

Perturbations of Black Holes Surrounded by Anisotropic Matter Field

M.SC. 4TH SEM PROJECT REPORT

Submitted in partial fulfilment of the requirements for the degree of

MASTER OF SCIENCE in PHYSICS

by

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DECLARATION

I *Sagar J C*, bearing the register number 226PH028 of National Institute of Technology Karnataka, Surathkal, hereby solemnly declare that the M.Sc. research thesis entitled “*Perturbations of Black Holes Surrounded by Anisotropic Matter Field* ” submitted by me in partial fulfilment of the requirements for the award of the Degree of *Master of Science* in *Physics* is a *bonafide report of the M.Sc. Major project carried out by me*. I also declare that the work embodied in the paper, either in part or in full, has not been previously submitted to any University for the award of any degree.

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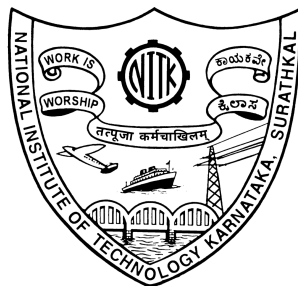
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CERTIFICATE OF COMPLETION

This is to certify that **Mr. Sagar J C**, bearing the register number: 226PH028 has submitted the research thesis entitled “*Perturbations of Black Holes Surrounded by Anisotropic Matter Field*” as the record of the project work carried out by him under the guidance and supervision of **Dr. Ajith K M**. It is *accepted* as the *M.Sc. Major project Thesis submission* in partial fulfilment of the requirements for the award of the degree of **Master of Science**.

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ABSTRACT

I have conducted research with the objective of examining the anisotropic matter field around black holes using quasinormal modes (QNMs). This study specifically delves into a group of black hole solutions to Einstein's equations that expand the Reissner-Nordström space-time to encompass an anisotropic matter field. Alongside mass and charge, this particular type of black hole exhibits an additional hair induced by the negative radial pressure of the anisotropic matter. The presence of an Anisotropic Black Hole results in the splitting of the effective potential in the form of Zeeman Splitting. The splitting caused by ' K ' is symmetrical in relation to the Schwarzschild case, while the splitting caused by ' w ' is asymmetric and approaches the Schwarzschild case as $w \rightarrow \infty$. For the calculation of the QNMs, I used the optimal order WKB method, which selects the order with the least error. My primary focus has been on exploring the perturbations of the massless scalar field, electromagnetic field, and space-time structure within the backdrop of black hole space-time with an anisotropic matter field and determining the QNMs.

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Introduction

The recent developments in observational astrophysics ushered new interests in black hole physics, transmuting the subject from mere theoretical aspects to an observational perspective. The first direct evidence for the existence of astrophysical black holes came through the detection of gravitational waves originating from the collision of black holes in a binary black hole system [1–3]. On the other hand, the physical origin and mechanism of black hole shadow gained attention after its observation by the Event Horizon Telescope [4–6]. These two great discoveries opened up new windows for probing the universe through observations, and also called for a need in devising and revisiting the theoretical tools of black hole physics. An astrophysical black hole must be a rotating one as most of the cosmic objects are characterised by definite angular momentum. Furthermore, in a realistic scenario, a stellar black hole resides in the background of matters or fields, which is to say that black hole solution must be encompassed with the coexistence of matter field with it.

The spacetime geometry of a rotating black hole gets modified in the presence of matter or fields around it. Isotropic fluids have been studied substantially in gravity theories for a long time, whereas, anisotropic matter fields gained considerable attention recently. However, a plethora of studies with anisotropic matter field have appeared in the context of compact stars, relativistic stellar objects, self-gravitating systems, stellar objects constituting quark stars, and black holes etc. Recently a simple anisotropic matter field was incorporated into a static spherically symmetric black hole solution [7]. Soon, a rotating case was proposed with details of thermodynamics and energy extraction process [8]. The rotating solution is obtained from static solution by using the Newman-Janis algorithm, which is a generalisation of Kerr-Newman spacetime to incorporate an anisotropic matter. The new solution has an additional hair which arises from the negative radial pressure of the anisotropic matter, in addition to mass, charge and spin. The properties of the solution has considerable deviation from Kerr and Kerr-Newman

solutions due to the density and anisotropy of the surrounding matter field, which are described by the parameters K and w , respectively. As the presence of fields or fluids modify the spacetime geometry of the black hole, the corresponding effect is expected to change the shape of the black hole as seen by a distant observer. In subsequent studies, the shadow produced by rotating black holes with anisotropic matter field were investigated and a deviation from the Kerr-Newmann spacetime was reported [9], which also discusses the observational prospects of these effects.

The first direct detection of gravitational waves (GWs) on September 14, 2015, stood as a pivotal moment, confirming a crucial aspect of Einstein's theory and introducing a new perspective to explore the universe [10]. This breakthrough enabled the direct observation of monumental events like collisions between black holes, offering pivotal insights into their nature and validating extensive theoretical and technological groundwork [11–17]. GWs are disturbances in space-time, that originate from interactions among colossal celestial bodies, notably merging black holes. By probing extreme astrophysical environments, they serve as laboratories for testing the fundamental laws of physics under extreme conditions. Also, GW detection from the universe's infancy might unravel important details about cosmic inflation and the early stages of the universe [18].

Astrophysical black holes rarely exist in isolation, as evidenced by the images of the black hole in the centre of M87 elliptical galaxy [19] and the Sagittarius A* black hole [20] by the Event Horizon Telescope Collaboration. They are typically enveloped by various structures like accretion disks, photon spheres, particle jets, corona, and more [21–24]. Examining black holes within these matter fields provides outcomes that align closely with observed data. Thus, our study focuses on non-rotating charged black holes surrounded by anisotropic matter field whose energy-momentum tensor T_μ^ν is given by [25],

$$T_\mu^\nu = (\rho + P_2)U_\mu U^\nu + (P_1 - P_2)x_\mu x^\nu + P_2\delta_\mu^\nu \quad (1)$$

Here the $\rho \rightarrow \rho(r)$ is a radial function that represents the energy density, U_μ and x_μ are arbitrary time-like and space-like four vectors that satisfy $U_\mu \propto (\partial_t)_\mu$ and $x_\mu \propto (\partial_r)_\mu$ [26] respectively. P_1 and P_2 represent the pressures and are related to the energy density by the barotropic equation of state of the form,

$$P_i = w_i \rho \quad i = \{1, 2\} \quad (2)$$

Upon simplification, as done by [25] the energy-momentum tensor for the anisotropic field can be written concisely as,

$$T_{\mu}^{\nu} = \text{diag}(P_t, P_r, P_{\theta}, P_{\phi}) = \text{diag}(-\rho, P_1, P_2, P_2) = \rho \text{diag}(-1, w_1, w_2, w_2) \quad (3)$$

The distinctive feature observed in the solutions of space-time near black holes is the existence of an event horizon at $r = r_{\text{eh}}$, where $f(r_{\text{eh}}) = 0$. Across the horizon, the sign of $f(r)$ changes which in turn causes a change in the signs of g_{00} and g_{11} across it. If $f(r) < 0$, then the roles of t and r swap: r becomes time-like while t becomes space-like. Therefore, $-\rho$ represents the pressure along the spatial t direction, and $-P_r$ signifies the energy density along the temporal r direction. To maintain the continuity of energy density and pressure across the event horizon, $w_1 = -1$ is necessary. This negative radial pressure, $-\rho$, facilitates the dispersion of the anisotropic matter field across the entire spatial range ($0 < r < \infty$) [27–29]. Throughout this study, we adopt $w_1 = -1$ for this very reason. w_2 from now denoted as w is a free parameter that characterises the anisotropy in the matter field. Thus the final form of energy-momentum tensor is,

$$T_{\mu}^{\nu} = \rho(r) \text{diag}(-1, -1, w, w) \quad (4)$$

When a black hole initially in a stable equilibrium state is perturbed by any means, will undergo oscillations and produces characteristic vibrational modes called Quasi Normal Modes (QNMs) [30–33]. Think of a bell being struck and producing a specific sound; similarly, black holes, when perturbed, exhibit these characteristic oscillations with specific frequencies that decays. Similarly, QNMs describe the way a black hole responds to such disturbances, emitting gravitational waves at specific frequencies that decay. The interesting aspect about QNMs is that they are solely determined by the intrinsic properties of the black hole, such as its mass, charge, and angular momentum, and in our case it includes the property of the surrounding medium, rather than the specific way it was disturbed. Hence, they are also called "fingerprints" of black holes [30, 34, 35]. They serve as encoded messages, revealing essential parameters of black holes and providing indirect measurements, making them vital for astronomers trying to study black holes. Hence in this paper, we focus on calculating the QNMs of Charged black holes surrounded by anisotropic matter field since their analysis can provide insights into the behavior of matter near the event horizon.

These modes are termed "quasi-normal" due to their resemblance to normal modes observed in various systems, possessing a frequency and decay rate. However, their distinctiveness arises from their interaction with the black hole's event horizon, making them exclusive to black hole physics. The spectrum of QNMs of a stable black hole comprises an infinite number of complex frequencies depicting normal mode vibrations with damping effects. The real part of the QNM gives the oscillation frequency and the imaginary part gives the decay rate of the gravitational wave. Over time, these oscillations reveal a trend where those with small damping factors become more prominent, while the ones with higher damping fade away. Studying the signal during the phase of quasi-normal ringing in the merging of black holes allows scientists to pinpoint specific modes distinguished by their minimal imaginary parts. The concept of QNM was initially introduced by C. V. Vishveshwara [36] in the context of scattered gravitational waves around a Schwarzschild black hole. This era also marked the calculation of lower quasi-normal modes for Schwarzschild [37] and later Kerr black holes [38]. Later explorations extended to Reissner-Nordström black holes [39], examining their linked gravitational and electromagnetic perturbations.

Any type of black hole can be perturbed in three major ways and QNMs can be calculated numerically in many methods [40–58]. A black hole can be perturbed by adding a massless scalar (Klein-Gordon) field as studied in the section(2) or we can add an EM field and study the perturbations as done in section(3), or we can perturb the space-time metric itself and follow the Regge-Wheeler formalism to get the perturbation equation as done in section(4). But in all three ways, the final perturbation equation will be of the form,

$$\frac{d^2\psi(\tau)}{d\tau^2} + [\omega^2 - V(\tau)] \psi(\tau) = 0 \quad (5)$$

Here ‘ τ ’ is the tortoise coordinate that can take values from $-\infty$ to $+\infty$. And $\psi(\tau)$ is the perturbation that represents the quasinormal mode wave. And $V(\tau)$ is the perturbation potential. ω is the quasinormal mode frequency. Eq.(5) resembles the well-known Schrodinger equation, so the ω values are like the energy eigenvalues of the Schrodinger equation. We need to look for appropriate boundary conditions to solve the above equation. Now, if GW is very close to the event horizon, i.e., for $\tau \rightarrow -\infty$ then the gravitational wave has to be purely ongoing. Hence we get,

$$\psi(\tau) \approx e^{i\omega(\tau+t)} \quad (6)$$

But very far from the black hole, the GW will not be reflected. Hence, as $r \rightarrow +\infty$, i.e., for $\tau \rightarrow +\infty$, the gravitational wave is purely outgoing. Thus,

$$\psi(\tau) \approx e^{-i\omega(\tau-t)} \quad (7)$$

Thus we need to solve Eq.(5) with these boundary conditions in mind as an eigenvalue problem. Hence, we can use the known approximation/Numerical methods to find the ω values. In this paper, we employ the 3rd-order WKB approximation method to compute the QNMs. For an effective potential $V(\tau)$ whose nature is in the form of a barrier potential, we can use the WKB approximation to get the QNMs. By using the formulas given in [59], we get,

$$\omega^2 = \left[V_0 + (-2V_0^{(2)})^{\frac{1}{2}} L_2 \right] - i\alpha(-2V_0^{(2)})^{\frac{1}{2}}(1 + L_3) \quad (8)$$

$$L_2 = \frac{1}{(-2V_0^{(2)})^{\frac{1}{2}}} \left[\frac{1}{8} \left(\frac{V_0^{(4)}}{V_0^{(2)}} \right) \left(\frac{1}{4} + \alpha^2 \right) - \frac{1}{288} \left(\frac{V_0^{(3)}}{V_0^{(2)}} \right)^2 (7 + 60\alpha^2) \right] \quad (9)$$

$$\begin{aligned} L_3 = \frac{1}{(-2V_0^{(2)})^{\frac{1}{2}}} & \left[\frac{5}{6912} \left(\frac{V_0^{(3)}}{V_0^{(2)}} \right)^4 (77 + 188\alpha^2) - \frac{1}{384} \left(\frac{(V_0^{(3)})^2 V_0^{(4)}}{(V_0^{(2)})^3} \right) (51 + 100\alpha^2) \right. \\ & + \frac{1}{2304} \left(\frac{V_0^{(4)}}{V_0^{(2)}} \right)^2 (67 + 68\alpha^2) + \frac{1}{288} \left(\frac{V_0^{(3)} V_0^{(5)}}{(V_0^{(2)})^2} \right) (19 + 28\alpha^2) \\ & \left. - \frac{1}{288} \left(\frac{V_0^{(6)}}{V_0^{(2)}} \right) (5 + 4\alpha^2) \right] \quad (10) \end{aligned}$$

Here, $\alpha = n + \frac{1}{2}$ where n is the overtone number and takes values $n = 0, 1, 2, 3 \dots$ and $V_0 = V(\tau_0)$ where τ_0 is the critical point, defined as $\left. \frac{dV}{d\tau} \right|_{\tau=\tau_0} = 0$ and $V_0^{(m)} = \left. \frac{d^m V}{d\tau^m} \right|_{\tau=\tau_0}$ which is the value of m^{th} derivative at the critical point τ_0 .

Similarly, the 6th order WKB formula is given in [60],

$$\frac{i(\omega^2 - V_0)}{\sqrt{-2V_0^{(2)}}} = \alpha + \sum_{i=2}^6 L_i \quad (11)$$

The correction terms L_2 and L_3 are as shown before, and L_4 , L_5 and L_6 are given in [60]. Usually higher the order of WKB approximation, it gives more and more accurate results. But if the order of WKB is more than 8 the error dominates.

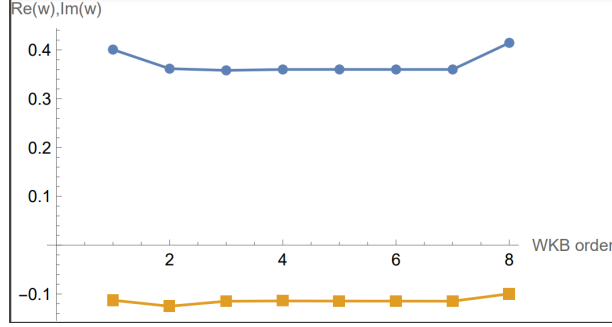


Figure 1: Scalar Perturbation: $M = 1$, $Q = 0$, $K = -0.5$, $w = 2/3$, $\ell = 1$, $n = 0$

The error of a particular order QNM is got by the formula.

$$\Delta\omega_i = \frac{Abs(\omega_{i+1} - \omega_{i-1})}{2} \quad (12)$$

where ω_i represents the QNM value got by the i^{th} order WKB approximation. Then the optimal order is that where the error is the minimum.

WKB Order	QNM	Error
12	$4521.08 + 37.7897 i$	107538.
11	$412.276 + 414.407 i$	2260.5
10	$0.0744842 + 41.9693 i$	291.027
9	$1.79433 + 1.74218 i$	21.0324
8	$0.439556 - 0.0938015 i$	1.17313
7	$0.359935 - 0.114551 i$	0.0411405
6	$0.359932 - 0.114543 i$	0.0000561181
5	$0.360039 - 0.114509 i$	0.000155831
4	$0.359951 - 0.114232 i$	0.000959753
3	$0.358143 - 0.114808 i$	0.00546165
2	$0.361552 - 0.125037 i$	0.021308

The optimal order is got for WKB order 6 with the most minimum error of 0.0000561181 with the QNM value of $0.359932 - 0.114543i$

Chapter 1

Solutions of Charged Black Hole in Anisotropic Matter Field

1.1 Deriving the metric tensor

Such an energy-momentum tensor of the form given in Eq.(4) demands a spherically symmetric and a static space-time metric of the form,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + [f(r)]^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 \quad (1.1)$$

For this metric the Einstein tensor $G_\mu{}^\nu$ is diagonal given by,

$$G_0{}^0 = G_1{}^1 = \frac{1}{r} \frac{df(r)}{dr} + \frac{f(r)}{r^2} - \frac{1}{r^2} \quad (1.2)$$

$$G_2{}^2 = G_3{}^3 = \frac{1}{2} \frac{d^2f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} \quad (1.3)$$

Even if $f \rightarrow f(t, r)$ was taken, the components $G_0{}^1$ and $G_1{}^0$ would be,

$$G_0{}^1 = -\frac{1}{r} \frac{\partial f(t, r)}{\partial t} \quad \text{and} \quad G_1{}^0 = -\frac{1}{r f^2(t, r)} \frac{\partial f(t, r)}{\partial t} \quad (1.4)$$

And then according to Einstein's field equations [61, 62] we have,

$$G_\mu{}^\nu = 8\pi T_\mu{}^\nu \quad (1.5)$$

Thus $T_0{}^1 = T_1{}^0 = 0$ makes the function 'f' to not depend on the time-like coordinate. Hence, the space-time will be static where f will be a function of just r .

Now, we have two Einstein equations given by,

$$\frac{1}{r} \frac{df(r)}{dr} + \frac{f(r)}{r^2} - \frac{1}{r^2} = -8\pi\rho(r) \quad (1.6)$$

$$\frac{1}{2} \frac{d^2 f(r)}{dr^2} + \frac{1}{r} \frac{df(r)}{dr} = 8\pi w\rho(r) \quad (1.7)$$

The first equation can be rearranged to give a simple 1st order differential equation,

$$\frac{df(r)}{dr} + \frac{f(r)}{r} = \frac{1}{r} - 8\pi r\rho(r) \quad (1.8)$$

The integration factor $I.F.$ is $e^{\int \frac{1}{r} dr} = r$. Thus the solution is given by,

$$\begin{aligned} r f(r) &= \int r \left(\frac{1}{r} - 8\pi r\rho(r) \right) dr = \int dr - 2 \int 4\pi r^2 \rho(r) dr = r - 2m(r) \\ \Rightarrow f(r) &= 1 - \frac{2m(r)}{r} \end{aligned} \quad (1.9)$$

where $m(r)$ is the mass function which is given by,

$$m(r) = \int 4\pi r^2 \rho(r) dr \quad (1.10)$$

Now, substituting Eq.(1.9) in Eq.(1.7) we get,

$$\frac{d^2 m(r)}{dr^2} = -8\pi w r \rho(r) \quad (1.11)$$

$$\begin{aligned} \frac{d^2 m(r)}{dr^2} &= \frac{d^2}{dr^2} \int 4\pi r^2 \rho(r) dr = \frac{d}{dr} 4\pi r^2 \rho(r) = 4\pi \left(r^2 \frac{d\rho(r)}{dr} + 2r\rho(r) \right) = -8\pi w r \rho(r) \\ \Rightarrow \frac{r}{2} \frac{d\rho(r)}{dr} + \rho(r) &= -w\rho(r) \end{aligned} \quad (1.12)$$

The above equation can be rearranged to get $\frac{d\rho}{\rho} + \frac{2(1+w)dr}{r} = 0 \Rightarrow \rho = \frac{C}{r^{2(1+w)}}$ where C is some arbitrary constant which is chosen such that the energy density is given by,

$$\rho(r) = \frac{(1-2w)K}{8\pi r^{2(1+w)}} = \frac{1}{8\pi r^2} \left(\frac{r_0}{r} \right)^{2w} \quad (1.13)$$

Here, K controls the energy density of the anisotropic matter field and r_0 is a charge-like quantity defined by $r_0^{2w} = (1-2w)K$.

Then the mass function $m(r)$ can be got using Eq.(1.10) as,

$$m(r) = 4\pi \int r^2 \frac{(1-2w)K}{8\pi r^{2(1+w)}} dr = \int \frac{(1-2w)K}{2r^{2w}} dr = \frac{Kr}{2r^{2w}} + M \quad (1.14)$$

$$\Rightarrow m(r) = M + \frac{Kr}{2r^{2w}} \quad (1.15)$$

where M is the integration constant that represents the ADM mass of the Black hole. This gives $f(r)$ as,

$$f(r) = 1 - \frac{2M}{r} - \frac{K}{r^{2w}} \quad (1.16)$$

Consider a static, isolated black hole that has a charge Q . The energy-momentum tensor near a charged black hole is given by,

$$T_\mu{}^\nu = \frac{1}{4\pi} \left(F_{\mu\alpha} F^{\nu\alpha} - \frac{1}{4} \delta_\mu^\nu F_{\alpha\beta} F^{\alpha\beta} \right) \quad (1.17)$$

where F is the Faraday tensor. Since the point-charge of the black hole produces only a radial electric field ($E(r)$) and no magnetic field, we have the non-zero components of the Faraday tensor to be $F_{10} = -F^{01} = E(r)$ and $F^{01} = -F_{10} = E(r)$. Then the only non-zero components of the energy-momentum tensor are $T_\mu{}^\nu = (E^2(r)/8\pi) \text{diag}(-1, -1, 1, 1)$. The Electric field due to a charge Q is given by $E(r) = Q/r^2$. The energy-momentum tensor for the charged black hole is,

$$T_\mu{}^\nu = \frac{Q^2}{8\pi r^4} \text{diag}(-1, -1, 1, 1) \quad \Rightarrow \quad \rho(r) = \frac{Q^2}{8\pi r^2} \quad \text{and} \quad w = 1 \quad (1.18)$$

Also, when the Einstein equations Eq.(1.5) for the charged black hole are solved for the above energy-momentum tensor with the metric ansatz similar to that of Eq.(??) we get the Reissner–Nordström metric,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad (1.19)$$

Now in the anisotropic matter field solution if we choose $w = 1$ then,

$$r_0^2 = (1-2)K = -K$$

Then $f(r)$ for $w = -1$ becomes,

$$f(r) = 1 - \frac{2M}{r} + \frac{r_0^2}{r^2} \quad (1.20)$$

Thus remarkably, $w = 1$ in anisotropic matter field solution represents the Reissner–Nordström (RN) geometry with charge r_0 . Hence, we can say that the RN geometry is a special case of the anisotropic matter field solution. This also emphasises that the energy around the charged black hole is anisotropic.

Now we try to generalize the anisotropic matter field solution when there are other additional factors. As shown in [63] for a number of parameters q_n called the quintessential state parameters and their normalizing factors r_n , the solution for the metric Eq.(??) is given by,

$$f(r) = 1 - \frac{2M}{r} - \sum_n \left(\frac{r_n}{r} \right)^{3q_n+1} \quad (1.21)$$

We need to consider both the charge and then the anisotropic matter field. Hence there are two parameters q_{charge} and q_{aniso} with normalizing factors r_{charge} and r_{aniso} . Thus $f(r)$ is,

$$f(r) = 1 - \frac{2M}{r} - \left(\frac{r_{\text{charge}}}{r} \right)^{3q_{\text{charge}}+1} - \left(\frac{r_{\text{aniso}}}{r} \right)^{3q_{\text{aniso}}+1} \quad (1.22)$$

For the anisotropic matter field we have $3q_{\text{aniso}} + 1 = 2w$ which gives $(r_{\text{aniso}})^{2w} = K$ and for the charge parameter, $3q_{\text{charge}} + 1 = 2$ with $(r_{\text{charge}})^2 = -Q^2$. Thus the final solution for the charged black hole surrounded by the anisotropic matter field is given by [64],

$$\boxed{f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{K}{r^{2w}}} \quad (1.23)$$

In this case we have the total energy density to be $\rho = \rho_{\text{charge}} + \rho_{\text{aniso}}$ and the energy-momentum tensor is given by $T_\mu^\nu = \rho(r) \text{diag}(-1, -1, w, w)$, where,

$$\rho(r) = \frac{Q}{8\pi r^4} + \frac{(1-2w)K}{8\pi r^{2(1+w)}} \quad (1.24)$$

The mass function $m(r)$ in this case can be found using Eq.(1.10), on performing the integral with M as integration constant we get,

$$m(r) = M + \int 4\pi r^2 \left(\frac{Q}{8\pi r^4} + \frac{(1-2w)K}{8\pi r^{2(1+w)}} \right) dr = M - \frac{Q^2}{2r} + \frac{Kr}{2r^{2w}} \quad (1.25)$$

$$\Rightarrow m(r) = M - \frac{Q^2}{2r} + \frac{Kr}{2r^{2w}} \quad (1.26)$$

We could also get the same result through the Variational Principle by Extremizing the action. Here is an action which leads to the field equations corresponding to black hole surrounded by anisotropic matter field [8].

$$\mathcal{I} = \frac{1}{16\pi G} \int_{\mathcal{M}} \sqrt{-g} d^4x [(R - F_{\mu\nu}F^{\mu\nu}) + \mathcal{L}_m] + \mathcal{I}_S \quad (1.27)$$

where R is the Ricci scalar of the spacetime manifold $\mathcal{M}$, $F_{\mu\nu}$ is the electromagnetic field tensor and $\mathcal{L}_m$ is the Lagrangian density corresponding to the effective anisotropic matter fields. The term corresponding to anisotropic matter field can be a result of an extra $U(1)$ field or other diverse dark matters. With out loss of generality, we set $G = 1$. In the above action, $\mathcal{I}_S$ is the Gibbon-Hawking boundary term which is essential to make the variational principle well-defined [65, 66]. The variation of the action (1.27) leads to the Einstein equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu} \quad (1.28)$$

and the Maxwell equations

$$\nabla_{\mu}F^{\mu\nu} = \frac{1}{\sqrt{-g}}[\partial_{\mu}(\sqrt{-g}F^{\mu\nu})] = 0 \quad (1.29)$$

The energy momentum tensor in (1.28) is sourced by the maxwell field and the anisotropic matter field, which is given by,

$$T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\alpha}F_{\nu}^{\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \right) - 2\frac{\partial\mathcal{L}_m}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_m \quad (1.30)$$

Starting from a spherically symmetric and a static space-time metric ansatz of the form the field equations leads to the following metric [7, 67],

$$g_{\mu\nu} = \text{diag} \left(-f(r), \frac{1}{f(r)}, r^2, r^2 \sin^2 \theta \right) \quad (1.31)$$

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{K}{r^{2w}} \quad (1.32)$$

Where, M is the ADM mass and Q is the electric charge of the black hole. The parameter K controls the density of the fluid surrounding the black hole, whereas, w represents its anisotropy [7, 8, 67].

The function $f(r)$ defines the horizons for the anisotropic matter field charged black hole solution and also whether the space-time is flat or curved. To find

the exact analytic expressions for the horizons is not possible for $f(r)$ given in Eq.(1.23). For the value of $w \geq 0$, the space-time is asymptotically flat. And due to $1/r$ and $1/r^2$ dependencies of $f(r)$, $r = 0$ defines the black hole singularity. Now, if $K \geq 0$ then $f(r)$ asymptotes to 1 from below and if $K < 0$ then $f(r)$ asymptotes to 1 from above. For the values of $w < 0$ then there exists a singularity at infinity.

1.2 True Singularities

The Kretschmann invariant for the metric of the form in Eq.(??) is given by,

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \left(\frac{d^2f(r)}{dr^2}\right)^2 + \frac{4}{r^2} \left(\frac{df(r)}{dr}\right)^2 + \frac{4}{r^4}(1-f(r))^2 \quad (1.33)$$

Thus for anisotropic matter field by substituting $f(r)$ of Eq.(1.23) in the above equation we get,

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = AK^2 + BK + C \quad (1.34)$$

$$A = \frac{4r^{-4w}(4w^4 + 4w^3 + 5w^2 + 1)}{r^4} \quad (1.35)$$

$$B = 8r^{-2w} \frac{(w+1)}{r^5} \left(2M(2w+1) - \frac{Q^2}{r}(6w+1) \right) \quad (1.36)$$

$$C = \frac{48M^2}{r^6} - \frac{96MQ^2}{r^7} + \frac{56Q^4}{r^8} \quad (1.37)$$

Now, there exists singularity wherever the Kretschmann invariant diverges [64]. Thus we see that there is no singularity other than $r = 0$ and $r = \infty$. According to the C factor, there is a singularity at $r = 0$. In factor A , we can see that if $w < -1$ then it diverges when $r \rightarrow \infty$. Finally for factor B , if $w < -5/2$ ($w \neq -1$) then also there exists singularity at infinity.

1.3 Analyzing the Black hole solution $f(r)$

Even though the space-time metric for the charged black hole in anisotropic matter field is complicated, we can infer its geometry graphically by analyzing $f(r)$. Without the loss of generality, we choose $M = 0.8$ and $Q = 0.4$ in this particular section unless otherwise stated and see the effects of varying K and w .

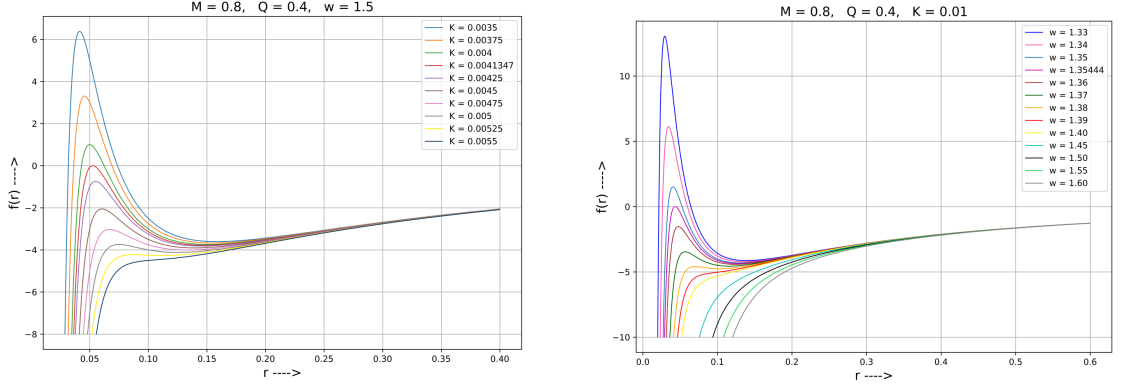


Figure 1.1: Positive values of K and for $w > 1$

Case 1 - For positive values of K and $w > 1$

In the above (left) graph, we have chosen very small positive K values and $w = 1.5$. For the values of $K < 0.0041347$, there exist three horizons. And for all $K > 0.0041347$, there is only one horizon. As K increases the size of the outermost horizon increases. In the adjacent graph, we put $K = 0.01$ and vary w . Again there is a possibility of three horizons, with a similar variation of $f(r)$. And for $1 < w < 1.354447635$ there are three horizons and for $w > 1.354447635$ there is only one horizon. If the K value is increased then this critical value of w decreases. For example, if $K = 0.011$ then the critical value of $w = 1.339327514$.

Case 2 - For positive values of K and $0 < w < 1$

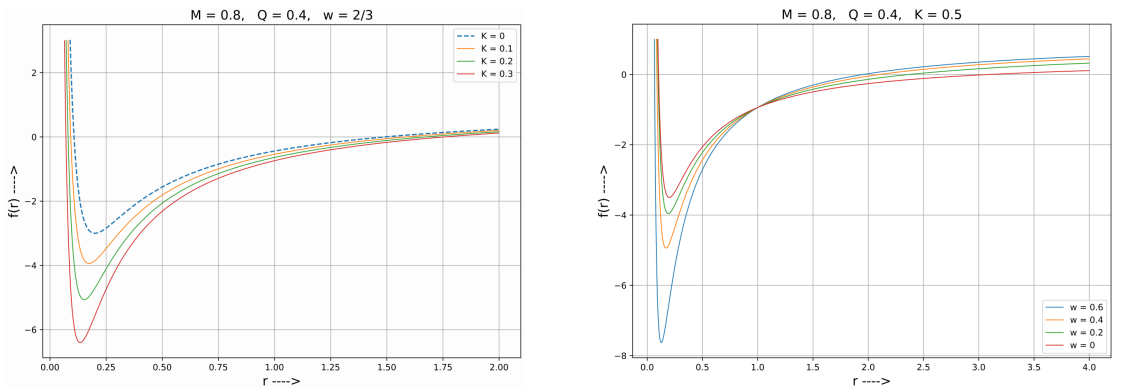


Figure 1.2: Positive values of K and for $0 < w < 1$

Here there are two event horizons with $f(r_{h-}) = f(r_{h+}) = 0$ where in between the two event horizons the sign of $f(r)$ is negative. But for $r > r_{h+}$ and $r < r_{h-}$, $f(r)$ is positive. This is an acceptable trend that closely resembles the *Reissner-Nordström* black hole. As the K value increases, the $f(r)$ value between the

two horizons decreases. Also, if we vary w , then as its value increases, $f(r)$ value decreases steeply between the two horizons. Also, both r_{h-} and r_{h+} decrease but r_{h+} decreases at a higher rate than r_{h-} .

Case 3 - For $M^2 < Q^2$ and varying values of K and w

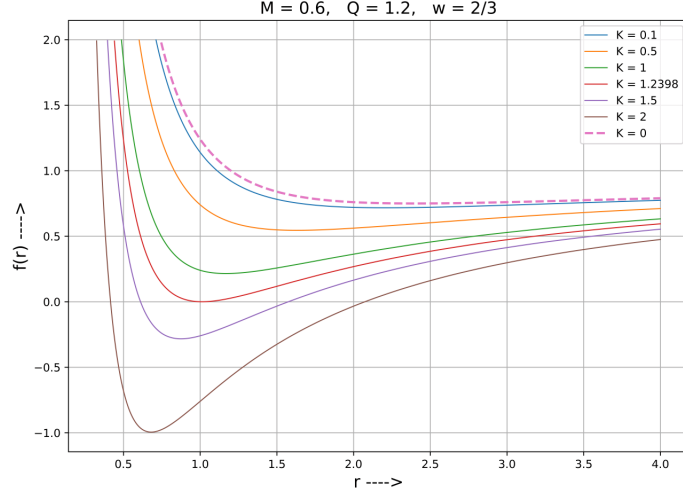


Figure 1.3: Positive values of K with $M^2 < Q^2$

When $Q^2 > M^2$ we see that there is a possibility of naked singularity without an event horizon (where $f(r)$ has imaginary roots). This type of black hole has no physical significance according to the Cosmic Censorship Hypothesis [68]. And to avoid that we have chosen $M = 0.8$ and $Q = 0.4$. As shown in the above graph for $0 < w < 1$ and $K > 0$, if we choose $M = 0.6$, $Q = 1.2$, $w = 2/3$ there are naked singularities. But if K is sufficiently large ($K > 1.23984486045$) then the possibility of naked singularity is evaded where the black hole has two event horizons. In the case of $w > 1$ and $K > 0$ then there is no naked singularity as there is at least one horizon. And when $K < 0$ then there is always a naked singularity for any $\pm w$.

Case 4 - For negative values of K and $w > 0$

When $K < 0$, even when $M^2 > Q^2$ for $M = 0.8$ and $Q = 0.4$ we see that there are possibilities of naked singularities as shown below. In graph(a), we fix $w = 1.5$ and vary the K values. For $K < -0.440282855654$ there are naked singularities and for $-0.440282855654 < K \leq 0$, the black hole has two event horizons. Also, as seen the case 1, for the same values of M , Q and w when $0 < K < 0.0041347156372588$ there are three event horizons and for $K > 0.0041347156372588$, there exists only

one event horizon. In graph(b), we fix $K = -0.1$ and observe that as the w value increases the minima of $f(r)$ will increase but there is no naked singularity for any value of w . In graph(c), we fix $w = 2/3$, for $K < -0.62195556429$ there are naked singularities, and for $K > -0.62195556429$ there are two event horizons. In graph(d), we fix $K = -0.8$ and when $0.5 < w < 3.3580867141$ it always forms naked singularities and for other values of w , there are no naked singularities. Generally, when $K < -0.440282855654$, there exist two critical values of w where in between those values of w , the solution is a naked singularity.

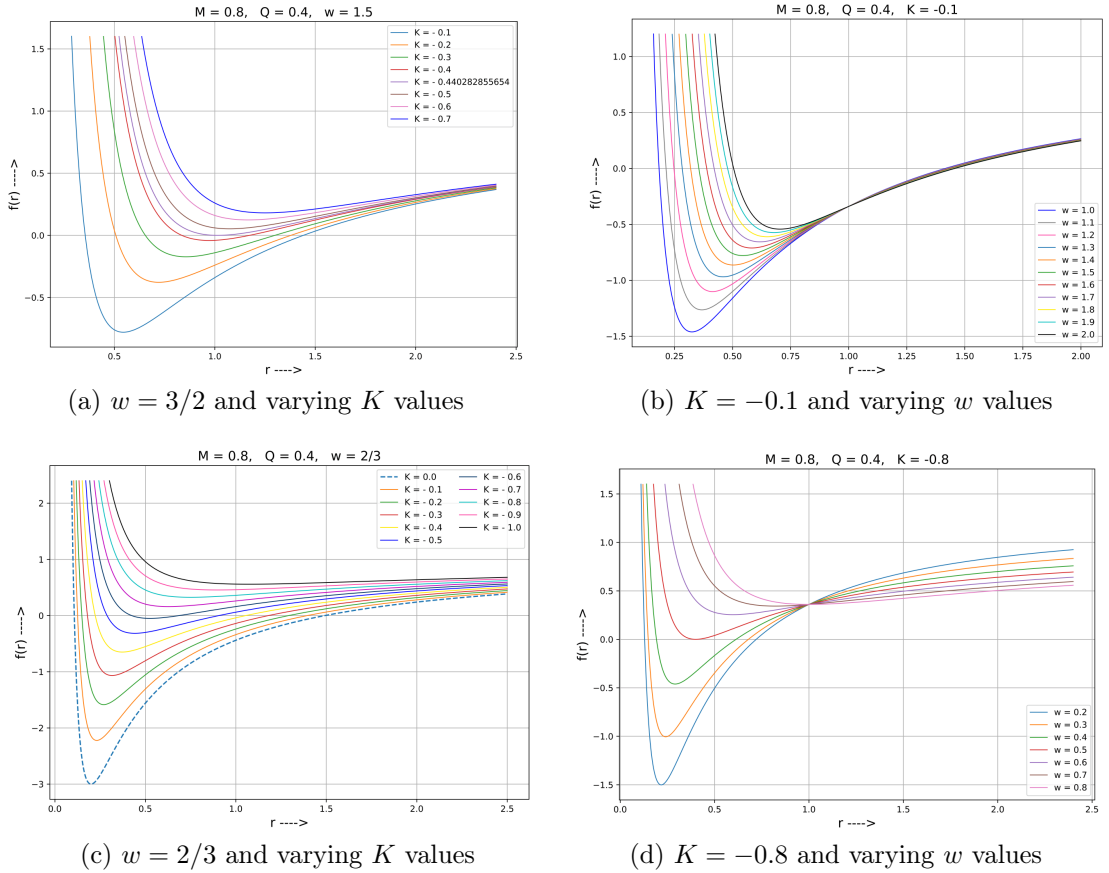


Figure 1.4: Trends for negative values of K

Case 5 - For negative values of $w < 0$

At large values of r , the mass term and charge term become negligible as they have $1/r$ and $1/r^2$ dependency, respectively. But when $w < 0$, then the anisotropic term $-K r^{-2w} \rightarrow \pm \infty$ as $r \rightarrow \infty$ depending on whether K is negative or positive respectively. Hence as $r \rightarrow \infty$, we see that $f(r) \rightarrow \pm \infty$. Thus, when $w < 0$, there is a coordinate singularity at infinity. This singularity has no physical significance.

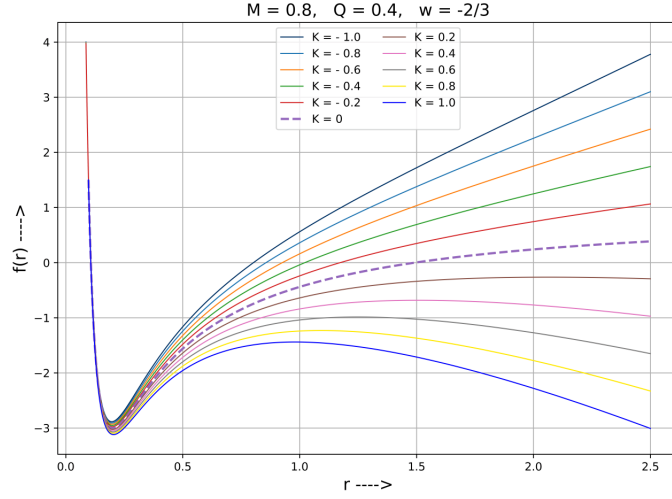
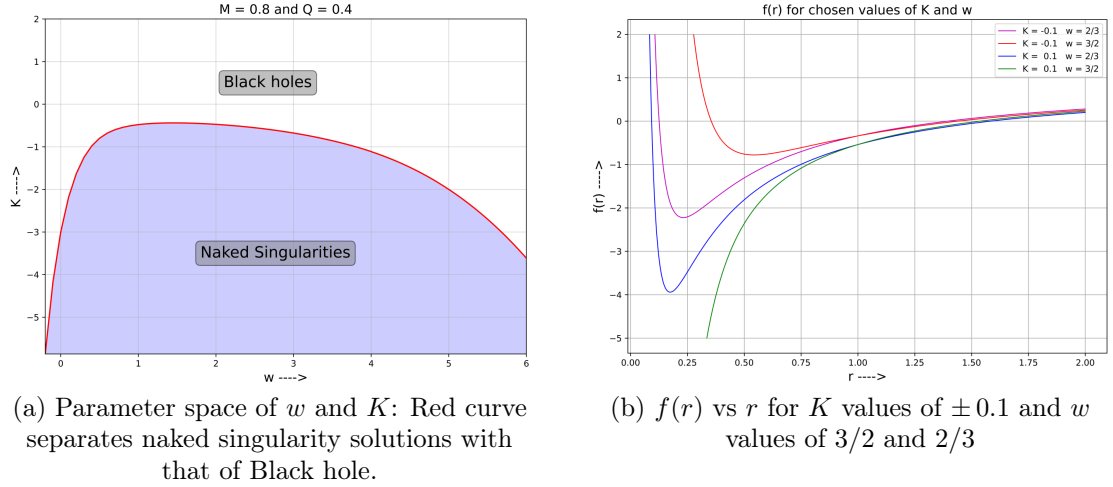


Figure 1.5: Negative value of w and varying value of K from -1 to $+1$

Chosen values of K and w for Black hole solutions



(a) Parameter space of w and K : Red curve separates naked singularity solutions with that of Black hole.

(b) $f(r)$ vs r for K values of ± 0.1 and w values of $3/2$ and $2/3$

Figure 1.6: Parameters for final Black Hole Solution

To avoid the nonphysical singularity at infinity we focus on the case where $w > 0$ in this article, which ensures asymptotically flat geometry. And also as shown in [25], for $0 < w < 1/2$ the mass function $m(r)$ given in Eq.(1.26) tends to $\pm\infty$ as $r \rightarrow \infty$ and hence cannot describe a localized object. Thus we will not consider the case where $w < 1/2$. Also, the case of $w = 1/2$ is very similar to the Reissner–Nordström black hole with the new mass parameter, $\tilde{M} = 2M + K$. Hence, we focus on the case where $w > 1/2$. The case of $w = 1$ represents an extra charge-like parameter r_0 given by $r_0^2 = -K$ this also describes a Reissner–Nordström black hole with a new charge parameter, $\tilde{Q}^2 = Q^2 + r_0^2$.

Chapter 2

Scalar perturbation

2.1 Massless Klein-Gordon Equation

A massless scalar field $\Psi(t, \vec{r})$ is the solution of the Klein-Gordon (KG) equation, $\square\Psi(t, \vec{r}) = 0$. In curved space-time, the D'Alembertian operator ($\square$) is given by, $\square = \nabla_\mu \nabla^\mu$ where, ∇_μ is the covariant derivative. When the D'Alembertian operator is expanded, the Klein Gordon equation becomes [40],

$$\square\Psi(t, \vec{r}) = \nabla_\mu \nabla^\mu \Psi(t, \vec{r}) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi(t, \vec{r})) = 0 \quad (2.1)$$

The metric tensor is given by Eq.(1.31) with $e^\nu = f(r)$ as in Eq.(1.23). In the above Eq.(2.1), ' g ' is the determinant of the metric tensor which is $g = r^4 \sin^2(\theta)$. Then, on expanding the Klein-Gordon equation, we get,

$$f \frac{\partial^2 \Psi}{\partial r^2} + \left[\frac{\partial f}{\partial r} + \frac{2f}{r} \right] \frac{\partial \Psi}{\partial r} - \frac{1}{f} \frac{\partial^2 \Psi}{\partial t^2} + \frac{1}{r^2} \left[\frac{\partial^2 \Psi}{\partial \theta^2} + \frac{\cos(\theta)}{\sin(\theta)} \frac{\partial \Psi}{\partial \theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2 \Psi}{\partial \phi^2} \right] = 0 \quad (2.2)$$

2.2 Effective Potential of for Scalar Perturbation

Due to the spherical symmetry in space-time, we can perform the separation of variables by splitting the scalar field function $\Psi(t, r, \theta, \phi)$ into radial and angular parts where,

$$\Psi(t, r, \theta, \phi) = \sum_{l,m} \frac{\psi}{r} Y_l^m(\theta, \phi) \quad (2.3)$$

Here, $Y_l^m(\theta, \phi)$ is the spherical harmonics, defined by the below operator,

$$\left(\frac{\partial^2}{\partial \theta^2} + \frac{\cos(\theta)}{\sin(\theta)} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \right) Y_l^m(\theta, \phi) = -l(l+1)Y_l^m(\theta, \phi) \quad (2.4)$$

On substituting $\Psi(t, r, \theta, \phi)$ given in Eq.(2.3) into the KG Eq.(2.2) we get,

$$Y_l^m \left(f(r) \frac{\partial^2}{\partial r^2} + \left[\frac{\partial f(r)}{\partial r} + \frac{2}{r} f(r) \right] \frac{\partial}{\partial r} - \frac{1}{f(r)} \frac{\partial^2}{\partial t^2} \right) \frac{\psi}{r} - l(l+1) \frac{\psi Y_l^m}{r^3} = 0 \quad (2.5)$$

After expanding the first term by performing the differentiation we get,

$$Y_l^m \left[\frac{1}{r} \frac{\partial f}{\partial r} \frac{\partial \psi(t, r)}{\partial r} + \frac{f}{r} \frac{\partial^2 \psi(t, r)}{\partial r^2} - \frac{1}{rf} \frac{\partial^2 \psi(t, r)}{\partial t^2} - \frac{\psi(t, r)}{r^2} \frac{\partial f}{\partial r} \right] - l(l+1) \frac{\psi Y_l^m}{r^3} = 0$$

Then multiplying throughout by $r/Y_l^m f$, expanding $\partial f/\partial r$ and rearranging, we get the radial differential equation,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{f'}{f} \frac{\partial \psi}{\partial r} - \frac{1}{f^2} \frac{\partial^2 \psi}{\partial t^2} - \left(\frac{l(l+1)}{r^2} + \frac{f'}{r} \right) \frac{1}{f} \psi = 0 \quad (2.6)$$

To remove the first derivative $\partial \psi/\partial r$ we perform the coordinate transformation of ' $r \rightarrow r(\tau)$ ' (where ' τ ' is the tortoise coordinate) defined by,

$$\frac{d\tau}{dr} = \frac{1}{f(r)} \quad \Rightarrow \quad \tau = \int \frac{1}{f(r)} dr \quad (2.7)$$

Then the differential equation in terms of t and τ becomes,

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial \tau^2} + \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right) f(r) \psi = 0 \quad (2.8)$$

The effective potential $V_{SC}(r(\tau))$ (where ' r ' is a function of tortoise coordinate ' τ ') is the coefficient of $\psi(\tau)$ in Eq.(2.8) which is given by,

$$\boxed{V_{SC}(r) = \left(\frac{l(l+1)}{r^2} + \frac{f'(r)}{r} \right) f(r)} \quad (2.9)$$

We go to the frequency domain by substituting $\psi(t, \tau) = e^{i\omega t} \psi(\tau)$ in Eq.(2.8), where ω is the **QNM** frequency. Thus we get,

$$(i\omega)^2 e^{i\omega t} \psi(\tau) - e^{i\omega t} \frac{d^2 \psi(\tau)}{d\tau^2} + V_{SC} e^{i\omega t} \psi(\tau) = 0 \quad \Rightarrow \quad \frac{d^2 \psi(\tau)}{d\tau^2} + [\omega^2 - V_{SC}] \psi(\tau) = 0 \quad (2.10)$$

For charged black hole in anisotropic matter field we have $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{K}{r^{2w}}$. Then the effective potential of scalar field perturbation becomes,

$$V_{SC}(r) = \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} + \frac{2Kw}{r^{2(w+1)}} \right) \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{K}{r^{2w}} \right) \quad (2.11)$$

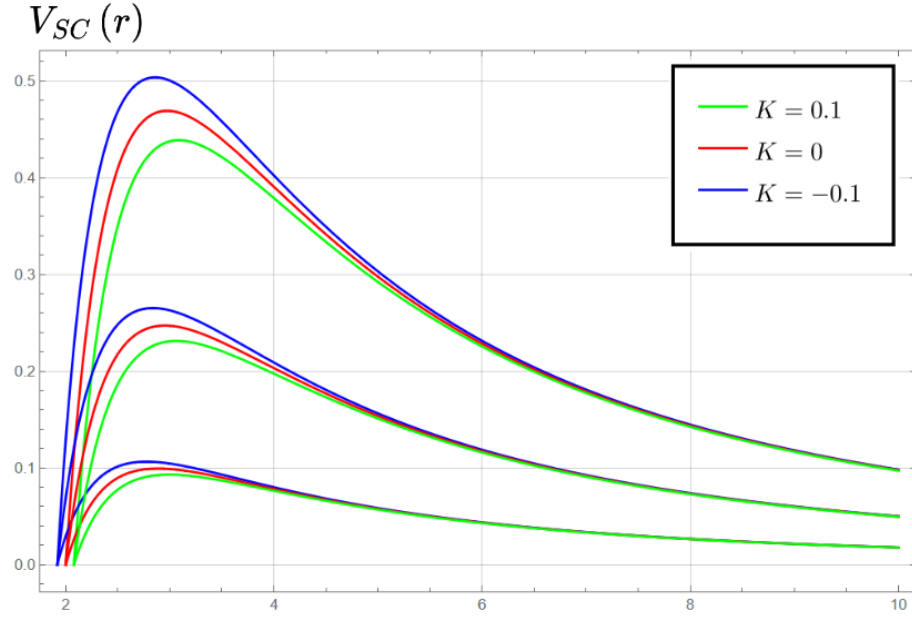


Figure 2.1: Symmetric splitting of Effective Potential due to ‘ K ’ for $M = 1$ $Q = 0$ $w = 2/3$ with $\ell = 0, 1, 2$

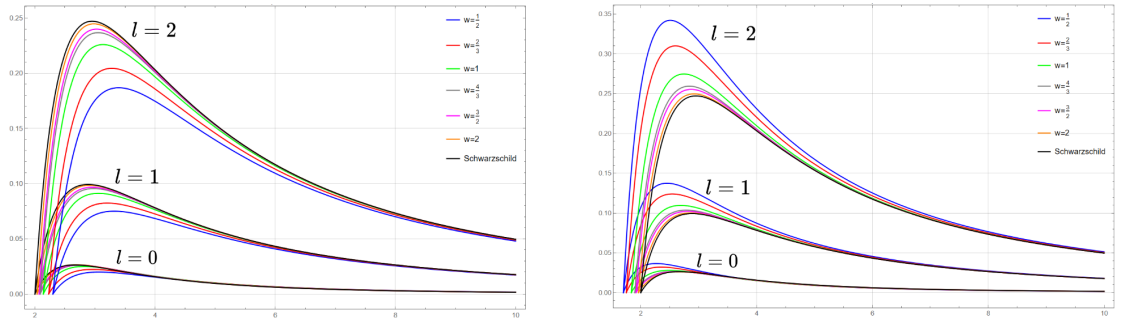


Figure 2.2: Asymmetric splitting of Effective Potential due to ‘ w ’ for $M = 1$ $Q = 0$ $K = 0.3$ (left) and $K = -0.3$ (right) with $\ell = 0, 1, 2$

2.3 QNMs of massless scalar field perturbations

$M = 1$	$Q = 0$	$w = 2/3$	$l = 0$	$n = 0$
K	order	QNM	error	
0.5	6	0.0951168 -0.0889349 i	0.00381082	
0.4	6	0.0977266 -0.0910186 i	0.00379095	
0.3	6	0.100542 -0.0932287 i	0.00378674	
0.2	6	0.103586 -0.0955815 i	0.00416315	
0.1	6	0.106873 -0.0981126 i	0.00418054	
0.01	6	0.110093 -0.100537 i	0.00369649	
0.001	6	0.110436 -0.100782 i	0.00437215	
0	7	0.112922 -0.103501 i	0.00294289	
-0.001	6	0.110507 -0.100842 i	0.00398027	
-0.01	6	0.110845 -0.101097 i	0.00357343	
-0.1	5	0.109207 -0.108663 i	0.00464097	
-0.2	6	0.118744 -0.106872 i	0.00366302	
-0.3	5	0.118277 -0.115209 i	0.00469072	
-0.4	5	0.123651 -0.118883 i	0.00467726	
-0.5	5	0.129757 -0.122868 i	0.00498554	

$M = 1$	$Q = 0$	$w = 2/3$	$l = 1$	$n = 0$
K	order	QNM	error	
0.5	7	0.251564 -0.0860734 i	0.000081579	
0.4	7	0.258578 -0.0880991 i	0.0000936773	
0.3	7	0.266133 -0.0902544 i	0.0000956014	
0.2	7	0.274304 -0.092554 i	0.0000847134	
0.1	7	0.283181 -0.0950124 i	0.0000665287	
0.01	6	0.2919002 -0.0974897 i	0.0000984449	
0.001	7	0.292772 -0.0976254 i	0.0000578768	
0	9	0.292934 -0.0976419 i	0.0000202195	
-0.001	7	0.292974 -0.0976785 i	0.0000841749	
-0.01	6	0.293928 -0.0980356 i	0.0000981714	
-0.1	7	0.303517 -0.100491 i	0.0000680406	
-0.2	6	0.315317 -0.103658 i	0.0000908603	
-0.3	6	0.328426 -0.106978 i	0.0000839193	
-0.4	6	0.343168 -0.110593 i	0.0000625846	
-0.5	6	0.359932 -0.114543 i	0.0000561181	

$M = 1$	$Q = 0$	$w = 3/2$	$l = 1$	$n = 0$
K	order	QNM	error	
0.5	6	0.285014 -0.0991376 i	0.0000169756	
0.4	6	0.286487 -0.0989253 i	0.0000265346	
0.3	6	0.288008 -0.0986865 i	0.0000409372	
0.2	6	0.289583 -0.0984168 i	0.0000590972	
0.1	6	0.291215 -0.0981107 i	0.0000791691	
0.01	6	0.292737 -0.0977987 i	0.000097626	
0.001	6	0.292892 -0.0977653 i	0.0000977669	
0	9	0.292934 -0.0976419 i	0.0000202195	
-0.001	6	0.292927 -0.0977579 i	0.0000990285	
-0.01	6	0.293083 -0.097724 i	0.000101121	
-0.1	7	0.29463 -0.0972363 i	0.0000918677	
-0.2	7	0.296461 -0.0967617 i	0.000108501	
-0.3	6	0.29842 -0.0963638 i	0.000147487	
-0.4	6	0.300419 -0.0957381 i	0.000156524	
-0.5	6	0.302507 -0.0950028 i	0.000156299	

$M = 1$	$Q = 0$	$w = 2$	$l = 1$	$n = 0$
K	order	QNM	error	
0.5	5	0.289789 -0.0993224 i	0.000208655	
0.4	5	0.29043 -0.0990397 i	0.000130619	
0.3	5	0.291078 -0.0987389 i	0.000063821	
0.2	6	0.291755 -0.0984113 i	0.0000206957	
0.1	6	0.292325 -0.0981 i	0.0000439141	
0.01	6	0.292851 -0.0977967 i	0.0000953591	
0.001	6	0.292904 -0.0977652 i	0.0000974345	
0	9	0.292934 -0.0976419 i	0.0000202195	
-0.001	6	0.292916 -0.0977581 i	0.0000984021	
-0.01	7	0.292932 -0.0976141 i	0.0000704331	
-0.1	6	0.293509 -0.0973938 i	0.000143085	
-0.2	6	0.294121 -0.096994 i	0.000175189	
-0.3	6	0.294747 -0.0965595 i	0.000199151	
-0.4	6	0.295384 -0.0960876 i	0.000215491	
-0.5	6	0.296029 -0.0955759 i	0.000220041	

$M = 1$	$Q = 0$	$K = 0.3$	$l = 0$	$n = 0$
w	Order	QNM	error	
$\frac{1}{2}$	6	0.0960612 -0.0876637 i	0.00336733	
$\frac{2}{3}$	6	0.100542 -0.0932287 i	0.00378674	
$\frac{3}{4}$	6	0.102115 -0.0952006 i	0.00433782	
1	6	0.105233 -0.0990899 i	0.00441774	
$\frac{4}{3}$	6	0.107198 -0.101996 i	0.00409733	
$\frac{5}{2}$	6	0.107586 -0.102954 i	0.00323123	
2	6	0.108775 -0.101809 i	0.0040388	
3	4	0.108851 -0.113658 i	0.0062496	
10	6	0.110431 -0.10084 i	0.0036706	

$M = 1$	$Q = 0$	$K = -0.3$	$l = 0$	$n = 0$
w	Order	QNM	error	
$\frac{1}{2}$	6	0.129973 -0.118597 i	0.00443451	
$\frac{2}{3}$	5	0.118277 -0.115209 i	0.00469072	
$\frac{3}{4}$	5	0.116301 -0.112042 i	0.00447246	
1	6	0.117028 -0.102127 i	0.00393814	
$\frac{4}{3}$	5	0.10998 -0.103805 i	0.00366143	
$\frac{5}{2}$	5	0.109464 -0.103473 i	0.00332688	
2	5	0.109115 -0.103797 i	0.00247706	
3	5	0.10908 -0.105333 i	0.00130766	
10	6	0.11034 -0.100941 i	0.00368622	

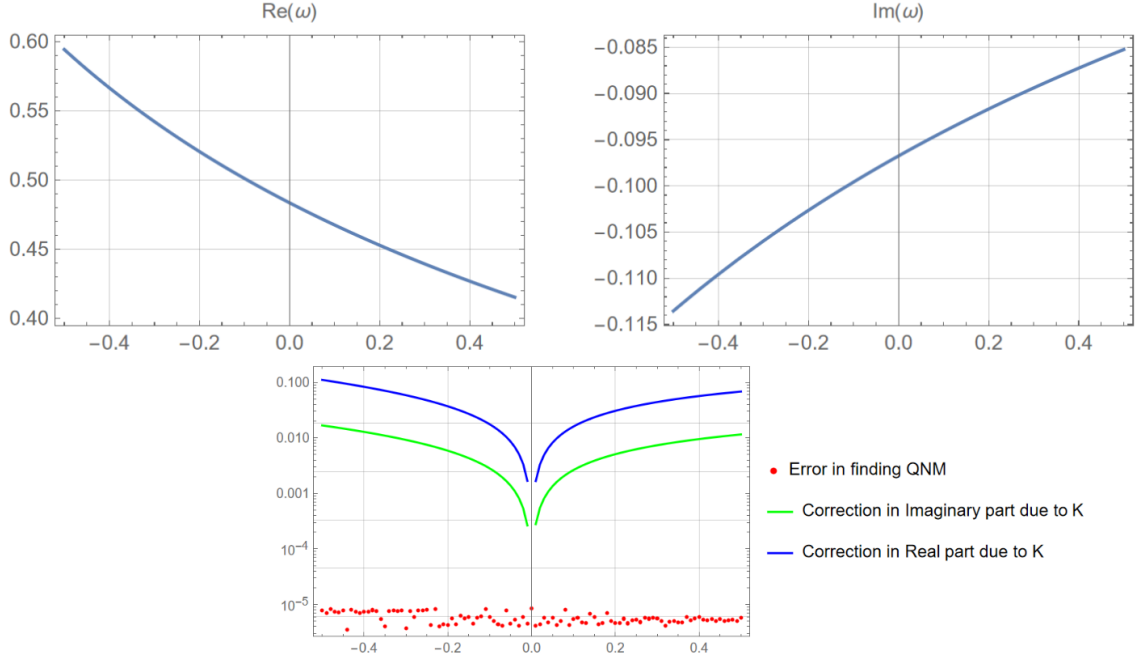


Figure 2.3: Variation of QNM due to ‘K’ with $w = 2/3$, $\ell = 2$ and $n = 0$

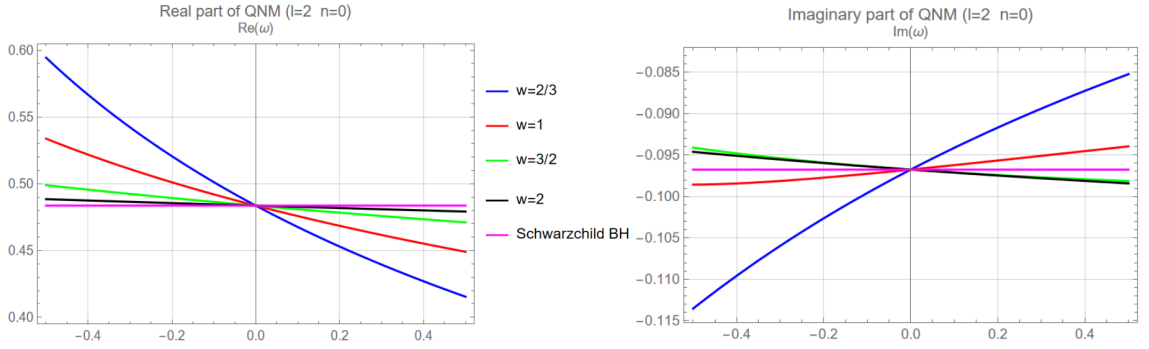


Figure 2.4: Variation of QNM due to ‘K’ with different w values for $\ell = 2$ and $n = 0$

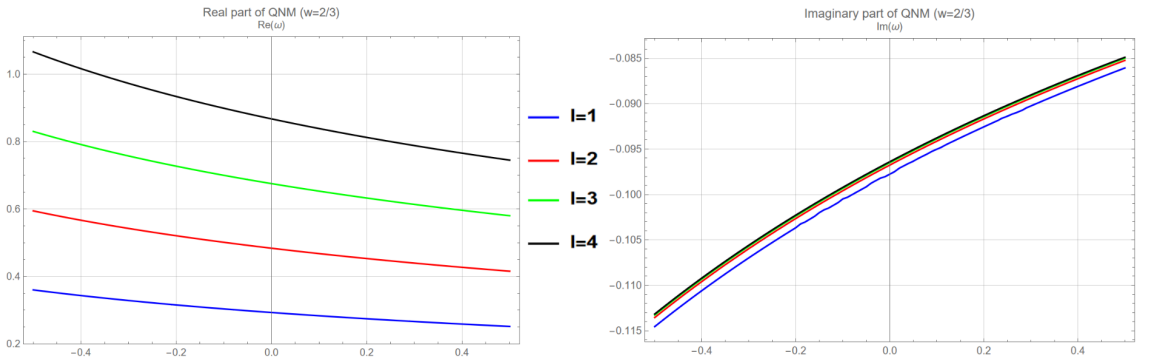


Figure 2.5: Variation of QNM due to ‘K’ with different ℓ values for $w = 2/3$ and $n = 0$

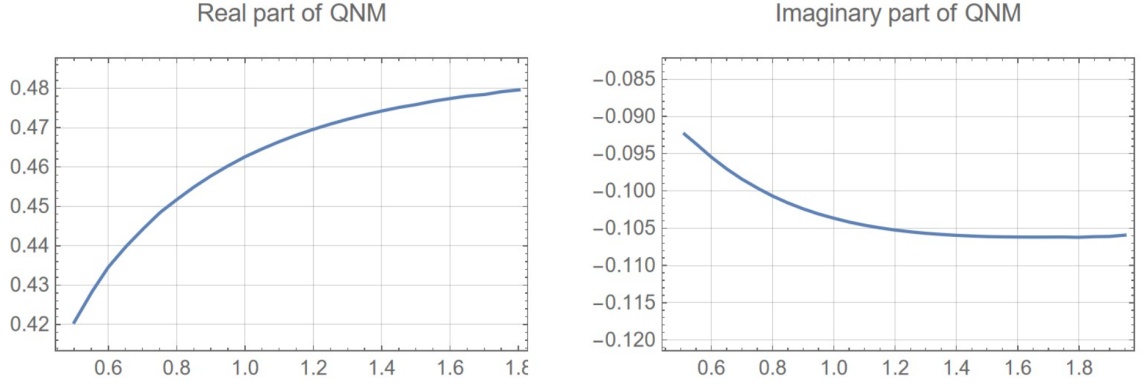


Figure 2.6: Variation of QNM due to ' w ' with $K = 0.3$ and $n = 0$

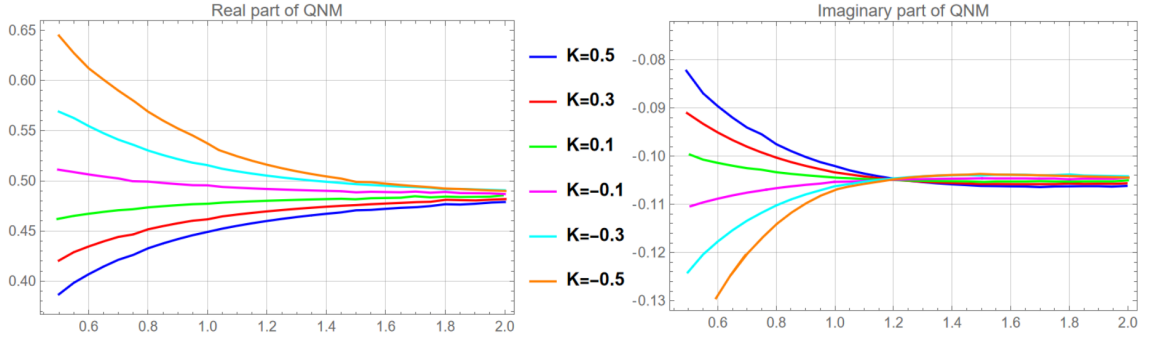


Figure 2.7: Variation of QNM due to ' w ' with different K values for $\ell = 2$ and $n = 0$

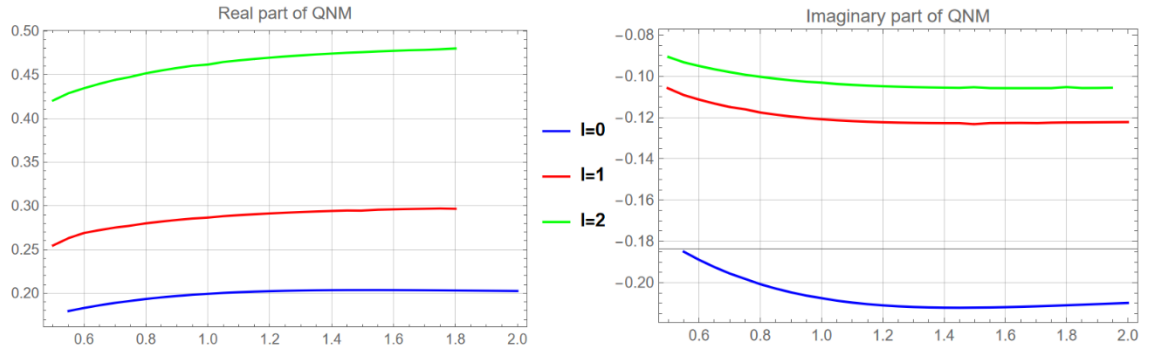


Figure 2.8: Variation of QNM due to ' w ' with different ℓ values for $K = 0.3$ and $n = 0$

Chapter 3

Electromagnetic perturbation

3.1 Maxwell's equation in curved space-time

Maxwell's equations in curved space-time are given by,

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu \quad (3.1)$$

where ∇_μ denotes the covariant derivative with respect to μ^{th} coordinate. And $F^{\mu\nu}$ are the contravariant components of the Electromagnetic Stress Tensor (Faraday Tensor). J^ν is the four current density and μ_0 is the permeability of vacuum. The expansion of $\nabla_\mu F^{\mu\nu}$ is $\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} F^{\mu\nu})$ where g is the determinant of metric tensor. In a source-free region i.e., where $J^\nu = 0$, the Maxwell's equations become,

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} F^{\mu\nu}) = 0 \quad (3.2)$$

The covariant components $F_{\mu\nu}$ is defined in terms of the four potential A_μ as,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3.3)$$

Since the background space-time is spherically symmetric, we can use the vector spherical harmonics to describe the four potential A_μ whose components are,

$$A_\mu = \sum_{l,m} \left[\begin{pmatrix} 0 \\ 0 \\ \frac{b(t,r)}{\sin(\theta)} \partial_\phi Y_l^m(\theta, \phi) \\ -b(t,r) \sin(\theta) \partial_\theta Y_l^m(\theta, \phi) \end{pmatrix} + \begin{pmatrix} f(t,r) Y_l^m(\theta, \phi) \\ h(t,r) Y_l^m(\theta, \phi) \\ k(t,r) \partial_\theta Y_l^m(\theta, \phi) \\ k(t,r) \partial_\phi Y_l^m(\theta, \phi) \end{pmatrix} \right] \quad (3.4)$$

Where b, f, h, k are functions of t and r . The left part of A_μ has a parity of $(-1)^{l+1}$ results in the axial EM perturbation and the right part of A_μ has a parity of $(-1)^l$ results in the polar EM perturbation. But focusing on the axial EM perturbations and denoting $b(t, r)$ by $\psi(t, r)$ we get the four potential A_μ as,

$$A_\mu = \sum_{l,m} \begin{pmatrix} 0 \\ 0 \\ \frac{\psi(t, r)}{\sin(\theta)} \partial_\phi Y_l^m(\theta, \phi) \\ -\psi(t, r) \sin(\theta) \partial_\theta Y_l^m(\theta, \phi) \end{pmatrix} \quad (3.5)$$

This results in the electromagnetic stress tensor $F_{\mu\nu}$ components as,

$$[F_{\mu\nu}] = \begin{bmatrix} 0 & 0 & \frac{1}{\sin(\theta)} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial \psi}{\partial t} & -\sin(\theta) \frac{\partial Y_l^m}{\partial \theta} \frac{\partial \psi}{\partial t} \\ 0 & 0 & \frac{1}{\sin(\theta)} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial \psi}{\partial r} & -\sin(\theta) \frac{\partial Y_l^m}{\partial \theta} \frac{\partial \psi}{\partial r} \\ asym & asym & 0 & l(l+1) \sin(\theta) Y_l^m \psi \\ asym & asym & asym & 0 \end{bmatrix} \quad (3.6)$$

Here ‘*asym*’ indicates that the particular component is found by $F_{\mu\nu} = -F_{\nu\mu}$ since the Faraday tensor is asymmetric. The components, F_{23} and F_{32} were simplified using Eq.(2.4). The contravariant components of Faraday Tensor is $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$. Since the metric tensor is diagonal, its contravariant components are reciprocal of covariant components, and are given by,

$$g^{\mu\nu} = \text{diag}(-1/f(r), f(r), 1/r^2, 1/(r^2 \sin^2 \theta))$$

$$F^{02} = g^{00} g^{22} F_{02}$$

$$F^{12} = g^{11} g^{22} F_{12}$$

$$F^{23} = g^{22} g^{33} F_{23}$$

$$F^{30} = g^{33} g^{00} F_{30}$$

$$F^{31} = g^{33} g^{11} F_{31}$$

Thus the independent non-zero components of the Faraday tensor are,

$$F_{02} = \frac{1}{\sin(\theta)} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial \psi}{\partial t} \quad F^{02} = -\frac{e^{-\nu(r)}}{r^2 \sin(\theta)} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial \psi}{\partial t} \quad (3.7)$$

$$F_{12} = \frac{e^{-\nu(r)}}{r^2 \sin(\theta)} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial \psi}{\partial r} \quad F^{12} = \frac{1}{\sin(\theta)} \frac{\partial Y_l^m}{\partial \phi} \frac{\partial \psi}{\partial r} \quad (3.8)$$

$$F_{23} = l(l+1) \sin(\theta) Y_l^m \psi \quad F^{23} = \frac{l(l+1) Y_l^m \psi}{r^4 \sin(\theta)} \quad (3.9)$$

$$F_{30} = \sin(\theta) \frac{\partial Y_l^m}{\partial \theta} \frac{\partial \psi}{\partial t} \quad F^{30} = -\frac{e^{-\nu(r)}}{r^2 \sin(\theta)} \frac{\partial Y_l^m}{\partial \theta} \frac{\partial \psi}{\partial t} \quad (3.10)$$

$$F_{31} = \sin(\theta) \frac{\partial Y_l^m}{\partial \theta} \frac{\partial \psi}{\partial r} \quad F^{31} = \frac{e^{\nu(r)}}{r^2 \sin(\theta)} \frac{\partial Y_l^m}{\partial \theta} \frac{\partial \psi}{\partial r} \quad (3.11)$$

3.2 Effective Potential of EM Perturbation

On expanding Eq.(3.2) by substituting the Faraday tensor components, we get,

$$\frac{1}{r^2 \sin(\theta)} \left(\frac{\partial Y_l^m}{\partial \theta} - \frac{\partial Y_l^m}{\partial \phi} \right) \underbrace{\left(f \frac{\partial^2 \psi}{\partial r^2} + f' \frac{\partial \psi}{\partial r} - \frac{1}{f} \frac{\partial^2 \psi}{\partial t^2} - \frac{l(l+1)}{r^2} \psi \right)}_{\text{Radial Equation}} = 0 \quad (3.12)$$

Focusing on the radial equation, to remove the first derivative term, we employ the coordinate transformation $r \rightarrow \tau$ where the tortoise coordinate τ is defined according to Eq.(2.7), which can be written as,

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial \tau^2} + \frac{l(l+1)f(r)}{r^2} \psi = 0 \quad (3.13)$$

Where $r \rightarrow r(\tau)$ and $\psi \rightarrow \psi(t, \tau)$. The effective potential for electromagnetic perturbation is given by the coefficient of ψ which is,

$$\boxed{V_{EM}(r) = \frac{l(l+1)f(r)}{r^2}} \quad (3.14)$$

If $\psi(t, \tau) = e^{i\omega t} \psi(\tau)$ in Eq.(3.13) $\Rightarrow (-i\omega)^2 e^{i\omega t} \psi(\tau) - e^{i\omega t} \frac{\partial^2 \psi(\tau)}{\partial \tau^2} + V_{EM} e^{i\omega t} \psi(\tau) = 0$

$$\Rightarrow \boxed{\frac{d^2 \psi}{d\tau^2} + [\omega^2 - V_{EM}(r)] \psi = 0} \quad (3.15)$$

Using $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{K}{r^{2w}}$ for the metric near a charged black hole in anisotropic matter field, the effective potential of EM perturbation becomes,

$$V_{EM}(r) = \frac{l(l+1)}{r^2} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{K}{r^{2w}} \right) \quad (3.16)$$

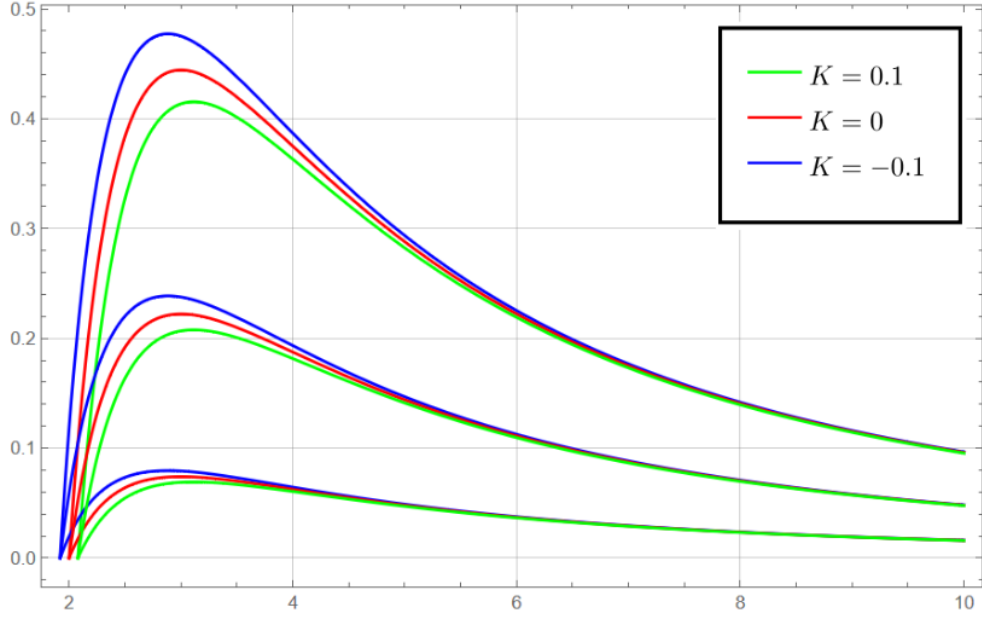


Figure 3.1: Symmetric splitting of Effective Potential due to ‘ K ’ for $M = 1$ $Q = 0$ $w = 2/3$ with $\ell = 0, 1, 2$

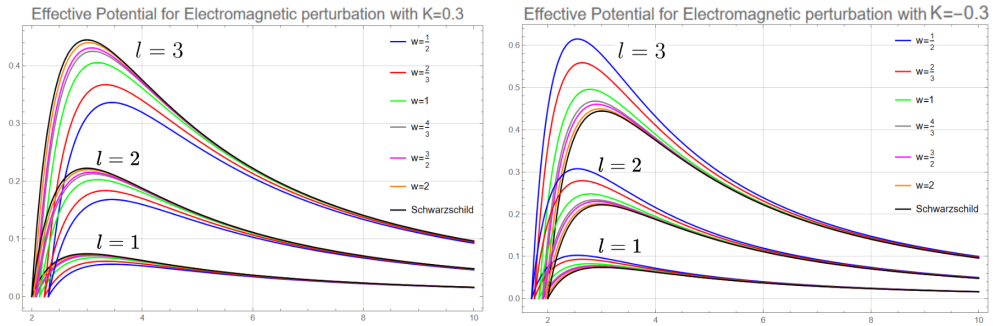


Figure 3.2: Asymmetric splitting of Effective Potential due to ‘ w ’ for $M = 1$ $Q = 0$ $K = 0.3$ (left) and $K = -0.3$ (right) with $\ell = 0, 1, 2$

3.3 QNMs of Electromagnetic perturbation

$M = 1$		$Q = 0$	$w = 2/3$	$l = 1$	$n = 0$
K	order	QNM		error	
0.5	7	0.211924−0.081163 i		0.00012081	
0.4	6	0.218115−0.0833272 i		0.000138212	
0.3	6	0.224714−0.0854211 i		0.000139989	
0.2	6	0.231868−0.0876587 i		0.000140546	
0.1	6	0.23966−0.0900574 i		0.000140646	
0.01	6	0.247302−0.0923703 i		0.000140889	
0.001	6	0.248102−0.0926102 i		0.00014239	
0	12	0.248238−0.0924756 i		0.0000353713	
−0.001	7	0.248204−0.0924564 i		0.000111401	
−0.01	7	0.249013−0.0926984 i		0.000124956	
−0.1	6	0.25759−0.0954213 i		0.000139933	
−0.2	6	0.268016−0.098438 i		0.000139713	
−0.3	6	0.279679−0.101721 i		0.000130778	
−0.4	6	0.292855−0.105308 i		0.000104688	
−0.5	6	0.307909−0.109249 i		0.000104718	

$M = 1$		$Q = 0$	$w = 2/3$	$l = 2$	$n = 0$
K	order	QNM		error	
0.5	7	0.392339−0.0835923 i		4.0429×10^{-6}	
0.4	7	0.403399−0.0855847 i		4.4118×10^{-6}	
0.3	7	0.415321−0.0877061 i		4.7699×10^{-6}	
0.2	6	0.428227−0.0899792 i		6.8520×10^{-6}	
0.1	6	0.442258−0.0924051 i		6.9393×10^{-6}	
0.01	7	0.455994−0.0947344 i		4.5315×10^{-6}	
0.001	7	0.457431−0.0949765 i		3.8757×10^{-6}	
0	12	0.457595−0.0950044 i		5.3497×10^{-8}	
−0.001	7	0.457752−0.0950305 i		4.0999×10^{-6}	
−0.01	6	0.459207−0.0952825 i		6.9498×10^{-6}	
−0.1	6	0.474454−0.0978204 i		7.0486×10^{-6}	
−0.2	6	0.493117−0.10086 i		7.4915×10^{-6}	
−0.3	6	0.51394−0.104162 i		7.5930×10^{-6}	
−0.4	6	0.537396−0.107765 i		9.0435×10^{-6}	
−0.5	6	0.564117−0.111712 i		6.4961×10^{-6}	

$M = 1$		$Q = 0$	$w = 3/2$	$l = 1$	$n = 0$
K	order	QNM		error	
0.5	6	0.239832−0.0932887 i		0.000169184	
0.4	6	0.24137−0.0932204 i		0.000111921	
0.3	6	0.242969−0.0931259 i		0.0000603321	
0.2	6	0.244635−0.093001 i		0.0000460723	
0.1	6	0.246374−0.0928403 i		0.0000889145	
0.01	6	0.248006−0.0926595 i		0.000137588	
0.001	6	0.248173−0.0926393 i		0.000139768	
0	12	0.248238−0.0924756 i		0.0000353713	
−0.001	6	0.24821−0.0926347 i		0.000141842	
−0.01	6	0.248378−0.092614 i		0.000152232	
−0.1	6	0.250095−0.0923828 i		0.000201455	
−0.2	6	0.252094−0.0920669 i		0.000262861	
−0.3	6	0.254196−0.0916759 i		0.000320562	
−0.4	6	0.256413−0.0911919 i		0.000378538	
−0.5	6	0.258755−0.0905914 i		0.000429459	

$M = 1$		$Q = 0$	$w = 2$	$l = 1$	$n = 0$
K	order	QNM		error	
0.5	6	0.244643−0.0937604 i		0.00022081	
0.4	7	0.245291−0.0935095 i		0.0000933026	
0.3	6	0.24601−0.0933782 i		0.000106356	
0.2	5	0.246669−0.0931748 i		0.0000326109	
0.1	6	0.247446−0.0929097 i		0.000104077	
0.01	7	0.248041−0.0924641 i		0.000115113	
0.001	6	0.248184−0.0926398 i		0.000151737	
0	12	0.248238−0.0924756 i		0.0000353713	
−0.001	6	0.248199−0.0926342 i		0.000142213	
−0.01	6	0.248267−0.0926082 i		0.000142792	
−0.1	6	0.248957−0.0923349 i		0.000190005	
−0.2	6	0.249742−0.092 i		0.000232034	
−0.3	6	0.250547−0.0916287 i		0.000271507	
−0.4	6	0.251372−0.0912167 i		0.000310425	
−0.5	6	0.252216−0.0907598 i		0.000338825	

$M = 1$		$Q = 0$	$K = 0.3$	$l = 1$	$n = 0$
w	Order	QNM		error	
$\frac{1}{2}$	7	0.215754−0.0803811 i		0.000115726	
$\frac{2}{3}$	6	0.224714−0.0854211 i		0.000139989	
$\frac{3}{4}$	7	0.22803−0.0869942 i		0.000123453	
1	6	0.235466−0.090695 i		0.000102452	
$\frac{4}{3}$	6	0.241171−0.0927225 i		0.0000350898	
$\frac{5}{4}$	6	0.242969−0.0931259 i		0.0000603321	
2	6	0.24601−0.0933782 i		0.000106356	
3	5	0.247912−0.0930989 i		0.000216339	
10	6	0.248191−0.0926367 i		0.000140674	

$M = 1$		$Q = 0$	$K = -0.3$	$l = 1$	$n = 0$
w	Order	QNM		error	
$\frac{1}{2}$	6	0.29199−0.108985 i		0.000168079	
$\frac{2}{3}$	6	0.279679−0.101721 i		0.000130778	
$\frac{3}{4}$	6	0.27484−0.099145 i		0.000153213	
1	6	0.264323−0.094383 i		0.000207927	
$\frac{4}{3}$	7	0.256381−0.0916834 i		0.00024382	
$\frac{5}{4}$	6	0.254196−0.0916759 i		0.000320562	
2	6	0.250547−0.0916287 i		0.000271507	
3	5	0.248793−0.0920096 i		0.000264998	
10	6	0.24819−0.0926376 i		0.000141573	

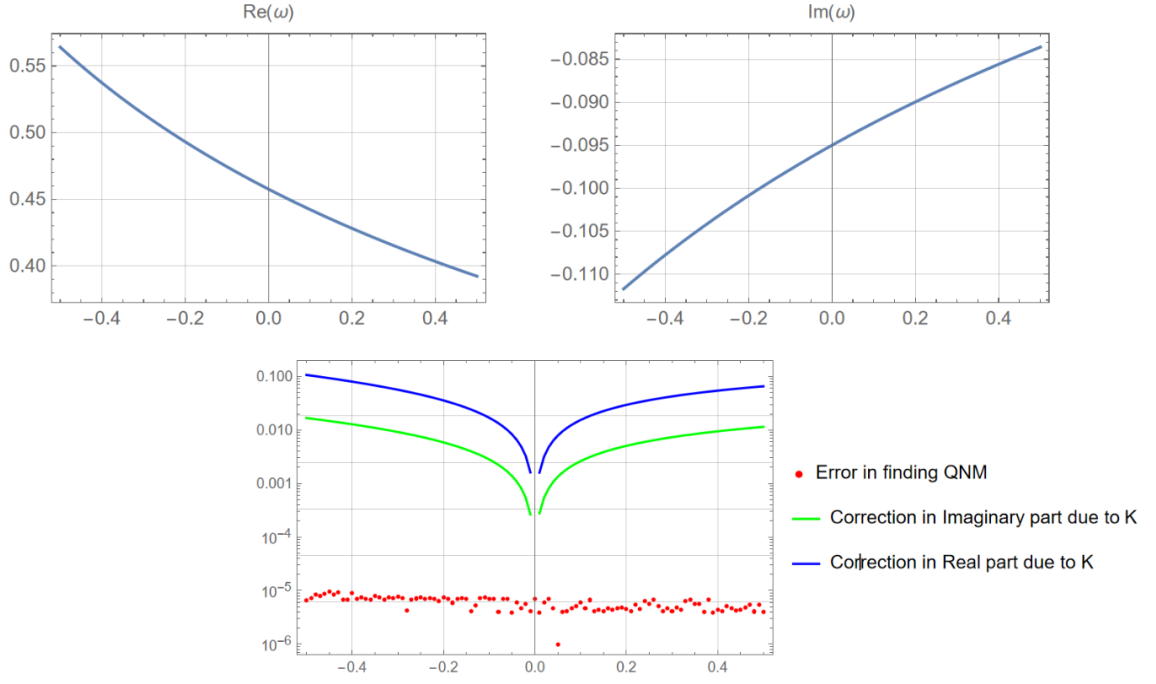


Figure 3.3: Variation of QNM due to ‘ K ’ with $w = 2/3$, $\ell = 2$ and $n = 0$

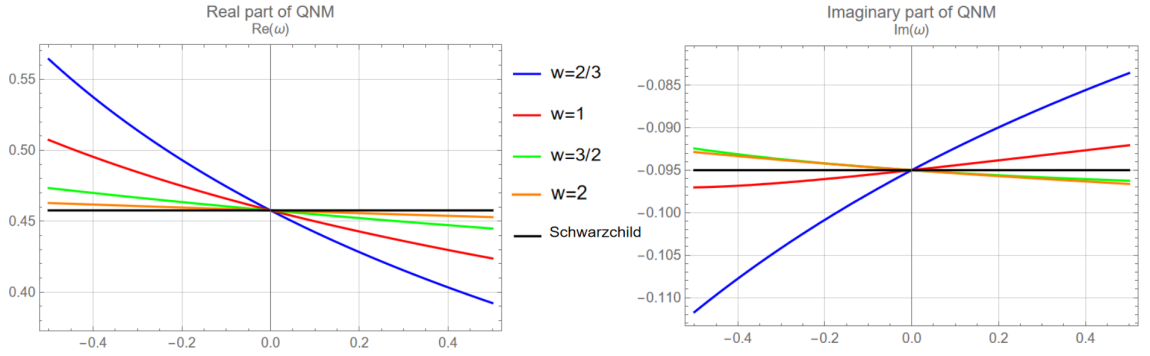


Figure 3.4: Variation of QNM due to ‘ K ’ with different w values for $\ell = 2$ and $n = 0$

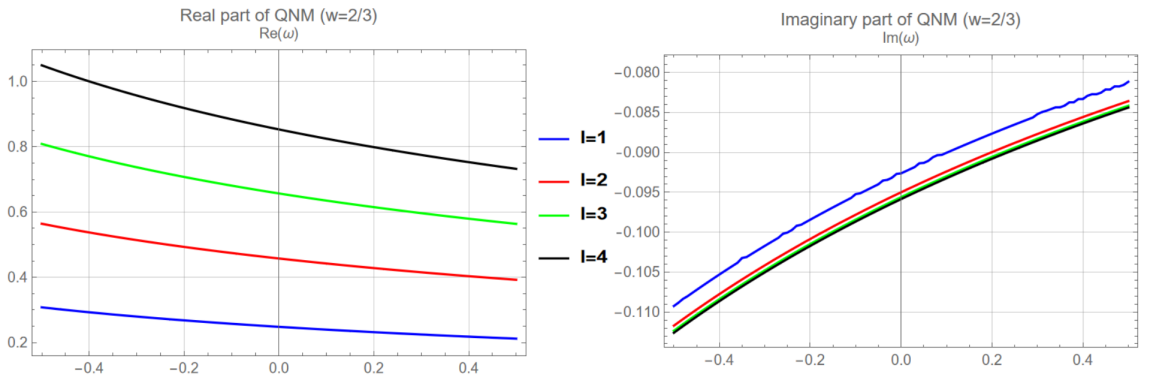


Figure 3.5: Variation of QNM due to ‘ K ’ with different ℓ values for $w = 2/3$ and $n = 0$

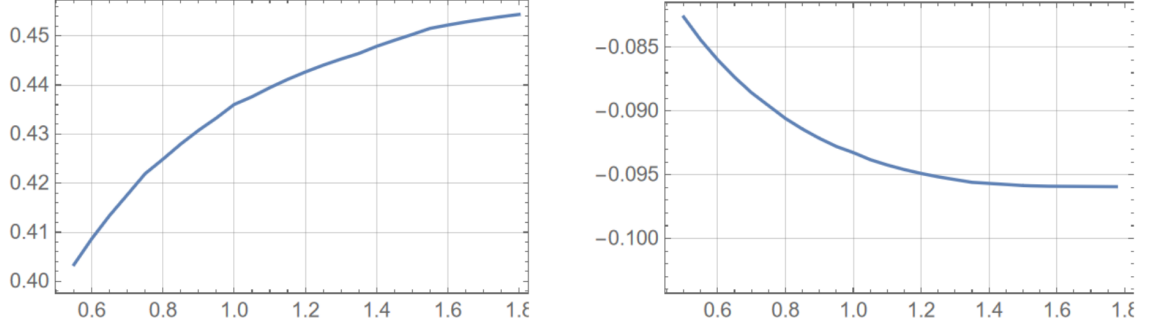


Figure 3.6: Variation of QNM due to ‘ w ’ with $\ell = 2$ and $n = 0$

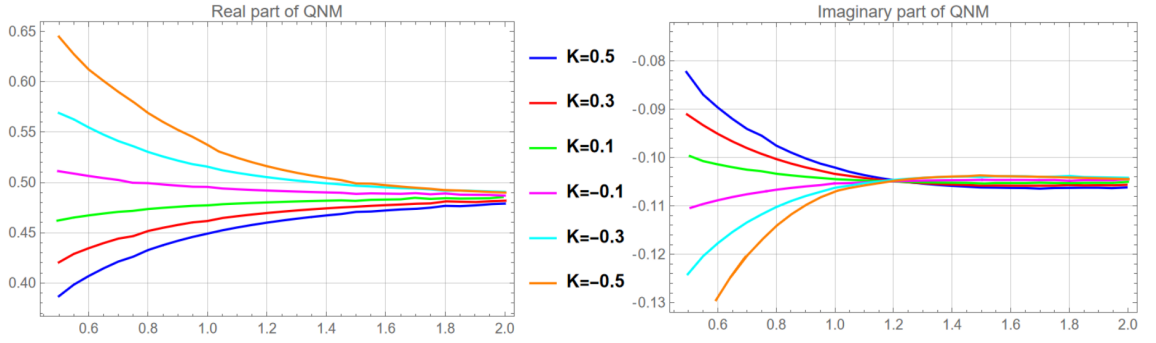


Figure 3.7: Variation of QNM due to ‘ w ’ with different K values for $\ell = 2$ and $n = 0$

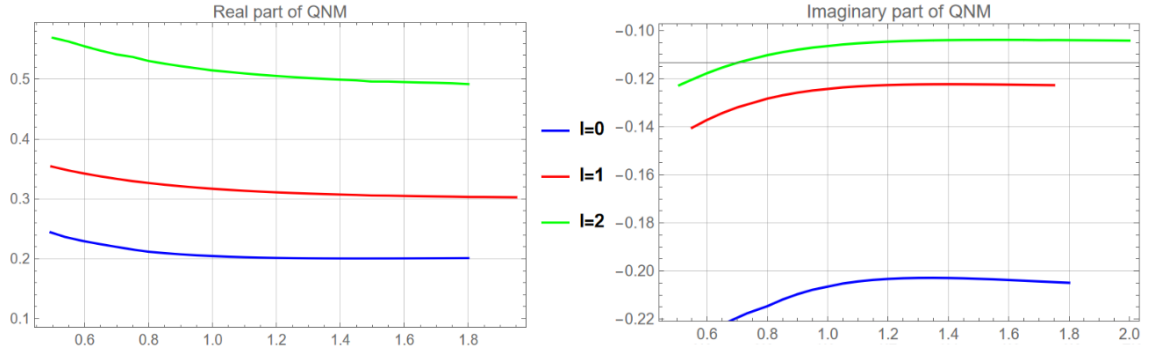


Figure 3.8: Variation of QNM due to ‘ w ’ with different $K = 0.3$ (left) and $K = -0.3$ (right) with different ℓ values and $n = 0$

Chapter 4

Gravitational Perturbation

4.1 Regge-Wheeler Formalism

Einstein's Field Equations are differential equations of the metric $[g_{\mu\nu}]$ where the curvature $G_{\mu\nu}$ of space-time is related to the mass-energy $[T_{\mu\nu}]$. They are [61, 62],

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (4.1)$$

In Regge-Wheeler (RW) formalism, we sought to perturb the space-time (gravity) itself by adding perturbations to the background metric $[g_{\mu\nu}^{(0)}]$ which results in,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \underbrace{\varepsilon h_{\mu\nu}^{(1)} + \varepsilon^2 h_{\mu\nu}^{(2)} + \dots}_{\text{perturbations}} \quad (4.2)$$

But we perturb the system very so slightly that $|\varepsilon| \ll 1$ and for orders higher than one the perturbation coefficients ε goes to zero, $\varepsilon^m \rightarrow 0$ if $m > 1$.

$$\Rightarrow g_{\mu\nu} = g_{\mu\nu}^{(0)} + \varepsilon h_{\mu\nu} \quad (4.3)$$

The background metric satisfies the background einstein equation given by,

$$R_{\mu\nu}^{(0)} - \frac{1}{2}g_{\mu\nu}^{(0)}R = 8\pi T_{\mu\nu}^{(0)} \quad (4.4)$$

After perturbation we get the perturbed Ricci Tensor as,

$$R_{\mu\nu} = R_{\mu\nu}^{(0)} + \varepsilon R_{\mu\nu}^{(1)} \quad (4.5)$$

and the perturbed Ricci Scalar as

$$R = R^{(0)} + \varepsilon R^{(1)} \quad (4.6)$$

But if we assume a static Energy-momentum tensor [69] of the surrounding matter then,

$$T_{\mu\nu} = T_{\mu\nu}^{(0)} \quad (4.7)$$

Then the Einstein equation for the 1st order perturbation correction satisfies,

$$R_{\mu\nu}^{(1)} - \frac{1}{2}[g_{\mu\nu}^{(0)}R^{(1)} + h_{\mu\nu}R^{(0)}] = 0 \quad (4.8)$$

In the Regge-Wheeler formalism, spherically symmetric tensorial perturbations $[h_{\mu\nu}]$ are split into odd wave (axial perturbations) and even wave (polar perturbations). The axial perturbations in RW canonical gauge are [45],

$$h_{\mu\nu\text{odd}} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} e^{-i\omega t} \sin(\theta) \frac{\partial}{\partial\theta} (P_l(\cos(\theta))) \quad (4.9)$$

$$h_{\mu\nu\text{even}} = \begin{pmatrix} f(r)H_0(r) & H_1(r) & 0 & 0 \\ H_1(r) & [f(r)]^{-1}H_2(r) & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)K(r) \end{pmatrix} e^{-i\omega t} P_l(\cos(\theta)) \quad (4.10)$$

Then the perturbed metric is given by,

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \varepsilon(h_{\mu\nu\text{Even}} + \eta h_{\mu\nu\text{Odd}}) \quad (4.11)$$

Here ε and η takes care of the odd and even perturbation terms. If we choose the background metric given in Eq.(1.31) and the Axial (odd) & Even (polar) perturbation given in Eq.(4.9) and Eq.(4.10) and by substituting these in Eq.(4.3) we get the resultant metric to be,

$$[g_{\mu\nu}] = \begin{bmatrix} -f(r)[1 - \varepsilon H_0(r)e^{-i\omega t}P_l] & \varepsilon H_1(r)e^{-i\omega t}P_l & 0 & \varepsilon \eta h_0(r)e^{-i\omega t}\sin(\theta)\frac{dP_l}{d\theta} \\ \varepsilon H_1(r)e^{-i\omega t}P_l & [f(r)]^{-1}(1 + \varepsilon H_2(r)e^{-i\omega t}P_l) & 0 & \varepsilon \eta h_1(r)e^{-i\omega t}\sin(\theta)\frac{dP_l}{d\theta} \\ 0 & 0 & r^2[1 + \varepsilon K(r)e^{-i\omega t}P_l] & 0 \\ \varepsilon \eta h_0(r)e^{-i\omega t}\sin(\theta)\frac{dP_l}{d\theta} & \varepsilon \eta h_1(r)e^{-i\omega t}\sin(\theta)\frac{dP_l}{d\theta} & 0 & r^2\sin^2(\theta)[1 + \varepsilon K(r)e^{-i\omega t}P_l] \end{bmatrix} \quad (4.12)$$

Then to get the inverse metric tensor components $[g^{\mu\nu}]$ we invert the metric tensor

treating it as a 4×4 matrix. Then the linearization operation where each component undergoes a series expansion in terms of ε and then the terms only up to the first order of ε survive. Then the affine metric connections $\Gamma_{\mu\nu}^\rho$ are found using the perturbed metric components by the below formula.

$$\Gamma_{\mu\nu}^\rho = \frac{g^{\rho\sigma}}{2} \left(\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right) \quad (4.13)$$

Then the Ricci tensor components are found using the below formula.

$$R_{\mu\nu} = \partial_\alpha \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\alpha}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta \quad (4.14)$$

The resulting Ricci Tensor components are of the form,

$$R_{\mu\nu} = R_{\mu\nu}^{(0)} + \varepsilon \left(R_{\mu\nu_{even}}^{(1)} + \eta R_{\mu\nu_{odd}}^{(1)} \right) \quad (4.15)$$

The resulting Ricci Scalar is of the form,

$$R = R^{(0)} + \varepsilon \left(R_{even}^{(1)} + \eta R_{odd}^{(1)} \right) \quad (4.16)$$

Then we collect all the odd perturbation terms and even perturbation terms separately to get the equation,

$$R_{\mu\nu_{odd}}^{(1)} - \frac{1}{2} [g_{\mu\nu}^{(0)} R_{odd}^{(1)} + h_{\mu\nu_{odd}} R^{(0)}] = 0 \quad (4.17)$$

$$R_{\mu\nu_{even}}^{(1)} - \frac{1}{2} [g_{\mu\nu}^{(0)} R_{even}^{(1)} + h_{\mu\nu_{even}} R^{(0)}] = 0 \quad (4.18)$$

Then by following the algorithm given by [43] and to remove the first derivative transform the radial coordinate $r \rightarrow r(\tau)$ and $\psi(r) \rightarrow \psi(r(\tau)) \rightarrow \psi(\tau)$ (i.e. in terms of the tortoise coordinate) as shown in Eq.(2.7) we can get effective potentials for the odd and even perturbations as follows,

$$V_{\text{Odd}}(r) = f(r) \left(\frac{l(l+1) + 2[f(r) - 1]}{r^2} + \frac{f'(r)}{r} + f''(r) \right) \quad (4.19)$$

$$V_{\text{Even}}(r) = \frac{f}{r^2 (3rf' + 2\sigma)^2} \left[(3rf' + 2\sigma) (r(rf'') (\sigma - 3rf') - 3f'(rf' - 2\sigma)) \right. \\ \left. + 4\sigma^2 \right) + f \left(3r \left(3r^3 (f'')^2 - 2r\sigma f'' + 8\sigma f' + 6r(f')^2 - r^2 f''' (3rf' + 2\sigma) \right) + 8\sigma^2 \right) \right]$$

where σ is given by the relation $\sigma = \frac{1}{2} (l-1)(l+2)$

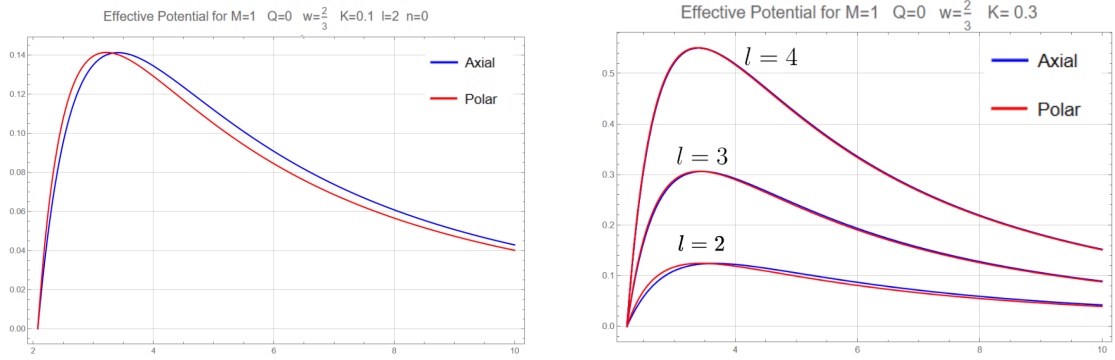


Figure 4.1: Axial and Polar effective potentials

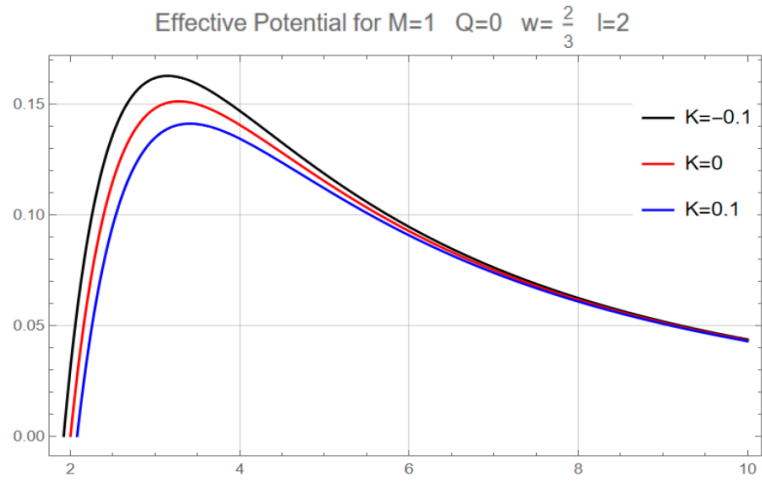


Figure 4.2: Symmetric splitting of Axial effective potential due to 'K'

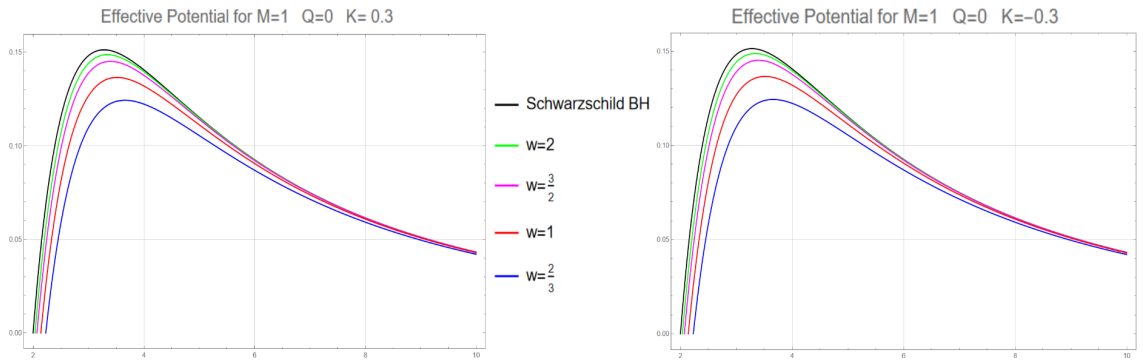


Figure 4.3: Asymmetric splitting of Axial effective potential due to 'w' with $K = 0.3$ (left) and $K = -0.3$ (right)

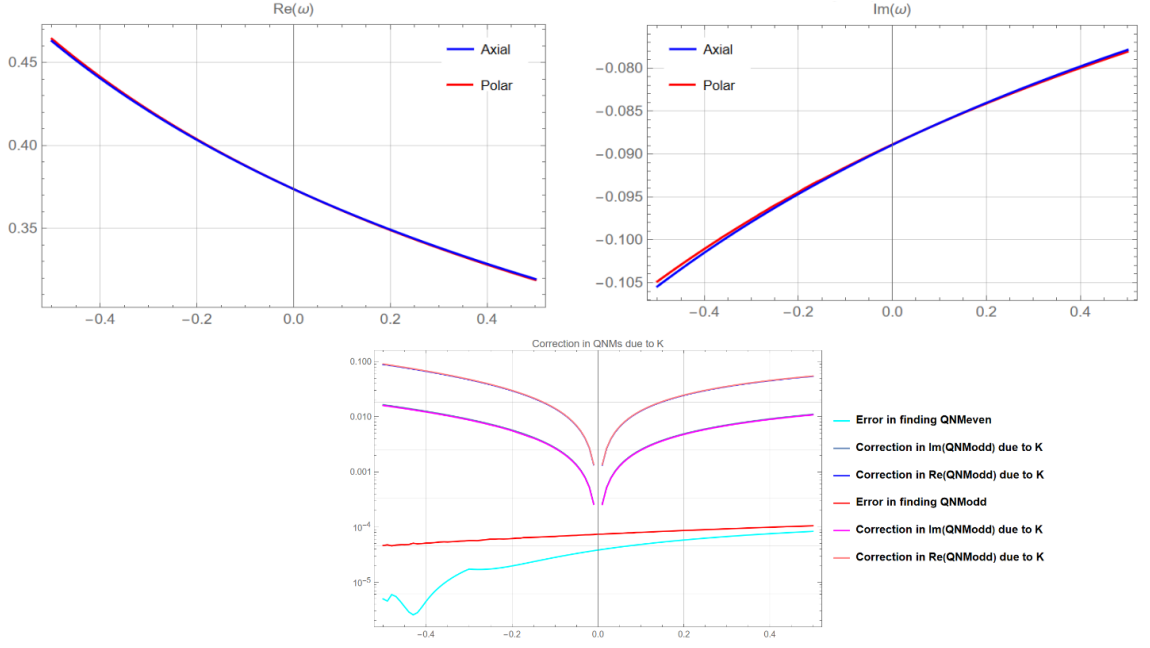


Figure 4.4: Variation of QNM due to 'K' with $w = 2/3$, $\ell = 2$ and $n = 0$

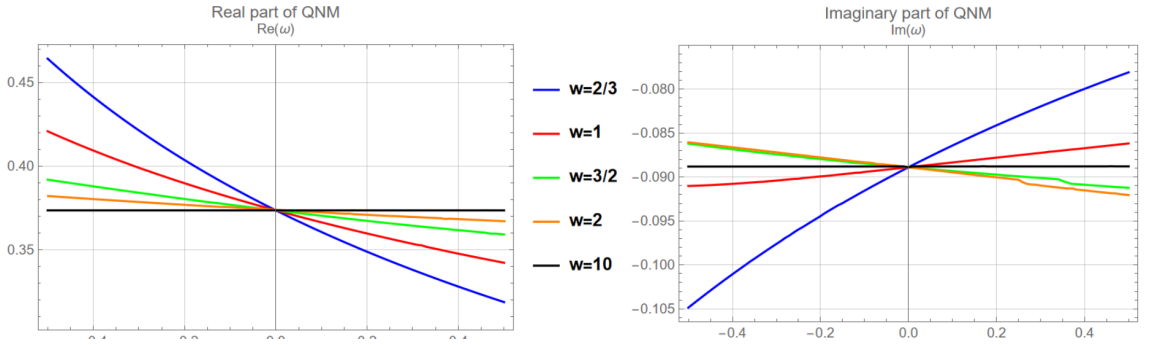


Figure 4.5: Variation of Axial QNM due to 'K' with different w values for $\ell = 2$ and $n = 0$

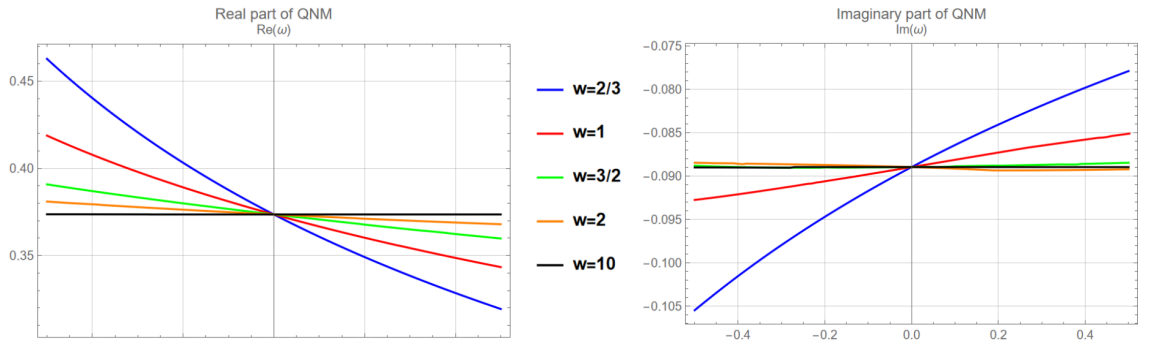


Figure 4.6: Variation of Polar QNM due to 'K' with different w values for $\ell = 2$ and $n = 0$

Chapter 5

Dynamics of Perturbations

5.1 Prony's method

From [70], the Prony method allows us to find z_i in terms of the known x_k , and to calculate out the quasinormal modes ω_i since v is also known. The dominant mode is then selected as the one that has the greatest amplitude C_i among the p modes we obtained. From [71] The fitting in the range formally gives $30 < t < 85$ gives two modes $\omega_0 = 4.14628 - 0.431359i$ and $\omega_1 = 6.29882 - 0.629939i$ which is the reflection of the two stages of oscillations.

5.2 Scattering and Greybody factors

In this section, we analyze Reflection coefficients $R(\omega)$ and Transmission coefficients $T(\omega)$ for various values of black hole parameters and in different types of perturbations. Radiation originating near a black hole event horizon is emitted into the surrounding space, necessitating traversal through a non-trivial, curved space-time geometry before detection by an observer, such as one situated at asymptotic infinity in an asymptotically flat spacetime. Consequently, the surrounding space-time acts as a potential barrier for the radiation, resulting in a deviation from the original radiation spectrum observed by an asymptotic observer. This deviation, quantified by the relative factor between the asymptotic radiation spectrum and the emitted radiation spectrum, is commonly referred to as the greybody factor. Understanding this concept is straightforward by an examination of the behavior of the effective potential. For both scalar and electromagnetic perturbations, a finite potential barrier exists between the event horizon and asymptotic infinity. As a result, any wave traversing black hole spacetime encounters these finite pos-

itive potential barriers as obstacles, leading to partial reflection and transmission of the wave. As demonstrated previously, the radial perturbation equations can be deduced to a Schrödinger-like equation, which characterizes the scattering of waves in black hole spacetimes. This equation yields asymptotic solutions,

$$\psi(\tau) = T(\omega)e^{-i\omega\tau} \quad \tau \rightarrow -\infty \quad (5.1)$$

$$\psi(\tau) = e^{-i\omega\tau} + R(\omega)e^{i\omega\tau} \quad \tau \rightarrow +\infty \quad (5.2)$$

Next, we proceed to compute the square of the amplitude of the wave function, which is partially transmitted and partially reflected by the potential barrier. The conservation of probability dictates that,

$$|R(\omega)|^2 + |T(\omega)|^2 = 1 \quad (5.3)$$

We employ the WKB approximation to calculate the transmission and reflection coefficients. Specifically, the reflection amplitude is expressed as [72],

$$R(\omega) = \frac{1}{\sqrt{1 + e^{-2\pi i\alpha}}}. \quad (5.4)$$

Considering higher-order WKB approximations, α in the above equation is [73],

$$\alpha = i \frac{\omega_n^2 - V(r_0)}{\sqrt{-2V''(r_0)}} - \sum_{i=2}^6 \Lambda_i \quad (5.5)$$

The greybody factor is defined as $\gamma_l = |T(\omega)|^2$. Substituting (5.3), we obtain,

$$\gamma_l = |T(\omega)|^2 = 1 - \left| \frac{1}{\sqrt{1 + e^{-2\pi i\alpha}}} \right|^2 \quad (5.6)$$

Depending on the frequency and height of the potential barrier, various scenarios may arise. When $\omega^2 \gg V(r_0)$, indicating that the frequency of the wave is significantly greater than the height of the barrier, reflection of the wave by the barrier does not occur. Consequently, one would anticipate the reflection coefficient to approach zero, as the wave frequency permits it to cross the barrier. Hence, under these conditions, $|T(\omega)|^2$ is expected to be close to 1. Conversely, when $\omega^2 \ll V(r_0)$, signifying that the frequency is much smaller than the barrier height, the wave will be reflected back by the barrier. Also, depending on the values of ω and $V(r_0)$, some portion of the wave may tunnel through the barrier.

Conclusion

We got the space-time metric for a charged black hole surrounded by an anisotropic matter field by solving Einstein's field equations with the negative radial pressure which is barotropic. Then we also perturbed this black hole three types, namely, by a massless scalar field, then by an Electromagnetic field, and finally, we perturbed the gravity itself by Regge-Wheeler formalism. In all these types of perturbation, the final perturbation equation was similar to a Schrodinger-like differential equation with appropriate boundary conditions. Where the eigenvalues which are the Quasinormal Mode frequencies were also calculated using optimal order WKB approximation. QNMs are complex numbers of the form $\omega_r + i\omega_i$ that gave information about the frequency of the gravitational wave being emitted ($\omega_r > 0$) and the decay constant of the gravitational wave ($\omega_i < 0$). When we add the spin, how the dynamics change will be my upcoming research. The trend shown by QNMs for all the type of perturbation is exactly the same that proves the fact that QNMs depend only on the BH property and not how it was perturbed.

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