{-n} \hat{\alpha}_n^+}$$

The light cone coordinates and momenta are,

$$X^{\pm} = \frac{X^0 \pm X^1}{\sqrt{2}} \quad \& \quad k^{\pm} = \frac{k^0 \pm k^1}{\sqrt{2}}$$

Now,

$$k^0 = E_k = \sqrt{m^2 + |\vec{k}|^2} = \sqrt{m^2 + (k^1)^2 + (k^{\perp})^2}$$

↘ space-time mass ↘

where,

k^1 = space-time momenta along X^1 -axis.

$(k^{\perp})^2 \rightarrow$ Magnitude of momenta along transverse directions.

We can always choose a frame of reference whose ' k^+ ' component is not zero.

Consider, $k^+ = \frac{k^0 + k^1}{\sqrt{2}} = \frac{E_k + k^1}{\sqrt{2}}$

For any particle ' $k^0 = E_k > 0$ '.

$$k^1 = \pm \sqrt{E_k^2 - m^2 - (k^{\perp})^2}$$

$$\rightarrow k^1 = \pm E_k \sqrt{1 - \left(\frac{m^2 + (k^\perp)^2}{E_k^2} \right)}$$

$$k^1 = \pm E_k \left[1 - \left(\frac{m^2 + (k^\perp)^2}{E_k^2} \right) - \frac{1}{8} \left(\frac{m^2 + (k^\perp)^2}{E_k^2} \right)^2 \dots \right]$$

$$k^+ = \frac{1}{\sqrt{2}} \left[E_k \pm E_k \mp \left(\frac{m^2 + (k^\perp)^2}{E_k} \right) \mp \frac{E_k}{8} \left(\frac{m^2 + (k^\perp)^2}{E_k^2} \right)^2 \dots \right]$$

For particle moving along +ve X^1 -axis, k^1 is +ve. So,

$$k^+ = \frac{1}{\sqrt{2}} \left[2E_k - \left(\frac{m^2 + (k^\perp)^2}{E_k} \right) - \frac{E_k}{8} \left(\frac{m^2 + (k^\perp)^2}{E_k^2} \right)^2 \dots \right]$$

Thus, for particle moving along +ve X^1 axis, k^+ is always non-zero.

For particle moving along -ve X^1 -axis, k^1 is -ve. So,

$$k^+ = \frac{1}{\sqrt{2}} \left[\left(\frac{m^2 + (k^\perp)^2}{E_k} \right) + \frac{E_k}{8} \left(\frac{m^2 + (k^\perp)^2}{E_k^2} \right)^2 \dots \right]$$

For massive particle, whose spacetime mass is not zero ($m \neq 0$), k^+ is never zero.

But for massless particle ($m = 0$),

$$k^+ = \frac{E_k}{\sqrt{2}} \left[\left(\frac{k^\perp}{E_k} \right)^2 + \frac{1}{8} \left(\frac{k^\perp}{E_k} \right)^4 \dots \right]$$

$$\rightarrow k^+ = \frac{1}{\sqrt{2}} \left[E_k - \sqrt{E_k^2 - (k^\perp)^2} \right]$$

Thus, for a massless particle moving along -ve X^1 only, without any ' $k^\perp$ ', k^+ is zero & $k^- = \sqrt{2} E_k$. But we can choose a frame with a rotated spatial axes such that we exchange $X^1 \longleftrightarrow -X^1$ & thus $X^\pm \longleftrightarrow X^\mp$. Hence, $k^\pm \longleftrightarrow k^\mp$ & consequently $k^+ \neq 0$.

Now, we have,

$$\hat{Q}_1 = -\sqrt{2\alpha'} k^+ \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{c}_n)$$

$$\hat{Q}_{-1} = -\sqrt{2\alpha'} k^- \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{c}_n)$$

Define an operator,

$$\hat{R} = \frac{1}{\sqrt{2\alpha'} k^+} \sum_{n=1}^{\infty} (\hat{b}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{b}_n)$$

Both $\hat{Q}_1$ & $\hat{R}$ annihilate the ground state.

Consider, $\hat{\mathcal{S}} = \{\hat{Q}_1, \hat{R}\}$, it also should annihilate the ground state.

$$\hat{\mathcal{S}} = \hat{Q}_1 \hat{R} + \hat{R} \hat{Q}_1$$

$$= - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[(\hat{c}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{c}_n) (\hat{b}_{-m} \hat{\alpha}_m^+ + \hat{\alpha}_{-m}^+ \hat{b}_m) \right. \\ \left. + (\hat{b}_{-m} \hat{\alpha}_m^+ + \hat{\alpha}_{-m}^+ \hat{b}_m) (\hat{c}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{c}_n) \right]$$

$$\hat{\mathcal{G}} = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\begin{aligned} & \underline{\hat{c}_{-n} \hat{b}_{-m} \hat{\alpha}_n^- \hat{\alpha}_m^+} + \underline{\hat{c}_{-n} \hat{b}_m \hat{\alpha}_n^- \hat{\alpha}_{-m}^+} \\ & + \underline{\hat{c}_n \hat{b}_{-m} \hat{\alpha}_{-n}^- \hat{\alpha}_m^+} + \underline{\hat{c}_n \hat{b}_m \hat{\alpha}_{-n}^- \hat{\alpha}_{-m}^+} \\ & + \underline{\hat{b}_{-m} \hat{c}_{-n} \hat{\alpha}_m^+ \hat{\alpha}_n^-} + \underline{\hat{b}_{-m} \hat{c}_n \hat{\alpha}_m^+ \hat{\alpha}_{-n}^-} \\ & + \underline{\hat{b}_m \hat{c}_n \hat{\alpha}_{-m}^+ \hat{\alpha}_{-n}^-} + \underline{\hat{b}_m \hat{c}_{-n} \hat{\alpha}_{-m}^+ \hat{\alpha}_n^-} \end{aligned} \right]$$

$$= - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\begin{aligned} & (\hat{c}_{-n} \hat{b}_{-m} + \hat{b}_{-m} \hat{c}_{-n}) \hat{\alpha}_n^- \hat{\alpha}_m^+ \quad \xrightarrow{\text{blue } 0} \\ & + (\hat{b}_m \hat{c}_n + \hat{c}_n \hat{b}_m) \hat{\alpha}_{-n}^- \hat{\alpha}_{-m}^+ \quad \xrightarrow{\text{purple } 0} \\ & + \hat{c}_{-n} \hat{b}_m \hat{\alpha}_n^- \hat{\alpha}_{-m}^+ \\ & + (\delta_{mn} - \hat{c}_{-n} \hat{b}_m) \hat{\alpha}_{-m}^+ \hat{\alpha}_n^- \\ & + \hat{c}_n \hat{b}_{-m} \hat{\alpha}_{-n}^- \hat{\alpha}_m^+ \\ & + (\delta_{mn} - \hat{c}_n \hat{b}_{-m}) \hat{\alpha}_m^+ \hat{\alpha}_{-n}^- \end{aligned} \right]$$

$$\hat{\mathcal{G}} = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\begin{aligned} & \hat{c}_{-n} \hat{b}_m (\hat{\alpha}_n^- \hat{\alpha}_{-m}^+ - \hat{\alpha}_{-m}^+ \hat{\alpha}_n^-) \\ & + \delta_{mn} \hat{\alpha}_{-m}^+ \hat{\alpha}_n^- \\ & + \hat{c}_n \hat{b}_{-m} (\hat{\alpha}_{-n}^- \hat{\alpha}_m^+ - \hat{\alpha}_m^+ \hat{\alpha}_{-n}^-) \\ & + \delta_{mn} \hat{\alpha}_m^+ \hat{\alpha}_{-n}^- \end{aligned} \right]$$

$$\hat{\mathcal{S}} = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\begin{aligned} &\hat{c}_{-n} \hat{b}_m [\hat{\alpha}_{-n}^-, \hat{\alpha}_{-m}^+] \\ &+ \hat{c}_n \hat{b}_{-m} [\hat{\alpha}_{-n}^-, \hat{\alpha}_m^+] \\ &+ \delta_{mn} \hat{\alpha}_m^+ \hat{\alpha}_{-n}^- + \delta_{mn} \hat{\alpha}_{-m}^+ \hat{\alpha}_n^- \end{aligned} \right]$$

use $[\hat{\alpha}_{-n}^-, \hat{\alpha}_m^+] = m \delta_{mn}$ & $[\hat{\alpha}_n^-, \hat{\alpha}_{-m}^+] = -m \delta_{mn}$

$$\hat{\mathcal{S}} = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\begin{aligned} &-m \hat{c}_{-n} \hat{b}_m \delta_{mn} + m \hat{c}_n \hat{b}_{-m} \delta_{mn} \\ &+ \delta_{mn} \hat{\alpha}_m^+ \hat{\alpha}_{-n}^- + \delta_{mn} \hat{\alpha}_{-m}^+ \hat{\alpha}_n^- \end{aligned} \right]$$

$$= - \sum_{n=1}^{\infty} \left[-n \hat{c}_{-n} \hat{b}_n + n \hat{c}_n \hat{b}_{-n} + \hat{\alpha}_n^+ \hat{\alpha}_{-n}^- + \hat{\alpha}_{-n}^+ \hat{\alpha}_n^- \right]$$

$$= - \sum_{n=1}^{\infty} \left[-n \hat{c}_{-n} \hat{b}_n - n \hat{b}_{-n} \hat{c}_n + \cancel{n} + \hat{\alpha}_{-n}^+ \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{\alpha}_n^+ - \cancel{n} \right]$$

Zero normal ordering constant as $\hat{\mathcal{S}}$ annihilates the ground state.

$$\Rightarrow \hat{\mathcal{S}} = \sum_{n=1}^{\infty} \left[\begin{aligned} &n \hat{c}_{-n} \hat{b}_n + n \hat{b}_{-n} \hat{c}_n \\ &- \hat{\alpha}_{-n}^+ \hat{\alpha}_n^- - \hat{\alpha}_{-n}^- \hat{\alpha}_n^+ \end{aligned} \right]$$

$$-\frac{\hat{\alpha}_{-n}^+ \hat{\alpha}_n^-}{n} = N_n^+ \rightarrow \text{'+' light cone number operator}$$

$$-\frac{\hat{\alpha}_{-n}^- \hat{\alpha}_n^+}{n} = N_n^- \rightarrow \text{'-' light cone number operator}$$

$$\hat{c}_{-n} \hat{b}_n = N_n^c \rightarrow \text{c-ghost number operator}$$

$$\hat{b}_{-n} \hat{c}_n = N_n^b \rightarrow \text{b-ghost number operator}$$

$$\Rightarrow \boxed{\hat{\mathcal{L}} = \sum_{n=1}^{\infty} n (N_n^c + N_n^b + N_n^- + N_n^+)}$$

Also note that $\hat{Q}_1$ & $\hat{\mathcal{L}} = \{\hat{Q}_1, \hat{R}\}$ commute with each other.

$$[\hat{Q}_1, \hat{\mathcal{L}}] = [\hat{Q}_1, \{\hat{Q}_1, \hat{R}\}]$$

$$= [\hat{Q}_1, \hat{Q}_1 \hat{R} + \hat{R} \hat{Q}_1]$$

$$= [\hat{Q}_1, \hat{Q}_1 \hat{R}] + [\hat{Q}_1, \hat{R} \hat{Q}_1]$$

$$= \underbrace{\hat{Q}_1^2 \hat{R}}_0 - \cancel{\hat{Q}_1 \hat{R} \hat{Q}_1} + \cancel{\hat{Q}_1 \hat{R} \hat{Q}_1} - \underbrace{\hat{R} \hat{Q}_1^2}_0$$

$$\Rightarrow \underline{[\hat{Q}_1, \hat{\mathcal{L}}] = 0}$$

So $\hat{Q}_1$ and $\hat{\mathcal{S}}$ can be diagonalized simultaneously. Now, consider the $\hat{Q}_1$ -closed states, i.e. those states ' $|\psi\rangle$ ' which satisfy

$$' \hat{Q}_1 |\psi\rangle = 0 ' \implies |\psi\rangle \text{ is } \hat{Q}_1\text{-closed}$$

Since ' $|\psi\rangle$ ' is some zero Eigen vector, it should also be an Eigen vector of $\hat{\mathcal{S}}$.

Let $\hat{\mathcal{S}} |\psi\rangle = s |\psi\rangle$. If $s \neq 0$, then,

$$\begin{aligned} |\psi\rangle &= \frac{1}{s} \hat{\mathcal{S}} |\psi\rangle = \frac{1}{s} \{ \hat{Q}_1, \hat{\mathcal{R}} \} |\psi\rangle \\ &= \frac{1}{s} (\hat{Q}_1 \hat{\mathcal{R}} + \hat{\mathcal{R}} \hat{Q}_1) |\psi\rangle \end{aligned}$$

$$|\psi\rangle = \frac{1}{s} \hat{Q}_1 \hat{\mathcal{R}} |\psi\rangle = \hat{Q}_1 \left(\frac{1}{s} \hat{\mathcal{R}} |\psi\rangle \right)$$

$\implies |\psi\rangle$ is of the form $\hat{Q}_1 |\chi\rangle$.

Thus, if $\hat{\mathcal{S}} |\psi\rangle = s |\psi\rangle$ with $s \neq 0$ and $|\psi\rangle$ is $\hat{Q}_1$ -closed, then $|\psi\rangle$ is $\hat{Q}_1$ -exact.

So, the states in $\hat{Q}_1$ -cohomology should satisfy $\hat{\mathcal{S}}|\psi\rangle = 0$ and $\hat{Q}_1|\psi\rangle = 0$.

The states with zero $\hat{\mathcal{S}}$ eigen value have ' $\sum_{n=1}^{\infty} n(N_n^c + N_n^b + N_n^- + N_n^+)$ ' to be zero.

Since $\hat{\mathcal{S}}$ is a counting for ghosts & light cone excitations, it implies that $\hat{Q}_1$ -cohomology has no ghosts & light cone excitations same as the states in $\mathbb{H}^\perp$.

Also, $\mathbb{H}^\perp$ cannot have $\hat{Q}_1$ -exact states. $\hat{Q}_1$ -exact states in $\mathbb{H}^\perp$ should be of the form $-\sqrt{2\alpha'} k^+ \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{c}_n) |\chi\rangle$

where $|\chi\rangle \in \mathbb{H}^\perp$. But every $|\chi\rangle$ in $\mathbb{H}^\perp$ is annihilated by $\hat{\alpha}_{n>0}^-$ & $\hat{c}_{n>0}$. So, for $|\chi\rangle \in \mathbb{H}^\perp$, $\hat{Q}_1|\chi\rangle = 0$.

Hence, $\hat{Q}_1$ -cohomology $\simeq \mathbb{H}^\perp$

Another abstract way of proving is as follows. The states in $\mathbb{H}^\perp$ are built by the transverse excitations on ground states of the form $|0, k^\mu, I, \downarrow\rangle$ whose, $k^\mu k_\mu = -m^2$.

The ghost number of ground states is $-\frac{1}{2}$.

$$\text{Since, } N^g |0, k^\mu, I, \downarrow\rangle = -\frac{1}{2} |0, k^\mu, I, \downarrow\rangle$$

For the zero eigenvalue states of $\hat{\mathcal{S}}$, if $\hat{\mathcal{S}}|\psi\rangle = 0$. Due to the absence of ghost excitations, the ghost number of $|\psi\rangle$ should still be $-\frac{1}{2}$. When $\hat{\mathcal{S}}|\psi\rangle = 0$

$$\Rightarrow \hat{Q}_1 \hat{\mathcal{S}}|\psi\rangle = 0 \Rightarrow \hat{\mathcal{S}} \hat{Q}_1 |\psi\rangle = 0$$

$$\text{Now, } \hat{Q}_1 = -\sqrt{2\alpha'} k^+ \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{C}_n)$$

$\hat{Q}_1$ has ghost number of $+1$. So, $\hat{Q}_1 |\psi\rangle$ has ghost number of $+\frac{1}{2}$.

Now, if the Eigen value under $\hat{S}$ of $\hat{Q}_1|\psi\rangle$ is not zero, then we can create a $\hat{S}^{-1}$ matrix without the zero eigen vectors of $\hat{S}$ & thus, when,

$$\begin{aligned}\hat{S}|\psi\rangle = 0 &\Rightarrow \hat{Q}_1\hat{S}|\psi\rangle = 0 \Rightarrow \hat{S}\hat{Q}_1|\psi\rangle = 0 \\ &\Rightarrow \hat{S}^{-1}\hat{S}\hat{Q}_1|\psi\rangle = 0 \Rightarrow \hat{Q}_1|\psi\rangle = 0.\end{aligned}$$

Thus states with $\hat{S}|\psi\rangle = 0$ whose $S=0$, are $\hat{Q}_1$ closed. And they are all free of ghosts & light cone excitations same as the states in $H^\perp$.

Hence, we proved, that $\hat{Q}_1$ -cohomology comes only if $S=0$ & then we also proved, the states with $S=0$ are $\hat{Q}_1$ -invariant, i.e, $\hat{Q}_1$ -closed. And they are also not $\hat{Q}_1$ -exact. Hence, the states with $S=0$ forms $\hat{Q}_1$ -cohomology.

Next, we have to show that the $\hat{Q}_1$ -cohomology is isomorphic to the $\hat{Q}_B$ -cohomology.

To prove $\hat{Q}_1$ -cohomology $\simeq \hat{Q}_B$ -cohomology

Now, consider $\hat{T} = \{\hat{Q}_B, \hat{R}\}$. It annihilates the ground state.

$$\hat{T} = \{\hat{Q}_B, \hat{R}\} = \{\hat{Q}_1 + \hat{Q}_0 + \hat{Q}_{-1}, \hat{R}\}$$

$$= \underbrace{\{\hat{Q}_1, \hat{R}\}}_{\hat{\mathcal{S}}} + \underbrace{\{\hat{Q}_0 + \hat{Q}_{-1}, \hat{R}\}}_{\hat{\mathcal{U}}}$$

$$\Rightarrow \underline{\hat{T} = \hat{\mathcal{S}} + \hat{\mathcal{U}} = \{\hat{Q}_B, \hat{R}\}}$$

$$\text{where } \hat{\mathcal{U}} = \{\hat{Q}_0 + \hat{Q}_{-1}, \hat{R}\}$$

$$\text{Then, } [N^{l.c}, \hat{\mathcal{U}}] = [N^{l.c}, \{\hat{Q}_0 + \hat{Q}_{-1}, \hat{R}\}]$$

$$= [N^{l.c}, \hat{Q}_0 \hat{R} + \hat{Q}_{-1} \hat{R}] + [N^{l.c}, \hat{R} \hat{Q}_0 + \hat{R} \hat{Q}_{-1}]$$

$$\begin{aligned}
[N^{L.C}, \hat{U}] &= [N^{L.C}, \hat{Q}_0 \hat{R}] + [N^{L.C}, \hat{Q}_{-1} \hat{R}] \\
&\quad + [N^{L.C}, \hat{R} \hat{Q}_0] + [N^{L.C}, \hat{R} \hat{Q}_{-1}] \\
&= [N^{L.C}, \hat{Q}_0] \hat{R} + \hat{Q}_0 [N^{L.C}, \hat{R}] + [N^{L.C}, \hat{Q}_{-1}] \hat{R} \\
&\quad + \hat{Q}_{-1} [N^{L.C}, \hat{R}] + \hat{R} [N^{L.C}, \hat{Q}_0] + [N^{L.C}, \hat{R}] \hat{Q}_0 \\
&\quad + \hat{R} [N^{L.C}, \hat{Q}_{-1}] + [N^{L.C}, \hat{R}] \hat{Q}_{-1}
\end{aligned}$$

use $[N^{L.C}, \hat{Q}_j] = j \hat{Q}_j$, where $j = \{1, 0, -1\}$

$$\begin{aligned}
[N^{L.C}, \hat{R}] &= \frac{1}{\sqrt{2\alpha'} k^+} \sum_{n=1}^{\infty} [N^{L.C}, (\hat{b}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{b}_n)] \\
&= \frac{1}{\sqrt{2\alpha'} k^+} \sum_{n=1}^{\infty} \left(\hat{b}_{-n} \underbrace{[N^{L.C}, \hat{\alpha}_n^+]}_{-\hat{\alpha}_n^+ (1 - \delta_{n0})} + \underbrace{[N^{L.C}, \hat{\alpha}_{-n}^+]}_{-\hat{\alpha}_{-n}^+ (1 - \delta_{n0})} \hat{b}_n \right) \\
&= \frac{-1}{\sqrt{2\alpha'} k^+} \sum_{n=1}^{\infty} (\hat{b}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{b}_n) \\
&\rightarrow \underline{[N^{L.C}, \hat{R}] = -\hat{R}}
\end{aligned}$$

$$\begin{aligned} \rightarrow [N^{l.c}, \hat{U}] &= -\hat{Q}_0 \hat{R} - \hat{Q}_{-1} \hat{R} - \hat{Q}_{-1} \hat{R} \\ &\quad - \hat{R} \hat{Q}_0 - \hat{R} \hat{Q}_{-1} - \hat{R} \hat{Q}_{-1} \end{aligned}$$

$$\begin{aligned} \rightarrow [N^{l.c}, \hat{U}] &= -\{\hat{Q}_0, \hat{R}\} - 2\{\hat{Q}_{-1}, \hat{R}\} \\ &= -\hat{U} - \{\hat{Q}_{-1}, \hat{R}\} \end{aligned}$$

So, $\hat{U}$ lowers the light-cone number by one unit or two units.

And since $\hat{\mathcal{S}} = \{\hat{Q}_1, \hat{R}\}$

$$\rightarrow [N^{l.c}, \hat{\mathcal{S}}] = [N^{l.c}, \{\hat{Q}_1, \hat{R}\}]$$

$$= [N^{l.c}, \hat{Q}_1] \hat{R} + \hat{Q}_1 [N^{l.c}, \hat{R}]$$

$$+ \hat{R} [N^{l.c}, \hat{Q}_1] + [N^{l.c}, \hat{R}] \hat{Q}_1$$

$$= \cancel{\hat{Q}_1 \hat{R}} - \cancel{\hat{Q}_1 \hat{R}} + \cancel{\hat{R} \hat{Q}_1} - \cancel{\hat{R} \hat{Q}_1}$$

$$\rightarrow [N^{l.c}, \hat{\mathcal{S}}] = 0$$

Since, $[N^{l.c}, \hat{U}] = -\{\hat{Q}_0, \hat{R}\} - 2\{\hat{Q}_{-1}, \hat{R}\}$
 if we write $\hat{U}$ in a light cone number basis, then $\hat{U}$ maps a basis vector with a light-cone number 'l' to linear combination of two states with light-cone numbers $l-1$ & $l-2$. Thus, in this basis ' $\hat{U}$ ' is a strictly lower triangular matrix.

Similarly, as $[N^{l.c}, \hat{\mathcal{S}}] = 0$, $\hat{\mathcal{S}}$ maps a basis vector with a light-cone number 'l' to the same basis vector with same 'l'. Thus, $\hat{\mathcal{S}}$ is diagonal in this basis.

So, $\hat{T} = \hat{\mathcal{S}} + \hat{U}$ is a lower triangular matrix. Now, we can show that the Kernel of $\hat{\mathcal{S}} + \hat{U}$ is isomorphic to Kernel of $\hat{\mathcal{S}}$.

Let $\hat{\mathcal{S}}|\psi_0\rangle = 0$. Then, consider

$$|\psi\rangle = \left(1 - \hat{\mathcal{S}}^{-1}\hat{U} + (\hat{\mathcal{S}}^{-1}\hat{U})^2 - (\hat{\mathcal{S}}^{-1}\hat{U})^3 \dots\right)|\psi_0\rangle$$

$\hat{\mathcal{S}}^{-1}$ makes sense as they always act on states whose $N^{l.c} < 0$. So ' $\hat{\mathcal{S}}$ ' is invertible.

$$\text{Then, } \hat{T}|\psi\rangle = (\hat{\mathcal{S}} + \hat{U})|\psi\rangle$$

$$= (\hat{\mathcal{S}} + \hat{U}) \left(1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots \right) |\psi_0\rangle$$

$$= \hat{\mathcal{S}} \left(1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots \right) |\psi_0\rangle \\ + \hat{U} \left(1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots \right) |\psi_0\rangle$$

$$= (\hat{\mathcal{S}} - \cancel{\hat{U}} + \cancel{\hat{U} \hat{\mathcal{S}}^{-1}} \hat{U} - \cancel{\hat{U} \hat{\mathcal{S}}^{-1} \hat{U} \hat{\mathcal{S}}^{-1}} \hat{U} \dots) |\psi_0\rangle \\ + (\cancel{\hat{U}} - \cancel{\hat{U} \hat{\mathcal{S}}^{-1}} \hat{U} + \cancel{\hat{U} \hat{\mathcal{S}}^{-1} \hat{U} \hat{\mathcal{S}}^{-1}} \hat{U} \dots) |\psi_0\rangle$$

$$= \hat{\mathcal{S}} |\psi_0\rangle = 0$$

Thus,

$$(\hat{\mathcal{S}} + \hat{U}) \left(1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots \right) = \hat{\mathcal{S}}$$

$$(\hat{\mathcal{S}} + \hat{U}) \left(1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots \right) |\psi_0\rangle = \hat{\mathcal{S}} |\psi_0\rangle$$

Hence, for all $|\psi_0\rangle \in \text{Kernel}(\hat{\mathcal{S}})$, there exists,

$$|\psi\rangle = \left(1 - \hat{\mathcal{S}}^{-1}\hat{U} + (\hat{\mathcal{S}}^{-1}\hat{U})^2 - (\hat{\mathcal{S}}^{-1}\hat{U})^3 + \dots\right)|\psi_0\rangle$$

Such that $|\psi\rangle \in \text{Kernel}(\hat{\mathcal{S}} + \hat{U})$

$$\implies \text{Kernel}(\hat{\mathcal{S}} + \hat{U}) \simeq \text{Kernel}(\hat{\mathcal{S}})$$

We can see that, $\hat{T} = \hat{\mathcal{S}} + \hat{U} = \{\hat{Q}_B, \hat{R}\}$
& $\hat{Q}_B$ commute with each other.

$$[\hat{Q}_B, \hat{T}] = [\hat{Q}_B, \{\hat{Q}_B, \hat{R}\}]$$

$$= \hat{Q}_B \{\hat{Q}_B, \hat{R}\} - \{\hat{Q}_B, \hat{R}\} \hat{Q}_B$$

$$= \hat{Q}_B (\hat{Q}_B \hat{R} + \hat{R} \hat{Q}_B) - (\hat{Q}_B \hat{R} + \hat{R} \hat{Q}_B) \hat{Q}_B$$

$$= \cancel{\hat{Q}_B^2} \hat{R} + \hat{Q}_B \cancel{\hat{R}} \hat{Q}_B - \hat{Q}_B \cancel{\hat{R}} \hat{Q}_B - \hat{R} \cancel{\hat{Q}_B^2}$$

$$\implies [\hat{Q}_B, \hat{T}] = 0$$

Now, we can diagonalize $\hat{Q}_B$ & $\hat{T}$ simultaneously.

Consider $\hat{Q}_B$ -closed states $|\psi\rangle$.

$$\text{i.e. } \hat{Q}_B|\psi\rangle = 0 \text{ \& } \hat{T}|\psi\rangle = t|\psi\rangle$$

$$\text{If } t \neq 0, \quad |\psi\rangle = \frac{1}{t} \hat{T}|\psi\rangle$$

$$= \frac{1}{t} \{\hat{Q}_B, \hat{R}\}|\psi\rangle$$

$$= \frac{1}{t} (\hat{Q}_B \hat{R}|\psi\rangle + \underbrace{\hat{R} \hat{Q}_B|\psi\rangle}_0)$$

$$\Rightarrow |\psi\rangle = \hat{Q}_B \left(\frac{1}{t} \hat{R}|\psi\rangle \right)$$

Thus, if $\hat{Q}_B|\psi\rangle = 0$ & $\hat{T}|\psi\rangle = t|\psi\rangle$ with $t \neq 0$, $\Rightarrow |\psi\rangle$ is a $\hat{Q}_B$ -exact state.

So, any physical state has to belong to $t=0$ eigen space or the Kernel($\hat{S} + \hat{U}$). Only then those states are not $\hat{Q}_B$ -exact.

Conversely, when, $\hat{T}|\psi\rangle = 0$

$$\rightarrow \hat{Q}_B \hat{T}|\psi\rangle = 0$$

$$\rightarrow \hat{T} \hat{Q}_B |\psi\rangle = 0$$

$$\rightarrow (\hat{\mathcal{S}} + \hat{U}) \hat{Q}_B |\psi\rangle = 0$$

$$\rightarrow \frac{1}{(1 + \hat{\mathcal{S}}^{-1} \hat{U})} (\hat{\mathcal{S}} + \hat{U}) \hat{Q}_B |\psi\rangle = 0$$

$$\Rightarrow (1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots) (\hat{\mathcal{S}} + \hat{U}) \hat{Q}_B |\psi\rangle = 0$$

$$\Rightarrow \hat{\mathcal{S}} \hat{Q}_B |\psi\rangle = 0 \Rightarrow \hat{\mathcal{S}}^{-1} \hat{\mathcal{S}} \hat{Q}_B |\psi\rangle = 0$$

$$\Rightarrow \underline{\hat{Q}_B |\psi\rangle = 0}$$

We have used the fact that $\hat{T} \neq \hat{\mathcal{S}}$ have inverses except at ghost no. $-\frac{1}{2}$. And $\hat{Q}_B |\psi\rangle$ has ghost no. of $+\frac{1}{2}$. Since $\hat{T}|\psi\rangle = 0$ implies $\hat{\mathcal{S}}|\psi\rangle$ as their kernels are isomorphic.

Thus, we conclude that, Physical states that belong to $\hat{Q}_B$ -cohomology are those states from $\text{Kernel}(\hat{T})$.

Since, if $\hat{Q}_B|\psi\rangle = 0$ & $\hat{T}|\psi\rangle = t|\psi\rangle$ with $t \neq 0$, $\Rightarrow |\psi\rangle$ is a $\hat{Q}_B$ -exact state. So, any physical state has to belong to $t=0$ eigen space or the $\text{Kernel}(\hat{\mathcal{S}} + \hat{U})$ & conversely, $\hat{T}|\psi\rangle = 0 \Rightarrow \hat{Q}_B|\psi\rangle = 0$.

Thus,

$$\hat{Q}_B\text{-cohomology} \simeq \text{Kernel}(\hat{\mathcal{S}} + \hat{U}) \simeq \text{Kernel}(\hat{\mathcal{S}})$$

$$\& \text{Kernel}(\hat{\mathcal{S}}) \simeq \hat{Q}_1\text{-cohomology} \simeq \mathbb{H}^\perp$$

$$\Rightarrow \boxed{\hat{Q}_B\text{-cohomology} \simeq \mathbb{H}^\perp}$$

Then, lastly we should show that, the inner products are positive in $\hat{Q}_B$ -cohomology.

We know that all states in the cohomology of $\hat{Q}_2$ is of the form

$$|\psi\rangle = \left(1 - \hat{\mathcal{L}}^{-1} \hat{U} + (\hat{\mathcal{L}}^{-1} \hat{U})^2 - (\hat{\mathcal{L}}^{-1} \hat{U})^3 \dots\right) |\psi_0\rangle$$

where,

$$|\psi_0\rangle \in \text{Kernel}(\hat{\mathcal{L}}) \simeq \hat{Q}_1\text{-cohomology} \simeq \mathbb{H}^1$$

Now, the factor below,

$$\left(1 - \hat{\mathcal{L}}^{-1} \hat{U} + (\hat{\mathcal{L}}^{-1} \hat{U})^2 - (\hat{\mathcal{L}}^{-1} \hat{U})^3 \dots\right)$$

has a -ve light-cone number since $\hat{U}$ reduces it by one or two units. If we denote the above in light cone basis,

$$|\psi\rangle = |\psi_0\rangle + a_{-1} |\psi_{-1}\rangle + a_{-2} |\psi_{-2}\rangle + a_{-3} |\psi_{-3}\rangle \dots$$

This is an orthonormal basis.

$$\langle\psi|\psi\rangle = \langle\psi_0|\psi_0\rangle + \sum_{m,n=1}^{\infty} a_{-m}^* a_{-n} \langle\psi_{-m}|\psi_{-n}\rangle$$

$$\text{But } \langle\psi_m|\psi_n\rangle = \delta_{m+n,0}$$

$$\Rightarrow \langle \psi \| \psi \rangle = \langle \psi_0 \| \psi_0 \rangle + \sum_{m,n=1}^{\infty} a_{-m}^* a_{-n} \underbrace{\delta_{m+n,0}}_{\substack{\text{not possible} \\ \text{when } m,n > 0}}$$

$$\text{Thus } \sum_{m,n=1}^{\infty} a_{-m}^* a_{-n} \delta_{m+n,0} = 0$$

$$\Rightarrow \underline{\langle \psi \| \psi \rangle = \langle \psi_0 \| \psi_0 \rangle}$$

But $|\psi_0\rangle \in \mathbb{H}^\perp$ & hence has no any -ve norm state. Thus, $\langle \psi_0 \| \psi_0 \rangle$ is +ve. So, the $\langle \psi \| \psi \rangle$ in $\hat{Q}_B$ -cohomology also has +ve innerproduct.

$\Rightarrow \hat{Q}_B$ -cohomology $\simeq \mathbb{H}^\perp$ holds as a Hilbert Space (i.e as a Vector Space and as well as an innerproduct Space).

Now, $\mathbb{H}^\perp$ has no light cone or ghost excitations. So, it is isomorphic to $\mathbb{H}_{l.c.}$

This is the $\mathbb{H}_{BRST} - \mathbb{H}_{l.c}$ equivalence.

To prove $H_{BRST} - H_{OCG}$ equivalence

To a state $|\psi^M\rangle$ in the matter Hilbert space, associate a state $|\psi^M, \downarrow\rangle$ from the full matter plus Ghost theory.

$$|\psi^M, \downarrow\rangle = |\psi^M\rangle \otimes |\downarrow\rangle$$

where $\hat{b}_{n>0}|\downarrow\rangle = 0 = \hat{c}_{n>0}|\downarrow\rangle$. Now,

$$\begin{aligned}\hat{Q}_B &= \sum_{n \in \mathbb{Z}} \hat{c}_{-n} \hat{L}_n^M + \sum_{m, n \in \mathbb{Z}} \frac{(m-n)}{2} : \hat{c}_m \hat{c}_n \hat{b}_{-m-n} : - \hat{c}_0 \\ &= \hat{c}_0 (\hat{L}_0^M - 1) + \sum_{n=1}^{\infty} (\hat{c}_{-n} \hat{L}_n^M + \hat{L}_{-n}^M \hat{c}_n) \\ &\quad + \sum_{m, n \in \mathbb{Z}} \frac{(m-n)}{2} : \hat{c}_m \hat{c}_n \hat{b}_{-m-n} : \end{aligned}$$

We see that

$$\sum_{m, n \in \mathbb{Z}} \frac{(m-n)}{2} : \hat{c}_m \hat{c}_n \hat{b}_{-m-n} : |\psi^M, \downarrow\rangle = 0$$

&

$$\sum_{n=1}^{\infty} \hat{L}_{-n}^M \hat{c}_n |\psi^M, \downarrow\rangle = 0$$

So, when $\hat{Q}_B$ acts on $|\psi^M, \downarrow\rangle$ we get,

$$\hat{Q}_B |\psi^M, \downarrow\rangle = \hat{C}_0 (\hat{L}_0^M - 1) |\psi^M, \downarrow\rangle + \sum_{n=1}^{\infty} \hat{C}_{-n} \hat{L}_n^M |\psi^M, \downarrow\rangle$$

$$\Rightarrow \hat{Q}_B |\psi^M, \downarrow\rangle = \sum_{n=0}^{\infty} \hat{C}_{-n} (\hat{L}_n^M - \delta_{n,0}) |\psi^M, \downarrow\rangle$$

Thus, if $|\psi^M, \downarrow\rangle$ has to be a $\hat{Q}_B$ -closed state, then the matter state $|\psi^M\rangle$ should satisfy, $(\hat{L}_n^M - \delta_{n,0}) |\psi^M\rangle = 0 \quad \forall n \geq 0$

$$\Rightarrow \underline{\hat{L}_{n>0}^M |\psi^M\rangle = 0 \quad \& \quad (\hat{L}_0^M - 1) |\psi^M\rangle = 0}$$

Thus, if the matter state $|\psi^M\rangle$ satisfies the OCQ Virasoro constraints, then the state $|\psi^M, \downarrow\rangle$ automatically satisfies BRST closedness.

Now, we have to establish the equivalence relations b/w OCQ & BRST states.

If $|\psi^M\rangle \neq |\psi'^M\rangle$ represents two states from same OCQ physical equivalence class, where $|\psi^M\rangle - |\psi'^M\rangle$ is a OCQ null state with zero norm. Then, we have to show that the states $|\psi^M, \downarrow\rangle \neq |\psi'^M, \downarrow\rangle$ belong to the same BRST equivalence class.

Let, $|\psi^M\rangle - |\psi'^M\rangle = |\chi^M\rangle$. Define

$$|\chi^M, \downarrow\rangle = |\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle$$

$$\Rightarrow \hat{Q}_B |\chi^M, \downarrow\rangle = \underbrace{\hat{Q}_B |\psi^M, \downarrow\rangle}_0 - \underbrace{\hat{Q}_B |\psi'^M, \downarrow\rangle}_0 = 0$$

$$\text{Then, } \langle \chi^M, \downarrow | \chi^M, \downarrow \rangle = \langle \chi^M | \chi^M \rangle = 0$$

But by the no-ghost theorem, we have, any state which is $\hat{Q}_B$ -closed & has zero norm, should be a $\hat{Q}_B$ -exact state.

$$\text{Thus, } |\chi^M, \downarrow\rangle = |\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle = \hat{Q}_B |\Lambda\rangle$$

Now we have to show that this map is isomorphic. To show the map is one-one, we should show that if $|\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle$ is $\hat{Q}_B$ -exact state, then $|\psi^M\rangle - |\psi'^M\rangle$ is an OCQ null state.

If $|\psi^M, \downarrow\rangle \neq |\psi'^M, \downarrow\rangle$ are $\hat{Q}_B$ -closed, then their ghost numbers should be $-\frac{1}{2}$. And if $|\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle = \hat{Q}_B |\Lambda\rangle$, then only those terms in $\hat{Q}_B |\Lambda\rangle$ which have ghost number $-\frac{1}{2}$ will remain.

The ghost number of

- $\hat{b}_n$ is -1 .
- $\hat{c}_n$ is $+1$.
- $\hat{Q}_B$ is $+1$.

Since $\hat{Q}_B |\Lambda\rangle$ has ghost number $-\frac{1}{2}$, $|\Lambda\rangle$ should have ghost number $+\frac{3}{2}$.

$$\Rightarrow |\Lambda\rangle = \sum_{n=1}^{\infty} \hat{b}_{-n} |\Lambda_n^M, \downarrow\rangle + \sum_{n,m,k=1}^{\infty} \hat{b}_{-n} \hat{b}_{-m} \hat{c}_{-k} |\Lambda_{nmk}^M, \downarrow\rangle + \dots$$

where $|\Lambda_n^M, \downarrow\rangle, |\Lambda_{nmk}^M, \downarrow\rangle, \dots$ are $\hat{Q}_B$ -closed.

$$\begin{aligned} \Rightarrow \hat{Q}_B |\Lambda\rangle &= \sum_{n=1}^{\infty} \hat{Q}_B \hat{b}_{-n} |\Lambda_n^M, \downarrow\rangle \\ &+ \sum_{n,m,k=1}^{\infty} \hat{Q}_B \hat{b}_{-n} \hat{b}_{-m} \hat{c}_{-k} |\Lambda_{nmk}^M, \downarrow\rangle + \dots \end{aligned}$$

use $\{\hat{Q}_B, \hat{b}_{-n}\} = \hat{L}_{-n}^M + \hat{L}_{-n}^g$

$$\begin{aligned} \Rightarrow \hat{Q}_B \hat{b}_{-n} |\Lambda_n^M, \downarrow\rangle &= \hat{L}_{-n}^M |\Lambda_n^M, \downarrow\rangle + \hat{L}_{-n}^g |\Lambda_n^M, \downarrow\rangle \\ &\quad - \underbrace{\hat{b}_{-n} \hat{Q}_B |\Lambda_n^M, \downarrow\rangle}_{\text{Zero since } |\Lambda_n^M, \downarrow\rangle \text{ is physical.}} \end{aligned}$$

$$\Rightarrow \hat{Q}_B \hat{b}_{-n} |\Lambda_n^M, \downarrow\rangle = \hat{L}_{-n}^M |\Lambda_n^M, \downarrow\rangle + \hat{L}_{-n}^g |\Lambda_n^M, \downarrow\rangle$$

Since $|\psi^M, \downarrow\rangle \neq |\psi'^M, \downarrow\rangle$ ghost ground state, only relevant terms that can remain in $|\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle = \hat{Q}_B |\Lambda\rangle$ are

$$\Rightarrow |\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle = \sum_{n=1}^{\infty} \hat{L}_{-n}^M |\Lambda_n^M, \downarrow\rangle$$

The terms with ghost excitations should all vanish separately.

$$\Rightarrow |\psi^M\rangle - |\psi'^M\rangle = \underbrace{\sum_{n=1}^{\infty} \hat{L}_{-n}^M |\Lambda_n^M\rangle}_{\text{This is OCQ null.}}$$

Thus, $|\psi^M, \downarrow\rangle - |\psi'^M, \downarrow\rangle = \text{BRST exact}$ implies $|\psi^M\rangle - |\psi'^M\rangle = \text{OCQ null}$. So, the map b/w BRST & OCQ equivalence classes is one one.

Now, we should show that this map is onto. i.e., every BRST equivalence class contains atleast one of the form $|\psi^M, \downarrow\rangle$ where $|\psi^M\rangle$ is OCQ physical state.

Consider the quantum number operator,

$$N' = 2N^- + N^b + N^c$$

For the below operator,

$$\hat{R} = \frac{1}{\sqrt{2\alpha'} k^+} \sum_{n=1}^{\infty} (\hat{b}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{b}_n)$$

The first term creates one 'b'-excitation & annihilates a '-'-excitation.

Since $[\hat{\alpha}_n^+, \hat{\alpha}_{-n}^-] = -n$, it implies ' $\hat{\alpha}_n^+$ ' acts as a annihilation of ' $\hat{\alpha}_{-n}^-$ ' excitation.

So $\hat{b}_{-n} \hat{\alpha}_n^+$ has $N^b = +1$ & $N^- = -1$ & $N^c = 0$. Thus, $N' = 2(-1) + 1 + 0 = -1$

The second term annihilates one 'c'-excitation & creates a '+'-excitation.

Since $\{\hat{c}_{-n}, \hat{b}_n\} = 1$, it implies ' $\hat{b}_n$ ' acts as a annihilation of ' $\hat{c}_{-n}$ ' excitation.

So $\hat{\alpha}_{-n}^+ \hat{b}_n$ has $N^c = -1$ & $N^- = 0$ & $N^b = 0$. Thus, $N' = 2(0) - 1 + 0 = -1$

$$\rightarrow \boxed{[N', \hat{R}] = -\hat{R}}$$

Then, consider $\hat{Q}_{-1}$, $\hat{Q}_1$ & $\hat{Q}_0$

$$\hat{Q}_{-1} = -\sqrt{2\alpha'} k^- \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{C}_n)$$

The first term creates one 'c'-excitation & annihilates a '-'-excitation.

So $\hat{C}_{-n} \hat{\alpha}_n^+$ has $N^c = +1$ & $N^- = -1$ & $N^b = 0$. Thus, $N' = 2(-1) + 1 + 0 = -1$

The second term creates one '+'-excitation & annihilates a 'b'-excitation.

So $\hat{\alpha}_{-n}^+ \hat{C}_n$ has $N^b = -1$ & $N^- = 0$ & $N^c = 0$. Thus, $N' = 2(0) - 1 + 0 = -1$

$$\rightarrow \boxed{[N', \hat{Q}_{-1}] = -\hat{Q}_{-1}}$$

$$\text{Then, } \hat{Q}_1 = -\sqrt{2\alpha'} k^+ \sum_{n=1}^{\infty} (\hat{C}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{C}_n)$$

The first term creates one 'c'-excitation & annihilates a '+'-excitation.

$\hat{C}_{-n} \hat{\alpha}_n^-$ has $N' = 2(0) + 0 + 1 = +1$

The second term creates one '-'-excitation & annihilates a 'b'-excitation.

$$\hat{\alpha}_{-n}^{\dagger} \hat{c}_n \text{ has } N' = 2(+1) - 1 + 0 = +1$$

$$\text{Thus, overall } [N', \hat{Q}_1] = \hat{Q}_1$$

Now, we should find the N' number of ' $\hat{Q}_0$ '.

$$\hat{Q}_0 = \hat{Q}_2 - \hat{Q}_1 - \hat{Q}_{-1}$$

$$\begin{aligned} \hat{Q}_0 = & -\frac{1}{2} \sum_{n,m \in \mathbb{Z}} \hat{c}_{-n} \left(:\hat{\alpha}_{n-m}^+ \hat{\alpha}_m^-: + :\hat{\alpha}_{n-m}^- \hat{\alpha}_m^+: \right) \\ & + \frac{1}{2} \sum_{n,m \in \mathbb{Z}} \sum_{i=2}^{d-1} \hat{c}_{-n} :\hat{\alpha}_{n-m}^i \hat{\alpha}_m^i: \\ & + \sum_{n \in \mathbb{Z}} \hat{c}_{-n} \hat{L}_n^{\mathcal{H}} + \sum_{m,n \in \mathbb{Z}} \frac{(m-n)}{2} :\hat{c}_m \hat{c}_n \hat{b}_{-m-n}: - \hat{c}_0 \\ & + \sqrt{2\alpha'} k^+ \sum_{n=1}^{\infty} \left(\hat{c}_{-n} \hat{\alpha}_n^- + \hat{\alpha}_{-n}^- \hat{c}_n \right) \\ & + \sqrt{2\alpha'} k^- \sum_{n=1}^{\infty} \left(\hat{c}_{-n} \hat{\alpha}_n^+ + \hat{\alpha}_{-n}^+ \hat{c}_n \right) \end{aligned}$$

Explicitly,

$$\begin{aligned}\hat{Q}_0 = & -\frac{1}{2} \sum_{\{n,m \in \mathbb{Z}, n \neq 0, m \neq 0, n \neq m\}} \hat{C}_{-n} \left(:\hat{\alpha}_{n-m}^+ \hat{\alpha}_m^-: + :\hat{\alpha}_{n-m}^- \hat{\alpha}_m^+: \right) \\ & + \frac{1}{2} \sum_{n,m \in \mathbb{Z}} \sum_{i=2}^{d-1} \hat{C}_{-n} :\hat{\alpha}_{n-m}^i \hat{\alpha}_m^i: \\ & + \sum_{n \in \mathbb{Z}} \hat{C}_{-n} \hat{L}_n^{\mathcal{K}} + \sum_{m,n \in \mathbb{Z}} \frac{(m-n)}{2} :\hat{C}_m \hat{C}_n \hat{b}_{-m-n}: - \hat{C}_0\end{aligned}$$

We shall see N' number of each $\hat{Q}_0$ terms as follows.

- $\frac{1}{2} \sum_{n,m \in \mathbb{Z}} \sum_{i=2}^{d-1} \hat{C}_{-n} :\hat{\alpha}_{n-m}^i \hat{\alpha}_m^i:$ has $N' = 0, \pm 1$

Since $N^- = 0$ for any n, m .

$N^c = +1$ for $n > 0$, any m .

$N^b = -1$ for $n < 0$, any m .

$N^- = N^c = N^b = 0$ for $n = 0$, any m .

Thus, this term has $N' = 0, \pm 1$

- $\sum_{n \in \mathbb{Z}} \hat{C}_{-n} \hat{L}_n^{\mathcal{H}}$ has $N' = 0, \pm 1$

since $N^- = 0$ for any n .

$$N^c = +1 \text{ for } n > 0.$$

$$N^b = -1 \text{ for } n < 0.$$

$$N^- = N^c = N^b = 0 \text{ for } n = 0.$$

Thus, this term has $N' = 0, \pm 1$

- $\hat{C}_0$ has $N' = 0$

- $\sum_{m, n \in \mathbb{Z}} \frac{(m-n)}{2} \circ \hat{C}_m \hat{C}_n \hat{b}_{-m-n} \circ$ has $N' = 0, \pm 1$

since $N^- = 0$ for any n, m .

$$N^c = 0, N^b = -1 \text{ for } n > 0, m > 0.$$

$$N^c = +1, N^b = 0 \text{ for } n < 0, m < 0.$$

$$N^c = 0, N^b = -1 \text{ for } n > 0, m < 0, \\ \& -m-n > 0$$

$$N^c = +1, N^b = 0 \text{ for } n > 0, m < 0, \\ \& -m-n < 0$$

Thus, this term has $N' = 0, \pm 1$

- $\frac{1}{2} \sum_{\{n, m \in \mathbb{Z}, n \neq 0, m \neq 0, n \neq m\}} \hat{C}_{-n} (:\hat{\alpha}_{n-m}^+ \hat{\alpha}_m^-: + :\hat{\alpha}_{n-m}^- \hat{\alpha}_m^+ :)$
has $N' = \pm 1$

We can get $N^- = -1$ when $n > m > 0$
 This makes $N^c = +1$. So,

$$N' = 2(-1) + 1 + 0 = -1$$

We can get $N^- = +1$ when $n < m < 0$
 This makes $N^b = -1$.

So, $N' = 2(+1) - 1 + 0 = +1$

And whenever $N^- = 0$, $\hat{C}_{-n}$ always has
 $N' = \pm 1$ as $n \neq 0, m \neq 0$

Thus, this term has $N' = \pm 1$.

So, ' $\hat{Q}_0$ ' is not eigen operator of N' ,
 & it can't increase N' greater than '+1'.

So, ' $\hat{Q}_0 + \hat{Q}_{-1}$ ' has $N' \leq +1$

And as seen before $\hat{R}$ has $N' = -1$.

Then, $\hat{U} = \{\hat{Q}_0 + \hat{Q}_{-1}, \hat{R}\}$ has

$$N'(\hat{U}) = N'(\hat{R}) + N'(\hat{Q}_0 + \hat{Q}_{-1})$$

$$N'(\hat{R}) + N'(\hat{Q}_0 + \hat{Q}_{-1}) \leq -1 + 1$$

$$\Rightarrow \boxed{N'(\hat{U}) \leq 0}$$

So, ' $\hat{U}$ ' cannot increase ' N' ' number.

Now, any BRST equivalence class or the $\hat{Q}_B$ -cohomology has states of the form,

$$|\psi\rangle = \left(1 - \hat{\mathcal{S}}^{-1} \hat{U} + (\hat{\mathcal{S}}^{-1} \hat{U})^2 - (\hat{\mathcal{S}}^{-1} \hat{U})^3 \dots\right) |\psi_0\rangle$$

where $\hat{\mathcal{S}} |\psi_0\rangle = 0$. ' $|\psi_0\rangle$ ' has no any ghost or light-cone excitations.

$$\Rightarrow \underline{N'|\psi_0\rangle = 0}$$

Now, $\hat{\mathcal{S}}$ & N' commute as they are all number operators. Thus,

$$[N', \hat{\mathcal{S}}] = 0 \implies N'(\hat{\mathcal{S}}) = 0$$

Also, ' $\hat{U}$ ' can't increase the ' N' ' eigen value of any state. So, $N'(\hat{U}|\psi_0\rangle) = 0$ as eigen value of any number operator on a ket has to be 0 or +ve. So,

$$\boxed{N'|\psi\rangle = 0} \text{ where } |\psi\rangle \in \hat{Q}_B\text{-cohomology.}$$

So, the corrections $1 - \hat{\mathcal{S}}^{-1}\hat{U} + (\hat{\mathcal{S}}^{-1}\hat{U})^2 - (\hat{\mathcal{S}}^{-1}\hat{U})^3 \dots$ will not add any unwanted excitations. And $|\psi_0\rangle$ is already absent of '+' excitation.

So, $\hat{Q}_B$ -cohomology has no light-cone (' $\pm$ ') & ghost ('b' or 'c') excitations.

Thus, there is atleast one state in the $\hat{Q}_B$ -cohomology of the form $|\psi^M, \downarrow\rangle$ which is called as 'OCQ type state'.

So every BRS T class contains atleast one state in which the ghost modes are un-excited. We call these as 'OCQ type states'.

This OCQ type state should have conformal weight of '1' with respect to matter Virasoro 0^{th} -mode ' $\hat{L}_0^M |\psi^M\rangle = 1 |\psi^M\rangle$ ' & highest weight vector with respect to Virasoro algebra.
i.e. ' $\hat{L}_{n>0}^M |\psi^M\rangle = 0$ '

The corresponding vertex operator is a weight-1 primary tensor ' $V_\psi^M(z)$ ' which is constructed from the matter fields. Then the vertex operator corresponding to $|\psi^M, \downarrow\rangle$ is given by ' $C(z) V_\psi^M(z)$ '.